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# Dynamical Continuous High Gain Observer For Sampled Measurements Systems

Cheikh A.B. Hann, Vincent Van Assche, Naveena Crasta, and Françoise Lamnabhi-Lagarigue

**Abstract**—This paper presents the design of an observer for a class of nonlinear Lipschitz sampled-data systems. The proposed observer uses a predictor of the output between the sampling times. This predictor is re-initialized at each sampling time. Besides, unlike the conventional high gain observer, the observer introduced herein has a dynamic observation gain. Using a Lyapunov approach, we will derive an explicit relation between the bound on maximum allowable sampling period and the parameters of the observer in order to guarantee an asymptotic convergence of the observation error.

## I. INTRODUCTION

The study of observers for continuous-time systems with sampled output has attracted a lot of attention during the last decades. For linear time invariant systems, the solution is more obvious since it is possible to use the available discrete model of the continuous time system in order to design a discrete time observer [1]. For other classes of systems, the exact discrete model is generally not available. Therefore, one can resort to the use of a consistent approximation (such Euler approximation) of the exact discrete-time model, see for instance [2], [3], [4] and [5], [6] for the control design. However, this method does not take into account the behavior of the system between the sampling instants. Furthermore, it just guarantees a semi-global practical stability of the observation error.

Another approach to circumvent the need of the exact discrete model is to use a hybrid observer. This observer is formed of two parts: a prediction part that consists of ‘copying’ system dynamics with no correction term between the sampling instants and an update part, at the sampling times, when the error between the system and the observer output is used to correct the estimate state trajectory. Most of this kind of observers is based on the extended Kalman filter techniques. Unlike the discrete time approach, this one provides in most cases global and asymptotic stability [7], [8], [9], [10].

Recently, in [11], the authors have proposed a novel hybrid observer that consists of continuous state estimation combined with an output predictor. One can point out that only the output predictor is updated at each sampling instant.

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Therefore, the estimate of the state is continuous. This approach has been applied to several classes of systems such linear detectable systems and triangular globally Lipschitz nonlinear systems. The exponential convergence of the observation error has been derived by using a small gain approach.

In the present work, we present a hybrid high gain observer for a class of nonlinear triangular systems with sampled measurements. There is a large literature concerning the design of high gain observers see for instance [12], [13], [14]. In [15], [16], dynamical high gain observers for continuous time systems are designed in order to address the issue of the sensitivity to measurements error inherent to this kind of observers. The observer proposed herein includes a dynamical observation gain and uses an output predictor that is updated at each sampling time. The asymptotic convergence of the observer is derived using, as in [17], a suitable Lyapunov Krasovskii function and an explicit relation between the bound on maximum allowable sampling period ( $\tau_{MASP}$ ) and the parameters of the observer is given.

The paper is organized as follows: Section II gives the preliminary definitions. Section III describes the class of considered systems. A dynamical high gain observer for nonlinear triangular systems with sampled measurements is designed in Section IV, the asymptotic convergence of the observation error is proved and an explicit bound on the maximum allowable sampling period ( $\tau_{MASP}$ ) depending on the observer parameters is given. In Section V, an academic example is provided in order to illustrate our result.

## II. NOMENCLATURE

Throughout this paper, the following mathematical notations are used. Let  $\mathbf{R} = (-\infty, +\infty)$ ,  $\mathbf{R}_+ = (0, +\infty)$ ,  $\mathbf{R}_0^+ = [0, +\infty)$ . The Euclidian norm is defined by  $|\cdot|$ . For  $p, q, n, m \in \mathbf{N}$ ,  $\mathbf{R}^{p \times q}$  represents the set of real matrices of order  $p \times q$  and  $\mathbf{I}_p$  stands for the identity matrix of order  $p \times p$ .  $D \stackrel{\text{def}}{=} \text{diag}\{d_1, \dots, d_n\}$ ,  $d_1, \dots, d_n \in \mathbb{R}$  denotes the diagonal matrix of size  $n$ . If  $P \in \mathbf{R}^{p \times p}$ ,  $P > 0$  means that  $P$  is positive definite. The notation  $|P|$  represents the  $L_2$ -norm of  $P$ . If  $\mathcal{X} \subset \mathbf{R}^{p \times q}$  and  $\mathcal{Y} \subset \mathbf{R}^{n \times m}$ ,  $\mathcal{C}(\mathcal{X}, \mathcal{Y})$  denotes the space of all continuous functions mapping  $\mathcal{X} \rightarrow \mathcal{Y}$ .  $\lambda_{\min}(P)$  (resp.  $\lambda_{\max}(P)$ ), for  $P \in \mathbf{R}^{p \times p}$ , is the minimal (resp. maximal) eigenvalue of  $P(t)$ ,  $\forall t \in \mathbf{R}_0^+$ . In all this study, the initial time is called  $t_0 \in \mathbf{R}_0^+$ .

### III. SYSTEM DESCRIPTION

We consider the following class of nonlinear systems described by

$$\begin{cases} \dot{x}(t) = Ax(t) + \phi(x(t), u) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where  $x := [x_1 \ \cdots \ x_n]^T \in \mathbb{R}^n$  is the state vector of the system, the input to the system  $u = [u_1 \ \cdots \ u_m]^T \in \Omega \subseteq \mathbb{R}^m$ , with  $\Omega$  a compact subset of  $\mathbb{R}^m$ .  $y(t) \in \mathbf{R}$  represents the output of the system.

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

$$\phi(x(t), u) := \begin{bmatrix} \phi_1(x_1, u) \\ \phi_2(x_1, x_2, u) \\ \vdots \\ \phi_n(x_1, \dots, x_n, u) \end{bmatrix} \in \mathbb{R}^n,$$

$$C := [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^n.$$

Furthermore, we make the following assumption.

*Assumption 1:* For every  $i \in \{1, \dots, n\}$ , the triangular functions  $\phi_i: \mathbb{R}^i \times \mathbb{R}^m \rightarrow \mathbb{R}$  are globally Lipschitz with a Lipschitz constant  $L \geq 0$ , that is: there exists  $L \geq 0$  such that for every  $x, z \in \mathbb{R}^n$  and  $u \in \Omega$ ,

$$|\phi(x, u) - \phi(z, u)| \leq L|x - z|. \quad (2)$$

We consider the output of the system only at the sampling time such that we have the sequence of the output  $\{y_{t_k}\}_{k \geq 0}$ , where  $\{t_k\}_{k \geq 0}$  represents a strictly increasing sequence that models the sampling instants, such that  $\lim_{k \rightarrow \infty} t_k = \infty$ . Besides the sampling period, either uniform or nonuniform, is such that  $0 < \tau = t_{k+1} - t_k \leq \tau_{MASP}$  for every  $k \in \mathbf{N}$ . The question is whether we can still preserve the convergence property of the observer when we only have the sequence  $\{y_{t_k}\}_{k \geq 0}$  as output of the system.

### IV. SAMPLED HIGH-GAIN OBSERVER

#### A. Observer structure

We consider a continuous-time adaptive high-gain observer for the system (1) without sampling given by

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \phi(\hat{x}(t), u) \\ \quad - \theta \Delta_\theta^{-1} K (C\hat{x}(t) - y(t)) \\ \hat{y}(t) = C\hat{x}(t) \\ \dot{\theta} = l(\theta) \quad \theta(t_0) > 1 \end{cases} \quad (3)$$

where  $\hat{x} \in \mathbb{R}^n$ ,  $\Delta_\theta = \text{diag}\{1, \theta^{-1}, \dots, \theta^{1-n}\}$ ,  $K \in \mathbb{R}^n$  is such that  $A - KC$  is Hurwitz,  $\theta \geq 1$ , and  $l: \mathbb{R} \rightarrow \mathbb{R}$  is sufficiently smooth function and is chosen such that  $\theta(t) > 1$  for all  $t \in \mathbf{R}_0^+$ .

Let us consider the following sampled-data observer with a high-gain continuous-time observer and an inter sample output predictor, that is,

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + \phi(\hat{x}(t), u) \\ \quad - \theta \Delta_\theta^{-1} K (C\hat{x}(t) - w(t)) \\ \dot{w}(t) = CA\hat{x}(t) + C\phi(\hat{x}(t), u) \\ w(t_{k+1}) = y(t_{k+1}) \\ \dot{\theta} = l(\theta) \quad \theta(t_0) > 1 \end{cases} \quad (4)$$

for every  $k \in \mathbf{N}$ . The output predictor is represented by  $w(t) \in \mathbf{R}$ .

Let us define the state error  $\tilde{x}(t) \stackrel{\text{def}}{=} \hat{x}(t) - x(t)$ . The time derivative of the state error is given by  $\dot{\tilde{x}}(t) = \dot{\hat{x}}(t) - \dot{x}(t)$ . Let  $e_w(t)$  denote the prediction error between the predictor and the output:  $e_w(t) = w(t) - y(t)$ . Using (1) and (4), we obtain:

$$\begin{aligned} \dot{\tilde{x}}(t) &= (A - \theta \Delta_\theta^{-1} KC) \tilde{x}(t) \\ &\quad + \phi(\hat{x}(t), u) - \phi(x(t), u) + \theta \Delta_\theta^{-1} K e_w(t) \end{aligned} \quad (5)$$

We define:  $\tilde{\phi}(x(t), \hat{x}(t), u) \stackrel{\text{def}}{=} \phi(\hat{x}(t), u) - \phi(x(t), u)$ .

Let us introduce the change of coordinate

$$z(t) \stackrel{\text{def}}{=} \frac{1}{\theta^b} \Delta_\theta \tilde{x}(t) \quad (6)$$

for some real constant  $b > 1$ , that will be defined further. We should point out that the statement  $\tilde{x}(t) = 0$  is equivalent to  $z(t) = 0$ . Thus, by deriving (6), we have:

$$\dot{z}(t) = \left( \frac{1}{\theta^b} \Delta_\theta \right) \dot{\tilde{x}}(t) + \frac{d}{dt} \left( \frac{1}{\theta^b} \Delta_\theta \right) \tilde{x}(t) \quad (7)$$

On one hand, we have:

$$\begin{aligned} \frac{1}{\theta^b} \Delta_\theta \dot{\tilde{x}}(t) &= \Delta_\theta (A - \theta \Delta_\theta^{-1} KC) \Delta_\theta^{-1} z(t) \\ &\quad + \frac{1}{\theta^b} \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) + \frac{1}{\theta^{b-1}} K e_w(t) \end{aligned} \quad (8)$$

We introduce these following identities:

$$\Delta_\theta A \Delta_\theta^{-1} = \theta A \quad \text{and} \quad C \Delta_\theta^{-1} = C \quad (9)$$

Using the above identities, we get:

$$\begin{aligned} \frac{1}{\theta^b} \Delta_\theta \dot{\tilde{x}}(t) &= \theta (A - KC) z(t) \\ &\quad + \frac{1}{\theta^b} \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) + \frac{1}{\theta^{b-1}} K e_w(t) \end{aligned} \quad (10)$$

On the other hand,

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{\theta^b} \Delta_\theta \right) \tilde{x}(t) &= \frac{d}{dt} \text{diag}\{\theta^{-b}, \theta^{-b-1}, \dots, \theta^{-b-n+1}\} \\ &= -\frac{\dot{\theta}}{\theta^{b+1}} \text{diag}\{b, b-1, \dots, b-n+1\} \\ &\quad \times \Delta_\theta \tilde{x}(t) \end{aligned} \quad (11)$$

$$\begin{aligned} &= -\frac{\dot{\theta}}{\theta^{b+1}} \text{diag}\{b, b-1, \dots, b-n+1\} \\ &\quad \times \Delta_\theta \tilde{x}(t) \end{aligned} \quad (12)$$

$$= -\frac{\dot{\theta}}{\theta} D \left( \frac{1}{\theta^b} \Delta_\theta \tilde{x}(t) \right) \quad (13)$$

where  $D \stackrel{\text{def}}{=} \text{diag}\{b, b+1, \dots, b+n-1\}$ . Note that  $D = b\mathbf{I}_n + E$ , where

$$E \stackrel{\text{def}}{=} \text{diag}\{0, 1, \dots, n-1\} \quad (14)$$

From (6) and (13), we deduce that:

$$\frac{d}{dt} \left( \frac{1}{\theta^b} \Delta_\theta \right) \tilde{x}(t) = -\frac{\dot{\theta}}{\theta} Dz(t) \quad (15)$$

Thus we have

$$\begin{aligned} \dot{z}(t) &= \theta(A - KC)z(t) + \frac{1}{\theta^b} \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) \\ &+ \frac{1}{\theta^{b-1}} Ke_w(t) - \frac{\dot{\theta}}{\theta} Dz(t) \end{aligned} \quad (16)$$

### B. Technical results

Now, we shall present some useful technical results that are necessary for our next result. We recall Jensen's Inequality [18, Prop B.8, pg. 316].

*Proposition 1:* For any constant matrix  $M \in \mathbf{R}^{m \times m}$ ,  $M = M^T > 0$ , scalar  $\gamma > 0$ , vector function  $\omega: [0, \gamma] \rightarrow \mathbf{R}^m$  such that the integrations concerned are well defined, then

$$\left( \int_0^\gamma \omega(s) ds \right)^T M \left( \int_0^\gamma \omega(s) ds \right) \leq \gamma \int_0^\gamma \omega(s)^T M \omega(s) ds. \quad (17)$$

*Lemma 1:* Consider the output of the system and the predictor as defined in (1) and (4), the prediction error  $e_w(t)$  can be written as follows:

$$|e_w(t)|^2 \leq \theta^{2b} (\theta + \sqrt{n}L)^2 \tau_{MASP} \int_{t-\tau_{MASP}}^t |z(s)|^2 ds \quad (18)$$

for every  $t \in [t_k, t_{k+1})$  and  $k \in \{1, \dots, s-1\}$ .

*Proof:* To prove this lemma, we first note that from (1) and (4), we have:

$$\dot{e}_w(t) = CA\tilde{x}(s) + C\tilde{\phi}(\hat{x}(s), x(s), u) \quad (19)$$

for  $t \in [t_k, t_{k+1})$ .

Therefore, by integrating the above equation between  $t_k$  and  $t$ , we get:

$$e_w(t) = \int_{t_k}^t CA\tilde{x}(s) + C\tilde{\phi}(\hat{x}(s), x(s), u) ds \quad (20)$$

and then, using the change of coordinates (6) and the identities (9), we obtain:

$$\begin{aligned} |e_w(t)| &\leq \int_{t_k}^t \left| C\Delta_\theta A \Delta_\theta^{-1} \theta^b z(s) \right. \\ &\quad \left. + C\Delta_\theta \tilde{\phi}(\hat{x}(s), x(s), u) \right| ds \end{aligned} \quad (21)$$

Thus, by triangle inequality and assumption 1, we can state that:

$$|e_w(t)| \leq \int_{t_k}^t \left| \theta^{b+1} CAz(s) + |\sqrt{n}L\theta^b z(s)| \right| ds \quad (22)$$

As a result, we get:

$$|e_w(t)| \leq \theta^b (\theta + \sqrt{n}L) \int_{t_k}^t |z(s)| ds \quad (23)$$

Since we have  $\tau_{MASP} \geq t_{k+1} - t_k$  for every  $k \in \{1, \dots, s-1\}$ , we can assert that:

$$|e_w(t)| \leq \theta^b (\theta + \sqrt{n}L) \int_{t-\tau_{MASP}}^t |z(s)| ds \quad (24)$$

Invoking Jensen's inequality, we have

$$\left( \int_{t-\tau_{MASP}}^t |z(s)| ds \right)^2 \leq \tau_{MASP} \int_{t-\tau_{MASP}}^t |z(s)|^2 ds \quad (25)$$

Hence, we come to the following conclusion:

$$|e_w(t)|^2 \leq \theta^{2b} (\theta + \sqrt{n}L)^2 \tau_{MASP} \int_{t-\tau_{MASP}}^t |z(s)|^2 ds \quad (26)$$

This completes the proof.  $\blacksquare$

### C. Stability Analysis

Inspired by the work of [19], let us choose the following Lyapunov function:

$$\begin{aligned} V(t, z(t)) &\stackrel{\text{def}}{=} z(t)^T Pz(t) \\ &+ \frac{1}{\tau_{MASP}} \int_{t-\tau_{MASP}}^t \int_s^t |z(\xi)|^2 d\xi ds \end{aligned} \quad (27)$$

where  $P$  is a symmetric positive definite matrix such that, for some real  $q \in (0, 1]$ ,  $q\mathbf{I}_n \leq P \leq \mathbf{I}_n$ . Furthermore,  $P$  satisfies the following inequality:

$$(A - KC)^T P + P(A - KC) \leq -\mu \mathbf{I}_n \quad (28)$$

for some  $\mu > 0$ .

We can remark that for all  $t \in \mathbf{R}_0^+$ ,  $V(t, 0) = 0$  and  $V(t, z(t)) > 0$  for all  $z(t) \neq 0_n$ .

Let us decompose  $V$  into two functions  $V_1$  and  $V_2$ :

$$V_1(z(t)) \stackrel{\text{def}}{=} z(t)^T Pz(t) \quad (29)$$

and

$$V_2(t, z(t)) \stackrel{\text{def}}{=} \frac{1}{\tau_{MASP}} \int_{t-\tau_{MASP}}^t \int_s^t |z(\xi)|^2 d\xi ds \quad (30)$$

We can assume that:

$$V(t_k, z(t_k)) = V(t_k^-, z(t_k^-)), \quad (31)$$

since  $z(t)$  and therefore  $V_1(z(t))$  and  $V_2(t, z(t))$  are continuous functions of the time  $t$ .

In order to demonstrate the asymptotic convergence of  $z(t)$  towards zero and thereby that of the observation error, it suffices now to show that  $\dot{V}(t, z(t)) \leq -W(z(t))$  for  $t \in [t_k, t_{k+1}) \forall k \in \mathbf{N}$ , where  $W: \mathbf{R}^n \rightarrow \mathbf{R}$  is a continuous and nonnegative definite function.

Now, let us consider  $t \in [t_k, t_{k+1})$ .

The time derivative of the Lyapunov function  $V(t, z(t))$  is given by  $\dot{V}(t, z(t)) = \dot{V}_1(z(t)) + \dot{V}_2(t, z(t))$ , where

$$\dot{V}_1(z(t)) = z(t)^T P\dot{z}(t) + \dot{z}(t)^T Pz(t) \quad (32)$$

$$\dot{V}_2(t, z(t)) = |z(t)|^2 - \frac{1}{\tau_{MASP}} \int_{t-\tau_{MASP}}^t |z(s)|^2 ds \quad (33)$$

Using (16), we can compute  $\dot{V}_1(z)$  so that we have:

$$\begin{aligned} \dot{V}_1 &= z(t)^T P \\ &\times \left[ \theta(A - KC)z(t) + \frac{1}{\theta^b} \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) \right. \\ &\quad \left. + \frac{1}{\theta^{b-1}} K e_w(t) - \frac{\dot{\theta}}{\theta} D z(t) \right] \\ &+ \left[ \theta(A - KC)z(t) + \frac{1}{\theta^b} \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) \right. \\ &\quad \left. + \frac{1}{\theta^{b-1}} K e_w(t) - \frac{\dot{\theta}}{\theta} D z(t) \right]^T P z(t) \end{aligned} \quad (34)$$

$$\begin{aligned} \dot{V}_1 &= \theta z(t)^T \left[ P(A - KC) + (A - KC)^T P \right] z(t) \\ &+ \frac{2}{\theta^b} z(t)^T P \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) \\ &+ \frac{2}{\theta^{b-1}} z(t)^T P K e_w(t) - 2 \frac{\dot{\theta}}{\theta} z(t)^T P D z(t) \end{aligned} \quad (35)$$

Using (28), we get:

$$\begin{aligned} \dot{V}_1 &\leq - \left( \theta \mu + 2b \frac{\dot{\theta}}{\theta} \right) z(t)^T P z(t) \\ &+ \frac{2}{\theta^b} z(t)^T P \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) \\ &+ \frac{2}{\theta^{b-1}} z(t)^T P K e_w(t) - 2 \frac{\dot{\theta}}{\theta} z(t)^T P E z(t) \end{aligned} \quad (36)$$

Besides  $b > 1$  is chosen such that it satisfies the following relation:

$$-bP \leq PE + EP \leq bP \quad (37)$$

from which it follows that

$$-2 \frac{\dot{\theta}}{\theta} z(t)^T P E z(t) \leq b \frac{|\dot{\theta}|}{\theta} z(t)^T P z(t) \quad (38)$$

Therefore, we have:

$$\begin{aligned} \dot{V}_1 &\leq - \left( \theta \mu + 2b \frac{\dot{\theta}}{\theta} - b \frac{|\dot{\theta}|}{\theta} \right) z(t)^T P z(t) \\ &+ \frac{2}{\theta^b} z(t)^T P \Delta_\theta \tilde{\phi}(x(t), \hat{x}(t), u) \\ &+ \frac{2}{\theta^{b-1}} z(t)^T P K e_w(t) \end{aligned} \quad (39)$$

From (6) and assumption (1), we can derive that:

$$\begin{aligned} \dot{V}_1 &\leq - \left( \mu \theta + 2b \frac{\dot{\theta}}{\theta} - b \frac{|\dot{\theta}|}{\theta} - 2\sqrt{n}L \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right) V_1 \\ &+ \frac{2}{\theta^{b-1}} z(t)^T P K e_w(t) \end{aligned} \quad (40)$$

Then, using Young inequality, we get:

$$\begin{aligned} \dot{V}_1 &\leq - \left( \mu \theta + 2b \frac{\dot{\theta}}{\theta} - b \frac{|\dot{\theta}|}{\theta} - 2\sqrt{n}L \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right) V_1 \\ &+ \frac{\mu \theta \lambda_{\min}(P)}{2} |z(t)|^2 \\ &+ \frac{2}{\mu \theta \lambda_{\min}(P)} \frac{\lambda_{\max}(P)^2}{\theta^{2b-2}} |K|^2 |e_w(t)|^2 \end{aligned} \quad (41)$$

It follows that:

$$\begin{aligned} \dot{V}_1 &\leq - \left( \mu \frac{\theta}{2} + 2b \frac{\dot{\theta}}{\theta} - b \frac{|\dot{\theta}|}{\theta} - 2\sqrt{n}L \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right) V_1 \\ &+ \frac{2}{\mu \theta^{2b-1}} \frac{\lambda_{\max}(P)^2}{\lambda_{\min}(P)} |K|^2 |e_w(t)|^2 \end{aligned} \quad (42)$$

Which, using lemma (1), leads us to

$$\begin{aligned} \dot{V}_1 &\leq - \left( \mu \frac{\theta}{2} + 2b \frac{\dot{\theta}}{\theta} - b \frac{|\dot{\theta}|}{\theta} - 2\sqrt{n}L \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \right) V_1 \\ &+ \tau_{MASP} \frac{2\theta \lambda_{\max}(P)^2}{\mu \lambda_{\min}(P)} |K|^2 \\ &\times (\theta + \sqrt{n}L)^2 \int_{t-\tau_{MASP}}^t |z(s)|^2 ds \end{aligned} \quad (43)$$

From (43) and (33), we can deduce that:

$$\begin{aligned} \dot{V} &\leq - \left( \mu \frac{\theta}{2} + 2b \frac{\dot{\theta}}{\theta} - b \frac{|\dot{\theta}|}{\theta} - \frac{2\sqrt{n}L \lambda_{\max}(P) + 1}{\lambda_{\min}(P)} \right) V_1 \\ &- \left( \frac{1}{\tau_{MASP}} - \tau_{MASP} \frac{2\theta \lambda_{\max}(P)^2}{\mu \lambda_{\min}(P)} |K|^2 \right. \\ &\quad \left. \times (\theta + \sqrt{n}L)^2 \right) \int_{t-\tau_{MASP}}^t |z(s)|^2 ds. \end{aligned} \quad (44)$$

We can choose the maximum allowable sampling period  $\tau_{MASP}$  such that:

$$\tau_{MASP} \leq \frac{\sqrt{\mu \lambda_{\min}(P)}}{\sqrt{2\theta_{\max} \lambda_{\max}(P)} |K| (\theta_{\max} + \sqrt{n}L)}, \quad (45)$$

where  $\theta_{\max}$  is the maximum value of  $\theta$  for all  $t \in \mathbf{R}_+$ .

Thus, we obtain:

$$\dot{V} \leq - \left( \mu \frac{\theta}{2} + 2b \frac{\dot{\theta}}{\theta} - b \frac{|\dot{\theta}|}{\theta} - \frac{2\sqrt{n}L \lambda_{\max}(P) + 1}{\lambda_{\min}(P)} \right) V_1 \quad (46)$$

Following [20], let us choose  $\dot{\theta}$  as:

$$\dot{\theta} = -\frac{1}{b} \theta \left[ \frac{\mu}{6} (\theta - 1) - \frac{2\sqrt{n} \lambda_{\max}(P) L + 1}{\lambda_{\min}(P)} \right]. \quad (47)$$

Therefore, we can deduce from (45), (46), and (47) that:

$$\dot{V} \leq -\frac{\mu}{2} V_1 \quad (48)$$

which completes the proof.

*Remark 1:* Choosing  $\dot{\theta}$  as in (47), let us consider the two following cases: First, if  $\dot{\theta} > 0$ , we have:

$$\dot{V} \leq - \left( \mu \frac{\theta}{2} + b \frac{\dot{\theta}}{\theta} - \frac{2\sqrt{n}L\lambda_{\max}(P) + 1}{\lambda_{\min}(P)} \right) V_1 \quad (49)$$

$$\leq -\frac{\mu}{6} (2\theta + 1) V_1 \quad (50)$$

$$\leq -\frac{\mu}{2} V_1, \quad (51)$$

since  $\theta(t_0) > 1$  and  $\theta$  is an increasing function of the time ( $\dot{\theta} > 0$ ).

Now we consider the case when  $\dot{\theta} < 0$ , we get:

$$\dot{V} \leq - \left( \mu \frac{\theta}{2} + 3b \frac{\dot{\theta}}{\theta} - \frac{2\sqrt{n}L\lambda_{\max}(P) + 1}{\lambda_{\min}(P)} \right) V_1 \quad (52)$$

$$\leq - \left( \frac{\mu}{2} - \frac{4\sqrt{n}L\lambda_{\max}(P) + 2}{\lambda_{\min}(P)} \right) V_1 \quad (53)$$

$$\leq -\frac{\mu}{2} V_1, \quad (54)$$

since  $\frac{4\sqrt{n}L\lambda_{\max}(P) + 2}{\lambda_{\min}(P)} > 0$ .

Furthermore, we can remark for every given  $\theta(t_0) \in \mathbf{R}$ , the final value of  $\theta$  remains  $\theta_{\infty} := 1 + \frac{12\sqrt{n}\lambda_{\max}(P)L + 6}{\mu\lambda_{\min}(P)}$ . Besides,  $\theta_{\max}$  and  $\theta_{\min}$  are such that:

$$\theta_{\max} = \begin{cases} \theta(t_0), & \text{for } \theta(t_0) \geq \theta_{\infty} \\ \theta_{\infty}, & \text{for } \theta(t_0) < \theta_{\infty} \end{cases} \quad (55)$$

$$\theta_{\min} = \begin{cases} \theta_{\infty}, & \text{for } \theta(t_0) \geq \theta_{\infty} \\ \theta(t_0), & \text{for } \theta(t_0) < \theta_{\infty} \end{cases} \quad (56)$$

$$\theta_{\min} = \begin{cases} \theta_{\infty}, & \text{for } \theta(t_0) \geq \theta_{\infty} \\ \theta(t_0), & \text{for } \theta(t_0) < \theta_{\infty} \end{cases} \quad (57)$$

$$\theta_{\min} = \begin{cases} \theta_{\infty}, & \text{for } \theta(t_0) \geq \theta_{\infty} \\ \theta(t_0), & \text{for } \theta(t_0) < \theta_{\infty} \end{cases} \quad (58)$$

It should be noted that for  $\theta(t_0) > 1$ , we have  $\theta(t) > 1$  for all  $t > t_0$ .

## V. APPLICATION

Consider the nonlinear dynamical system described by

$$\begin{aligned} \dot{x}_1 &= x_2 + \sin x_1 \\ \dot{x}_2 &= -x_1 \\ y &= x_1 \end{aligned} \quad (59)$$

where the state  $x \in \mathbb{R}^2$  and output  $y \in \mathbb{R}$ . Note that  $(A, C)$  is observable and hence there exists  $K = [k_1 \ k_2]^T \in \mathbb{R}^2$  such that

$$A - KC = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}$$

is Hurwitz, where  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $C = [1 \ 0]$ .

The sampled data dynamical high gain observer is given by

$$\begin{aligned} \dot{\hat{x}}_1(t) &= \hat{x}_2(t) + \sin \hat{x}_1(t) - \theta k_1(\hat{x}_1(t) - w(t)) \\ \dot{\hat{x}}_2(t) &= -\hat{x}_1(t) - \theta^2 k_2(\hat{x}_1(t) - w(t)), \\ \dot{w}(t) &= \hat{x}_2(t) + \sin \hat{x}_1(t) \quad t \in [t_k, t_{k+1}), \\ w(t_{i+1}) &= x_1(t_{k+1}), \\ \dot{\theta} &= -\frac{1}{b} \theta \left[ \frac{\mu}{6} (\theta - 1) - \frac{2\sqrt{n}\lambda_{\max}(P)L + 1}{\lambda_{\min}(P)} \right]. \end{aligned}$$

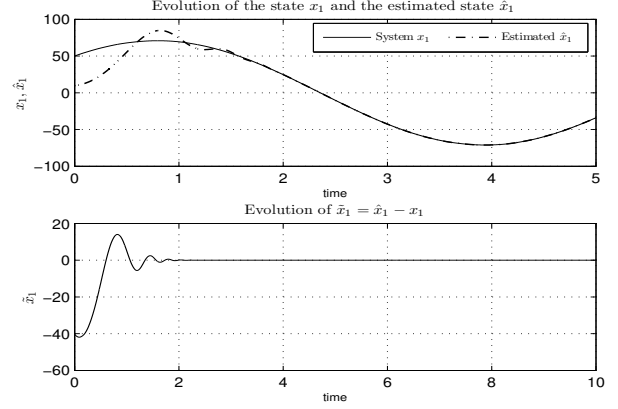


Fig. 1. Evolution of the state  $x_1$  and its estimate  $\hat{x}_1$

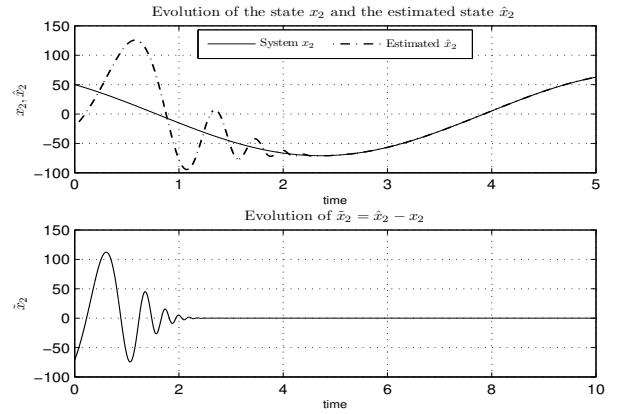


Fig. 2. Evolution of the state  $x_2$  and its estimate  $\hat{x}_2$

The simulations have been performed with the initial conditions of the system chosen as:  $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 50 \\ 50 \end{pmatrix}$ . Note that the Lipschitz constant  $L = 2$ . The results presented below are obtained by fixing the design parameters of the observer (4) at:

$$\begin{aligned} \hat{x}_0 &= [10 \quad -20]^T, \\ K &= [0.5 \quad 1.2]^T, \\ w_0 &= \hat{x}_{2_0} + \sin(\hat{x}_{1_0}) = -20.5440, \\ b &= 8, \\ P &= \begin{bmatrix} 0.8360 & -0.1900 \\ -0.1900 & 0.7758 \end{bmatrix} \end{aligned}$$

In order to show the convergence of the observer, the following simulations display the evolution of the system for a sampling period chosen constant and equal to  $\tau = 0.005$ .

## VI. CONCLUSION

We have presented a hybrid high gain observer with dynamical observation gain for a class of continuous nonlinear triangular sampled data systems. This observer uses an inter-sample output predictor. Using a Lyapunov approach, an

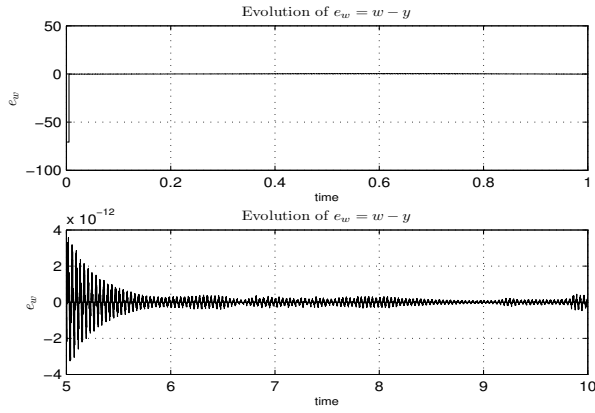


Fig. 3. Evolution of the prediction error

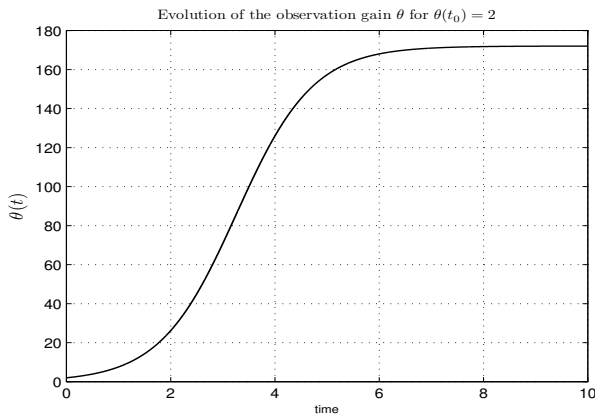


Fig. 4. Evolution of the dynamic observation gain

explicit relation between the maximum allowable sampling time and the parameters of the proposed observer is given in order to ensure the asymptotic convergence of the observation error towards zero. An example is also presented to illustrate this result.

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