The Geometry of Cauchy's Fluxes

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Dedicated to the memory of my grandfather
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Abstract

A formulation of the Cauchy theory for balance laws of scalar valued quantities is considered from a general geometric point of view. It is assumed only that the ambient space is an orientable m-dimensional manifold. The analog of the usual flux vector field is an $(m-1)$-differential form. Both the Cauchy theorem and differential version of the balance laws are formulated in this context.

1. Introduction

The category of differentiable manifolds and differentiable mappings is a framework wherein one should be able to formulate continuum mechanics. As early as 1957, a relatively short time after differentiable manifolds had been introduced by Hassler Whitney, NOLL [11], used them in his definition of bodies. While Noll noted that a 3-dimensional manifold representing a body has a very simple topology – it may be covered by a single chart, neither a metric nor a connection can be naturally assigned to it. The definition of bodies as general manifolds whose charts are provided by the configurations, or placements, in a 3-dimensional Euclidean space representing the physical space, is repeated in [12] and [13].

Nevertheless, Cauchy's theory of fluxes and forces, has been formulated traditionally using a particular configuration of the body – a chart in the geometrical language (e.g., [16] and [7]). As such, the fluxes and stresses resulting from the theory are not invariant entities and do not correspond to the body itself. This is analogous to the situation where one knows the transformation rules for the components of stresses without knowing they represent a linear mapping. A generalization of Cauchy's theory to differentiable manifolds will render such arbitrary choice unnecessary. For example, the Cauchy flux and the Piola-Kirchhoff flux will be two representations of the same geometrical object.

In recent decades, the method of continuum mechanics was extended to media with microstructure, or media having an internal state specified by some order

parameters. Various phases of liquid crystals, continua with voids, liquids with bubbles, continua with spin (two- or three-dimensional), and Cosserat continua are but examples of structured media (see [10, 1, 9]). The $\mathbb{R}^n$-valued order parameters are in many cases physical quantities that do not have an invariant geometric meaning. They are coordinates on a differentiable manifold (see *loc. cit.*). Thus, in order to apply the method of continuum mechanics to such structured continua, the basic notions of continuum mechanics should be generalized to manifold geometry (see [2, 3] for such attempts).

In addition to the extension to structured continua, the methods of continuum mechanics have been applied in recent years by GURTIN (e.g., [4]) to two-dimensional manifolds representing the interfaces in multiphase bodies. Again, these interfaces are generalized geometrical objects – shapes.

This paper considers the generalization of Cauchy's theorem for fluxes to differentiable manifolds. Such a generalization seems to be a natural step towards a better understanding of the foundations of continuum mechanics of simple bodies, and an essential step needed for the application of the continuum mechanics method to generalized media such as described above.

Cauchy's theorem for fluxes considers a balance equation of a property P in the form

$$P(\mathcal{R}) = \int_{\mathcal{R}} b_{\mathcal{R}}\, dv + \int_{\partial\mathcal{R}} t_{\mathcal{R}}\, da = \int_{\mathcal{R}} p_{\mathcal{R}}\, dv,$$

where $\mathcal{R}$ is a region in $\mathbb{R}^3$, $b_{\mathcal{R}}$ is interpreted as the rate at which the density of the property changes, $t_{\mathcal{R}}$ is interpreted as the density with respect to the surface area at which the property leaves the region at the boundary, and $p_{\mathcal{R}}$ is interpreted as the density of the sources producing P. Specifically, the theory is concerned with the way the fields $b_{\mathcal{R}}$, $t_{\mathcal{R}}$ and $p_{\mathcal{R}}$ depend on the the subregion $\mathcal{R}$. The basic assumption made, Cauchy's postulate, is that the fields $b_{\mathcal{R}}$, $p_{\mathcal{R}}$ do not depend on the region $\mathcal{R}$ and that the field $t_{\mathcal{R}}$ depends on $\mathcal{R}$ only through the normal to the boundary. This is usually written as $t_{\mathcal{R}} = t(\mathbf{n})$. In addition, one assumes regularity of the density fields and regularity of the dependence of $t_{\mathcal{R}}$ on $\mathbf{n}$. The resulting Cauchy's theorem states that with these assumptions, there is a unique vector field v such that

$$t = v \cdot \mathbf{n}.$$

Thus, the balance of the scalar property together with Cauchy's postulate generate a vector field. This is how, for example, a balance of heat will result in the existence of the heat flux vector field. Note that we consider here only scalar valued quantities.

Over the years, numerous attempts have been made to make the Cauchy theory rigorous and to generalize the framework in which it applies. Following Hamel, NOLL [11], (see [16] pp. 156–172) has replaced the assumption for the dependence of $t_{\mathcal{R}}$ on the normal by assumptions of boundedness of the interaction forces – the Hamel-Noll theorem. GURTIN & WILLIAMS [6] gave an axiomatic framework to the Cauchy theory and GURTIN & MARTINS [5] relaxed some of the regularity assumptions for the fields. MARSDEN & HUGHES [9] proved the theorem for 3-dimensional metric manifolds. SEGEV [14] gives a general formulation on manifolds based on a global weak setting for the balance law and SEGEV & DE BOTTON [15]

give, in the geometrical setting of $\mathbb{R}^3$, a detailed analysis of the weak framework where the fluxes (stresses in that paper) may be as irregular as measures. In these weak formulations, fluxes appear as measures representing linear functionals and the geometric construction of Cauchy's theorem, in which a balance of the property is applied to a tetrahedron, is avoided. In another approach to the proof, FOSDICK & VIRGA [8] prove the theorem using a variational approach.

Here, Cauchy's theory is formulated in the general setting of an m-dimensional orientable manifold $\mathcal{S}$ without any further geometrical structure. The analytic side of the theory is traditional and the fields are assumed to be smooth. The formulation and proof are analogous to the Cauchy construction and the classical tetrahedron is replaced by a simplex. The basic idea is to replace the densities by differential forms. Thus, the flux density $t_{\mathcal{R}}$ is replaced by an $(m-1)$-form $\tau_{\mathcal{R}}$ on the boundary of the region whose integral gives the total flux. It is noted that at any point on the boundary, an $(m-1)$-dimensional manifold, the value of such an $(m-1)$-form is an element of a 1-dimensional vector space. Similarly, the flux field replacing the vector field v, is an $(m-1)$-form σ on the m-dimensional ambient manifold. Roughly, the flux field form, whose value at a point has m components, is generated by using the components of the flux density forms on m hyperplanes – $(m-1)$-dimensional subspaces of the tangent space to $\mathcal{S}$.

Assuming that an orientation is given in $\mathcal{S}$, the equation $t = v \cdot \mathbf{n}$ assumes the form

$$\tau = \mathcal{I}^*(\sigma),$$

where $\mathcal{I}^*$ is the restriction of $(m-1)$-forms on $T\mathcal{S}$ to forms on the tangent bundle of the boundary. The differential version of the balance law $\operatorname{div} v + b = p$ takes the form

$$d\sigma + \beta = \pi,$$

where β and π are the m-forms corresponding to the densities b and p above and $d\sigma$ is the exterior derivative of σ, an m-form.

2. Integral Balance Laws for Scalars

We consider an ambient orientable manifold $\mathcal{S}$ where the balance laws are to be formulated. The ambient manifold may be thought of as either the space manifold of continuum mechanics where physical phenomena take place or as the material manifold containing the material points. The following formulation is independent of the dimension of $\mathcal{S}$ (assuming it is finite) and we use m to denote it. Compact m-dimensional submanifolds with corners of $\mathcal{S}$ will be referred to as *regions*. Regions may be thought of as either control volumes or bodies according to the interpretation of $\mathcal{S}$.

It is assumed that for every region $\mathcal{R}$ one is given an m-form $\beta_{\mathcal{R}}$ on $\mathcal{R}$, an $(m-1)$-form $\tau_{\mathcal{R}}$ on $\partial\mathcal{R}$, an m-form $\pi_{\mathcal{R}}$ on $\mathcal{R}$, and balance law in the form

$$\int_{\mathcal{R}} \beta_{\mathcal{R}} + \int_{\partial\mathcal{R}} \tau_{\mathcal{R}} = \int_{\mathcal{R}} \pi_{\mathcal{R}}.$$

The form $\beta_{\mathcal{R}}$ is interpreted as the rate of change of the density of an extensive property whose balance we are considering. Thus, at each point $x \in \mathcal{R}$, $\beta_{\mathcal{R}}(x)$ is a completely anti-symmetric m-linear mapping of tangent vectors at x, whose evaluation $(\beta_{\mathcal{R}})(x)(v_1, v_2, \ldots, v_m)$ on m independent tangent vectors $v_1, v_2, \ldots, v_m$, represents the rate of change of the property in the infinitesimal region defined by the m vectors. Hence, the first integral above is interpreted as the rate of change of the total measure of the property enclosed within the region $\mathcal{R}$. The form $\tau_{\mathcal{R}}$ is interpreted as the density of the rate at which the property under consideration is leaving $\mathcal{R}$ through its boundary. Thus, for each $x \in \partial\mathcal{R}$, $\tau_{\mathcal{R}}(x)$ is an anti-symmetric $(m-1)$-linear form whose evaluation $\tau_{\mathcal{R}}(v_1, v_2, \ldots, v_{m-1})$ on $m-1$ vectors $v_1, v_2, \ldots, v_{m-1}$ that are tangent at x to the boundary $\partial\mathcal{R}$, may be thought of as the rate at which the property is leaving the region through the infinitesimal boundary element determined by the tangent vectors. Thus, the second integral is the total rate at which the property leaves the region through its boundary. The form $\pi_{\mathcal{R}}$ is interpreted as the density at which the property is being produced inside $\mathcal{R}$, and hence the integral on the right-hand side is the total production rate of the property. The balance law states that the production rate is balanced by the rate of change of the total measure of the property and the rate at which the property leaves at the boundary.

3. The Generalized Cauchy Postulates

The generalized Cauchy postulates restrict the dependence of the forms $\beta_{\mathcal{R}}$, $\tau_{\mathcal{R}}$, and $\pi_{\mathcal{R}}$ on $\mathcal{R}$. The following assumption are made.

GC1 The values of the forms $\beta_{\mathcal{R}}$ and $\pi_{\mathcal{R}}$ at any point $x \in \mathcal{S}$ do not depend on $\mathcal{R}$. Thus, we will omit the $\mathcal{R}$ index in what follows.

GC2 The value of $\tau_{\mathcal{R}}$ at any point $x \in \partial\mathcal{R}$ depends on the region $\mathcal{R}$ only through its tangent space at x including its (inwards versus outwards) orientation. That is, if $T_x\mathcal{R}_1 = T_x\mathcal{R}_2$ for the two regions $\mathcal{R}_1, \mathcal{R}_2$ whose boundaries contain x that are situated on the same side of the common tangent space, then

$$\tau_{\mathcal{R}_1}(x) = \tau_{\mathcal{R}_2}(x).$$

In order to specify the dependence of $\tau_{\mathcal{R}}$ on $\mathcal{R}$ explicitly, the previous assumption is reformulated. Below we will refer to an $(m-1)$-dimensional subspace H of $T_x\mathcal{S}$ as a *hyperplane* and use $\mathcal{I}_H : H \hookrightarrow T_x\mathcal{S}$ to denote its inclusion in $T_x\mathcal{S}$. The dual mapping $\mathcal{I}_H^*$ is the restriction of forms on $T_x\mathcal{S}$ to forms on H. Each hyperplane H defines a one-dimensional subspace $H^+$of the dual space $T_x^*\mathcal{S}$ containing the annihilators of H. An orientation on H, relative to an orientation on $T_x\mathcal{S}$ as determined by an m-form ω, is induced by a choice of a half space $H^\perp$ of H^+. The orientation of an $(m-1)$-form ω_H on H is that of $\phi \wedge \omega_H$ for any form $\phi \in H^\perp$. If the condition holds, one says that ω_H is *positively oriented* with respect to ω. The form ϕ can be thought of as giving positive values to vectors pointing "outwards".

Clearly, by normalizing ϕ (say by using a metric in a neighborhood of x) the collection of oriented hyperplanes at any $x \in \mathcal{S}$ may be identified with the $(m-1)$-sphere. (This is in contrast with the regular construction of the projective spaces

where orientation is ignored.) Thus, we have the *bundle of oriented hyperplanes*

$$G_{m-1}^{\perp}\mathcal{S} \to \mathcal{S},$$

whose fibers are diffeomorphic to the $(m-1)$-sphere and any particular element is an equivalence class of 1-forms (under multiplication by a positive number). Moreover, on each fiber there is an operation of orientation inversion corresponding to multiplication by a negative number.

We may associate the vector space $\bigwedge^{m-1} H^*$ of $(m-1)$-forms on H with any oriented hyperplane $H \in G_{m-1}^{\perp}$. Thus, we have a vector bundle

$$\bigwedge G_{m-1}^{\perp}\mathcal{S} \to G_{m-1}^{\perp}\mathcal{S},$$

whose fiber over H is $\bigwedge^{m-1} H^*$.

Using this notation we may reformulate GC2 as follows.

GC2' There is a section

$$\tau\colon G_{m-1}^{\perp}\mathcal{S} \to \bigwedge G_{m-1}^{\perp}\mathcal{S},$$

such that for each $\mathcal{R}$

$$\tau_{\mathcal{R}}(x) = \tau(T_x\partial\mathcal{R}) \in \bigwedge^{m-1} (T_x^*\partial\mathcal{R}).$$

GC3 The section τ is C^r for some integer $r \geqq 0$.

4. The Generalized Cauchy Theorem

Let $-H$ denote the subspace of inverse orientation to that of H so that if $\phi \in H^{\perp}$ represents H, then $a\phi,\ a < 0$, represents $-H$.

Proposition 4.1. $\tau(-H) = -\tau(H)$.

Proof. Let H be an oriented hyperplane at $x_0 \in \mathcal{S}$ defined by a form $\phi \in H^{\perp}$. Let x^i be a coordinate system in a neighborhood of x such that ϕ is represented by dx^1 and $x^i(x_0) = 0$ for all $i = 1, \ldots, m$. For $t > 0$ we consider a singular cube $c\colon [0,1]^m \to \mathcal{S}$ such that $c(0) = x_0$ and $c(z^1, \ldots, z^m)$ is represented by $(t^2 z^1, tz^2, tz^3, \ldots, tz^m)$. We note that the volume integrals over the image(c) are of order t^{m+1}, the flux over the faces where $z^i = 0$ or $z^i = 1$, $i > 1$, are of order t^m, and the flux over the faces where $z^1 = 0$ or $z^1 = 1$ are of order t^{m-1}. Thus, denoting the image of the cube by $\mathcal{R}$, the balance law implies

$$\lim_{t\to 0}\left\{\frac{1}{t^{m-1}}\left(\int_{z^1=0} \tau_{\mathcal{R}} + \int_{z^1=1} \tau_{\mathcal{R}}\right)\right\} = 0.$$

We may use $\boldsymbol{\tau}_0$, $\boldsymbol{\tau}_1$ to denote the local representatives in the $\{x^i\}$ coordinates of $\tau_{\mathcal{R}}$ on the faces $x^1 = 0$ and $x^1 = t^2$ respectively, so that we have

$$\lim_{t\to 0}\left\{\frac{1}{t^{m-1}}\left(\int\limits_{z^1=0}\boldsymbol{\tau}_0 + \int\limits_{z^1=1}\boldsymbol{\tau}_1\right)\right\} = 0.$$

By the mean value theorem for integrals, for each value of t there are points $q_{t,0},\ q_{t,1}$ on the faces $x^1 = 0$ and $x^1 = t^2$ respectively, such that

$$\int\limits_{z^1=0}\boldsymbol{\tau}_0 = t^{m-1}\boldsymbol{\tau}_0(q_{t,0}), \qquad \int\limits_{z^1=1}\boldsymbol{\tau}_1 = t^{m-1}\boldsymbol{\tau}_1(q_{t,1}).$$

Thus,

$$\lim_{t\to 0}\left\{\boldsymbol{\tau}_0(q_{t,0}) + \boldsymbol{\tau}_1(q_{t,1})\right\} = 0.$$

Now GC2' allows us to write the last limit in the form

$$\lim_{t\to 0}\left\{\boldsymbol{\tau}(\phi_{t,0}) + \boldsymbol{\tau}(\phi_{t,1})\right\} = 0,$$

where $\boldsymbol{\tau}$ is the local representative of the section τ and $\phi_{t,0} \in (T_{q_{t,0}}\partial\mathcal{R})^{\perp}$, $\phi_{t,1} \in (T_{q_{t,1}}\partial\mathcal{R})^{\perp}$ are the forms representing the oriented hypeplanes tangent to the boundary of $\mathcal{R}$ at the points $q_{t,0},\ q_{t,1}$ respectively. However, as $t \to 0$, $\phi_{t,0} \to -\phi$ and $\tau_{t,1} \to \phi$ as both points approach x_0. The assertion follows now from GC3. □

We will say that a collection $H_1, \ldots, H_j$ of oriented hyperplanes are independent if any collection of forms $\{e^i \in H_i^{\perp}\}$ are linearly independent in $T_x^*\mathcal{S}$. Clearly, the particular choice of each form e^i among the annihilators of H_i will not affect the independence or dependence of the hyperplanes. In particular, if we are given m hyperplanes $H_1, \ldots, H_m$ the corresponding forms $e^1, \ldots, e^m$ generate a basis for $T_x^*\mathcal{S}$ and a corresponding (dual) basis $e_1, \ldots, e_m$ for $T_x\mathcal{S}$. Given i, since $e^j(e_i) = \delta_i^j$, $e_j \in H_i$ for all $j \neq i$, and $e_i \notin H_i$. Thus, $\{e_1, \ldots, \widehat{e_i}, \ldots, e_m\}$ form a basis for H_i, where the "hat" denotes an omitted element.

Since $\bigwedge^{m-1} H_i^*$ is one dimensional, we may write

$$\tau_i = \tau(H_i) = \tau_i(e_1, \ldots, \widehat{e_i}, \ldots, e_m)e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^m,$$

where, $\tau_i(e_1, \ldots, \widehat{e_i}, \ldots, e_m)$ is the single component of τ_i with respect to the given basis. Below we will use the notation $\tau_{\widehat{i}} = \tau_i(e_1, \ldots, \widehat{e_i}, \ldots, e_m)$

Proposition 4.2. *Let $H_1, \ldots, H_m$ be m independent oriented hyperplanes of $T_x\mathcal{S}$, and for each i, let*

$$\tau_i = \tau(H_i) \in \bigwedge^{m-1} H_i^*$$

be the corresponding flux density. Then, there exists a unique $(m-1)$-form

$$\sigma \in \bigwedge^{m-1}(T_x^*\mathcal{S})$$

such that

$$\tau_i = \mathcal{I}_i^*(\sigma), \quad \text{for all } i = 1, \ldots, m.$$

Proof. Given the oriented hyperplanes we can form bases $\{e_1, \ldots, e_m\}$ of $T_x\mathcal{S}$, and $\{e^1, \ldots, e^m\}$ of $T_x^*\mathcal{S}$. We recall that $\bigwedge^{m-1}(T_x^*\mathcal{S})$ is m-dimensional and any form in it may be written as

$$\sigma = \sum_{i=1}^{m} \sigma_i \, e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^m,$$

and

$$\sigma_i = \sigma(e_1, \ldots, \widehat{e_i}, \ldots, e_m).$$

Noting that each term $\sigma_i \, e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^m$ in the sum above is an $(m-1)$-form on H_i, we may set

$$\sigma_i = \tau_i(e_1, \ldots, \widehat{e_i}, \ldots, e_m) = \tau_{\widehat{\imath}} \qquad \text{for all } i = 1, \ldots, m$$

so

$$\sigma = \sum_{i=1}^{m} \tau_{\widehat{\imath}} \, e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^m.$$

The relation

$$\phi^1 \wedge \ldots \wedge \phi^n(v_1, \ldots, v_n) = \det\left[\phi^i(v_j)\right],$$

for any two collections of n 1-forms ϕ^i and n vectors v_i, implies that for any collection $v_1 \ldots v_{m-1} \in T_x\mathcal{S}$,

$$\begin{aligned} e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^m(v_1, \ldots, v_{m-1}) &= \varepsilon^{1\ldots\widehat{i}\ldots m}_{i_1\ldots i_{m-1}} (v_1)^{i_1} \ldots (v_{m-1})^{i_{m-1}} \\ &= \det \begin{pmatrix} (v_1)^1 & (v_2)^1 & \ldots & (v_{m-1})^1 \\ \vdots & \vdots & \vdots & \vdots \\ (\widehat{v_1})^i & (\widehat{v_2})^i & \ldots & (\widehat{v_{m-1}})^i \\ \vdots & \vdots & \vdots & \vdots \\ (v_1)^m & (v_2)^m & \ldots & (v_{m-1})^m \end{pmatrix}, \end{aligned}$$

where $(v_j)^k$ denotes the k-th component of v_j with respect to the basis $\{e_k\}$. In particular,

$$e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^m(e_1, \ldots, \widehat{e_j}, \ldots, e_m) = \delta^i_j.$$

Thus,

$$\sigma(v_1, \ldots, v_{m-1}) = \sum_{i=1}^{m} \varepsilon^{1\ldots\widehat{i}\ldots m}_{i_1\ldots i_{m-1}} \tau_{\widehat{\imath}} (v_1)^{i_1} \ldots (v_{m-1})^{i_{m-1}}.$$

To show that σ restricts to τ_i on H_i one has only to use the fact that $\bigwedge^{m-1} H_i^*$ is 1-dimensional and show that $\sigma(e_1, \ldots, \widehat{e_i}, \ldots, e_m) = \tau_{\widehat{\imath}}$. It is a simple calculation to show that picking another basis comprising annihilators of the given hyperplanes while retaining their orientations will result in the same form σ. $\square$

Remark 4.3. In the last proposition, the form σ will reverse its sign if we change the orientation of any one element e_j of the basis $\{e_i\}$ to $e'_j = -e_j$. Thus, we consider another basis $\{e'_i\}$ such that $e'_i = e_i$ for all $i \neq j$, and $e'_j = -e_j$. In the expression for

$$\sigma' = \sum_{i'=1}^{m} \tau'_i(e'_1, \ldots, \widehat{e}'_i, \ldots, e'_m) e'^1 \wedge \cdots \wedge \widehat{e'^i} \wedge \cdots \wedge e'^m,$$

the term containing $\tau'_j(e'_1, \ldots, \widehat{e}'_j, \ldots, e'_m)$ will reverse its sign because $\tau(-\phi) = -\tau(\phi)$ implies

$$\begin{aligned}\tau'_j(e'_1, \ldots, \widehat{e'_j}, \ldots, e'_m) e'^1 \wedge \cdots \wedge \widehat{e'^j} \wedge \cdots \wedge e'^m \\ = -\tau_j(e_1, \ldots, \widehat{e}_j, \ldots, e_m) e^1 \wedge \cdots \wedge \widehat{e}^j \wedge \cdots \wedge e^m,\end{aligned}$$

while for $i \neq j$, the τ'_i will not be affected. In addition, for $i \neq j$, the terms

$$e'^1 \wedge \cdots \wedge \widehat{e'^i} \wedge \cdots \wedge e'^m = -e^1 \wedge \cdots \wedge \widehat{e}^i \wedge \cdots \wedge e^m$$

since they contain $e'^j = -e^j$.

If an orientation of $T_x\mathcal{S}$ is given by an m-form ω, then the basis $\{e_i\}$ may be either positively or negatively oriented relative to that orientation according to the sign of $\omega(e_1, \ldots, e_m)$. Thus, if we fix the m hyperplanes but vary their orientations, those having positive orientations with respect to ω will determine a preferred "sign" for σ. In other words, an orientation of $T_x\mathcal{S}$ fixes a "sign" for σ. Henceforth, we assume that an orientation is given on $\mathcal{S}$. An oriented hyperplane will have the orientation induced by the orientation of $T_x\mathcal{S}$ and the form $\phi \in H^{\perp}$. It is noted that the assumption that the basis $\{e_i\}$ is positively oriented also implies that we can use $e^1 \wedge \ldots \wedge e^m$ instead of the form ω.

Remark 4.4. In what follows, we will refer to the form σ as the *flux field form*. Once σ is given, it is possible to restrict it to any non-oriented hyperplane $H \subset T_x\mathcal{S}$ to obtain

$$\tau_H = \mathcal{I}_H^*(\sigma) \in \bigwedge^{m-1} H^*.$$

The expression for $\sigma(v_1, \ldots, v_{m-1})$ implies that

$$\tau_H(v_1, \ldots, v_{m-1}) = \sum_{i=1}^{m} \varepsilon^{1 \ldots \widehat{i} \ldots m}_{i_1 \ldots i_{m-1}} \tau_{\widehat{i}}\, (v_1)^{i_1} \ldots (v_{m-1})^{i_{m-1}}.$$

However, in the Cauchy theory one has to take into account the orientation of the hyperplane. Thus, given σ we set its oriented restriction to an oriented hyperplane H to be given by the above relation if the vectors $v_1, \ldots, v_{m-1}$ are positively oriented (with respect to the orientation induced on H), and to be given by the inverse of the relation if the vectors are negatively oriented. Hence,

Definition 1. Let $T_x\mathcal{S}$ be oriented by a form ω. Given an oriented hyperplane H specified by a 1-form ϕ, the *oriented restriction*

$$\mathcal{I}_H^{*\perp}\colon \bigwedge^{m-1}(T_x\mathcal{S})^* \to \bigwedge^{m-1} H^*$$

is given by

$$\mathcal{I}_H^{*\perp}(\sigma)(v_1,\ldots,v_{m-1}) = \operatorname{sign}\{\omega(\mathcal{I}_H(v_0),\mathcal{I}_H(v_1),\ldots,\mathcal{I}_H(v_{m-1}))\}\mathcal{I}_H^*(\sigma),$$

where v_0 is any outwards pointing vector, i.e., $\phi(v_0) > 0$.

Remark 4.5. With the previous definition the flux field form σ induces flux density τ_H on any oriented hyperplane by

$$\tau_H = \mathcal{I}_H^{*\perp}(\sigma)$$

and explicitly

$$\tau_H(v_1,\ldots,v_{m-1}) = \operatorname{sign}\{\omega(v_0,v_1,\ldots,v_{m-1})\}\sum_{i=1}^{m}\varepsilon^{1\ldots\hat{i}\ldots m}_{i_1\ldots i_{m-1}}\tau_{\hat{i}}(v_1)^{i_1}\ldots(v_{m-1})^{i_{m-1}}.$$

Remark 4.6. Ignoring momentarily the complications due to orientation, the foregoing construction may also be described in the language of multi-vectors. Recall that the evaluation of a form σ to a collection $v_1,\ldots,v_{m-1}$ of vectors can be replaced by the evaluation of the form on the exterior product of the vectors, the multi-vector, $v_1\wedge\ldots\wedge v_{m-1}$. In addition, given the basis $\{e_k\}$, the multi-vectors $e_1\wedge\ldots\wedge\widehat{e_i}\wedge\ldots\wedge e_m$ form a basis of vector space $\bigwedge^{m-1}T_x\mathcal{S}$. Hence, in the dual space $(\bigwedge^{m-1}T_x\mathcal{S})^* = \bigwedge^{m-1}T_x^*\mathcal{S}$ we can express $(m-1)$-forms as linear combinations of elements

$$e^1\wedge\ldots\wedge\widehat{e^i}\wedge\ldots\wedge e^m \in \bigwedge^{m-1}H_i^* = \left(\bigwedge^{m-1}H_i\right)^*$$

of the dual space.

Thus, the construction of σ using the various τ_i is simply the construction of an element in $(\bigwedge^{m-1}T_x\mathcal{S})^*$ using its components.

Remark 4.7. Clearly, the above construction is not limited to the form constructed in the proposition. Given an $(m-1)$ differential form σ, one can assign to any region $\mathcal{R}$ a flux density form

$$\tau_{\mathcal{R}} = \mathcal{I}_{T(\partial\mathcal{R})}^{*\perp}(\sigma),$$

where $\mathcal{I}_{T(\partial\mathcal{R})}\colon T(\partial\mathcal{R}) \hookrightarrow T\mathcal{S}$ is the inclusion.

Next we consider the question of consistency, i.e., whether the value $\tau(H)$ for any hyperplane H may be obtained using σ as above.

Definition 2. The section

$$\tau : G_{m-1}^{\perp}\mathcal{S} \to \bigwedge G_{m-1}^{\perp}\mathcal{S}$$

is *consistent* if there is an $(m-1)$-form $\sigma \in \bigwedge^{m-1} T^*\mathcal{S}$ such that

$$\tau(H) = \mathcal{I}_H^{*\perp}(\sigma)$$

for all $H \in G_{m-1}\mathcal{S}$.

Remark 4.8. By Proposition 4.2, if τ is consistent, then the form σ that satisfies the condition of the definition is unique.

Proposition 4.9. *The set function τ is consistent if* GC1 *and* GC3 *hold.*

Proof. Consider an arbitrary hyperplane H at a point $x \in \mathcal{S}$ determined together with its orientation by the form ϕ, and let $\tau_H = \tau(H)$ be the corresponding flux density. Choose any collection of m independent oriented hyperplanes $\{H_i\}$ and let $\{\phi^i\}$ be corresponding annihilators. For simplicity, we assume that ϕ is linearly independent of any $m-1$ sub-collection of the ϕ^is. Choose a basis of $T_x\mathcal{S}$ using the following procedure. Let e_1 be any vector satisfying $\phi^j(e_1) = 0$ for all $j \neq 1$. For $j \neq 1$ determine e_j by the m equations $\phi^i(e_j) = 0$ for all $i \neq j$ and $\phi(e_j - e_1) = 0$. Thus, e_i is on the intersection $H_1 \cap \ldots \cap \widehat{H_i} \cap \ldots \cap H_m$ and $v_{i-1} = e_i - e_1 \in H$ for all $i \neq 1$. The m-simplex constructed is analogous to the traditional tetrahedron used in the proof of Cauchy's theorem.

Without loss of generality we may assume that the basis $\{e_i\}$ and the vectors $\{v_j\}$ are positively oriented (or otherwise we can calculate $\tau(-H)$ and use Proposition 4.1).

By Remark 4.7 and the fact that $\bigwedge^{m-1} H^*$ is one-dimensional, we have to show that GC1 and GC3 imply that

$$\tau_H(v_1, \ldots, v_{m-1}) = \sum_{i=1}^{m} \varepsilon_{i_1 \ldots i_{m-1}}^{1 \ldots \widehat{i} \ldots m} \tau_{\widehat{i}}\,(v_1)^{i_1} \ldots (v_{m-1})^{i_{m-1}}$$

for one collection of $m-1$ linearly independent vectors $v_1, \ldots, v_{m-1} \in H$. In particular, we can use the collection of vectors as defined above. The construction of the vectors $\{v_k\}$ and basis $\{e_j\}$ implies that $(v_k)^j = \delta_k^{j+1} - \delta_1^j$ so the determinants

in the sum above satisfy

$$\varepsilon^{1\ldots\widehat{i}\ldots m}_{i_1\ldots i_{m-1}}(v_1)^{i_1}\ldots(v_{m-1})^{i_{m-1}} = \begin{vmatrix} & & & & & \overbrace{}^{i} & & & \\ -1 & -1 & -1 & \ldots & -1 & -1 & \ldots & \ldots & -1 \\ 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \ldots & \ddots & \vdots & \vdots & \ldots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots \end{vmatrix}$$

$$= (-1)^{i-1}\begin{vmatrix} -1 & -1 & -1 & \ldots & -1 \\ 0 & 1 & 0 & \ldots & \ldots \\ 0 & 0 & 1 & \ldots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}$$

$$= (-1)^{i}.$$

Hence, we have to show that

$$\tau_H(v_1,\ldots,v_{m-1}) = \sum_{i=1}^{m}(-1)^i\,\tau_{\widehat{\tau}}.$$

We now construct a family of simplexes as follows. Consider a chart $\{z_k\}$ in a neighborhood of x such that $z_k(x) = 0$ and $e_k = \frac{\partial}{\partial z^k}$ so the base vectors are tangent to the coordinate curves at x. For some positive t_0, the collection of points whose coordinates satisfy $|z^k| \leqq t_0$ are contained in the coordinate neighborhood. For $0 < t < t_0$ let $\mathcal{R}_t$ be the region containing the points whose coordinates satisfy $t \geqq z^k \geqq 0$ and $\sum_{k=1}^m z^k \leqq t$. Thus, $\mathcal{R}_t$ is the image of the "shrunk" standard simplex $\Delta^t_p = \{0 \leqq x^i \leqq t,\ \sum_{i=1}^p \leqq t\}$ under the embedding $s\colon \Delta^t_p \to \mathcal{S}$ that is represented locally by $z^i(x^1,\ldots,x^m) = x^i$.

Denote by $S_i,\ i = 1,\ldots,m$, the face containing points whose i-th coordinates vanish and by S_0 the face containing the points with $\sum_{k=1}^m z^k = t$.

By the definition of the integral of a form over a chain and the definition of a boundary of a simplex,

$$\int\limits_{\partial\mathcal{R}_t}\tau_{\mathcal{R}_t} = \sum_{k=0}^{m}(-1)^k\int\limits_{s\circ\Phi_k}\tau_{\mathcal{R}_t},$$

where Φ_i is the standard linear embedding of the $m-1$ standard simplex in the i-th face of the m-standard simplex.

Let $\{\mathbf{e}_i\}$ be the standard basis of $\mathbb{R}^m$ and Ts the tangent mapping to s. By Lemma 4.10, Corollary 4.11, Remark 4.12 and the construction of the various vectors, for each value of t and each $k = 0, \dots, m$, there are points $q_t^k \in S_k$ such that

$$\begin{aligned}\frac{1}{A_{m-1}t^{m-1}} \int_{S_k} \tau_{\mathcal{R}_t} &= \tau_{\mathcal{R}_t}(q_t^k)\Big(T_{s^{-1}(q_t^k)}s(\mathbf{e}_1), \dots, \\ &\qquad \dots, \widehat{T_{s^{-1}(q_t^k)}s(\mathbf{e}_k)}, \dots, T_{s^{-1}(q_t^k)}s(\mathbf{e}_m)\Big) \\ &= \tau_{\mathcal{R}_t}(q_t^k)(e_1, \dots, \widehat{e_k}, \dots, e_m) \quad \text{for } k \neq 0,\end{aligned}$$

while

$$\begin{aligned}\frac{1}{A_{m-1}t^{m-1}} \int_{S_0} \tau_{\mathcal{R}_t} &= \tau_{\mathcal{R}_t}(q_t^0)\Big(T_{s^{-1}(q_t^0)}s(\mathbf{e}_2 - \mathbf{e}_1), \dots, T_{s^{-1}(q_t^k)}s(\mathbf{e}_m - \mathbf{e}_1)\Big) \\ &= \tau_{\mathcal{R}_t}(q_t^0)(v_1, \dots, v_{m-1}) \quad \text{for } k = 0\end{aligned}$$

where $A_m \in \mathbb{R}$ is constant for every m. Hence,

$$\begin{aligned}&\frac{1}{A_{m-1}t^{m-1}} \int_{\partial\mathcal{R}_t} \tau_{\mathcal{R}_t} \\ &\qquad = \tau_{\mathcal{R}_t}(q_t^0)(v_1, \dots, v_{m-1}) + \sum_{i=1}^{m} (-1)^i \tau_{\mathcal{R}_t}(q_t^i)(e_1, \dots, \widehat{e_i}, \dots, e_m).\end{aligned}$$

Next we note that the balance law implies that

$$\lim_{t\to 0} \frac{\int_{\partial\mathcal{R}_t} \tau_{\mathcal{R}_t}}{t^{m-1}} = 0$$

as the integrals of $\beta_{\mathcal{R}_t}$ and $\pi_{\mathcal{R}_t}$ over $\mathcal{R}_t$ are of order t^m. Thus,

$$\lim_{t\to 0}\left\{\tau_{\mathcal{R}_t}(q_t^0)(v_1, \dots, v_{m-1}) + \sum_{i=1}^{m} (-1)^i \tau_{\mathcal{R}_t}(q_t^i)(e_1, \dots, \widehat{e_i}, \dots, e_m)\right\} = 0.$$

In addition, for all $k = 0, \dots, m$, $\lim_{t\to 0} q_t^k = x$. For $i \neq 0$, $\lim_{t\to 0} T_{q_t^i}(\partial\mathcal{R}_t)$ is annihilated by e^i and $\lim_{t\to 0} T_{q_t^0}(\partial\mathcal{R}_t)$ is determined by the form ϕ. Moreover, for $i \neq 0$, the points q_t^i are on the i-th face of the simplex whose orientation outwards from the simplex is opposite to the orientation determined by e^i. It follows that

$$\begin{aligned}\lim_{t\to 0}\{\tau_{\mathcal{R}_t}(q_t^i)\} &= \tau(-e^i) \\ &= -\tau(e^i),\end{aligned}$$

so

$$\lim_{t\to 0}\left\{\tau_{\mathcal{R}_t}(q_t^i)(e_1,\ldots,\widehat{e_i},\ldots,e_m)\right\} = -\tau_{\hat{\imath}}.$$

The balance can be rewritten now as

$$\tau(H)(v_1,\ldots,v_{m-1}) = \sum_{i=1}^{m}(-1)^i\tau_{\hat{\imath}}. \qquad \square$$

Lemma 4.10 (Mean Value Theorem for Integration of Forms**).** *Let ω be a p-form on a C^r singular p-simplex $s\colon \Delta_p \to M$ in a manifold M, where Δ_p is the standard p-simplex. Denote by A_p the integral $\int_{\Delta_p} dx^1\ldots dx^p$. Then, there is a point $q \in \text{image}(s)$ such that*

$$\int_s \omega = w(s^{-1}(q))A_p = \omega(q)\big(T_{s^{-1}(q)}s(\mathbf{e}_1),\ldots,T_{s^{-1}(q)}s(\mathbf{e}_p)\big)A_p,$$

where, $w(x^i)\,dx^1\wedge\cdots\wedge dx^p$ is $Ts^*(\omega)$ and $\{\mathbf{e}_i\}$ is the standard basis of $\mathbb{R}^p$.

If Δ_p^t is the simplex $\{0 \leqq x^i \leqq t,\ \sum_{i=1}^p \leqq t\}$ obtained by expanding Δ_p by t, and s^t is the restriction of the mapping s, then there is a corresponding point q^t satisfying

$$\int_{s^t}\omega = w(s^{-1}(q^t))A_p\,t^p = \omega(q^t)\big(T_{s^{-1}(q^t)}s(\mathbf{e}_1),\ldots,T_{s^{-1}(q^t)}s(\mathbf{e}_p)\big)\,A_p t^p.$$

Proof. By the definition of the integral of a form on a simplex and the mean value theorem for integrals (in $\mathbb{R}^p$), we have

$$\begin{aligned}\int_s \omega &= \int_{\Delta_p} w(x^i)\,dx^1\ldots dx^p\\ &= w(\mathbf{q})\int_{\Delta_p} dx^1\ldots dx^p\\ &= w(\mathbf{q})A_p,\end{aligned}$$

for some point $\mathbf{q}\in\Delta_p$. However,

$$\begin{aligned}w(\mathbf{q}) &= w(\mathbf{q})dx^1\wedge\ldots\wedge dx^p(\mathbf{e}^1,\ldots,\mathbf{e}^p)\\ &= Ts^*_{s^{-1}(q)}(\omega)(\mathbf{e}^1,\ldots,\mathbf{e}^p)\\ &= \omega(q)(T_{s^{-1}(q)}s(\mathbf{e}_1),\ldots,T_{s^{-1}(q)}s(\mathbf{e}_p)),\end{aligned}$$

where $q = s(\mathbf{q})$.

The second statement follows by scaling. $\square$

Corollary 4.11. *Let $s\colon \Delta_p \to \mathbb{R}^p$ be a simplex and let S_i, $i = 0, 1, \ldots, p$ be the image of i-th face. That is, $S_i = \operatorname{image}(s \circ \Phi_i)$, where the standard embedding $\Phi_i\colon \Delta_{p-1} \to \Delta_p$ of the i-th face in Δ_p, is given by*

$$\Phi_0(x^1, \ldots, x^{p-1}) = \big(1 - \textstyle\sum_{k=1}^{p-1} x^k, x^1, \ldots, x^{p-1}\big),$$
$$\Phi_i(x^1, \ldots, x^{p-1}) = \big(x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{p-1}\big), \quad \text{for } i \neq 0.$$

Then, for a form τ_i defined on S_i, there is a point q_i such that

$$\int_{S_i} \tau_i = \tau(q_i)\Big(T_{s^{-1}(q_i)} s\big(T\Phi_i(\mathbf{e}_1)\big), \ldots, T_{s^{-1}(q_i)} s\big(T\Phi_i(\mathbf{e}_{p-1})\big)\Big) A_{p-1}.$$

Hence, for a $(p-1)$-form τ whose restriction to S_i is denoted by τ_i, there exist points $q_i \in S_i$, such that

$$\int_{\partial s} \tau = A_{p-1} \sum_{i=0}^{p} (-1)^i \tau_i(q_i)\Big(T_{s^{-1}(q_i)} s\big(T\Phi_i(\mathbf{e}_1)\big), \ldots, T_{s^{-1}(q_i)} s\big(T\Phi_i(\mathbf{e}_{p-1})\big)\Big).$$

Proof. The definition of the boundary of the simplex as

$$\partial s = \sum_{i=0}^{p} (-1)^i s \circ \Phi_i$$

implies

$$\begin{aligned}
\int_{\partial s} \tau &= \int_{\sum_i (-1)^i s\circ\Phi_i} \tau \\
&= \sum_{i=0}^{p} (-1)^i \int_{s\circ\Phi_i} \tau \\
&= A_{p-1} \sum_{i=0}^{p} (-1)^i \tau_i(q_i)\Big(T_{(s\circ\Phi_i)^{-1}(q_i)}(s \circ \Phi_i)(\mathbf{e}_1), \ldots, \\
&\qquad\qquad \ldots, T_{(s\circ\Phi_i)^{-1}(q_i)}(s \circ \Phi_i)(\mathbf{e}_{p-1})\Big) \\
&= A_{p-1} \sum_{i=0}^{p} (-1)^i \tau_i(q_i)\Big(T_{s^{-1}(q_i)} s\big(T\Phi_i(\mathbf{e}_1)\big), \ldots, T_{s^{-1}(q_i)} s\big(T\Phi_i(\mathbf{e}_{p-1})\big)\Big)
\end{aligned}$$

for some points $q_i \in S_i$. □

Remark 4.12. Note that the definition of the mappings $\{\Phi_i\}$ implies that

$$\big(T\Phi_i(\mathbf{e}_1), \ldots, T\Phi_i(\mathbf{e}_{p-1})\big) = (\mathbf{e}_1, \ldots, \widehat{\mathbf{e}_i}, \ldots, \mathbf{e}_p), \quad \text{for } i \neq 0$$

and $T\Phi_0(\mathbf{e}_i) = \mathbf{e}_{i+1} - \mathbf{e}_1$.

5. Additional Remarks

Consider the situation where a flux field form σ is given on the oriented $\mathcal{S}$. For each orientable region $\mathcal{R}$, one can use the positive orientation on $\partial\mathcal{R}$ so that $\mathcal{I}^{*\perp} = \mathcal{I}^*$. Thus, the flux density on $\partial\mathcal{R}$ is induced by $\tau_{\mathcal{R}}(x) = \mathcal{I}^*_{T_x\mathcal{S}}(\sigma(x))$, or in concise notation $\tau_{\mathcal{R}} = \mathcal{I}^*\sigma$.

It follows that the balance equation may be rewritten as

$$\int_{\mathcal{R}} \beta + \int_{\partial\mathcal{R}} \mathcal{I}^*\sigma = \int_{\mathcal{R}} \pi.$$

The Stokes' theorem implies that this is equivalent to

$$\int_{\mathcal{R}} \beta + \int_{\mathcal{R}} d\sigma = \int_{\mathcal{R}} \pi,$$

where $d\sigma$ is the exterior derivative of the $(m-1)$-form σ – an m-form. Since the balance holds for an arbitrary oriented region, one can state

Proposition 5.1. *If the flux density forms $\tau_{\mathcal{R}}$ are consistent with the flux field $(m-1)$-differential form σ, then*

$$d\sigma + \beta = \pi$$

– the differential version of the balance equation.

Remark 5.2. It is noted that in the case where a volume element θ is given on $\mathcal{S}$, any flux field form σ is associated with a unique vector field v satisfying the equation $v \lrcorner \theta = \sigma$, where $v \lrcorner \theta$ is the contraction (interior product) of the form θ with the vector field v to produce an $(m-1)$-form. Thus, given a volume element, it is possible to replace the flux field form by a *flux vector field*.

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