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Stability properties for quasilinear parabolic equations with measure data

Marie-Françoise BIDAUT-VERON*

Quoc-Hung NGUYEN†

Abstract

Let Ω be a bounded domain of \mathbb{R}^N , and $Q = \Omega \times (0, T)$. We study problems of the model type

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $p > 1$, $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Our main result is a *stability theorem* extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators $u \mapsto \mathcal{A}(u) = \text{div}(A(x, t, \nabla u))$.

1 Introduction

Let Ω be a bounded domain of \mathbb{R}^N , and $Q = \Omega \times (0, T)$, $T > 0$. We denote by $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_b(Q)$ the sets of bounded Radon measures on Ω and Q respectively. We are concerned with the problem

$$\begin{cases} u_t - \text{div}(A(x, t, \nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$ and A is a Caratheodory function on $Q \times \mathbb{R}^N$, such that for *a.e.* $(x, t) \in Q$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x, t, \xi) \cdot \xi \geq \Lambda_1 |\xi|^p, \quad |A(x, t, \xi)| \leq a(x, t) + \Lambda_2 |\xi|^{p-1}, \quad \Lambda_1, \Lambda_2 > 0, a \in L^{p'}(Q), \quad (1.2)$$

$$(A(x, t, \xi) - A(x, t, \zeta)) \cdot (\xi - \zeta) > 0 \quad \text{if } \xi \neq \zeta, \quad (1.3)$$

for $p > 1$. This includes the model problem where $\text{div}(A(x, t, \nabla u)) = \Delta_p u$, where Δ_p is the p -Laplacian.

The corresponding elliptic problem:

$$-\Delta_p u = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

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with $\mu \in \mathcal{M}_b(\Omega)$, was studied in [9, 10] for $p > 2 - 1/N$, leading to the existence of solutions in the sense of distributions. For any $p > 1$, and $\mu \in L^1(\Omega)$, existence and uniqueness are proved in [4] in the class of *entropy solutions*. For any $\mu \in \mathcal{M}_b(\Omega)$ the main work is done in [14, Theorems 3.1, 3.2], where not only existence is proved in the class of *renormalized solutions*, but also a stability result, fundamental for applications.

Concerning problem (1.1), the first studies concern the case $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, where existence and uniqueness are obtained by variational methods, see [19]. In the general case $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$, the pionner results come from [9], proving the existence of solutions in the sense of distributions for

$$p > p_1 = 2 - \frac{1}{N+1}, \quad (1.4)$$

see also [11]. The approximated solutions of (1.1) lie in Marcinkiewicz spaces $u \in L^{p_c, \infty}(Q)$ and $|\nabla u| \in L^{m_c, \infty}(Q)$, where

$$p_c = p - 1 + \frac{p}{N}, \quad m_c = p - \frac{N}{N+1}. \quad (1.5)$$

This condition (1.4) ensures that u and $|\nabla u|$ belong to $L^1(Q)$, since $m_c > 1$ means $p > p_1$ and $p_c > 1$ means $p > 2N/(N+1)$. Uniqueness follows in the case $p = 2$, $A(x, t, \nabla u) = \nabla u$, by duality methods, see [21].

For $\mu \in L^1(Q)$, uniqueness is obtained in new classes of *entropy solutions*, and *renormalized solutions*, see [5, 26, 27].

A larger set of measures is studied in [15]. They introduce a notion of parabolic capacity initiated and inspired by [24], used after in [22, 23], defined by

$$c_p^Q(E) = \inf_{E \subset U} \inf_{\text{open } C \subset Q} \{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \},$$

for any Borel set $E \subset Q$, where setting $X = L^p((0, T); W_0^{1,p}(\Omega) \cap L^2(\Omega))$,

$$W = \{ z : z \in X, z_t \in X' \}, \text{ embedded with the norm } \|u\|_W = \|u\|_X + \|u_t\|_{X'}.$$

Let $\mathcal{M}_0(Q)$ be the set of Radon measures μ on Q that do not charge the sets of zero c_p^Q -capacity:

$$\forall E \text{ Borel set } \subset Q, \quad c_p^Q(E) = 0 \implies |\mu|(E) = 0.$$

Then existence and uniqueness of renormalized solutions of (1.1) hold for any measure $\mu \in \mathcal{M}_b(Q) \cap \mathcal{M}_0(Q)$, called *soft (or diffuse, or regular) measure*, and $u_0 \in L^1(\Omega)$, and $p > 1$. The equivalence with the notion of entropy solutions is shown in [16]. For such a soft measure, an extension to equations of type $(b(u))_t - \Delta_p u = \mu$ is given in [6]; another formulation is used in [23] for solving a perturbed problem from (1.1) by an absorption term.

Next consider an *arbitrary measure* $\mu \in \mathcal{M}_b(Q)$. Let $\mathcal{M}_s(Q)$ be the set of all bounded Radon measures on Q with support on a set of zero c_p^Q -capacity, also called *singular*. Let $\mathcal{M}_b^+(Q), \mathcal{M}_0^+(Q), \mathcal{M}_s^+(Q)$ be the positive cones of $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$. From [15], μ can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \quad \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \quad \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q), \quad (1.6)$$

and $\mu_0 \in \mathcal{M}_0(Q)$ admits (at least) a decomposition under the form

$$\mu_0 = f - \text{div } g + h_t, \quad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in X, \quad (1.7)$$

and we write $\mu_0 = (f, g, h)$. Conversely, any measure of this form, *such that* $h \in L^\infty(Q)$, lies in $\mathcal{M}_0(Q)$, see [23, Proposition 3.1]. The solutions of (1.1) are searched in a renormalized sense linked to this decomposition, introduced in [15, 22]. In the range (1.4) the existence of a renormalized solution relative to the

decomposition (1.7) is proved in [22], using suitable approximations of μ_0 and μ_s . Uniqueness is still open, as well as in the elliptic case.

In *all the sequel* we suppose that p satisfies (1.4). Then the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is valid, that means

$$X = L^p((0, T); W_0^{1,p}(\Omega)), \quad X' = L^{p'}((0, T); W^{-1,p'}(\Omega)).$$

In Section 2 we recall the definition of renormalized solutions, given in [22], that we call R-solutions of (1.1), relative to the decomposition (1.7) of μ_0 , and study some of their properties. Our main result is a *stability theorem* for problem (1.1), proved in Section 3, extending to the parabolic case the stability result of [14, Theorem 3.4]. In order to state it, we recall that a sequence of measures $\mu_n \in \mathcal{M}_b(Q)$ converges to a measure $\mu \in \mathcal{M}_b(Q)$ in the *narrow topology* of measures if

$$\lim_{n \rightarrow \infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu \quad \forall \varphi \in C(Q) \cap L^\infty(Q).$$

Theorem 1.1 *Let $A : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfy (1.2), (1.3). Let $u_0 \in L^1(\Omega)$, and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q),$$

with $f \in L^1(Q)$, $g \in (L^{p'}(Q))^N$, $h \in X$ and $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q)$. Let $u_{0,n} \in L^1(\Omega)$,

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(Q),$$

with $f_n \in L^1(Q)$, $g_n \in (L^{p'}(Q))^N$, $h_n \in X$, and $\rho_n, \eta_n \in \mathcal{M}_b^+(Q)$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \quad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(Q)$, $\rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q)$. Assume that

$$\sup_n |\mu_n|(Q) < \infty,$$

and $\{u_{0,n}\}$ converges to u_0 strongly in $L^1(\Omega)$, $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, $\{h_n\}$ converges to h strongly in X , $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ converges to μ_s^- in the narrow topology; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(Q)$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $(L^{p'}(Q))^N$.

Let $\{u_n\}$ be a sequence of R-solutions of

$$\begin{cases} u_{n,t} - \operatorname{div}(A(x, t, \nabla u_n)) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases} \quad (1.8)$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $U_n = u_n - h_n$.

Then up to a subsequence, $\{u_n\}$ converges a.e. in Q to a R-solution u of (1.1), and $\{U_n\}$ converges a.e. in Q to $U = u - h$. Moreover, $\{\nabla u_n\}, \{\nabla U_n\}$ converge respectively to $\nabla u, \nabla U$ a.e. in Q , and $\{T_k(U_n)\}$ converge to $T_k(U)$ strongly in X for any $k > 0$.

In Section 4 we check that any measure $\mu \in \mathcal{M}_b(Q)$ can be approximated in the sense of the stability Theorem, hence we find again the existence result of [22]:

Corollary 1.2 *Let $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(Q)$. Then there exists a R -solution u to the problem (1.1) with data (μ, u_0) .*

Moreover we give more precise properties of approximations of $\mu \in \mathcal{M}_b(Q)$, fundamental for applications, see Propositions 4.1 and 4.2. As in the elliptic case, Theorem 1.1 is a key point for obtaining existence results for more general problems, and we give some of them in [2, 3, 20], for measures μ satisfying suitable capacity conditions. In [2] we study perturbed problems of order 0, of type

$$u_t - \Delta_p u + \mathcal{G}(u) = \mu \quad \text{in } Q, \quad (1.9)$$

where $\mathcal{G}(u)$ is an absorption or a source term with a growth of power or exponential type, and μ is a good in time measure. In [3] we use potential estimates to give other existence results in case of absorption with $p > 2$. In [20], one considers equations of the form

$$u_t - \operatorname{div}(A(x, t, \nabla u)) + \mathcal{G}(u, \nabla u) = \mu$$

under (1.2),(1.3) with $p = 2$, and extend in particular the results of [1] to nonlinear operators.

2 Renormalized solutions of problem (1.1)

2.1 Notations and Definition

For any function $f \in L^1(Q)$, we write $\int_Q f$ instead of $\int_Q f dx dt$, and for any measurable set $E \subset Q$, $\int_E f$ instead of $\int_E f dx dt$. For any open set ϖ of \mathbb{R}^m and $F \in (L^k(\varpi))^\nu$, $k \in [1, \infty]$, $m, \nu \in \mathbb{N}^*$, we set $\|F\|_{k, \varpi} = \|F\|_{(L^k(\varpi))^\nu}$.

We set $T_k(r) = \max\{\min\{r, k\}, -k\}$, for any $k > 0$ and $r \in \mathbb{R}$. We recall that if u is a measurable function defined and finite *a.e.* in Q , such that $T_k(u) \in X$ for any $k > 0$, there exists a measurable function w from Q into \mathbb{R}^N such that $\nabla T_k(u) = \chi_{|u| \leq k} w$, *a.e.* in Q , and for any $k > 0$. We define the gradient ∇u of u by $w = \nabla u$.

Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, and (f, g, h) be a decomposition of μ_0 given by (1.7), and $\widehat{\mu}_0 = \mu_0 - h_t = f - \operatorname{div} g$. In the general case $\widehat{\mu}_0 \notin \mathcal{M}(Q)$, but we write, for convenience,

$$\int_Q w d\widehat{\mu}_0 := \int_Q (fw + g \cdot \nabla w), \quad \forall w \in X \cap L^\infty(Q).$$

Definition 2.1 *Let $u_0 \in L^1(\Omega)$, $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$. A measurable function u is a **renormalized solution**, called **R-solution** of (1.1) if there exists a decomposition (f, g, h) of μ_0 such that*

$$U = u - h \in L^\sigma((0, T); W_0^{1, \sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega)), \quad \forall \sigma \in [1, m_c]; \quad T_k(U) \in X, \quad \forall k > 0, \quad (2.1)$$

and:

(i) for any $S \in W^{2, \infty}(\mathbb{R})$ such that S' has compact support on \mathbb{R} , and $S(0) = 0$,

$$-\int_\Omega S(u_0) \varphi(0) dx - \int_Q \varphi_t S(U) + \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(U) \varphi A(x, t, \nabla u) \cdot \nabla U = \int_Q S'(U) \varphi d\widehat{\mu}_0, \quad (2.2)$$

for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(\cdot, T) = 0$;

(ii) for any $\phi \in C(\overline{Q})$,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla U = \int_Q \phi d\mu_s^+ \quad (2.3)$$

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{-m \geq U > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla U = \int_Q \phi d\mu_s^-. \quad (2.4)$$

Remark 2.2 As a consequence, $S(U) \in C([0, T]; L^1(\Omega))$ and $S(U)(\cdot, 0) = S(u_0)$ in Ω ; and u satisfies the equation

$$(S(U))_t - \operatorname{div}(S'(U)A(x, t, \nabla u)) + S''(U)A(x, t, \nabla u) \cdot \nabla U = fS'(U) - \operatorname{div}(gS'(U)) + S''(U)g \cdot \nabla U, \quad (2.5)$$

in the sense of distributions in Q , see [22, Remark 3]. Moreover assume that $[-k, k] \supset \operatorname{supp} S'$. then from (1.2) and the Hölder inequality, we find easily that

$$\begin{aligned} \|S(U)_t\|_{X' + L^1(Q)} &\leq C \|S\|_{W^{2, \infty}(\mathbb{R})} (\|\nabla u\|^p \chi_{|U| \leq k} \|_{1, Q}^{1/p'} + \|\nabla u\|^p \chi_{|U| \leq k} \|_{1, Q} + \|\nabla T_k(U)\|_{p, Q}^p \\ &\quad + \|a\|_{p', Q} + \|a\|_{p', Q}^{p'} + \|f\|_{1, Q} + \|g\|_{p', Q} \|\nabla u\|^p \chi_{|U| \leq k} \|_{1, Q}^{1/p} + \|g\|_{p', Q}), \end{aligned} \quad (2.6)$$

where $C = C(p, \Lambda_2)$. We also deduce that, for any $\varphi \in X \cap L^\infty(Q)$, such that $\varphi_t \in X' + L^1(Q)$,

$$\begin{aligned} \int_\Omega S(U(T))\varphi(T)dx - \int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(U) + \int_Q S'(U)A(x, t, \nabla u) \cdot \nabla \varphi \\ + \int_Q S''(U)A(x, t, \nabla u) \cdot \nabla U \varphi = \int_Q S'(U)\varphi d\widehat{\mu}_0. \end{aligned} \quad (2.7)$$

Remark 2.3 Let u, U satisfy (2.1). It is easy to see that the condition (2.3) (resp. (2.4)) is equivalent to

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^+ \quad (2.8)$$

resp.

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \geq U > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^-. \quad (2.9)$$

In particular, for any $\varphi \in L^{p'}(Q)$ there holds

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |U| < 2m} |\nabla u| \varphi = 0, \quad \lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |U| < 2m} |\nabla U| \varphi = 0. \quad (2.10)$$

Remark 2.4 (i) Any function $U \in X$ such that $U_t \in X' + L^1(Q)$ admits a unique c_p^Q -quasi continuous representative, defined c_p^Q -quasi a.e. in Q , still denoted U . Furthermore, if $U \in L^\infty(Q)$, then for any $\mu_0 \in \mathcal{M}_0(Q)$, there holds $U \in L^\infty(Q, d\mu_0)$, see [22, Theorem 3 and Corollary 1].

(ii) Let u be any R - solution of problem (1.1). Then, $U = u - h$ admits a c_p^Q -quasi continuous functions representative which is finite c_p^Q -quasi a.e. in Q , and u satisfies definition 2.1 for every decomposition $(\tilde{f}, \tilde{g}, \tilde{h})$ such that $h - \tilde{h} \in L^\infty(Q)$, see [22, Proposition 3 and Theorem 4].

2.2 Steklov and Landes approximations

A main difficulty for proving Theorem 1.1 is the choice of admissible test functions (S, φ) in (2.2), valid for any R -solution. Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations:

Definition 2.5 Let $\varepsilon \in (0, T)$ and $z \in L^1_{loc}(Q)$. For any $l \in (0, \varepsilon)$ we define the **Steklov time-averages** $[z]_l, [z]_{-l}$ of z by

$$[z]_l(x, t) = \frac{1}{l} \int_t^{t+l} z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (0, T - \varepsilon),$$

$$[z]_{-l}(x, t) = \frac{1}{l} \int_{t-l}^t z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (\varepsilon, T).$$

The idea to use this approximation for R -solutions can be found in [7]. Recall some properties, given in [23]. Let $\varepsilon \in (0, T)$, and $\varphi_1 \in C_c^\infty(\overline{\Omega} \times [0, T])$, $\varphi_2 \in C_c^\infty(\overline{\Omega} \times (0, T])$ with $\text{Supp}\varphi_1 \subset \overline{\Omega} \times [0, T - \varepsilon]$, $\text{Supp}\varphi_2 \subset \overline{\Omega} \times [\varepsilon, T]$. There holds:

- (i) If $z \in X$, then $\varphi_1[z]_l$ and $\varphi_2[z]_{-l} \in W$.
- (ii) If $z \in X$ and $z_t \in X' + L^1(Q)$, then, as $l \rightarrow 0$, $(\varphi_1[z]_l)$ and $(\varphi_2[z]_{-l})$ converge respectively to $\varphi_1 z$ and $\varphi_2 z$ in X , and *a.e.* in Q ; and $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$ converge to $(\varphi_1 z)_t, (\varphi_2 z)_t$ in $X' + L^1(Q)$.
- (iii) If moreover $z \in L^\infty(Q)$, then from any sequence $\{l_n\} \rightarrow 0$, there exists a subsequence $\{l_\nu\}$ such that $\{[z]_{l_\nu}\}, \{[z]_{-l_\nu}\}$ converge to z, c_p^Q -quasi everywhere in Q .

Next we recall the approximation used in several articles [8, 12, 11], first introduced in [17].

Definition 2.6 Let $k > 0$, and $y \in L^\infty(\Omega)$ and $Y \in X$ such that $\|y\|_{L^\infty(\Omega)} \leq k$ and $\|Y\|_{L^\infty(Q)} \leq k$. For any $\nu \in \mathbb{N}$, a **Landes-time approximation** $\langle Y \rangle_\nu$ of the function Y is defined as follows:

$$\langle Y \rangle_\nu(x, t) = \nu \int_0^t Y(x, s) e^{\nu(s-t)} ds + e^{-\nu t} z_\nu(x), \quad \forall (x, t) \in Q.$$

where $\{z_\nu\}$ is a sequence of functions in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that $\|z_\nu\|_{L^\infty(\Omega)} \leq k$, $\{z_\nu\}$ converges to y *a.e.* in Ω , and $\nu^{-1} \|z_\nu\|_{W_0^{1,p}(\Omega)}^p$ converges to 0.

Therefore, we can verify that $(\langle Y \rangle_\nu)_t \in X$, $\langle Y \rangle_\nu \in X \cap L^\infty(Q)$, $\|\langle Y \rangle_\nu\|_{\infty, Q} \leq k$ and $\{\langle Y \rangle_\nu\}$ converges to Y strongly in X and *a.e.* in Q . Moreover, $\langle Y \rangle_\nu$ satisfies the equation $(\langle Y \rangle_\nu)_t = \nu(Y - \langle Y \rangle_\nu)$ in the sense of distributions in Q , and $\langle Y \rangle_\nu(0) = z_\nu$ in Ω . In this paper, we only use the **Landes-time approximation** of the function $Y = T_k(U)$, where $y = T_k(u_0)$.

2.3 First properties

In the sequel we use the following notations: for any function $J \in W^{1,\infty}(\mathbb{R})$, nondecreasing with $J(0) = 0$, we set

$$\overline{\mathcal{J}}(r) = \int_0^r J(\tau) d\tau, \quad \mathcal{J}(r) = \int_0^r J'(\tau) \tau d\tau. \quad (2.11)$$

It is easy to verify that $\mathcal{J}(r) \geq 0$,

$$\mathcal{J}(r) + \overline{\mathcal{J}}(r) = J(r)r, \quad \text{and} \quad \mathcal{J}(r) - \mathcal{J}(s) \geq s(J(r) - J(s)) \quad \forall r, s \in \mathbb{R}. \quad (2.12)$$

In particular we define, for any $k > 0$, and any $r \in \mathbb{R}$,

$$\overline{T}_k(r) = \int_0^r T_k(\tau) d\tau, \quad \mathcal{T}_k(r) = \int_0^r T'_k(\tau) \tau d\tau, \quad (2.13)$$

and we use several times a truncature used in [14]:

$$H_m(r) = \chi_{[-m, m]}(r) + \frac{2m - |s|}{m} \chi_{m < |s| \leq 2m}(r), \quad \overline{H}_m(r) = \int_0^r H_m(\tau) d\tau. \quad (2.14)$$

The next Lemma allows to extend the range of the test functions in (2.2).

Lemma 2.7 *Let u be a R -solution of problem (1.1). Let $J \in W^{1, \infty}(\mathbb{R})$ be nondecreasing with $J(0) = 0$, and \overline{J} defined by (2.11). Then,*

$$\begin{aligned} & \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla (\xi J(S(U))) + \int_Q S''(U) A(x, t, \nabla u) \cdot \nabla U \xi J(S(U)) \\ & - \int_{\Omega} \xi(0) J(S(u_0)) S(u_0) dx - \int_Q \xi_t \overline{J}(S(U)) \leq \int_Q S'(U) \xi J(S(U)) d\widehat{\mu}_0, \end{aligned} \quad (2.15)$$

for any $S \in W^{2, \infty}(\mathbb{R})$ such that S' has compact support on \mathbb{R} and $S(0) = 0$, and for any $\xi \in C^1(Q) \cap W^{1, \infty}(Q)$, $\xi \geq 0$.

Proof. Let \mathcal{J} be defined by (2.11). Let $\zeta \in C_c^1([0, T])$ with values in $[0, 1]$, such that $\zeta_t \leq 0$, and $\varphi = \zeta \xi [j(S(U))]_l$. Clearly, $\varphi \in X \cap L^\infty(Q)$; we choose the pair of functions (φ, S) as test function in (2.2). From the convergence properties of Steklov time-averages, we easily will obtain (2.15) if we prove that

$$\lim_{l \rightarrow 0, \zeta \rightarrow 1} \left(- \int_Q (\zeta \xi [j(S(U))]_l)_t S(U) \geq - \int_Q \xi_t \overline{J}(S(U)) \right).$$

We can write $- \int_Q (\zeta \xi [j(S(U))]_l)_t S(U) = F + G$, with

$$F = - \int_Q (\zeta \xi)_t [j(S(U))]_l S(U), \quad G = - \int_Q \zeta \xi S(U) \frac{1}{l} (j(S(U))(x, t + l) - j(S(U))(x, t)).$$

Using (2.12) and integrating by parts we have

$$\begin{aligned} G & \geq - \int_Q \zeta \xi \frac{1}{l} (\mathcal{J}(S(U))(x, t + l) - \mathcal{J}(S(U))(x, t)) = - \int_Q \zeta \xi \frac{\partial}{\partial t} ([\mathcal{J}(S(U))]_l) \\ & = \int_Q (\zeta \xi)_t [\mathcal{J}(S(U))]_l + \int_{\Omega} \zeta(0) \xi(0) [\mathcal{J}(S(U))]_l(0) dx \geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(U))]_l, \end{aligned}$$

since $\mathcal{J}(S(U)) \geq 0$. Hence,

$$- \int_Q (\zeta \xi [j(S(U))]_l)_t S(U) \geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(U))]_l + F = \int_Q (\zeta \xi)_t ([\mathcal{J}(S(U))]_l - [J(S(U))]_l) S(U).$$

Otherwise, $\mathcal{J}(S(U))$ and $J(S(U)) \in C([0, T]; L^1(\Omega))$, thus $\{(\zeta \xi)_t ([\mathcal{J}(S(u))]_l - [J(S(u))]_l) S(u)\}$ converges to $-(\zeta \xi)_t \overline{J}(S(u))$ in $L^1(Q)$ as $l \rightarrow 0$. Therefore,

$$\lim_{l \rightarrow 0, \zeta \rightarrow 1} \left(- \int_Q (\zeta \xi [J(S(U))]_l)_t S(U) \right) \geq \lim_{\zeta \rightarrow 1} \left(- \int_Q (\zeta \xi)_t \overline{J}(S(U)) \right) \geq - \int_Q \xi_t \overline{J}(S(U)),$$

which achieves the proof. \blacksquare

Next we give estimates of the function and its gradient, following the first ones of [11], inspired by the estimates of the elliptic case of [4]. In particular we extend and make more precise the a priori estimates of [22, Proposition 4] given for solutions with smooth data; see also [15, 18].

Proposition 2.8 *If u is a R -solution of problem (1.1), then there exists $C_1 = C_1(p, \Lambda_1, \Lambda_2)$ such that, for any $k \geq 1$ and $\ell \geq 0$,*

$$\int_{\ell \leq |U| \leq \ell+k} |\nabla u|^p + \int_{\ell \leq |U| \leq \ell+k} |\nabla U|^p \leq C_1 k M, \quad (2.16)$$

$$\|U\|_{L^\infty((0,T);L^1(\Omega))} \leq C_1(M + |\Omega|), \quad (2.17)$$

where $M = \|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q} + \|g\|_{p',Q}^{p'} + \|h\|_X^p + \|a\|_{p',Q}^{p'}$.
As a consequence, for any $k \geq 1$,

$$\text{meas}\{|U| > k\} \leq C_2 M_1 k^{-pc}, \quad \text{meas}\{|\nabla U| > k\} \leq C_2 M_2 k^{-mc}, \quad (2.18)$$

$$\text{meas}\{|u| > k\} \leq C_2 M_2 k^{-pc}, \quad \text{meas}\{|\nabla u| > k\} \leq C_2 M_2 k^{-mc}, \quad (2.19)$$

where $C_2 = C_2(N, p, \Lambda_1, \Lambda_2)$, and $M_1 = (M + |\Omega|)^{\frac{p}{N}} M$ and $M_2 = M_1 + M$.

Proof. Set for any $r \in \mathbb{R}$, and $m, k, \ell > 0$,

$$T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.$$

For $m > k + \ell$, we can choose $(J, S, \xi) = (T_{k,\ell}, \overline{H_m}, \xi)$ as test functions in (2.15), where $\overline{H_m}$ is defined in (2.14) and $\xi \in C^1([0, T])$ with values in $[0, 1]$, independent on x . Since $T_{k,\ell}(\overline{H_m}(r)) = T_{k,\ell}(r)$ for all $r \in \mathbb{R}$, we obtain

$$\begin{aligned} & - \int_{\Omega} \xi(0) T_{k,\ell}(u_0) \overline{H_m}(u_0) dx - \int_Q \xi_t \overline{T_{k,\ell}}(\overline{H_m}(U)) \\ & + \int_{\{\ell \leq |U| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla U - \frac{k}{m} \int_{\{m \leq |U| < 2m\}} \xi A(x, t, \nabla u) \cdot \nabla U \leq \int_Q H_m(U) \xi T_{k,\ell}(U) d\widehat{\mu}_0. \end{aligned}$$

And

$$\int_Q H_m(U) \xi T_{k,\ell}(U) d\widehat{\mu}_0 = \int_Q H_m(U) \xi T_{k,\ell}(U) f + \int_{\{\ell \leq |U| < \ell+k\}} \xi \nabla U \cdot g - \frac{k}{m} \int_{\{m \leq |U| < 2m\}} \xi \nabla U \cdot g.$$

Let $m \rightarrow \infty$; then, for any $k \geq 1$, since $U \in L^1(Q)$ and from (2.3), (2.4), and (2.10), we find

$$- \int_Q \xi_t \overline{T_{k,\ell}}(U) + \int_{\{\ell \leq |U| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla U \leq \int_{\{\ell \leq |U| < \ell+k\}} \xi \nabla U \cdot g + k(\|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q}). \quad (2.20)$$

Next, we take $\xi \equiv 1$. We verify that

$$A(x, t, \nabla u) \cdot \nabla U - \nabla U \cdot g \geq \frac{\Lambda_1}{4} (|\nabla u|^p + |\nabla U|^p) - c_1 (|g|^{p'} + |\nabla h|^p + |a|^{p'})$$

for some $c_1 = c_1(p, \Lambda_1, \Lambda_2) > 0$. Hence (2.16) follows. Thus, from (2.20) and the Hölder inequality, we get, for any $\xi \in C^1([0, T])$ with values in $[0, 1]$,

$$- \int_Q \xi_t \overline{T_{k,\ell}}(U) \leq c_2 k M$$

for some $c_2 = c_2(p, \Lambda_1, \Lambda_2) > 0$. Thus $\int_{\Omega} \overline{T_{k,\ell}}(U)(t) dx \leq c_2 k M$, for a.e. $t \in (0, T)$. We deduce (2.17) by taking $k = 1, \ell = 0$, since $\overline{T_{1,0}}(r) = \overline{T_1}(r) \geq |r| - 1$, for any $r \in \mathbb{R}$.

Next, from the Gagliardo-Nirenberg embedding Theorem, see [13, Proposition 3.1], we have

$$\int_Q |T_k(U)|^{\frac{p(N+1)}{N}} \leq c_3 \|U\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} \int_Q |\nabla T_k(U)|^p,$$

where $c_3 = c_3(N, p)$. Then, from (2.16) and (2.17), we get, for any $k \geq 1$,

$$\text{meas}\{|U| > k\} \leq k^{-\frac{p(N+1)}{N}} \int_Q |T_k(U)|^{\frac{p(N+1)}{N}} \leq c_3 \|U\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(U)|^p \leq c_4 M_1 k^{-p_c},$$

with $c_4 = c_4(N, p, \Lambda_1, \Lambda_2)$. We obtain

$$\begin{aligned} \text{meas}\{|\nabla U| > k\} &\leq \frac{1}{k^p} \int_0^{k^p} \text{meas}(\{|\nabla U|^p > s\}) ds \\ &\leq \text{meas}\{|U| > k^{\frac{N}{N+1}}\} + \frac{1}{k^p} \int_0^{k^p} \text{meas}\left(\{|\nabla U|^p > s, |U| \leq k^{\frac{N}{N+1}}\}\right) ds \\ &\leq c_4 M_1 k^{-m_c} + \frac{1}{k^p} \int_{|U| \leq k^{\frac{N}{N+1}}} |\nabla U|^p \leq c_5 M_2 k^{-m_c}, \end{aligned}$$

with $c_5 = c_5(N, p, \Lambda_1, \Lambda_2)$. Furthermore, for any $k \geq 1$,

$$\text{meas}\{|h| > k\} + \text{meas}\{|\nabla h| > k\} \leq c_6 k^{-p} \|h\|_X^p,$$

where $c_6 = c_6(N, p)$. Therefore, we easily get (2.19). ■

Remark 2.9 If $\mu \in L^1(Q)$ and $a \equiv 0$ in (1.2), then (2.16) holds for all $k > 0$ and the term $|\Omega|$ in inequality (2.17) can be removed, where $M = \|u_0\|_{1,\Omega} + |\mu|(Q)$. Furthermore, (2.19) is stated as follows:

$$\text{meas}\{|u| > k\} \leq C_2 M^{\frac{p+N}{N}} k^{-p_c}, \quad \text{meas}\{|\nabla u| > k\} \leq C_2 M^{\frac{N+2}{N+1}} k^{-m_c}, \quad \forall k > 0. \quad (2.21)$$

with $C_2 = C_2(N, p, \Lambda_1, \Lambda_2)$. To see last inequality, we do in the following way:

$$\begin{aligned} \text{meas}\{|\nabla U| > k\} &\leq \text{meas}\left\{|U| > M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas}\left\{|\nabla U|^p > s, |U| \leq M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} ds \\ &\leq C_2 M^{\frac{N+2}{N+1}} k^{-m_c}. \end{aligned}$$

Proposition 2.10 Let $\{\mu_n\} \subset \mathcal{M}_b(Q)$, and $\{u_{0,n}\} \subset L^1(\Omega)$, such that

$$\sup_n |\mu_n|(Q) < \infty, \quad \text{and} \quad \sup_n \|u_{0,n}\|_{1,\Omega} < \infty.$$

Let u_n be a R-solution of (1.1) with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ and $u_{0,n}$, relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$, and $U_n = u_n - h_n$. Assume that $\{f_n\}$ is bounded in $L^1(Q)$, $\{g_n\}$ bounded in $(L^{p'}(Q))^N$ and $\{h_n\}$ bounded in X .

Then, up to a subsequence, $\{U_n\}$ converges a.e. to a function $U \in L^\infty((0, T); L^1(\Omega))$, such that $T_k(U) \in X$ for any $k > 0$ and $U \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega))$ for any $\sigma \in [1, m_c)$. And

- (i) $\{U_n\}$ converges to U strongly in $L^\sigma(Q)$ for any $\sigma \in [1, m_c)$, and $\sup \|U_n\|_{L^\infty((0,T);L^1(\Omega))} < \infty$,
- (ii) $\sup_{k>0} \sup_n \frac{1}{k+1} \int_Q |\nabla T_k(U_n)|^p < \infty$,
- (iii) $\{T_k(U_n)\}$ converges to $T_k(U)$ weakly in X , for any $k > 0$,
- (iv) $\{A(x, t, \nabla (T_k(U_n) + h_n))\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$.

Proof. Take $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on \mathbb{R} and $S(0) = 0$. We combine (2.6) with (2.16), and deduce that $\{S(U_n)_t\}$ is bounded in $X' + L^1(Q)$ and $\{S(U_n)\}$ bounded in X . Hence, $\{S(U_n)\}$ is relatively compact in $L^1(Q)$. On the other hand, we choose $S = S_k$ such that $S_k(z) = z$, if $|z| < k$ and $S_k(z) = 2k \operatorname{sign} z$, if $|z| > 2k$. From (2.17), we obtain

$$\begin{aligned} \operatorname{meas} \{|U_n - U_m| > \sigma\} &\leq \operatorname{meas} \{|U_n| > k\} + \operatorname{meas} \{|U_m| > k\} + \operatorname{meas} \{|S_k(U_n) - S_k(U_m)| > \sigma\} \\ &\leq \frac{c}{k} + \operatorname{meas} \{|S_k(U_n) - S_k(U_m)| > \sigma\}, \end{aligned}$$

where c does not depend of n, m . Thus, up to a subsequence $\{u_n\}$ is a Cauchy sequence in measure, and converges *a.e.* in Q to a function u . Thus, $\{T_k(U_n)\}$ converges to $T_k(U)$ weakly in X , since $\sup_n \|T_k(U_n)\|_X < \infty$ for any $k > 0$. And $\{|\nabla (T_k(U_n) + h_n)|^{p-2} \nabla (T_k(U_n) + h_n)\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$. Furthermore, from (2.18), $\{U_n\}$ strongly converges to U in $L^\sigma(Q)$, for any $\sigma < p_c$. \blacksquare

3 The convergence theorem

We first recall some properties of the measures, see [22, Lemma 5], [14].

Proposition 3.1 *Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q)$, where μ_s^+ and μ_s^- are concentrated, respectively, on two disjoint sets E^+ and E^- of zero c_p^Q -capacity. Then, for any $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \setminus K_\delta^+) \leq \delta, \quad \mu_s^-(E^- \setminus K_\delta^-) \leq \delta,$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_c^1(Q)$ with values in $[0, 1]$, such that $\psi_\delta^+, \psi_\delta^- = 1$ respectively on K_δ^+, K_δ^- , and $\operatorname{supp}(\psi_\delta^+) \cap \operatorname{supp}(\psi_\delta^-) = \emptyset$, and

$$\|\psi_\delta^+\|_X + \|(\psi_\delta^+)_t\|_{X'+L^1(Q)} \leq \delta, \quad \|\psi_\delta^-\|_X + \|(\psi_\delta^-)_t\|_{X'+L^1(Q)} \leq \delta.$$

There exist decompositions $(\psi_\delta^+)_t = (\psi_\delta^+)_t^1 + (\psi_\delta^+)_t^2$ and $(\psi_\delta^-)_t = (\psi_\delta^-)_t^1 + (\psi_\delta^-)_t^2$ in $X' + L^1(Q)$, such that

$$\left\| (\psi_\delta^+)_t^1 \right\|_{X'} \leq \frac{\delta}{3}, \quad \left\| (\psi_\delta^+)_t^2 \right\|_{1,Q} \leq \frac{\delta}{3}, \quad \left\| (\psi_\delta^-)_t^1 \right\|_{X'} \leq \frac{\delta}{3}, \quad \left\| (\psi_\delta^-)_t^2 \right\|_{1,Q} \leq \frac{\delta}{3}. \quad (3.1)$$

Both $\{\psi_\delta^+\}$ and $\{\psi_\delta^-\}$ converge to 0, weak-* in $L^\infty(Q)$, and strongly in $L^1(Q)$ and up to subsequences, *a.e.* in Q , as δ tends to 0.

Moreover if ρ_n and η_n are as in Theorem 1.1, we have, for any $\delta, \delta_1, \delta_2 > 0$,

$$\int_Q \psi_\delta^- d\rho_n + \int_Q \psi_\delta^+ d\eta_n = \omega(n, \delta), \quad \int_Q \psi_\delta^- d\mu_s^+ \leq \delta, \quad \int_Q \psi_\delta^+ d\mu_s^- \leq \delta, \quad (3.2)$$

$$\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\rho_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\mu_s^+ \leq \delta_1 + \delta_2, \quad (3.3)$$

$$\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\eta_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\mu_s^- \leq \delta_1 + \delta_2. \quad (3.4)$$

Hereafter, if $n, \varepsilon, \dots, \nu$ are real numbers, and a function ϕ depends on $n, \varepsilon, \dots, \nu$ and eventual other parameters $\alpha, \beta, \dots, \gamma$, and $n \rightarrow n_0, \varepsilon \rightarrow \varepsilon_0, \dots, \nu \rightarrow \nu_0$, we write $\phi = \omega(n, \varepsilon, \dots, \nu)$, then this means that, for fixed $\alpha, \beta, \dots, \gamma$, there holds $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} |\phi| = 0$. In the same way, $\phi \leq \omega(n, \varepsilon, \delta, \dots, \nu)$ means $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} \phi \leq 0$, and $\phi \geq \omega(n, \varepsilon, \dots, \nu)$ means $-\phi \leq \omega(n, \varepsilon, \dots, \nu)$.

Remark 3.2 *In the sequel we recall a convergence property still used in [14]: If $\{b_{1,n}\}$ is a sequence in $L^1(Q)$ converging to b_1 weakly in $L^1(Q)$ and $\{b_{2,n}\}$ a bounded sequence in $L^\infty(Q)$ converging to b_2 , a.e. in Q , then $\lim_{n \rightarrow \infty} \int_Q b_{1,n} b_{2,n} = \int_Q b_1 b_2$.*

Next we prove Theorem 1.1.

Scheme of the proof. Let $\{\mu_n\}, \{u_{0,n}\}$ and $\{u_n\}$ satisfy the assumptions of Theorem 1.1. Then we can apply Proposition 2.10. Setting $U_n = u_n - h_n$, up to subsequences, $\{u_n\}$ converges a.e. in Q to some function u , and $\{U_n\}$ converges a.e. to $U = u - h$, such that $T_k(U) \in X$ for any $k > 0$, and $U \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega))$ for every $\sigma \in [1, m_c)$. And $\{U_n\}$ satisfies the conclusions (i) to (iv) of Proposition 2.10. We have

$$\begin{aligned} \mu_n &= (f_n - \operatorname{div} g_n + (h_n)_t) + (\rho_n^1 - \operatorname{div} \rho_n^2) - (\eta_n^1 - \operatorname{div} \eta_n^2) + \rho_{n,s} - \eta_{n,s} \\ &= \mu_{n,0} + (\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-, \end{aligned}$$

where

$$\mu_{n,0} = \lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \quad \text{with } \lambda_{n,0} = f_n - \operatorname{div} g_n + (h_n)_t, \quad \rho_{n,0} = \rho_n^1 - \operatorname{div} \rho_n^2, \quad \eta_{n,0} = \eta_n^1 - \operatorname{div} \eta_n^2. \quad (3.5)$$

Hence

$$\rho_{n,0}, \eta_{n,0} \in \mathcal{M}_b^+(Q) \cap \mathcal{M}_0(Q), \quad \text{and } \rho_n \geq \rho_{n,0}, \quad \eta_n \geq \eta_{n,0}. \quad (3.6)$$

Let E^+, E^- be the sets where, respectively, μ_s^+ and μ_s^- are concentrated. For any $\delta_1, \delta_2 > 0$, let $\psi_{\delta_1}^+, \psi_{\delta_2}^+$ and $\psi_{\delta_1}^-, \psi_{\delta_2}^-$ as in Proposition 3.1 and set

$$\Phi_{\delta_1, \delta_2} = \psi_{\delta_1}^+ \psi_{\delta_2}^+ + \psi_{\delta_1}^- \psi_{\delta_2}^-.$$

Suppose that we can prove the two estimates, near E

$$I_1 := \int_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla (U_n - \langle T_k(U) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2), \quad (3.7)$$

and far from E ,

$$I_2 := \int_{\{|U_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla (U_n - \langle T_k(U) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2). \quad (3.8)$$

Then it follows that

$$\overline{\lim}_{n, \nu} \int_{\{|U_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (U_n - \langle T_k(U) \rangle_\nu) \leq 0, \quad (3.9)$$

which implies

$$\overline{\lim}_{n \rightarrow \infty} \int_{\{|U_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (U_n - T_k(U)) \leq 0, \quad (3.10)$$

since $\{(T_k(U))_\nu\}$ converges to $T_k(U)$ in X . On the other hand, from the weak convergence of $\{T_k(U_n)\}$ to $T_k(U)$ in X , we verify that

$$\int_{\{|U_n| \leq k\}} A(x, t, \nabla(T_k(U) + h_n)) \cdot \nabla(T_k(U_n) - T_k(U)) = \omega(n).$$

Thus we get

$$\int_{\{|U_n| \leq k\}} (A(x, t, \nabla u_n) - A(x, t, \nabla(T_k(U) + h_n))) \cdot \nabla(u_n - (T_k(U) + h_n)) = \omega(n).$$

Then, it is easy to show that, up to a subsequence,

$$\{\nabla u_n\} \text{ converges to } \nabla u, \quad a.e. \text{ in } Q. \quad (3.11)$$

Therefore, $\{A(x, t, \nabla u_n)\}$ converges to $A(x, t, \nabla u)$ weakly in $(L^{p'}(Q))^N$; and from (3.10) we find

$$\overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) \leq \int_Q A(x, t, \nabla u) \nabla T_k(U).$$

Otherwise, $\{A(x, t, \nabla(T_k(U_n) + h_n))\}$ converges weakly in $(L^{p'}(Q))^N$ to some F_k , from Proposition 2.10, and we obtain that $F_k = A(x, t, \nabla(T_k(U) + h))$. Hence

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(U_n) + h_n)) \cdot \nabla(T_k(U_n) + h_n) \\ & \leq \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) + \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(U_n) + h_n)) \cdot \nabla h_n \\ & \leq \int_Q A(x, t, \nabla(T_k(U) + h)) \cdot \nabla(T_k(U) + h). \end{aligned}$$

As a consequence

$$\{T_k(U_n)\} \text{ converges to } T_k(U), \text{ strongly in } X, \quad \forall k > 0. \quad (3.12)$$

Then to finish the proof we have to check that u is a solution of (1.1). \blacksquare

In order to prove (3.7) we need a first Lemma, inspired of [14, Lemma 6.1]. It extends the results of [22, Lemma 6 and Lemma 7] relative to sequences of solutions with smooth data:

Lemma 3.3 *Let $\psi_{1,\delta}, \psi_{2,\delta} \in C^1(Q)$ be uniformly bounded in $W^{1,\infty}(Q)$ with values in $[0, 1]$, and such that $\int_Q \psi_{1,\delta} d\mu_s^- \leq \delta$ and $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$. Let $\{u_n\}$ satisfying the assumptions of Theorem 1.1, and $U_n = u_n - h_n$. Then*

$$\frac{1}{m} \int_{\{m \leq U_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{\{m \leq U_n < 2m\}} |\nabla U_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (3.13)$$

$$\frac{1}{m} \int_{-2m < U_n \leq -m} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{-2m < U_n \leq -m} |\nabla U_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad (3.14)$$

and for any $k > 0$,

$$\int_{\{m \leq U_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \int_{\{m \leq U_n < m+k\}} |\nabla U_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (3.15)$$

$$\int_{\{-m-k < U_n \leq -m\}} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \int_{\{-m-k < U_n \leq -m\}} |\nabla U_n|^p \psi_{1,\delta} = \omega(n, m, \delta). \quad (3.16)$$

Proof. (i) Proof of (3.13), (3.14). Set for any $r \in \mathbb{R}$ and any $m, \ell \geq 1$

$$S_{m,\ell}(r) = \int_0^r \left(\frac{-m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,2m+\ell]}(\tau) + \frac{4m + 2h - \tau}{2m + \ell} \chi_{(2m+\ell,4m+2h]}(\tau) \right) d\tau,$$

$$S_m(r) = \int_0^r \left(\frac{-m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,\infty)}(\tau) \right) d\tau.$$

Note that $S''_{m,\ell} = \chi_{[m,2m]}/m - \chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$. We choose $(\xi, J, S) = (\psi_{2,\delta}, T_1, S_{m,\ell})$ as test functions in (2.15) for u_n , and observe that, from (3.5),

$$\widehat{\mu_{n,0}} = \mu_{n,0} - (h_n)_t = \widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0} = f_n - \operatorname{div} g_n + \rho_{n,0} - \eta_{n,0}. \quad (3.17)$$

Thus we can write $\sum_{i=1}^6 A_i \leq \sum_{i=7}^{12} A_i$, where

$$A_1 = - \int_{\Omega} \psi_{2,\delta}(0) T_1(S_{m,\ell}(u_{0,n})) S_{m,\ell}(u_{0,n}) dx, \quad A_2 = - \int_Q (\psi_{2,\delta})_t \overline{T_1}(S_{m,\ell}(U_n)),$$

$$A_3 = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) A(x, t, \nabla u_n) \nabla \psi_{2,\delta}, \quad A_4 = \int_Q (S'_{m,\ell}(U_n))^2 \psi_{2,\delta} T'_1(S_{m,\ell}(U_n)) A(x, t, \nabla u_n) \nabla U_n,$$

$$A_5 = \frac{1}{m} \int_{\{m \leq U_n \leq 2m\}} \psi_{2,\delta} T_1(S_{m,\ell}(U_n)) A(x, t, \nabla u_n) \nabla U_n,$$

$$A_6 = - \frac{1}{2m + \ell} \int_{\{2m+\ell \leq U_n < 2(2m+\ell)\}} \psi_{2,\delta} A(x, t, \nabla u_n) \nabla U_n,$$

$$A_7 = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} f_n, \quad A_8 = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) g_n \cdot \nabla \psi_{2,\delta},$$

$$A_9 = \int_Q (S'_{m,\ell}(U_n))^2 T'_1(S_{m,\ell}(U_n)) \psi_{2,\delta} g_n \cdot \nabla U_n, \quad A_{10} = \frac{1}{m} \int_{m \leq U_n \leq 2m} T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} g_n \cdot \nabla U_n,$$

$$A_{11} = - \frac{1}{2m + \ell} \int_{\{2m+\ell \leq U_n < 2(2m+\ell)\}} \psi_{2,\delta} g_n \cdot \nabla U_n, \quad A_{12} = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} d(\rho_{n,0} - \eta_{n,0}).$$

Since $\|S_{m,\ell}(u_{0,n})\|_{1,\Omega} \leq \int_{\{m \leq u_{0,n}\}} u_{0,n} dx$, we find $A_1 = \omega(\ell, n, m)$. Otherwise

$$|A_2| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq U_n\}} U_n, \quad |A_3| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq U_n\}} (|a| + \Lambda_2 |\nabla u_n|^{p-1}),$$

which imply $A_2 = \omega(\ell, n, m)$ and $A_3 = \omega(\ell, n, m)$. Using (2.3) for u_n , we have

$$A_6 = - \int_Q \psi_{2,\delta} d(\rho_{n,s} - \eta_{n,s})^+ + \omega(\ell) = \omega(\ell, n, m, \delta).$$

Hence $A_6 = \omega(\ell, n, m, \delta)$, since $(\rho_{n,s} - \eta_{n,s})^+$ converges to μ_s^+ as $n \rightarrow \infty$ in the narrow topology, and $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$. We also obtain $A_{11} = \omega(\ell)$ from (2.10).

Now $\left\{ S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) \right\}_\ell$ converges to $S'_m(U) T_1(S_m(U))$, $\{S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n))\}_n$ converges to $S'_m(U) T_1(S_m(U))$, $\{S'_m(U) T_1(S_m(U))\}_m$ converges to 0, weak-* in $L^\infty(Q)$ and $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. From Remark 3.2, we obtain

$$\begin{aligned} A_7 &= \int_Q S'_m(U_n) T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} f_n + \omega(\ell) = \int_Q S'_m(U) T_1(S_m(U)) \psi_{2,\delta} f + \omega(\ell, n) = \omega(\ell, n, m), \\ A_8 &= \int_Q S'_m(U_n) T_1(S_{m,\ell}(U_n)) g_n \cdot \nabla \psi_{2,\delta} + \omega(\ell) = \int_Q S'_m(U) T_1(S_m(U)) g \nabla \psi_{2,\delta} + \omega(\ell, n) = \omega(\ell, n, m). \end{aligned}$$

Otherwise, $A_{12} \leq \int_Q \psi_{2,\delta} d\rho_n$, and $\left\{ \int_Q \psi_{2,\delta} d\rho_n \right\}$ converges to $\int_Q \psi_{2,\delta} d\mu_s^+$, thus $A_{12} \leq \omega(\ell, n, m, \delta)$. Using Holder inequality and the condition (1.2), we have

$$g_n \cdot \nabla U_n - A(x, t, \nabla u_n) \nabla U_n \leq c_1 \left(|g_n|^{p'} + |\nabla h_n|^p + |a|^{p'} \right)$$

with $c_1 = c_1(p, \Lambda_1, \Lambda_2)$, which implies

$$A_9 - A_4 \leq c_1 \int_Q (S'_{m,\ell}(U_n))^2 T_1'(S_{m,\ell}(U_n)) \psi_{2,\delta} \left(|g_n|^{p'} + |h_n|^p + |a|^{p'} \right) = \omega(\ell, n, m).$$

Similarly we also show that $A_{10} - A_5/2 \leq \omega(\ell, n, m)$. Combining the estimates, we get $A_5/2 \leq \omega(\ell, n, m, \delta)$. Using Holder inequality we have

$$A(x, t, \nabla u_n) \nabla U_n \geq \frac{\Lambda_1}{2} |\nabla u_n|^p - c_2 (|a|^{p'} + |\nabla h_n|^p).$$

with $c_2 = c_2(p, \Lambda_1, \Lambda_2)$, which implies

$$\frac{1}{m} \int_{\{m \leq U_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{m,\ell}(U_n)) = \omega(\ell, n, m, \delta).$$

Note that for all $m > 4$, $S_{m,\ell}(r) \geq 1$ for any $r \in [\frac{3}{2}m, 2m]$; hence $T_1(S_{m,\ell}(r)) = 1$. So,

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq U_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

Since $|\nabla U_n|^p \leq 2^{p-1} |\nabla u_n|^p + 2^{p-1} |\nabla h_n|^p$, there also holds

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq U_n < 2m\}} |\nabla U_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

We deduce (3.13) by summing on each set $\{(\frac{4}{3})^i m \leq U_n \leq (\frac{4}{3})^{i+1} m\}$ for $i = 0, 1, 2$. Similarly, we can choose $(\xi, \psi, S) = (\psi_{1,\delta}, T_1, \tilde{S}_{m,\ell})$ as test functions in (2.15) for u_n , where $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$, and we obtain (3.14).

(ii) Proof of (3.15), (3.16). We set, for any $k, m, \ell \geq 1$,

$$S_{k,m,\ell}(r) = \int_0^r \left(T_k(\tau - T_m(\tau)) \chi_{[m, k+m+\ell]} + k \frac{2(k+\ell+m) - \tau}{k+m+\ell} \chi_{(k+m+\ell, 2(k+m+\ell))} \right) d\tau$$

$$S_{k,m}(r) = \int_0^r T_k(\tau - T_m(\tau)) \chi_{[m,\infty)} d\tau.$$

We choose $(\xi, \psi, S) = (\psi_{2,\delta}, T_1, S_{k,m,\ell})$ as test functions in (2.15) for u_n . In the same way we also obtain

$$\int_{\{m \leq U_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{k,m,\ell}(U_n)) = \omega(\ell, n, m, \delta).$$

Note that $T_1(S_{k,m,\ell}(r)) = 1$ for any $r \geq m+1$, thus $\int_{\{m+1 \leq U_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta)$, which implies (3.15) by changing m into $m-1$. Similarly, we obtain (3.16). \blacksquare

Next we look at the behaviour near E .

Lemma 3.4 *Estimate (3.7) holds.*

Proof. There holds

$$I_1 = \int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) - \int_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(U) \rangle_\nu.$$

From Proposition 2.10, (iv), $\{A(x, t, \nabla (T_k(U_n) + h_n)) \cdot \nabla \langle T_k(U) \rangle_\nu\}$ converges weakly in $L^1(Q)$ to $F_k \nabla \langle T_k(U) \rangle_\nu$. And $\{\chi_{\{|U_n| \leq k\}}\}$ converges to $\chi_{|U| \leq k}$, a.e. in Q , and $\Phi_{\delta_1, \delta_2}$ converges to 0 a.e. in Q as $\delta_1 \rightarrow 0$, and $\Phi_{\delta_1, \delta_2}$ takes its values in $[0, 1]$. From Remark 3.2, we have

$$\begin{aligned} & \int_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(U) \rangle_\nu = \int_Q \chi_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla (T_k(U_n) + h_n)) \cdot \nabla \langle T_k(U) \rangle_\nu \\ & = \int_Q \chi_{|U| \leq k} \Phi_{\delta_1, \delta_2} F_k \cdot \nabla \langle T_k(U) \rangle_\nu + \omega(n) = \omega(n, \nu, \delta_1). \end{aligned}$$

Therefore, if we prove that

$$\int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) \leq \omega(n, \delta_1, \delta_2), \quad (3.18)$$

then we deduce (3.7). As noticed in [14, 22], it is precisely for this estimate that we need the double cut $\psi_{\delta_1}^+ \psi_{\delta_2}^+$. To do this, we set, for any $m > k > 0$, and any $r \in \mathbb{R}$,

$$\hat{S}_{k,m}(r) = \int_0^r (k - T_k(\tau)) H_m(\tau) d\tau,$$

where H_m is defined at (2.14). Hence $\text{supp } \hat{S}_{k,m} \subset [-2m, k]$; and $\hat{S}_{k,m}'' = -\chi_{[-k, k]} + \frac{2k}{m} \chi_{[-2m, -m]}$. We choose $(\varphi, S) = (\psi_{\delta_1}^+ \psi_{\delta_2}^+, \hat{S}_{k,m})$ as test functions in (2.2). From (3.17), we can write

$$A_1 + A_2 - A_3 + A_4 + A_5 + A_6 = 0,$$

where

$$\begin{aligned} A_1 &= - \int_Q (\psi_{\delta_1}^+ \psi_{\delta_2}^+)_t \hat{S}_{k,m}(U_n), \quad A_2 = \int_Q (k - T_k(U_n)) H_m(U_n) A(x, t, \nabla u_n) \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+), \\ A_3 &= \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla T_k(U_n), \quad A_4 = \frac{2k}{m} \int_{\{-2m < U_n \leq -m\}} \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla U_n, \\ A_5 &= - \int_Q (k - T_k(U_n)) H_m(U_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\widehat{\lambda}_{n,0}, \quad A_6 = \int_Q (k - T_k(U_n)) H_m(U_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d(\eta_{m,0} - \rho_{n,0}). \end{aligned}$$

We first estimate A_3 . As in [22, p.585], since $\{\hat{S}_{k,m}(U_n)\}$ converges to $\hat{S}_{k,m}(U)$ weakly in X , and $\hat{S}_{k,m}(U) \in L^\infty(Q)$, using (3.1), we find

$$A_1 = - \int_Q (\psi_{\delta_1}^+)_t \psi_{\delta_2}^+ \hat{S}_{k,m}(U) - \int_Q \psi_{\delta_1}^+ (\psi_{\delta_2}^+)_t \hat{S}_{k,m}(U) + \omega(n) = \omega(n, \delta_1).$$

Next consider A_2 . Notice that $U_n = T_{2m}(U_n)$ on $\text{supp}(H_m(U_n))$. From Proposition 2.10, (iv), the sequence $\{A(x, t, \nabla(T_{2m}(U_n) + h_n)) \cdot \nabla(\psi_{\delta_1}^+ \psi_{\delta_2}^+)\}$ converges to $F_{2m} \cdot \nabla(\psi_{\delta_1}^+ \psi_{\delta_2}^+)$ weakly in $L^1(Q)$. From Remark 3.2 and the convergence of $\psi_{\delta_1}^+ \psi_{\delta_2}^+$ in X to 0 as δ_1 tends to 0, we find

$$A_2 = \int_Q (k - T_k(U)) H_m(U) F_{2m} \cdot \nabla(\psi_{\delta_1}^+ \psi_{\delta_2}^+) + \omega(n) = \omega(n, \delta_1).$$

Then consider A_4 . Then for some $c_1 = c_1(p, \Lambda_2)$,

$$|A_4| \leq c_1 \frac{2k}{m} \int_{\{-2m < U_n \leq -m\}} (|\nabla u_n|^p + |\nabla U_n|^p + |a|^{p'}) \psi_{\delta_1}^+ \psi_{\delta_2}^+.$$

Since $\psi_{\delta_1}^+$ takes its values in $[0, 1]$, from Lemma 3.3, we get in particular $A_4 = \omega(n, \delta_1, m, \delta_2)$.

Now we estimate A_5 . The sequence $\{(k - T_k(U_n)) H_m(U_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+\}$ converges to $(k - T_k(U)) H_m(U) \psi_{\delta_1}^+ \psi_{\delta_2}^+$, weakly in X , and $\{(k - T_k(U_n)) H_m(U_n)\}$ converges to $(k - T_k(U)) H_m(U)$, weak-* in $L^\infty(Q)$ and *a.e.* in Q . Otherwise $\{f_n\}$ converges to f weakly in $L^1(Q)$ and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. From Remark 3.2 and the convergence of $\psi_{\delta_1}^+ \psi_{\delta_2}^+$ to 0 in X and *a.e.* in Q as $\delta_1 \rightarrow 0$, we deduce that

$$A_5 = - \int_Q (k - T_k(U_n)) H_m(U) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\hat{\nu}_0 + \omega(n) = \omega(n, \delta_1),$$

where $\hat{\nu}_0 = f - \text{div } g$.

Finally $A_6 \leq 2k \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\eta_n$; using (3.2) we also find $A_6 \leq \omega(n, \delta_1, m, \delta_2)$. By addition, since A_3 does not depend on m , we obtain

$$A_3 = \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla T_k(U_n) \leq \omega(n, \delta_1, \delta_2).$$

Arguing as before with $(\psi_{\delta_1}^- \psi_{\delta_2}^-, \check{S}_{k,m})$ as test function in (2.2), where $\check{S}_{k,m}(r) = -\hat{S}_{k,m}(-r)$, we get in the same way

$$\int_Q \psi_{\delta_1}^- \psi_{\delta_2}^- A(x, t, \nabla u_n) \nabla T_k(U_n) \leq \omega(n, \delta_1, \delta_2).$$

Then, (3.18) holds. ■

Next we look at the behaviour far from E .

Lemma 3.5 . *Estimate (3.8) holds.*

Proof. Here we estimate I_2 ; we can write

$$I_2 = \int_{\{|U_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \nabla (T_k(U_n) - \langle T_k(U) \rangle_\nu).$$

Following the ideas of [25], used also in [22], we define, for any $r \in \mathbb{R}$ and $\ell > 2k > 0$,

$$R_{n,\nu,\ell} = T_{\ell+k}(U_n - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U_n - T_k(U_n)).$$

Recall that $\|\langle T_k(U) \rangle_\nu\|_{\infty,Q} \leq k$, and observe that

$$R_{n,\nu,\ell} = 2k \operatorname{sign}(U_n) \quad \text{in } \{|U_n| \geq \ell + 2k\}, \quad |R_{n,\nu,\ell}| \leq 4k, \quad R_{n,\nu,\ell} = \omega(n, \nu, \ell) \text{ a.e. in } Q, \quad (3.19)$$

$$\lim_{n \rightarrow \infty} R_{n,\nu,\ell} = T_{\ell+k}(U - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U - T_k(U)), \quad \text{a.e. in } Q, \text{ and weakly in } X. \quad (3.20)$$

Next consider $\xi_{1,n_1} \in C_c^\infty([0, T])$, $\xi_{2,n_2} \in C_c^\infty((0, T])$ with values in $[0, 1]$, such that $(\xi_{1,n_1})_t \leq 0$ and $(\xi_{2,n_2})_t \geq 0$; and $\{\xi_{1,n_1}(t)\}$ (resp. $\{\xi_{2,n_2}(t)\}$) converges to 1, for any $t \in [0, T]$ (resp. $t \in (0, T]$); and moreover, for any $a \in C([0, T]; L^1(\Omega))$, $\left\{ \int_Q a(\xi_{1,n_1})_t \right\}$ and $\int_Q a(\xi_{2,n_2})_t$ converge respectively to $-\int_\Omega a(\cdot, T)dx$ and $\int_\Omega a(\cdot, 0)dx$. We set

$$\varphi = \varphi_{n,n_1,n_2,l_1,l_2,\ell} = \xi_{1,n_1}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell+k}(U_n - \langle T_k(U) \rangle_\nu)]_{l_1} - \xi_{2,n_2}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell-k}(U_n - T_k(U_n))]_{-l_2}.$$

We observe that

$$\varphi - (1 - \Phi_{\delta_1,\delta_2})R_{n,\nu,\ell} = \omega(l_1, l_2, n_1, n_2) \quad \text{in norm in } X \text{ and a.e. in } Q. \quad (3.21)$$

We can choose $(\varphi, S) = (\varphi_{n,n_1,n_2,l_1,l_2,\ell}, \overline{H_m})$ as test functions in (2.7) for u_n , where $\overline{H_m}$ is defined at (2.14), with $m > \ell + 2k$. We obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,$$

with

$$\begin{aligned} A_1 &= \int_\Omega \varphi(T) \overline{H_m}(U_n(T)) dx, & A_2 &= - \int_\Omega \varphi(0) \overline{H_m}(u_{0,n}) dx, & A_3 &= - \int_Q \varphi_t \overline{H_m}(U_n), \\ A_4 &= \int_Q H_m(U_n) A(x, t, \nabla u_n) \cdot \nabla \varphi, & A_5 &= \int_Q \varphi H'_m(U_n) A(x, t, \nabla u_n) \cdot \nabla U_n, \\ A_6 &= \int_Q H_m(U_n) \varphi d\widehat{\lambda}_{n,0}, & A_7 &= \int_Q H_m(U_n) \varphi d(\rho_{n,0} - \eta_{n,0}). \end{aligned}$$

Estimate of A_4 . This term allows to study I_2 . Indeed, $\{H_m(U_n)\}$ converges to 1, a.e. in Q ; From (3.21), (3.19) (3.20), we have

$$\begin{aligned} A_4 &= \int_Q (1 - \Phi_{\delta_1,\delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n,\nu,\ell} - \int_Q R_{n,\nu,\ell} A(x, t, \nabla u_n) \cdot \nabla \Phi_{\delta_1,\delta_2} + \omega(l_1, l_2, n_1, n_2, m) \\ &= \int_Q (1 - \Phi_{\delta_1,\delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n,\nu,\ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + \int_{\{|U_n| > k\}} (1 - \Phi_{\delta_1,\delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n,\nu,\ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + B_1 + B_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell), \end{aligned}$$

where

$$\begin{aligned} B_1 &= \int_{\{|U_n| > k\}} (1 - \Phi_{\delta,\eta}) (\chi_{|U_n - \langle T_k(U) \rangle_\nu| \leq \ell+k} - \chi_{|U_n| - k \leq \ell-k}) A(x, t, \nabla u_n) \cdot \nabla U_n, \\ B_2 &= - \int_{\{|U_n| > k\}} (1 - \Phi_{\delta_1,\delta_2}) \chi_{|U_n - \langle T_k(U) \rangle_\nu| \leq \ell+k} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(U) \rangle_\nu. \end{aligned}$$

Now $\{A(x, t, \nabla(T_{\ell+2k}(U_n) + h_n)) \cdot \nabla \langle T_k(U) \rangle_\nu\}$ converges to $F_{\ell+2k} \nabla \langle T_k(U) \rangle_\nu$, weakly in $L^1(Q)$. Otherwise $\{\chi_{|U_n|>k} \chi_{|U_n - \langle T_k(U) \rangle_\nu| \leq \ell+k}\}$ converges to $\chi_{|U|>k} \chi_{|U - \langle T_k(U) \rangle_\nu| \leq \ell+k}$, a.e. in Q . And $\{\langle T_k(U) \rangle_\nu\}$ converges to $T_k(U)$ strongly in X . From Remark 3.2 we get

$$\begin{aligned} B_2 &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|U|>k} \chi_{|U - \langle T_k(U) \rangle_\nu| \leq \ell+k} F_{\ell+2k} \cdot \nabla \langle T_k(U) \rangle_\nu + \omega(n) \\ &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|U|>k} \chi_{|U - T_k(U)| \leq \ell+k} F_{\ell+2k} \cdot \nabla T_k(U) + \omega(n, \nu) = \omega(n, \nu), \end{aligned}$$

since $\nabla T_k(U) \chi_{|U|>k} = 0$. Besides, we see that, for some $c_1 = c_1(p, \Lambda_2)$,

$$|B_1| \leq c_1 \int_{\{\ell-2k \leq |U_n| < \ell+2k\}} (1 - \Phi_{\delta_1, \delta_2}) (|\nabla u_n|^p + |\nabla U_n|^p + |a|^{p'}) .$$

Using (3.3) and (3.4) and applying (3.15) and (3.16) to $1 - \Phi_{\delta_1, \delta_2}$, we obtain, for $k > 0$,

$$\int_{\{m \leq |U_n| < m+4k\}} (|\nabla u_n|^p + |\nabla U_n|^p) (1 - \Phi_{\delta_1, \delta_2}) = \omega(n, m, \delta_1, \delta_2). \quad (3.22)$$

Thus, $B_1 = \omega(n, \nu, \ell, \delta_1, \delta_2)$, hence $B_1 + B_2 = \omega(n, \nu, \ell, \delta_1, \delta_2)$. Then

$$A_4 = I_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2). \quad (3.23)$$

Estimate of A_5 . For $m > \ell + 2k$, since $|\varphi| \leq 2\ell$, and (3.21) holds, we get, from the dominated convergence Theorem,

$$\begin{aligned} A_5 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} H'_m(U_n) A(x, t, \nabla u_n) \cdot \nabla U_n + \omega(l_1, l_2, n_1, n_2) \\ &= -\frac{2k}{m} \int_{\{m \leq |U_n| < 2m\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla U_n + \omega(l_1, l_2, n_1, n_2); \end{aligned}$$

here, the final equality followed from the relation, since $m > \ell + 2k$,

$$R_{n, \nu, \ell} H'_m(U_n) = -\frac{2k}{m} \chi_{m \leq |U_n| \leq 2m}, \quad a.e. \text{ in } Q. \quad (3.24)$$

Next we go to the limit in m , by using (2.3), (2.4) for u_n , with $\phi = (1 - \Phi_{\delta_1, \delta_2})$. There holds

$$A_5 = -2k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d((\rho_{n, s} - \eta_{n, s})^+ + (\rho_{n, s} - \eta_{n, s})^-) + \omega(l_1, l_2, n_1, n_2, m).$$

Then, from (3.3) and (3.4), we get $A_5 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of A_6 . Again, from (3.21),

$$\begin{aligned} A_6 &= \int_Q H_m(U_n) \varphi f_n + \int_Q g_n \cdot \nabla (H_m(U_n) \varphi) \\ &= \int_Q H_m(U_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} f_n + \int_Q g_n \cdot \nabla (H_m(U_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell}) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

Thus we can write $A_6 = D_1 + D_2 + D_3 + D_4 + \omega(l_1, l_2, n_1, n_2)$, where

$$\begin{aligned} D_1 &= \int_Q H_m(U_n)(1 - \Phi_{\delta_1, \delta_2})R_{n, \nu, \ell} f_n, & D_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2})R_{n, \nu, \ell} H'_m(U_n) g_n \cdot \nabla U_n, \\ D_3 &= \int_Q H_m(U_n)(1 - \Phi_{\delta_1, \delta_2})g_n \cdot \nabla R_{n, \nu, \ell}, & D_4 &= - \int_Q H_m(U_n)R_{n, \nu, \ell} g_n \cdot \nabla \Phi_{\delta_1, \delta_2}. \end{aligned}$$

Since $\{f_n\}$ converges to f weakly in $L^1(Q)$, and (3.19)-(3.20) hold, we get, from Remark 3.2,

$$D_1 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) (T_{\ell+k}(U - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U - T_k(U))) f + \omega(m, n) = \omega(m, n, \nu, \ell).$$

We deduce from (2.10) that $D_2 = \omega(m)$. Next consider D_3 . Note that $H_m(U_n) = 1 + \omega(m)$, and (3.20) holds, and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, and $\langle T_k(U) \rangle_\nu$ converges to $T_k(U)$ strongly in X . Then we obtain successively that

$$\begin{aligned} D_3 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(U - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U - T_k(U))) + \omega(m, n) \\ &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(U - T_k(U)) - T_{\ell-k}(U - T_k(U))) + \omega(m, n, \nu) \\ &= \omega(m, n, \nu, \ell). \end{aligned}$$

Similarly we also get $D_4 = \omega(m, n, \nu, \ell)$. Thus $A_6 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of A_7 . We have

$$\begin{aligned} |A_7| &= \left| \int_Q S'_m(U_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} d(\rho_{n,0} - \eta_{n,0}) \right| + \omega(l_1, l_2, n_1, n_2) \\ &\leq 4k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d(\rho_n + \eta_n) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

From (3.3) and (3.4) we get $A_7 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of $A_1 + A_2 + A_3$. We set

$$J(r) = T_{\ell-k}(r - T_k(r)), \quad \forall r \in \mathbb{R},$$

and use the notations $\bar{\mathcal{J}}$ and \mathcal{J} of (2.11). From the definitions of $\xi_{1, n_1}, \xi_{1, n_2}$, we can see that

$$\begin{aligned} A_1 + A_2 &= - \int_\Omega J(U_n(T)) \overline{H_m}(U_n(T)) dx - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu) \overline{H_m}(u_{0,n}) dx + \omega(l_1, l_2, n_1, n_2) \\ &= - \int_\Omega J(U_n(T)) U_n(T) dx - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu) u_{0,n} dx + \omega(l_1, l_2, n_1, n_2, m), \end{aligned} \quad (3.25)$$

where $z_\nu = \langle T_k(U) \rangle_\nu(0)$. We can write $A_3 = F_1 + F_2$, where

$$\begin{aligned} F_1 &= - \int_Q \left(\xi_{n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(U_n - \langle T_k(U) \rangle_\nu)]_{l_1} \right)_t \overline{H_m}(U_n), \\ F_2 &= \int_Q \left(\xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell-k}(U_n - T_k(U_n))] \right)_{-l_2} \overline{H_m}(U_n). \end{aligned}$$

Estimate of F_2 . We write $F_2 = G_1 + G_2 + G_3$, with

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t \xi_{n_2} [T_{\ell-k}(U_n - T_k(U_n))]_{-l_2} \overline{H_m}(U_n), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2})(\xi_{n_2})_t [T_{\ell-k}(U_n - T_k(U_n))]_{-l_2} \overline{H_m}(U_n), \\ G_3 &= \int_Q \xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) ([T_{\ell-k}(U_n - T_k(U_n))]_{-l_2})_t \overline{H_m}(U_n). \end{aligned}$$

We find easily that

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t J(U_n) U_n + \omega(l_1, l_2, n_1, n_2, m), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2})(\xi_{n_2})_t J(U_n) \overline{H_m}(U_n) + \omega(l_1, l_2) = \int_{\Omega} J(u_{0,n}) u_{0,n} dx + \omega(l_1, l_2, n_1, n_2, m). \end{aligned}$$

Next consider G_3 . Setting $b = \overline{H_m}(U_n)$, there holds from (2.13) and (2.12),

$$((J(b))_{-l_2})_t b(\cdot, t) = \frac{b(\cdot, t)}{l_2} (J(b)(\cdot, t) - J(b)(\cdot, t - l_2)).$$

Hence

$$([T_{\ell-k}(U_n - T_k(U_n))]_{-l_2})_t \overline{H_m}(U_n) \geq ([\mathcal{J}(\overline{H_m}(U_n))]_{-l_2})_t = ([\mathcal{J}(U_n)]_{-l_2})_t,$$

since \mathcal{J} is constant in $\{|r| \geq m + \ell + 2k\}$. Integrating by parts in G_3 , we find

$$\begin{aligned} G_3 &\geq \int_Q \xi_{2, n_2} (1 - \Phi_{\delta_1, \delta_2}) ([\mathcal{J}(U_n)]_{-l_2})_t = - \int_Q (\xi_{2, n_2} (1 - \Phi_{\delta_1, \delta_2}))_t [\mathcal{J}(U_n)]_{-l_2} + \int_{\Omega} \xi_{2, n_2}(T) [\mathcal{J}(U_n)]_{-l_2}(T) dx \\ &= - \int_Q (\xi_{2, n_2})_t (1 - \Phi_{\delta_1, \delta_2}) \mathcal{J}(U_n) + \int_Q \xi_{2, n_2} (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(U_n) + \int_{\Omega} \xi_{2, n_2}(T) \mathcal{J}(U_n(T)) dx + \omega(l_1, l_2) \\ &= - \int_{\Omega} \mathcal{J}(u_{0,n}) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(U_n) + \int_{\Omega} \mathcal{J}(U_n(T)) dx + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

Therefore, since $\mathcal{J}(U_n) - J(U_n)U_n = -\overline{\mathcal{J}}(U_n)$ and $\overline{\mathcal{J}}(u_{0,n}) = J(u_{0,n})u_{0,n} - \mathcal{J}(u_{0,n})$, we obtain

$$F_2 \geq \int_{\Omega} \overline{\mathcal{J}}(u_{0,n}) dx - \int_Q (\Phi_{\delta_1, \delta_2})_t \overline{\mathcal{J}}(U_n) + \int_{\Omega} \mathcal{J}(U_n(T)) dx + \omega(l_1, l_2, n_1, n_2, m). \quad (3.26)$$

Estimate of F_1 . Since $m > \ell + 2k$, there holds $T_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) = T_{\ell+k}(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu})$ on $\text{supp} \overline{H_m}(U_n)$. Hence we can write $F_1 = L_1 + L_2$, with

$$\begin{aligned} L_1 &= - \int_Q \left(\xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu}) \\ L_2 &= - \int_Q \left(\xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu})]_{l_1} \right)_t \langle T_k(\overline{H_m}(U)) \rangle_{\nu}. \end{aligned}$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$\begin{aligned}
L_2 &= \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (\langle T_k(\overline{H_m}(U)) \rangle_\nu)_t \\
&\quad + \int_\Omega \xi_{1,n_1} (0) [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (0) \langle T_k(\overline{H_m}(U)) \rangle_\nu (0) dx \\
&= \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) (T_k(U) - \langle T_k(U) \rangle_\nu) + \int_\Omega T_{\ell+k} (u_{0,n} - z_\nu) z_\nu dx + \omega(l_1, l_2, n_1, n_2).
\end{aligned} \tag{3.27}$$

We decompose L_1 into $L_1 = K_1 + K_2 + K_3$, where

$$\begin{aligned}
K_1 &= - \int_Q (\xi_{1,n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu) \\
K_2 &= \int_Q \xi_{1,n_1} (\Phi_{\delta_1, \delta_2})_t [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu) \\
K_3 &= - \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) ([T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1})_t (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu).
\end{aligned}$$

Then we check easily that

$$\begin{aligned}
K_1 &= \int_\Omega T_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) (T) (U_n - \langle T_k(U) \rangle_\nu) (T) dx + \omega(l_1, l_2, n_1, n_2, m), \\
K_2 &= \int_Q (\Phi_{\delta_1, \delta_2})_t T_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) (U_n - \langle T_k(U) \rangle_\nu) + \omega(l_1, l_2, n_1, n_2, m).
\end{aligned}$$

Next consider K_3 . Here we use the function \mathcal{T}_k defined at (2.13). We set $b = \overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu$. Hence from (2.12),

$$\begin{aligned}
([T_{\ell+k}(b)]_{l_1})_t(\cdot, t) &= \frac{b(\cdot, t)}{l_1} (T_{\ell+k}(b)(\cdot, t + l_1) - T_{\ell+k}(b)(\cdot, t)) \\
&\leq \frac{1}{l_1} (\mathcal{T}_{\ell+k}(b)((\cdot, t + l_1)) - \mathcal{T}_{\ell+k}(b)(\cdot, t)) = ([\mathcal{T}_{\ell+k}(b)]_{l_1})_t.
\end{aligned}$$

Thus

$$([T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1})_t (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu) \leq ([\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1})_t.$$

Then

$$\begin{aligned}
K_3 &\geq - \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) ([\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1})_t \\
&= \int_Q (\xi_{1,n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1} - \int_Q \xi_{1,n_1} (\Phi_{\delta_1, \delta_2})_t [\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1} \\
&\quad + \int_\Omega \xi_{1,n_1} (0) [\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1} (0) dx \\
&= - \int_\Omega \mathcal{T}_{\ell+k} (U_n(T) - \langle T_k(U) \rangle_\nu(T)) dx - \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) \\
&\quad + \int_\Omega \mathcal{T}_{\ell+k} (u_{0,n} - z_\nu) dx + \omega(l_1, l_2, n_1, n_2).
\end{aligned}$$

We find by addition, since $T_{\ell+k}(r) - \mathcal{T}_{\ell+k}(r) = \bar{T}_{\ell+k}(r)$ for any $r \in \mathbb{R}$,

$$\begin{aligned} L_1 &\geq \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) dx + \int_{\Omega} \bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) dx \\ &\quad + \int_Q (\Phi_{\delta_1, \delta_2})_t \bar{T}_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned} \quad (3.28)$$

We deduce from (3.28), (3.27), (3.26),

$$\begin{aligned} A_3 &\geq \int_{\Omega} \bar{J}(u_{0,n}) dx + \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) dx + \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) z_{\nu} dx \\ &\quad + \int_{\Omega} \bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) dx + \int_{\Omega} \mathcal{J}(U_n(T)) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U_n)) \\ &\quad + \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) (T_k(U) - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned} \quad (3.29)$$

Next we add (3.25) and (3.29). Note that $\mathcal{J}(U_n(T)) - J(U_n(T))U_n(T) = -\bar{J}(U_n(T))$, and also

$$\mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) - T_{\ell+k}(u_{0,n} - z_{\nu})(z_{\nu} - u_{0,n}) = -\bar{T}_{\ell+k}(u_{0,n} - z_{\nu}).$$

Then we find

$$\begin{aligned} A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{J}(u_{0,n}) - \bar{T}_{\ell+k}(u_{0,n} - z_{\nu})) dx + \int_{\Omega} (\bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) - \bar{J}(U_n(T))) dx \\ &\quad + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U_n)) \\ &\quad + \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) (T_k(U) - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned}$$

Notice that $\bar{T}_{\ell+k}(r-s) - \bar{J}(r) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$; thus

$$\int_{\Omega} (\bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) - \bar{J}(U_n(T))) dx \geq 0.$$

And $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$ and $\{U_n\}$ converges to U in $L^1(Q)$ from Proposition 2.10. Thus we obtain

$$\begin{aligned} A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{J}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U)) \\ &\quad + \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(U - \langle T_k(U) \rangle_{\nu}) (T_k(U) - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m, n). \end{aligned}$$

Moreover $T_{\ell+k}(r-s)(T_k(r) - s) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$, hence

$$\begin{aligned} A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{J}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U)) \\ &\quad + \omega(l_1, l_2, n_1, n_2, m, n). \end{aligned}$$

As $\nu \rightarrow \infty$, $\{z_\nu\}$ converges to $T_k(u_0)$, a.e. in Ω , thus we get

$$A_1 + A_2 + A_3 \geq \int_{\Omega} (\overline{J}(u_0) - \overline{T}_{\ell+k}(u_0 - T_k(u_0))) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\overline{T}_{\ell+k}(U - T_k(U)) - \overline{J}(U)) \\ + \omega(l_1, l_2, n_1, n_2, m, n, \nu).$$

Finally $|\overline{T}_{\ell+k}(r - T_k(r)) - \overline{J}(r)| \leq 2k|r|\chi_{\{|r| \geq \ell\}}$ for any $r \in \mathbb{R}$, thus

$$A_1 + A_2 + A_3 \geq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell).$$

Combining all the estimates, we obtain $I_2 \leq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$, which implies (3.8), since I_2 does not depend on $l_1, l_2, n_1, n_2, m, \ell$. \blacksquare

Next we conclude the proof of Theorem 1.1:

Lemma 3.6 *The function u is a R -solution of (1.1).*

Proof. (i) First show that u satisfies (2.2). Here we proceed as in [22]. Let $\varphi \in X \cap L^\infty(Q)$ such $\varphi_t \in X' + L^1(Q)$, $\varphi(\cdot, T) = 0$, and $S \in W^{2,\infty}(\mathbb{R})$, such that S' has compact support on \mathbb{R} , $S(0) = 0$. Let $M > 0$ such that $\text{supp} S' \subset [-M, M]$. Taking successively (φ, S) and $(\varphi\psi_\delta^\pm, S)$ as test functions in (2.2) applied to u_n , we can write

$$A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7, \quad A_{2,\delta,\pm} + A_{3,\delta,\pm} + A_{4,\delta,\pm} = A_{5,\delta,\pm} + A_{6,\delta,\pm} + A_{7,\delta,\pm},$$

where

$$A_1 = - \int_{\Omega} \varphi(0)S(u_{0,n})dx, \quad A_2 = - \int_Q \varphi_t S(U_n), \quad A_{2,\delta,\pm} = - \int_Q (\varphi\psi_\delta^\pm)_t S(U_n), \\ A_3 = \int_Q S'(U_n)A(x, t, \nabla u_n) \cdot \nabla \varphi, \quad A_{3,\delta,\pm} = \int_Q S'(U_n)A(x, t, \nabla u_n) \cdot \nabla (\varphi\psi_\delta^\pm), \\ A_4 = \int_Q S''(U_n)\varphi A(x, t, \nabla u_n) \cdot \nabla U_n, \quad A_{4,\delta,\pm} = \int_Q S''(U_n)\varphi\psi_\delta^\pm A(x, t, \nabla u_n) \cdot \nabla U_n, \\ A_5 = \int_Q S'(U_n)\varphi d\widehat{\lambda}_{n,0}, \quad A_6 = \int_Q S'(U_n)\varphi d\rho_{n,0}, \quad A_7 = - \int_Q S'(U_n)\varphi d\eta_{n,0}, \\ A_{5,\delta,\pm} = \int_Q S'(U_n)\varphi\psi_\delta^\pm d\widehat{\lambda}_{n,0}, \quad A_{6,\delta,\pm} = \int_Q S'(U_n)\varphi\psi_\delta^\pm d\rho_{n,0}, \quad A_{7,\delta,\pm} = - \int_Q S'(U_n)\varphi\psi_\delta^\pm d\eta_{n,0}.$$

Since $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$, and $\{S(U_n)\}$ converges to $S(U)$, strongly in X and weak-* in $L^\infty(Q)$, there holds, from (3.2),

$$A_1 = - \int_{\Omega} \varphi(0)S(u_0)dx + \omega(n), \quad A_2 = - \int_Q \varphi_t S(U) + \omega(n), \quad A_{2,\delta,\psi_\delta^\pm} = \omega(n, \delta).$$

Moreover $T_M(U_n)$ converges to $T_M(U)$, then $T_M(U_n) + h_n$ converges to $T_k(U) + h$ strongly in X , thus

$$A_3 = \int_Q S'(U_n)A(x, t, \nabla (T_M(U_n) + h_n)) \cdot \nabla \varphi = \int_Q S'(U)A(x, t, \nabla (T_M(U) + h)) \cdot \nabla \varphi + \omega(n) \\ = \int_Q S'(U)A(x, t, \nabla u) \cdot \nabla \varphi + \omega(n);$$

and

$$\begin{aligned} A_4 &= \int_Q S''(U_n) \varphi A(x, t, \nabla(T_M(U_n) + h_n)) \cdot \nabla T_M(U_n) \\ &= \int_Q S''(U) \varphi A(x, t, \nabla(T_M(U) + h)) \cdot \nabla T_M(U) + \omega(n) = \int_Q S''(U) \varphi A(x, t, \nabla u) \cdot \nabla U + \omega(n). \end{aligned}$$

In the same way, since ψ_δ^\pm converges to 0 in X ,

$$\begin{aligned} A_{3,\delta,\pm} &= \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla(\varphi \psi_\delta^\pm) + \omega(n) = \omega(n, \delta), \\ A_{4,\delta,\pm} &= \int_Q S''(U) \varphi \psi_\delta^\pm A(x, t, \nabla u) \cdot \nabla U + \omega(n) = \omega(n, \delta). \end{aligned}$$

And $\{g_n\}$ strongly converges to g in $(L^{p'}(\Omega))^N$, thus

$$\begin{aligned} A_5 &= \int_Q S'(U_n) \varphi f_n + \int_Q S'(U_n) g_n \cdot \nabla \varphi + \int_Q S''(U_n) \varphi g_n \cdot \nabla T_M(U_n) \\ &= \int_Q S'(U) \varphi f + \int_Q S'(U) g \cdot \nabla \varphi + \int_Q S''(U) \varphi g \cdot \nabla T_M(U) + \omega(n) \\ &= \int_Q S'(U) \varphi d\widehat{\mu}_0 + \omega(n). \end{aligned}$$

Now $A_{5,\delta,\pm} = \int_Q S'(U) \varphi \psi_\delta^\pm d\widehat{\lambda}_{n,0} + \omega(n) = \omega(n, \delta)$. Then $A_{6,\delta,\pm} + A_{7,\delta,\pm} = \omega(n, \delta)$. From (3.2) we verify that $A_{7,\delta,+} = \omega(n, \delta)$ and $A_{6,\delta,-} = \omega(n, \delta)$. Moreover, from (3.6) and (3.2), we find

$$|A_6 - A_{6,\delta,+}| \leq \int_Q |S'(U_n) \varphi| (1 - \psi_\delta^+) d\rho_{n,0} \leq \|S\|_{W^{2,\infty}(\mathbb{R})} \|\varphi\|_{L^\infty(Q)} \int_Q (1 - \psi_\delta^+) d\rho_n = \omega(n, \delta).$$

Similarly we also have $|A_7 - A_{7,\delta,-}| \leq \omega(n, \delta)$. Hence $A_6 = \omega(n)$ and $A_7 = \omega(n)$. Therefore, we finally obtain (2.2):

$$-\int_\Omega \varphi(0) S(u_0) dx - \int_Q \varphi_t S(U) + \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(U) \varphi A(x, t, \nabla u) \cdot \nabla U = \int_Q S'(U) \varphi d\widehat{\mu}_0. \quad (3.30)$$

(ii) Next, we prove (2.3) and (2.4). We take $\varphi \in C_c^\infty(Q)$ and take $((1 - \psi_\delta^-) \varphi, \overline{H_m})$ as test functions in (3.30), with $\overline{H_m}$ as in (2.14). We can write $D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m}$, where

$$\begin{aligned} D_{1,m} &= -\int_Q ((1 - \psi_\delta^-) \varphi)_t \overline{H_m}(U), & D_{2,m} &= \int_Q H_m(U) A(x, t, \nabla u) \cdot \nabla ((1 - \psi_\delta^-) \varphi), \\ D_{3,m} &= \int_Q H_m(U) (1 - \psi_\delta^-) \varphi d\widehat{\mu}_0, & D_{4,m} &= \frac{1}{m} \int_{m \leq U \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla U, \\ D_{5,m} &= -\frac{1}{m} \int_{-2m \leq U \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla U. \end{aligned} \quad (3.31)$$

Taking the same test functions in (2.2) applied to u_n , there holds $D_{1,m}^n + D_{2,m}^n = D_{3,m}^n + D_{4,m}^n + D_{5,m}^n$, where

$$\begin{aligned} D_{1,m}^n &= - \int_Q ((1 - \psi_\delta^-) \varphi)_t \overline{H_m}(U_n), & D_{2,m}^n &= \int_Q H_m(U_n) A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-) \varphi), \\ D_{3,m}^n &= \int_Q H_m(U_n) (1 - \psi_\delta^-) \varphi d(\widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0}), & D_{4,m}^n &= \frac{1}{m} \int_{m \leq U \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla U_n, \\ D_{5,m}^n &= - \frac{1}{m} \int_{-2m \leq U_n \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla U_n \end{aligned} \quad (3.32)$$

In (3.32), we go to the limit as $m \rightarrow \infty$. Since $\{\overline{H_m}(U_n)\}$ converges to U_n and $\{H_m(U_n)\}$ converges to 1, a.e. in Q , and $\{\nabla H_m(U_n)\}$ converges to 0, weakly in $(L^p(Q))^N$, we obtain the relation $D_1^n + D_2^n = D_3^n + D_4^n$, where

$$\begin{aligned} D_1^n &= - \int_Q ((1 - \psi_\delta^-) \varphi)_t U_n, & D_2^n &= \int_Q A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-) \varphi), & D_3^n &= \int_Q (1 - \psi_\delta^-) \varphi d\widehat{\lambda_{n,0}} \\ D_4^n &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_{n,0} - \eta_{n,0}) + \int_Q (1 - \psi_\delta^-) \varphi d((\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-) \\ &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_n - \eta_n). \end{aligned}$$

Clearly, $D_{i,m} - D_i^n = \omega(n, m)$ for $i = 1, 2, 3$. From Lemma (3.3) and (3.2)-(3.4), we obtain $D_{5,m} = \omega(n, m, \delta)$, and

$$\frac{1}{m} \int_{\{m \leq U < 2m\}} \psi_\delta^- \varphi A(x, t, \nabla u) \cdot \nabla U = \omega(n, m, \delta),$$

thus,

$$D_{4,m} = \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U + \omega(n, m, \delta).$$

Since $\left| \int_Q (1 - \psi_\delta^-) \varphi d\eta_n \right| \leq \|\varphi\|_{L^\infty} \int_Q (1 - \psi_\delta^-) d\eta_n$, it follows that $\int_Q (1 - \psi_\delta^-) \varphi d\eta_n = \omega(n, m, \delta)$ from (3.4). And $\left| \int_Q \psi_\delta^- \varphi d\rho_n \right| \leq \|\varphi\|_{L^\infty} \int_Q \psi_\delta^- d\rho_n$, thus, from (3.2), $\int_Q (1 - \psi_\delta^-) \varphi d\rho_n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$. Then $D^n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$. Therefore by subtraction, we get successively

$$\begin{aligned} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U &= \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta), \\ \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U &= \int_Q \varphi d\mu_s^+, \end{aligned} \quad (3.33)$$

which proves (2.3) when $\varphi \in C_c^\infty(Q)$. Next assume only $\varphi \in C^\infty(\overline{Q})$. Then

$$\begin{aligned} &\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U \\ &= \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi \psi_\delta^+ A(x, t, \nabla u) \cdot \nabla U + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla U \\ &= \int_Q \varphi \psi_\delta^+ d\mu_s^+ + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla U = \int_Q \varphi d\mu_s^+ + D, \end{aligned}$$

where

$$D = \int_Q \varphi(1 - \psi_\delta^+) d\mu_s^+ + \lim_{n \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi(1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla U = \omega(\delta).$$

Therefore, (3.33) still holds for $\varphi \in C^\infty(\overline{Q})$, and we deduce (2.3) by density, and similarly, (2.4). This completes the proof of Theorem 1.1. \blacksquare

4 Approximations of measures

Corollary 1.2 is a direct consequence of Theorem 1.1 and the following approximation property:

Proposition 4.1 *Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b^+(Q)$ with $\mu_0 \in \mathcal{M}_0^+(Q)$ and $\mu_s \in \mathcal{M}_s^+(Q)$.*

(i) Then, we can find a decomposition $\mu_0 = (f, g, h)$ with $f \in L^1(Q)$, $g \in (L^{p'}(Q))^N$, $h \in X$ such that

$$\|f\|_{1,Q} + \|g\|_{p',Q} + \|h\|_X + \mu_s(\Omega) \leq 2\mu(Q) \quad (4.1)$$

(ii) Furthermore, there exists sequences of measures $\mu_{0,n} = (f_n, g_n, h_n)$, $\mu_{s,n}$ such that $f_n, g_n, h_n \in C_c^\infty(Q)$ strongly converge to f, g, h in $L^1(Q)$, $(L^{p'}(Q))^N$ and X respectively, and $\mu_{s,n} \in (C_c^\infty(Q))^+$ converges to μ_s and $\mu_n := \mu_{0,n} + \mu_{s,n}$ converges to μ in the narrow topology, and satisfying $|\mu_n|(Q) \leq \mu(Q)$,

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X + \mu_{s,n}(Q) \leq 2\mu(Q). \quad (4.2)$$

Proof. (i) Step 1. Case where μ has a compact support in Q . By [15], we can find a decomposition $\mu_0 = (f, g, h)$ with f, g, h have a compact support in Q . Let $\{\varphi_n\}$ be sequence of mollifiers in \mathbb{R}^{N+1} . Then $\mu_{0,n} = \varphi_n * \mu_0 \in C_c^\infty(Q)$ for n large enough. We see that $\mu_{0,n}(Q) = \mu_0(Q)$ and $\mu_{0,n}$ admits the decomposition $\mu_{0,n} = (f_n, g_n, h_n) = (\varphi_n * f, \varphi_n * g, \varphi_n * h)$. Since $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, we have for n_0 large enough,

$$\|f - f_{n_0}\|_{1,Q} + \|g - g_{n_0}\|_{p',Q} + \|h - h_{n_0}\|_{L^p((0,T);W_0^{1,p}(\Omega))} \leq \frac{1}{2}\mu_0(Q).$$

Then we obtain a decomposition $\mu = (\hat{f}, \hat{g}, \hat{h}) = (\mu_{n_0} + f - f_{n_0}, g - g_{n_0}, h - h_{n_0})$, such that

$$\|\hat{f}\|_{1,Q} + \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X + \mu_s(Q) \leq \frac{3}{2}\mu(Q) \quad (4.3)$$

Step 2. General case. Let $\{\theta_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, a.e. in Q . Set $\tilde{\mu}_0 = \theta_0\mu$, and $\tilde{\mu}_n = (\theta_n - \theta_{n-1})\mu$, for any $n \geq 1$. Since $\tilde{\mu}_n = \tilde{\mu}_{0,n} + \tilde{\mu}_{s,n} \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ has compact support with $\tilde{\mu}_{0,n} \in \mathcal{M}_0(Q)$, $\tilde{\mu}_{s,n} \in \mathcal{M}_s(Q)$, by Step 1, we can find a decomposition $\tilde{\mu}_{0,n} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ such that

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X + \tilde{\mu}_{s,n}(\Omega) \leq \frac{3}{2}\tilde{\mu}_n(Q).$$

Let $\bar{f}_n = \sum_{k=0}^n \tilde{f}_k$, $\bar{g}_n = \sum_{k=0}^n \tilde{g}_k$, $\bar{h}_n = \sum_{k=0}^n \tilde{h}_k$ and $\bar{\mu}_{s,n} = \sum_{k=0}^n \tilde{\mu}_{s,k}$. Clearly, $\theta_n\mu_0 = (\bar{f}_n, \bar{g}_n, \bar{h}_n)$, $\theta_n\mu_s = \bar{\mu}_{s,n}$ and $\{\bar{f}_n\}, \{\bar{g}_n\}, \{\bar{h}_n\}$ and $\{\bar{\mu}_{s,n}\}$ converge strongly to some f, g, h , and μ_s respectively in $L^1(Q), (L^{p'}(Q))^N$, X and $\mathcal{M}_b^+(Q)$, and

$$\|\bar{f}_n\|_{1,Q} + \|\bar{g}_n\|_{p',Q} + \|\bar{h}_n\|_X + \bar{\mu}_{s,n}(Q) \leq \frac{3}{2}\mu(Q).$$

Therefore, $\mu_0 = (f, g, h)$, and (4.1) holds.

(ii) We take a sequence $\{m_n\}$ in \mathbb{N} such that $f_n = \varphi_{m_n} * \bar{f}_n$, $g_n = \varphi_{m_n} * \bar{g}_n$, $h_n = \varphi_{m_n} * \bar{h}_n$, $\varphi_{m_n} * \bar{\mu}_{s,n} \in (C_c^\infty(Q))^+$, $\int_Q \varphi_{m_n} * \bar{\mu}_{s,n} dxdt = \bar{\mu}_{s,n}(Q)$ and

$$\|f_n - \bar{f}_n\|_{1,Q} + \|g_n - \bar{g}_n\|_{p',Q} + \|h_n - \bar{h}_n\|_X \leq \frac{1}{n+2} \mu(Q).$$

Let $\mu_{0,n} = \varphi_{m_n} * (\theta_n \mu_0) = (f_n, g_n, h_n)$, $\mu_{s,n} = \varphi_{m_n} * \bar{\mu}_{s,n}$ and $\mu_n = \mu_{0,n} + \mu_{s,n}$. Therefore, $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively. And (4.2) holds. Furthermore, $\{\mu_{s,n}\}, \{\mu_n\}$ converge to μ_s, μ in the weak topology of measures, and $\mu_{s,n}(Q) = \int_Q \theta_n d\mu_s, \mu_n(Q) = \int_Q \theta_n d\mu$ converges to $\mu_s(Q), \mu(Q)$, thus $\{\mu_{s,n}\}, \{\mu_n\}$ converges to μ_s, μ in the narrow topology and $|\mu_n|(Q) \leq \mu(Q)$. ■

Observe that part (i) of Proposition 4.1 was used in [22], even if there was no explicit proof. Otherwise part (ii) is a *key point* for finding applications to the stability Theorem. Note also a very useful consequence for approximations by *nondecreasing* sequences:

Proposition 4.2 *Let $\mu \in \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$. Let $\{\mu_n\}$ be a nondecreasing sequence in $\mathcal{M}_b^+(Q)$ converging to μ in $\mathcal{M}_b(Q)$. Then, there exist $f_n, f \in L^1(Q)$, $g_n, g \in (L^{p'}(Q))^N$ and $h_n, h \in X$, $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$ such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \quad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, and $\{\mu_{n,s}\}$ converges to μ_s (strongly) in $\mathcal{M}_b(Q)$ and

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X + \mu_{n,s}(\Omega) \leq 2\mu(Q). \quad (4.4)$$

Proof. Since $\{\mu_n\}$ is nondecreasing, then $\{\mu_{n,0}\}, \{\mu_{n,s}\}$ are nondecreasing too. Clearly, $\|\mu - \mu_n\|_{\mathcal{M}_b(Q)} = \|\mu_0 - \mu_{n,0}\|_{\mathcal{M}_b(Q)} + \|\mu_s - \mu_{n,s}\|_{\mathcal{M}_b(Q)}$. Hence, $\{\mu_{n,s}\}$ converges to μ_s and $\{\mu_{n,0}\}$ converges to μ_0 (strongly) in $\mathcal{M}_b(Q)$. Set $\tilde{\mu}_{0,0} = \mu_{0,0}$, and $\tilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$ for any $n \geq 1$. By Proposition 4.1, (i), we can find $\tilde{f}_n \in L^1(Q)$, $\tilde{g}_n \in (L^{p'}(Q))^N$ and $\tilde{h}_n \in X$ such that $\tilde{\mu}_{n,0} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ and

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq 2\tilde{\mu}_{n,0}(Q)$$

Let $f_n = \sum_{k=0}^n \tilde{f}_k$, $G_n = \sum_{k=0}^n \tilde{g}_k$ and $h_n = \sum_{k=0}^n \tilde{h}_k$. Clearly, $\mu_{n,0} = (f_n, g_n, h_n)$ and the convergence properties hold with (4.4), since

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X \leq 2\mu_0(Q). \quad \blacksquare$$

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