Worldlines and body points associated with an extensive property

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ABSTRACT. Material structure of bodies that is usually assumed a-priory in continuum mechanics is constructed on the basis of a balance of a given extensive property on spacetime. Body points are identified with worldlines—the integral lines of the flux of the property. The geometric setting assumes that spacetime has only the structure of a differentiable manifold and no particular frame is assumed to be given.

Keywords. Continuum mechanics, balance laws, flux, Cauchy's theorem, differential forms, material structure, worldlines.

Dedicated to the memory of
AURELIA GRUNFELD (MARKOWITCH)
Brashov, 1930 – Beer-Sheva, 2000.

1. INTRODUCTION

This paper is concerned with the construction of material structure for bodies in continuum mechanics. Traditional continuum mechanics presupposes the existence of a material structure consisting of material points and subbodies of the material universe (see e.g., [10]).

Nevertheless, it would be desirable to extend continuum mechanics to include situations, like growing bodies, where the material points are not conserved. In previous works (e.g., [4], [6]) we used the notion of an organism in order to describe a body that has distinguishable body points although material is not conserved because of growth or destruction of cells.

In order to motivate the definition of an organism, we presented in [7] and [8] the way a body structure may be constructed on the basis of a balance of an extensive property. There is no requirement that the extensive property be conserved but its balance should satisfy Cauchy's postulates. The basic idea is to define body points as integral curves of a flux vector field whose existence is guaranteed by Cauchy's theorem.

While in [7] the construction was carried out for the Euclidean geometry, [8] presented an analogous construction in a setting where space is modeled by a differentiable manifold equipped with a volume element. The existence of a volume

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element may be motivated by interpreting it as the density of a positive extensive property in space. In addition, a frame on spacetime is used in the formulation of [8].

A formulation that uses a particular volume element is not general enough. If the volume element is interpreted physically as the density in space of the property under consideration, it need not be positive as in the case of electric charge. Furthermore, in order that we consider time dependent properties, a volume element is needed on spacetime. However, even if the volume element on spacetime θ is generated by a volume element ρ in space and the form dt induced by the "time axis" in the form $\theta = dt \wedge \rho$, this construction makes sense only in a particular frame that specifies that "time axis".

Here, we reformulate the theory without the use of a volume element or a particular frame. Section 2 presents the basic variables associated with an extensive property on a manifold and Section 3 presents Cauchy's theorem and the resulting flux form. Section 4 introduces the main tool of the formulation—the flux bundle and Section 5 motivates the form of the balance in spacetime. After the formulation of the balance of the property in spacetime in Section 6, Sections 7 introduces worldlines and Section 8 introduces body points, bodies and material frames.

Although the differential geometry used here is standard, some of the constructions of the standard proofs are adapted here to the current context with the purpose of making the paper more readable.

2. Scalar Valued Extensive Properties

We consider properties measured in an ambient physical space. The present setting is more general than classical counterparts as we assume that the physical space is an m-dimensional differentiable manifold $\mathcal{U}$ devoid of any additional structure. While one may interpret $\mathcal{U}$ as the "space manifold" of locations, or "geometrical" space, later $\mathcal{U}$ will be reinterpreted as spacetime—the collection of physical events.

A *control region* (control volume) $\mathcal{R}$ in $\mathcal{U}$ is an m-dimensional compact submanifold with boundary of $\mathcal{U}$.

It is assumed that to every control region, $\mathcal{R}$, there corresponds a *flux density*, $\tau_{\mathcal{R}}$, interpreted classically as the rate at which the extensive property under consideration leaves $\mathcal{R}$ through its boundary $\partial\mathcal{R}$. Mathematically, since $\partial\mathcal{R}$ is an $(m-1)$-dimensional manifold, we assume that $\tau_{\mathcal{R}}$ is an $(m-1)$-differential form (multi-linear and alternating) so its integration

$$\int_{\partial\mathcal{R}} \tau_{\mathcal{R}}$$

on $\partial\mathcal{R}$ can be carried out if an orientation is specified. Roughly, if the collection $\{v_1, \dots, v_{m-1}\}$ of vectors tangent to $\partial\mathcal{R}$ is positively oriented, then, the value $\tau_{\mathcal{R}}(v_1, \dots, v_{m-1})$ is interpreteted classically as the rate at which the property flows out of the region through the infinitesimal element generated by these tangent vectors. We will use $T\mathcal{U}$ to denote the tangent bundle of $\mathcal{U}$ and $T_x\mathcal{U}$ to denote the tangent space to $\mathcal{U}$ at the point x. The bundle of k-forms over $\mathcal{U}$ will be denoted by $\bigwedge^k(T^*\mathcal{U})$ and analogous notation will be used for the other manifolds considered.

In the case where $\mathcal{U}$ is orientable, one may choose one of its two orientations arbitrarily. This positive orientation, perceived as a sign rule, induces an orientation on the boundary of any control region $\mathcal{R}$ as follows. Consider a collection $\{v_1, \dots, v_{m-1}\}$ of vectors in $T_x\partial\mathcal{R}$, the tangent space to the boundary at $x \in \partial\mathcal{R}$. This collection is set to be positively oriented if the collection $\{v, v_1, \dots, v_{m-1}\}$ is positively oriented in $\mathcal{U}$ for any vector $v \in T_x\mathcal{U}$ transversal to the boundary and pointing outwards from $\mathcal{R}$. It is noted that the particular choice of such a vector v is immaterial. In case $\mathcal{U}$ is non-orientable, the theory of odd forms (see [3]) should be used. This will not affect the results presented here.

In the classical formulation of balance laws, $\mathcal{U}$ is interpreted as the "space manifold" and it is assumed that the extensive property has a density ρ, whose time rate is β. As both should be integrated over m-dimensional control regions, ρ and β are m-forms. To complete the balance of the property, the production density ς is introduced—again, an m-form on $\mathcal{U}$ such that

$$\int_{\mathcal{R}} \beta + \int_{\partial\mathcal{R}} \tau_{\mathcal{R}} = \int_{\mathcal{R}} \varsigma.$$

We will refer to such a balance law (under the interpretation of $\mathcal{U}$ as the space manifold) as the *classical balance law*.

Remark 2.1. Unlike the rest of this Section and much of the following material, the interpretation of β as the time derivative of the density of the property under consideration that we described in the last paragraph, involves differentiation of fields with respect to a time variable. Thus, this interpretation is meaningful only when the spacetime manifold, $\mathcal{E}$, is given a particular global frame

$$F \colon \mathcal{E} \to \mathbb{R} \times \mathcal{U}.$$

The time axis is of course represented by $\mathbb{R}$. Without such a frame (or a class of frames) that interpretation is meaningless. In the rest of the paper we will indicate the parts of the analysis that use a frame.

3. Cauchy's Theory and Kinetic Flux Forms

Cauchy's theory is concerned with the properties of the flux density $\tau_{\mathcal{R}}$ and its dependence on $\mathcal{R}$. The generalization of Cauchy's postulates to the geometrical setting considered here is given in [5] and may be summarized roughly as follows (see [5] for details).

GC1 There is an m-form θ on $\mathcal{U}$ such that

$$\left| \int_{\partial\mathcal{R}} \tau_{\mathcal{R}} \right| \leq \int_{\mathcal{R}} \theta.$$

GC2 The value of $\tau_{\mathcal{R}}$ at any point $x \in \partial\mathcal{R}$ depends on the region $\mathcal{R}$ only through its tangent space at x including its (inwards versus outwards) orientation. That is, if $T_x\mathcal{R}_1 = T_x\mathcal{R}_2$ for the two regions $\mathcal{R}_1$, $\mathcal{R}_2$ whose boundaries contain x that are situated on the same side of the common tangent space, then,

$$\tau_{\mathcal{R}_1}(x) = \tau_{\mathcal{R}_2}(x).$$

GC3 The value of the flux density $\tau_{\mathcal{R}}$ at the point $x \in \partial\mathcal{R}$ depends smoothly on both x and the tangent space $T_x\partial\mathcal{R}$.

Proposition 3.1 (Cauchy's Theorem)**.** If Cauchy's postulates hold then there is a unique $(m-1)$-form J on $\mathcal{U}$ such that

$$\tau_{\mathcal{R}}(v_1, \dots, v_{m-1}) = J(v_1, \dots, v_{m-1}).$$

See [5] for the proof and further details. The form J will be referred to as the *kinetic flux form* of the property under consideration.

Remark 3.2. The relation between $\tau_{\mathcal{R}}$ and J, *Cauchy's formula*, may be written in the form $\tau_{\mathcal{R}} = \iota^*(J)$. This follows from

$$\tau_{\mathcal{R}}(v_1, \dots, v_{m-1}) = J\big(\iota(v_1), \dots, \iota(v_{m-1})\big),$$

where $\iota\colon T(\partial\mathcal{R}) \to T\mathcal{U}$ is the natural inclusion of vectors (the tangent to the inclusion $\partial\mathcal{R} \to \mathcal{U}$).

Remark 3.3. Consider two control regions $\mathcal{R}$ and $\mathcal{R}'$ whose boundaries have the same tangent space P at x. In addition, assume that $\mathcal{R}$ and $\mathcal{R}'$ are situated on two opposite sides of P. In this case, the rate at which the property flows out of $\mathcal{R}$ through the infinitesimal element generated by the vectors $\{v_1, \dots, v_{m-1}\}$ is

$$\tau_{\mathcal{R}}(v_1, \dots, v_{m-1}) = J(v_1, \dots, v_{m-1}),$$

provided that the order of these vectors is positive with respect to the orientation of $\partial\mathcal{R}$. If this is the case, the vectors $\{v_1, \dots, v_{m-1}\}$ are negatively oriented relative to the orientation of $\partial\mathcal{R}'$ which is opposite to that of $\partial\mathcal{R}$. Hence, $\tau_{\mathcal{R}'}(v_1, \dots, v_{m-1})$ is minus the rate at which the property leaves $\mathcal{R}'$. Thus, although as forms $\tau_{\mathcal{R}} = \tau_{\mathcal{R}'}$, the rates at which the property leaves the corresponding control regions are opposite as expected.

Consider the situation where a kinetic flux field form J is given on the oriented $\mathcal{U}$. For each control region $\mathcal{R}$, we have using the generalized version of Cauchy's formula and Stokes' theorem

$$\int_{\partial\mathcal{R}} \tau_{\mathcal{R}} = \int_{\partial\mathcal{R}} \iota^*(J) = \int_{\mathcal{R}} dJ,$$

where d denotes the exterior derivative of differential forms. It follows that the classical balance equation may be written as

$$\int_{\mathcal{R}} \beta + \int_{\mathcal{R}} dJ = \int_{\mathcal{R}} \varsigma.$$

Since the balance holds for arbitrary control region, one obtains

$$dJ + \beta = \varsigma,$$

the differential version of the classical balance equation.

4. The Flux Bundle

In this Section we introduce the flux bundle on the basis of the mathematical notion of an enveloping subspace associated with a differential form (see [9] p. 25). Our aim it to show that although the kinetic flux is an $(m-1)$-form and not a vector field, it possesses, even in the general case, some properties one would expect from the velocity field. It is noted that the interpretation of the flux as a velocity field depends on the use of a frame. We first recall some facts about enveloping subspaces.

4.1. **Enveloping Subspaces.** Consider the value $J(x) \in \bigwedge^{m-1}\big(T_x^*\mathcal{U}\big)$ of the flux at $x \in \mathcal{U}$, where $\bigwedge^{m-1}\big(T_x^*\mathcal{U}\big)$ is the vector space of $(m-1)$-forms defined on $T_x\mathcal{U}$. A subspace $Y \subset T_x^*\mathcal{U}$, *envelops* $J(x)$ if $J(x) \in \bigwedge^{m-1} Y^*$. A basic property of enveloping subspaces associated with arbitrary alternating forms is that a minimal enveloping subspace always exists and its dimension is referred to as the rank of the form under consideration. The minimal enveloping subspace of $J(x)$ will be denoted by $E\big(J(x)\big)$. The minimal enveloping subspace is generated by the forms $z \lrcorner J$ for $(m-2)$-multi-vectors $z \in \bigwedge^{m-2} T_x\mathcal{U}$, where, $\lrcorner$ denotes the contraction of an form with a multi-vector.

Proposition 4.1. Let θ be a non-vanishing element of the one-dimensional space of m-forms $\bigwedge^m\big(T_x\mathcal{U}\big)$. Then, there is a unique vector $v \in T_x\mathcal{U}$ such that $J(x) = v \lrcorner \theta$.

Proof. We use the local coordinate system $\{x^1, \dots, x^m\}$ so θ is represented locally in the form

$$r(x^i)\, dx^1 \wedge \dots \wedge dx^m.$$

Then, for a vector v represented by its coordinates v^i, the contraction $v \lrcorner \theta$ is represented by

$$\sum_{i=1}^{m} (-1)^{i+1} r v^i\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m,$$

where the hat indicates the omission of dx^i.

Consider a local representation for the kinetic flux J in the form

$$\sum_{i=1}^{m} J_i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^m.$$

The condition $J(x) = v \lrcorner \theta$ is satisfied for the unique vector v represented by

$$v^i = \frac{(-1)^{i+1} J_i}{r}.$$

□

Remark 4.2. It is noted from the local representation that the relation between the value of the flow and vector v is linear and nonsingular. We will use

$$i_\theta \colon \bigwedge^{m-1} \big(T_x^*\mathcal{U}\big) \to T_x\mathcal{U}$$

to denote the resulting isomorphism.

Remark 4.3. From the local expression for the components of the vector v, it is clear that v depends on the choice of a volume element only through a multiplication by a number. Hence, allowing the volume element to vary, J determines a 1-dimensional subspace of $T_x\mathcal{U}$ that is isomorphic with the space of m-forms.

Proposition 4.4. There is a local coordinate system, $x^1, \dots, x^m$, such that $J(x)$ is represented by $dx^1 \wedge \dots \wedge dx^{m-1}$. In other words, J is decomposable.

Proof. Choose some nonzero m-form θ and let v be the vector such that $J(x) = v \lrcorner\, \theta$. Clearly, one can choose a local coordinate system $x^1, \dots, x^m$ such that v is tangent to the x^1 curve, i.e., $v = \frac{\partial}{\partial x^1}$. Then, θ is represented locally by $r(x^i)\, dx^1 \wedge \dots \wedge dx^m$ and $J(x) = v \lrcorner\, \theta$ is represented locally by

$$\frac{\partial}{\partial x^1} \lrcorner\, (r dx_1 \wedge \dots \wedge dx_m) = r\, dx^2 \wedge \cdots \wedge dx^m.$$

□

Remark 4.5. It follows from the proposition that if $J(x)$ is represented locally by $dx^1 \wedge \dots \wedge dx^{m-1}$, then the minimal enveloping subspace is spanned by the 1-forms $dx^1, \dots, dx^{m-1}$. In addition, the rank of J is $m - 1$.

4.2. **The Flux Bundle.** For a point $x \in \mathcal{U}$, consider the annihilator $E(J)^\perp_x \subset T_x\mathcal{U}$ of the minimal enveloping subspace, i.e.,

$$E(J)^\perp_x = \{v \in T_x\mathcal{U};\ \phi(v) = 0, \text{ for all } \phi \in E(J)_x\}.$$

We will refer to $E(J)^\perp_x$ as the *the flux space* of J at x. Clearly, the flux space is 1-dimensional.

Proposition 4.6. For a vector $v \in T_x\mathcal{U}$, $v \in E(J)^\perp_x$ if and only if $v \lrcorner\, J = 0$.

Proof. Write the flux in the form $J(x) = \phi^1 \wedge \dots \wedge \phi^{m-1}$, with $\phi^i \in E(J)_x$. Since the forms $\phi^1 \wedge \dots \wedge \phi^{m-1}$ are linearly independent, we add a form ϕ^m to generate a basis for $T^*_x\mathcal{U}$ and we denote its dual basis by $\{e_1, \dots, e_m\}$. Then, writing $v = v^k e_k$, so $v^k = \phi^k(v)$, we have

$$\begin{aligned} v \lrcorner\, J &= (v^k e_k) \lrcorner\, \phi^1 \wedge \dots \wedge \phi^{m-1} \\ &= \sum_{k=1}^{m-1} (-1)^{k+1} \phi^1 \wedge \dots \wedge \widehat{\phi^k} \wedge \dots \wedge \phi^{m-1} \phi^k(v). \end{aligned}$$

Assume that $v \in E(J)^\perp_x$. Then, $v \lrcorner\, J = 0$ as $\phi^k(v) = 0$ for all elements of $E(J)_x$. Conversely, if $v \lrcorner\, J = 0 \in \bigwedge^{m-2} E(J)_x$, then, since the elements

$$\{\phi^1 \wedge \dots \wedge \widehat{\phi^k} \wedge \dots \wedge \phi^{m-1}\}$$

are linearly independent in $\bigwedge^{m-2} E(J)_x$, $\phi^k(v) = 0$ for all $k = 1, \dots, m-1$ and $v \in E(J)^\perp_x$. □

Remark 4.7. Since $J = v \lrcorner \theta$ for some volume element θ and a corresponding vector v, the identity $v \lrcorner (v \lrcorner \theta) = 0$ implies that the flux space is the 1-dimensional subspace of $T_x\mathcal{U}$ containing v mentioned in Remark 4.3.

Assume that a non-vanishing kinetic flux J is given. From the foregoing discussion it follows that the flux through the infinitesimal surface element bounded by the vectors $v_1, \dots, v_{m-1}$ vanishes if one of the vectors is in the flux bundle. Moreover, referring to an $(m-1)$-dimensional subspace of $T_x\mathcal{U}$ as a hyperplane, the following is an immediate consequence.

Proposition 4.8. The flux through any hyperplane that contains the flux vector space vanishes.

Proof. Let $v_1, \dots, v_{m-1}$ be $m-1$ linearly independent vectors in a hyperplane H that contains the flux bundle. Choose a vector v in the flux bundle. Clearly, v can be expressed as a linear combination $v = a^i v_i$ of the vectors $v_1, \dots, v_{m-1}$. We can assume that $a_1 \neq 0$, or else we can reorder the vectors, so we can write

$$v_1 = \frac{1}{a} v - \frac{1}{a} \sum_{i=2}^{m-1} a^i v_i.$$

It follows that

$$\begin{aligned} J_x(v_1, \dots, v_{m-1}) &= J_x\left(\frac{1}{a} v - \frac{1}{a} \sum_{i=2}^{m-1} a^i v_i, v_2, \dots, v_{m-1}\right) \\ &= J_x\left(\frac{1}{a} v, v_2, \dots, v_{m-1}\right) - J_x\left(\frac{1}{a} \sum_{i=2}^{m-1} a^i v_i, v_2, \dots, v_{m-1}\right) \\ &= 0 \end{aligned}$$

as the first term contains v as an argument and the second term contains a linear combination of the rest $m-2$ vectors. $\square$

This is clearly a generalization of the analogous situation for the value of the velocity field of a body at x. If a surface element contains the velocity vector, the flux through that element vanishes.

The foregoing structure may be described using an alternative approach. Consider the space of multi-vectors $\bigwedge^{m-1} T_x\mathcal{U}$. Each $(m-1)$-multi-vector, represents an element in an oriented hyperplane. By the duality of the space of forms and the space of multi-vectors, the form $J(x)$ may be regarded as an element of $\left(\bigwedge^{m-1} T_x\mathcal{U}\right)^*$. For a multi-vector z, the value $J(x)(z)$ represents the flux through the surface element associated with z. Consider the annihilator $J_x^\perp$ of $J_x = J(x)$, i.e.,

$$J_x^\perp = \{z \in \bigwedge^{m-1} T_x\mathcal{U} \mid J_x(z) = 0\}.$$

An element of $J_x^\perp$ represents a hyperplane through which the flux vanishes. Since $\bigwedge^{m-1} T_x\mathcal{U}$ is m-dimensional, $J_x^\perp$ is $(m-1)$-dimensional. Thus, there are $m-1$

decomposable $(m-1)$-vectors that are linearly independent and on which J_x vanishes. With the notation of the proof of Proposition 4.6 these multi-vectors are

$$\{e_1\wedge\ldots\wedge\widehat{e}_k\wedge\ldots\wedge e_m\}, \qquad k\neq m.$$

These multi-vectors represent $m-1$ linearly independent hyperplanes whose intersection is the flux space.

Given a non-vanishing kinetic flux J, at each point $x\in\mathcal{U}$ we can construct the flux space and hence we have the *flux bundle*

$$E(J)^{\perp}=\bigcup_{x\in\mathcal{U}}E(J)^{\perp}_x$$

which is a 1-dimensional subbundle of $T\mathcal{U}$. In general, $E(J)^{\perp}$ is defined only on an open submanifold of $\mathcal{U}$.

We conclude with an immediate consequence of Proposition 4.1 and the sequel remark. In the particular case where a volume element is given on $\mathcal{U}$, there is a unique vector field v, to which we will refer as the *kinematic flux* such that $J=v\lrcorner\,\theta$. As mentioned earlier, $v\in E(J)^{\perp}$. Thus, it is the additional structure of a volume element that enables one to construct the analog of the velocity vector field (assuming we adopt the interpretation where a frame is given).

5. From Classical Balance to Spacetime Setting

In classical mechanics, balance laws are usually formulated on the "geometrical space" and a frame is utilized. In this Section we start with such a classical balance law and construct a balance law on spacetime that has a particular simple form. The form of the balance law formulated on spacetime is independent of the frame. Hence, it is adopted in the next Section as a general form of a balance law on spacetime.

We assume that spacetime $\mathcal{E}$ is given a frame $F\colon\mathcal{E}\to\mathcal{T}\times\mathcal{U}$ where we consider the time manifold to be modeled by $\mathbb{R}$ and $\mathcal{U}$ is the m-dimensional "geometrical space". It follows that at any event e in $\mathcal{E}$, $T_e\mathcal{E}=TF\big(\mathbb{R}\times T_{F_{\mathcal{U}}(e)}\mathcal{U}\big)$. We use $F_{\mathcal{T}}$ and $F_{\mathcal{U}}$ for the two components of F and in the sequel we will often use t and x for the two components of e. We start with a time-dependent balance law on $\mathcal{U}$. In general, the various fields and the flux are time-dependent. This implies that the differential balance equation is time-dependent.

Any time-dependent form on $\mathcal{U}$ induces through the pull-back $F^*_{\mathcal{U}}$ a form on $\mathcal{E}$. In order to simplify the notation we will use the same notation for both. Thus, any time-dependent r-form ϕ on $\mathcal{U}$ induces the $(r+1)$-form $\mathfrak{f}=dt\wedge\phi$ on $\mathcal{E}$, where dt is the natural 1-form on $\mathbb{R}$. We have

$$\begin{aligned}\frac{\partial}{\partial t}\lrcorner\,\mathfrak{f}&=\frac{\partial}{\partial t}\lrcorner\,(dt\wedge\phi)\\&=\left(\frac{\partial}{\partial t}\lrcorner\,dt\right)\wedge\phi-dt\wedge\left(\frac{\partial}{\partial t}\lrcorner\,\phi\right).\end{aligned}$$

As $TF_{\mathcal{U}}(\partial/\partial t)=0$, the second term vanishes, and we have

$$\frac{\partial}{\partial t}\lrcorner\,\mathfrak{f}=\phi.$$

For a classical balance law, we use the foregoing procedure to construct the form $\mathfrak{b} = dt \wedge \beta$ from the density rate and the form $\mathfrak{s} = dt \wedge \varsigma$ from the production rate. Similarly, we set $\mathfrak{J} = -dt \wedge J + \rho$, where we recall that ρ is the density of the property under consideration whose time derivative is β—an m-form on $\mathcal{E}$ induced by an m-form on $\mathcal{U}$ having the same notation. Thus, $\mathfrak{J}$ is represented locally by

$$-\sum_{i=1}^{m} J_i \, dt \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m + \rho_{1\ldots m} dx^1 \wedge \ldots \wedge dx^m.$$

Proposition 5.1. Given a classical balance law that satisfies Cauchy's postulates, $d\mathfrak{J} = \mathfrak{s}$, and for each control region $\mathfrak{R}$ in spacetime

$$\int_{\partial\mathfrak{R}} \mathfrak{t}_{\mathfrak{R}} = \int_{\mathfrak{R}} \mathfrak{s},$$

where, $\mathfrak{t}_{\mathfrak{R}} = \iota^*(\mathfrak{J})$.

Proof. We have,

$$\begin{aligned} d\mathfrak{J} &= d(-dt \wedge J) + d\rho \\ &= dt \wedge dJ + d\rho. \end{aligned}$$

Since $d\rho$ is represented by

$$\frac{\partial \rho_{1\ldots m}}{\partial t} \, dt \wedge dx^1 \wedge \ldots \wedge dx^m,$$

$d\rho = dt \wedge \beta = \mathfrak{b}$ and we have

$$\begin{aligned} d\mathfrak{J} &= dt \wedge (dJ + \beta) \\ &= dt \wedge \varsigma \\ &= \mathfrak{s} \end{aligned}$$

by the differential balance equation. We used above the fact that exterior differentiation commutes with the pull-back (natural with respect to mappings). It is clear now from Stokes' theorem that for any orientable $(m+1)$-submanifold with boundary $\mathfrak{R}$ in $\mathcal{E}$, we have

$$\int_{\partial\mathfrak{R}} \iota^*(\mathfrak{J}) = \int_{\mathfrak{R}} \mathfrak{s}.$$

For each such spacetime control region we set $\mathfrak{t}_{\mathfrak{R}} = \iota^*(\mathfrak{J})$ and in the sequel we will sometimes omit the $\mathfrak{R}$-subscript. We obtained a simple balance principle in spacetime for which the $(m+1)$-volume-form vanishes and for each control region

$$\int_{\partial\mathfrak{R}} \mathfrak{t}_{\mathfrak{R}} = \int_{\mathfrak{R}} \mathfrak{s}.$$

$\square$

Conversely, if we start with a balance equation in spacetime as above that satisfies Cauchy's postulates, the generalized Cauchy theorem implies that there is an m-form $\mathfrak{J}$ such that for each spacetime control region $\mathfrak{R}$, $\mathfrak{t}_{\mathfrak{R}} = \iota^*(\mathfrak{J})$.

Assume that $e \in \partial\mathfrak{R}$, with $F(e) = (x, t)$, and that $TF(T_e\partial\mathfrak{R}) = \{t\} \times T_x\mathcal{U}$. Then,

$$
\begin{aligned}
\mathfrak{t}(v_1, \ldots, v_m) &= \mathfrak{J}(v_1, \ldots, v_m) \\
&= \rho(v_1, \ldots, v_m) - (dt \wedge J)(v_1, \ldots, v_m) \\
&= \rho(v_1, \ldots, v_m),
\end{aligned}
$$

where the second term vanishes because $v_1, \ldots, v_m$ are all tangent to $\mathcal{U}$ and orthogonal to dt.

Similarly, consider a point on the boundary of $\partial\mathfrak{R}$ such that the tangent space at that point contains the vector $\partial/\partial t$. Then,

$$
\begin{aligned}
\mathfrak{t}(v_1, \ldots, v_m) &= \mathfrak{J}(v_1, \ldots, v_m) \\
&= \rho(v_1, \ldots, v_m) - (dt \wedge J)(v_1, \ldots, v_m) \\
&= -(dt \wedge J)(v_1, \ldots, v_m)
\end{aligned}
$$

as $\{TF_{\mathcal{U}}(v_1), \ldots, TF_{\mathcal{U}}(v_m)\}$ cannot contain m linearly independent vectors in $T_x\mathcal{U}$ if $\partial/\partial t$ was in the space spanned by $v_1, \ldots, v_m$. In the particular case where $v_1 = \partial/\partial t$, we have

$$
\begin{aligned}
\mathfrak{t}\left(\frac{\partial}{\partial t}, v_2, \ldots, v_m\right) &= -(dt \wedge J)\left(\frac{\partial}{\partial t}, v_2, \ldots, v_m\right) \\
&= -\frac{\partial}{\partial t} \lrcorner \big(dt \wedge J\big)(v_2, \ldots, v_m) \\
&= -J\big(TF_{\mathcal{U}}(v_2), \ldots, TF_{\mathcal{U}}(v_m)\big) \\
&= -\tau\big(TF_{\mathcal{U}}(v_2), \ldots, TF_{\mathcal{U}}(v_m)\big).
\end{aligned}
$$

Thus, $\mathfrak{t}$ contains the information on both ρ and J (or τ).

For a region $\mathcal{R}$ in $\mathcal{U}$ and for a time interval (t_0, t_1) consider the region $\mathfrak{R} = (t_0, t_1) \times \mathcal{R}$ in spacetime. Since

$$
\partial\mathfrak{R} = \{t_0\} \times \mathcal{R} \cup \{t_1\} \times \mathcal{R} \cup (t_0, t_1) \times \partial\mathcal{R},
$$

the previous observations imply that

$$
\int_{\{t_0\}\times\mathcal{R}} \rho \;+\; \int_{\{t_1\}\times\mathcal{R}} \rho \;-\; \int_{(t_0,t_1)\times\partial\mathcal{R}} \tau \;=\; \int_{(t_0,t_1)\times\mathcal{R}} dt \wedge \varsigma.
$$

Finally, note that by the classical balance equation and the general Cauchy theorem we have

$$
\int_{(t_0,t_1)\times\mathcal{R}} dt \wedge (\beta + dJ) \;=\; \int_{(t_0,t_1)\times\mathcal{R}} d\mathfrak{J} \;=\; \int_{(t_0,t_1)\times\mathcal{R}} dt \wedge \varsigma.
$$

Conversely, the differential balance equation in space is obtained by contracting $d\mathfrak{J} = \mathfrak{s}$ with $\partial/\partial t$.

We conclude that the classical balance principles are particular cases of balance principles in spacetime.

6. Balances on Spacetime

Following the conclusions of the previous section, we consider a balance principle in the form considered in Proposition 5.1. Specifically, we assume that on the m-dimensional spacetime $\mathcal{E}$ we have a balance of the form

$$\int_{\partial\mathfrak{R}} \mathfrak{t}_{\mathfrak{R}} = \int_{\mathfrak{R}} \mathfrak{s}$$

where, $\mathfrak{t}$ satisfies Cauchy's postulates. The corresponding flux on spacetime is $\mathfrak{J}$ so $\mathfrak{t}_{\mathfrak{R}} = \iota^*(\mathfrak{J})$ and the differential balance equation is $d\mathfrak{J} = \mathfrak{s}$. With the flux $\mathfrak{J}$ one can associate the flux bundle $E(\mathfrak{J})^{\perp}$ which is a 1-dimensional subbundle of $T\mathcal{E}$.

It is noted that for a balance on spacetime induced by a classical balance, $\mathfrak{J}$ does not vanish unless both J and ρ vanish. Hence, the flux bundle is defined on the open submanifold of $\mathcal{E}$ where either J or ρ does not vanish. In particular, if ρ is positive, i.e., it is a volume element, the flux bundle is defined on all $\mathcal{E}$.

When a volume element θ is given, it induces the kinematic flux v by the condition $J = v \lrcorner\, \theta$. The kinematic flux is clearly the analog of the 4-velocity. In this case, one may write the Lie derivative in the form

$$\mathcal{L}_v\theta = d(v \lrcorner\, \theta) + v \lrcorner\, d\theta,$$

so the differential balance equation assumes the form $\mathcal{L}_v\theta = \mathfrak{s}$. Hence, in the case where the property is conserved $\mathcal{L}_v\theta = 0$.

The previous discussion allows us to construct a natural generalization of the notion of a stream function of classical continuum mechanics, the stream form. Alternatively, stream forms generalize the maxwell 2-forms of electromagnetism (see [2] p. 98). Assume that the property is conserved so that $d\mathfrak{J} = 0$. Then we have locally some smooth form $\mathfrak{m}$, the *stream form* such that $\mathfrak{J} = d\mathfrak{m}$.

7. Worldlines

To show the way the flux bundle induces a generalized body structure, we recall first some basic notions concerning integrability of subbundles (see [1] and [9]). A subbundle V of the tangent bundle $T\mathcal{E}$ is also referred to as a distribution on $\mathcal{E}$. One says that V is integrable if for every $e \in \mathcal{E}$ there is a local submanifold Y of $\mathcal{E}$ such that $T_eY = V_e$, the fiber of V over e. In such a case Y is referred to as the integral manifold of V at e.

In our setting, the distribution under consideration is the 1-dimensional flux bundle. For an event $e \in \mathcal{E}$, let v be a section of the flux bundle $E(\mathfrak{J})^{\perp}$ that does not vanish in an open neighborhood of e. Let $\gamma\colon \mathbb{R} \to \mathcal{E}$ be the integral curve of v through e, i.e., $\gamma(0) = e$, and

$$\frac{d\gamma}{dt} = v \circ \gamma.$$

Then, the image of γ is a local integral manifold Y of the distribution $E(\mathfrak{J})^{\perp}$ at e. While the integral curve γ, i.e., the solution of the ordinary differential equation, depends on the choice of vector field v, its image Y is independent of this choice. For let v' be another such vector field in the flux bundle whose integral curve is γ', then, since the flux bundle is one dimensional, there is a real valued smooth function h that does not vanish in a neighborhood of e such that $v' = hv$. Consider the reparametrization $t \mapsto t'$ that satisfies the ordinary differential equation

$$\frac{dt'}{dt} = h\big(\gamma(t')\big), \qquad t'(0) = 0.$$

This reparametrization is well defined in an open neighborhood of $t = 0$ where h does not change sign due to the inverse function theorem. Specifically, the reparametrization is the inverse of the monotone mapping

$$t(t') = \int_0^{t'} \frac{dp}{h\big(\gamma(p)\big)}.$$

Define the reparametrized curve $\gamma'(t) = \gamma\big(t'(t)\big)$. By the chain rule, γ' is the integral curve of v'. Hence, the integral manifolds of v and v' coincide.

From a slightly different point of view, we can regard v and v' as the kinematic fluxes induced by the choice of two volume elements θ and θ', respectively. Thus, while the choice of a volume elements affects the resulting kinematic flux, it does not affect the integral manifolds for the flux bundle.

We may conclude therefore that the flux bundle is indeed integrable. We will refer to an integral manifold of the flux bundle as a *worldline*. Moreover, since the notion of a worldline is linked intuitively with the body point that travels along this worldline, we will associate a worldline with a body point in the next section.

8. The Foliation by Worldlines and Material Frames

With the foregoing construction, the global Frobenius theorem (see e.g., [1], p. 333) implies that the collection of worldlines form a foliation of spacetime. Specifically, at each event $e \in \mathcal{E}$ there is a chart (U, ψ), $\psi\colon U \to U' \times V' \subset \mathbb{R} \times \mathbb{R}^{m-1}$, whose two components will be denoted by ψ_1 and ψ_2, that has the following properties. Let Y be a worldline that intersects U and let X be any of the connected components of $U \cap Y$. Then, $\psi_2(e) \in \mathbb{R}^{m-1}$ is constant for all $x \in X$. In addition, for every connected component X of $U \cap Y$ for an arbitrary worldline Y, $\psi_1(X) = U'$—a fixed subset of $\mathbb{R}$. Thus, such a foliated chart parametrizes the connected components of the worldlines by the $X^2, \dots, X^m$ coordinates and assigns a fixed parametrization along all the the connected components of worldlines that intersect U.

Consider the equivalence relation $\sim$ such that $e \sim e'$ if e and e' are on the same worldline. Then, the collection of worldlines, or *material universe*, $\mathcal{B}$ can be identified with the quotient space $\mathcal{E}/{\sim}$. The theory of foliations also gives the following conditions for the equivalence relation $\sim$ to be regular, i.e., conditions such that $\mathcal{B}$ is a $(m-1)$-dimensional submanifold of $\mathcal{E}$ and for the natural projection $\mathcal{E} \to \mathcal{B} = \mathcal{E}/{\sim}$ to be a submersion. If at every event e there exists a local

$(m-1)$-dimensional submanifold P of $\mathcal{E}$ such that P intersects every worldline at one point at most and $T_e\mathcal{E} = T_eP \times T_eY$, where Y is the worldline through e, then, the foliation is regular. The $(m-1)$-dimensional submanifold P satisfying the unique intersection property is usually referred to as a *slice*. (Flows on the torus that twist around it either a rational or an irrational number of times per revolution may serve as examples for regular or irregular foliations, respectively.)

In case the foliation is indeed regular the material universe is a manifold and we can define a *body point* X as an element of $\mathcal{B} = \mathcal{E}/\sim$. One may also refer to an $(m-1)$-dimensional compact submanifold with a boundary of the material universe as a *body*. For a foliated chart (U, ψ), let B be a body of worldlines that intersect U. Then, each event e in U may be parametrized by a worldline, or a body point, and the number $\psi_1(e)$. We will refer to such a local mapping

$$F \colon U \subset \mathcal{E} \to U' \times B, \qquad U' \subset \mathbb{R},\ B \subset \mathcal{B}$$

as a *local material frame*. Thus, the extensive property on spacetime induces a class of local frames.

Remark 8.1. Note that a volume element θ induces a frame in a neighborhood of any event e and any particular slice containing this event. In a neighborhood of e, every other event e' will be given the unique "time" coordinate required to arrive at it from the slice along the kinematic flux vector field (the Straightening Out Theorem).

Remark 8.2. A local frame induces a non-vanishing local m-from θ by $\theta = dt \wedge \mathfrak{J}$. If $\mathfrak{J}$ is given in terms of an $(m-2)$-form J on a slice, as $\mathfrak{J} = -dt \wedge J + \rho$, then, $\theta = dt \wedge (-dt \wedge J + \rho) = dt \wedge \rho$ and $\frac{\partial}{\partial t} \lrcorner\, \theta = \rho$.

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