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► **To cite this version:**

Marie-Françoise Bidaut-Véron, Quoc-Hung Nguyen. Stability properties for quasilinear parabolic equations with measure data. *Journal of the European Mathematical Society*, 2015, 17, pp.2103-2135. 10.4171/JEMS/552 . hal-01060682

**HAL Id: hal-01060682**

**<https://hal.science/hal-01060682>**

Submitted on 4 Sep 2014

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# Stability properties for quasilinear parabolic equations with measure data

Marie-Françoise BIDAUT-VERON\*

Quoc-Hung NGUYEN†

## Abstract

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ , and  $Q = \Omega \times (0, T)$ . We study problems of the model type

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where  $p > 1$ ,  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in L^1(\Omega)$ . Our main result is a *stability theorem* extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators  $u \mapsto \mathcal{A}(u) = \text{div}(A(x, t, \nabla u))$ .

## 1 Introduction

Let  $\Omega$  be a bounded domain of  $\mathbb{R}^N$ , and  $Q = \Omega \times (0, T)$ ,  $T > 0$ . We denote by  $\mathcal{M}_b(\Omega)$  and  $\mathcal{M}_b(Q)$  the sets of bounded Radon measures on  $\Omega$  and  $Q$  respectively. We are concerned with the problem

$$\begin{cases} u_t - \text{div}(A(x, t, \nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where  $\mu \in \mathcal{M}_b(Q)$ ,  $u_0 \in L^1(\Omega)$  and  $A$  is a Caratheodory function on  $Q \times \mathbb{R}^N$ , such that for *a.e.*  $(x, t) \in Q$ , and any  $\xi, \zeta \in \mathbb{R}^N$ ,

$$A(x, t, \xi) \cdot \xi \geq \Lambda_1 |\xi|^p, \quad |A(x, t, \xi)| \leq a(x, t) + \Lambda_2 |\xi|^{p-1}, \quad \Lambda_1, \Lambda_2 > 0, a \in L^{p'}(Q), \quad (1.2)$$

$$(A(x, t, \xi) - A(x, t, \zeta)) \cdot (\xi - \zeta) > 0 \quad \text{if } \xi \neq \zeta, \quad (1.3)$$

for  $p > 1$ . This includes the model problem where  $\text{div}(A(x, t, \nabla u)) = \Delta_p u$ , where  $\Delta_p$  is the  $p$ -Laplacian.

The corresponding elliptic problem:

$$-\Delta_p u = \mu \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

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with  $\mu \in \mathcal{M}_b(\Omega)$ , was studied in [9, 10] for  $p > 2 - 1/N$ , leading to the existence of solutions in the sense of distributions. For any  $p > 1$ , and  $\mu \in L^1(\Omega)$ , existence and uniqueness are proved in [4] in the class of *entropy solutions*. For any  $\mu \in \mathcal{M}_b(\Omega)$  the main work is done in [14, Theorems 3.1, 3.2], where not only existence is proved in the class of *renormalized solutions*, but also a stability result, fundamental for applications.

Concerning problem (1.1), the first studies concern the case  $\mu \in L^{p'}(Q)$  and  $u_0 \in L^2(\Omega)$ , where existence and uniqueness are obtained by variational methods, see [19]. In the general case  $\mu \in \mathcal{M}_b(Q)$  and  $u_0 \in \mathcal{M}_b(\Omega)$ , the pionner results come from [9], proving the existence of solutions in the sense of distributions for

$$p > p_1 = 2 - \frac{1}{N+1}, \quad (1.4)$$

see also [11]. The approximated solutions of (1.1) lie in Marcinkiewicz spaces  $u \in L^{p_c, \infty}(Q)$  and  $|\nabla u| \in L^{m_c, \infty}(Q)$ , where

$$p_c = p - 1 + \frac{p}{N}, \quad m_c = p - \frac{N}{N+1}. \quad (1.5)$$

This condition (1.4) ensures that  $u$  and  $|\nabla u|$  belong to  $L^1(Q)$ , since  $m_c > 1$  means  $p > p_1$  and  $p_c > 1$  means  $p > 2N/(N+1)$ . Uniqueness follows in the case  $p = 2$ ,  $A(x, t, \nabla u) = \nabla u$ , by duality methods, see [21].

For  $\mu \in L^1(Q)$ , uniqueness is obtained in new classes of *entropy solutions*, and *renormalized solutions*, see [5, 26, 27].

A larger set of measures is studied in [15]. They introduce a notion of parabolic capacity initiated and inspired by [24], used after in [22, 23], defined by

$$c_p^Q(E) = \inf_{E \subset U} \inf_{\text{open } C \subset Q} \{ \|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \},$$

for any Borel set  $E \subset Q$ , where setting  $X = L^p((0, T); W_0^{1,p}(\Omega) \cap L^2(\Omega))$ ,

$$W = \{ z : z \in X, z_t \in X' \}, \text{ embedded with the norm } \|u\|_W = \|u\|_X + \|u_t\|_{X'}.$$

Let  $\mathcal{M}_0(Q)$  be the set of Radon measures  $\mu$  on  $Q$  that do not charge the sets of zero  $c_p^Q$ -capacity:

$$\forall E \text{ Borel set } \subset Q, \quad c_p^Q(E) = 0 \implies |\mu|(E) = 0.$$

Then existence and uniqueness of renormalized solutions of (1.1) hold for any measure  $\mu \in \mathcal{M}_b(Q) \cap \mathcal{M}_0(Q)$ , called *soft (or diffuse, or regular) measure*, and  $u_0 \in L^1(\Omega)$ , and  $p > 1$ . The equivalence with the notion of entropy solutions is shown in [16]. For such a soft measure, an extension to equations of type  $(b(u))_t - \Delta_p u = \mu$  is given in [6]; another formulation is used in [23] for solving a perturbed problem from (1.1) by an absorption term.

Next consider an *arbitrary measure*  $\mu \in \mathcal{M}_b(Q)$ . Let  $\mathcal{M}_s(Q)$  be the set of all bounded Radon measures on  $Q$  with support on a set of zero  $c_p^Q$ -capacity, also called *singular*. Let  $\mathcal{M}_b^+(Q), \mathcal{M}_0^+(Q), \mathcal{M}_s^+(Q)$  be the positive cones of  $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$ . From [15],  $\mu$  can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \quad \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \quad \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q), \quad (1.6)$$

and  $\mu_0 \in \mathcal{M}_0(Q)$  admits (at least) a decomposition under the form

$$\mu_0 = f - \text{div } g + h_t, \quad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in X, \quad (1.7)$$

and we write  $\mu_0 = (f, g, h)$ . Conversely, any measure of this form, *such that*  $h \in L^\infty(Q)$ , lies in  $\mathcal{M}_0(Q)$ , see [23, Proposition 3.1]. The solutions of (1.1) are searched in a renormalized sense linked to this decomposition, introduced in [15, 22]. In the range (1.4) the existence of a renormalized solution relative to the

decomposition (1.7) is proved in [22], using suitable approximations of  $\mu_0$  and  $\mu_s$ . Uniqueness is still open, as well as in the elliptic case.

In *all the sequel* we suppose that  $p$  satisfies (1.4). Then the embedding  $W_0^{1,p}(\Omega) \subset L^2(\Omega)$  is valid, that means

$$X = L^p((0, T); W_0^{1,p}(\Omega)), \quad X' = L^{p'}((0, T); W^{-1,p'}(\Omega)).$$

In Section 2 we recall the definition of renormalized solutions, given in [22], that we call R-solutions of (1.1), relative to the decomposition (1.7) of  $\mu_0$ , and study some of their properties. Our main result is a *stability theorem* for problem (1.1), proved in Section 3, extending to the parabolic case the stability result of [14, Theorem 3.4]. In order to state it, we recall that a sequence of measures  $\mu_n \in \mathcal{M}_b(Q)$  converges to a measure  $\mu \in \mathcal{M}_b(Q)$  in the *narrow topology* of measures if

$$\lim_{n \rightarrow \infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu \quad \forall \varphi \in C(Q) \cap L^\infty(Q).$$

**Theorem 1.1** *Let  $A : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  satisfy (1.2), (1.3). Let  $u_0 \in L^1(\Omega)$ , and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q),$$

*with  $f \in L^1(Q)$ ,  $g \in (L^{p'}(Q))^N$ ,  $h \in X$  and  $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q)$ . Let  $u_{0,n} \in L^1(\Omega)$ ,*

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(Q),$$

*with  $f_n \in L^1(Q)$ ,  $g_n \in (L^{p'}(Q))^N$ ,  $h_n \in X$ , and  $\rho_n, \eta_n \in \mathcal{M}_b^+(Q)$ , such that*

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \quad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

*with  $\rho_n^1, \eta_n^1 \in L^1(Q)$ ,  $\rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$  and  $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q)$ . Assume that*

$$\sup_n |\mu_n|(Q) < \infty,$$

*and  $\{u_{0,n}\}$  converges to  $u_0$  strongly in  $L^1(\Omega)$ ,  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$ ,  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ ,  $\{h_n\}$  converges to  $h$  strongly in  $X$ ,  $\{\rho_n\}$  converges to  $\mu_s^+$  and  $\{\eta_n\}$  converges to  $\mu_s^-$  in the narrow topology; and  $\{\rho_n^1\}, \{\eta_n^1\}$  are bounded in  $L^1(Q)$ , and  $\{\rho_n^2\}, \{\eta_n^2\}$  bounded in  $(L^{p'}(Q))^N$ .*

*Let  $\{u_n\}$  be a sequence of R-solutions of*

$$\begin{cases} u_{n,t} - \operatorname{div}(A(x, t, \nabla u_n)) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0, T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases} \quad (1.8)$$

*relative to the decomposition  $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$  of  $\mu_{n,0}$ . Let  $U_n = u_n - h_n$ .*

*Then up to a subsequence,  $\{u_n\}$  converges a.e. in  $Q$  to a R-solution  $u$  of (1.1), and  $\{U_n\}$  converges a.e. in  $Q$  to  $U = u - h$ . Moreover,  $\{\nabla u_n\}, \{\nabla U_n\}$  converge respectively to  $\nabla u, \nabla U$  a.e. in  $Q$ , and  $\{T_k(U_n)\}$  converge to  $T_k(U)$  strongly in  $X$  for any  $k > 0$ .*

In Section 4 we check that any measure  $\mu \in \mathcal{M}_b(Q)$  can be approximated in the sense of the stability Theorem, hence we find again the existence result of [22]:

**Corollary 1.2** *Let  $u_0 \in L^1(\Omega)$  and  $\mu \in \mathcal{M}_b(Q)$ . Then there exists a  $R$ -solution  $u$  to the problem (1.1) with data  $(\mu, u_0)$ .*

Moreover we give more precise properties of approximations of  $\mu \in \mathcal{M}_b(Q)$ , fundamental for applications, see Propositions 4.1 and 4.2. As in the elliptic case, Theorem 1.1 is a key point for obtaining existence results for more general problems, and we give some of them in [2, 3, 20], for measures  $\mu$  satisfying suitable capacity conditions. In [2] we study perturbed problems of order 0, of type

$$u_t - \Delta_p u + \mathcal{G}(u) = \mu \quad \text{in } Q, \quad (1.9)$$

where  $\mathcal{G}(u)$  is an absorption or a source term with a growth of power or exponential type, and  $\mu$  is a good in time measure. In [3] we use potential estimates to give other existence results in case of absorption with  $p > 2$ . In [20], one considers equations of the form

$$u_t - \operatorname{div}(A(x, t, \nabla u)) + \mathcal{G}(u, \nabla u) = \mu$$

under (1.2),(1.3) with  $p = 2$ , and extend in particular the results of [1] to nonlinear operators.

## 2 Renormalized solutions of problem (1.1)

### 2.1 Notations and Definition

For any function  $f \in L^1(Q)$ , we write  $\int_Q f$  instead of  $\int_Q f dx dt$ , and for any measurable set  $E \subset Q$ ,  $\int_E f$  instead of  $\int_E f dx dt$ . For any open set  $\varpi$  of  $\mathbb{R}^m$  and  $F \in (L^k(\varpi))^\nu$ ,  $k \in [1, \infty]$ ,  $m, \nu \in \mathbb{N}^*$ , we set  $\|F\|_{k, \varpi} = \|F\|_{(L^k(\varpi))^\nu}$ .

We set  $T_k(r) = \max\{\min\{r, k\}, -k\}$ , for any  $k > 0$  and  $r \in \mathbb{R}$ . We recall that if  $u$  is a measurable function defined and finite *a.e.* in  $Q$ , such that  $T_k(u) \in X$  for any  $k > 0$ , there exists a measurable function  $w$  from  $Q$  into  $\mathbb{R}^N$  such that  $\nabla T_k(u) = \chi_{|u| \leq k} w$ , *a.e.* in  $Q$ , and for any  $k > 0$ . We define the gradient  $\nabla u$  of  $u$  by  $w = \nabla u$ .

Let  $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$ , and  $(f, g, h)$  be a decomposition of  $\mu_0$  given by (1.7), and  $\widehat{\mu}_0 = \mu_0 - h_t = f - \operatorname{div} g$ . In the general case  $\widehat{\mu}_0 \notin \mathcal{M}(Q)$ , but we write, for convenience,

$$\int_Q w d\widehat{\mu}_0 := \int_Q (fw + g \cdot \nabla w), \quad \forall w \in X \cap L^\infty(Q).$$

**Definition 2.1** *Let  $u_0 \in L^1(\Omega)$ ,  $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$ . A measurable function  $u$  is a **renormalized solution**, called **R-solution** of (1.1) if there exists a decomposition  $(f, g, h)$  of  $\mu_0$  such that*

$$U = u - h \in L^\sigma((0, T); W_0^{1, \sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega)), \quad \forall \sigma \in [1, m_c]; \quad T_k(U) \in X, \quad \forall k > 0, \quad (2.1)$$

and:

(i) *for any  $S \in W^{2, \infty}(\mathbb{R})$  such that  $S'$  has compact support on  $\mathbb{R}$ , and  $S(0) = 0$ ,*

$$-\int_\Omega S(u_0) \varphi(0) dx - \int_Q \varphi_t S(U) + \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(U) \varphi A(x, t, \nabla u) \cdot \nabla U = \int_Q S'(U) \varphi d\widehat{\mu}_0, \quad (2.2)$$

*for any  $\varphi \in X \cap L^\infty(Q)$  such that  $\varphi_t \in X' + L^1(Q)$  and  $\varphi(\cdot, T) = 0$ ;*

(ii) for any  $\phi \in C(\overline{Q})$ ,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla U = \int_Q \phi d\mu_s^+ \quad (2.3)$$

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{-m \geq U > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla U = \int_Q \phi d\mu_s^-. \quad (2.4)$$

**Remark 2.2** As a consequence,  $S(U) \in C([0, T]; L^1(\Omega))$  and  $S(U)(\cdot, 0) = S(u_0)$  in  $\Omega$ ; and  $u$  satisfies the equation

$$(S(U))_t - \operatorname{div}(S'(U)A(x, t, \nabla u)) + S''(U)A(x, t, \nabla u) \cdot \nabla U = fS'(U) - \operatorname{div}(gS'(U)) + S''(U)g \cdot \nabla U, \quad (2.5)$$

in the sense of distributions in  $Q$ , see [22, Remark 3]. Moreover assume that  $[-k, k] \supset \operatorname{supp} S'$ . then from (1.2) and the Hölder inequality, we find easily that

$$\begin{aligned} \|S(U)_t\|_{X' + L^1(Q)} &\leq C \|S\|_{W^{2, \infty}(\mathbb{R})} ( \|\nabla u\|^p \chi_{|U| \leq k}\|_{1, Q}^{1/p'} + \|\nabla u\|^p \chi_{|U| \leq k}\|_{1, Q} + \|\nabla T_k(U)\|_{p, Q}^p \\ &\quad + \|a\|_{p', Q} + \|a\|_{p', Q}^{p'} + \|f\|_{1, Q} + \|g\|_{p', Q} \|\nabla u\|^p \chi_{|U| \leq k}\|_{1, Q}^{1/p} + \|g\|_{p', Q} ), \end{aligned} \quad (2.6)$$

where  $C = C(p, \Lambda_2)$ . We also deduce that, for any  $\varphi \in X \cap L^\infty(Q)$ , such that  $\varphi_t \in X' + L^1(Q)$ ,

$$\begin{aligned} \int_\Omega S(U(T))\varphi(T)dx - \int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(U) + \int_Q S'(U)A(x, t, \nabla u) \cdot \nabla \varphi \\ + \int_Q S''(U)A(x, t, \nabla u) \cdot \nabla U \varphi = \int_Q S'(U)\varphi d\widehat{\mu}_0. \end{aligned} \quad (2.7)$$

**Remark 2.3** Let  $u, U$  satisfy (2.1). It is easy to see that the condition (2.3) ( resp. (2.4) ) is equivalent to

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^+ \quad (2.8)$$

resp.

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \geq U > -2m\}} \phi A(x, t, \nabla u) \cdot \nabla u = \int_Q \phi d\mu_s^-. \quad (2.9)$$

In particular, for any  $\varphi \in L^{p'}(Q)$  there holds

$$\lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |U| < 2m} |\nabla u| \varphi = 0, \quad \lim_{m \rightarrow \infty} \frac{1}{m} \int_{m \leq |U| < 2m} |\nabla U| \varphi = 0. \quad (2.10)$$

**Remark 2.4** (i) Any function  $U \in X$  such that  $U_t \in X' + L^1(Q)$  admits a unique  $c_p^Q$ -quasi continuous representative, defined  $c_p^Q$ -quasi a.e. in  $Q$ , still denoted  $U$ . Furthermore, if  $U \in L^\infty(Q)$ , then for any  $\mu_0 \in \mathcal{M}_0(Q)$ , there holds  $U \in L^\infty(Q, d\mu_0)$ , see [22, Theorem 3 and Corollary 1].

(ii) Let  $u$  be any  $R$ - solution of problem (1.1). Then,  $U = u - h$  admits a  $c_p^Q$ -quasi continuous functions representative which is finite  $c_p^Q$ -quasi a.e. in  $Q$ , and  $u$  satisfies definition 2.1 for every decomposition  $(\tilde{f}, \tilde{g}, \tilde{h})$  such that  $h - \tilde{h} \in L^\infty(Q)$ , see [22, Proposition 3 and Theorem 4 ].

## 2.2 Steklov and Landes approximations

A main difficulty for proving Theorem 1.1 is the choice of admissible test functions  $(S, \varphi)$  in (2.2), valid for any  $R$ -solution. Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations:

**Definition 2.5** Let  $\varepsilon \in (0, T)$  and  $z \in L^1_{loc}(Q)$ . For any  $l \in (0, \varepsilon)$  we define the **Steklov time-averages**  $[z]_l, [z]_{-l}$  of  $z$  by

$$[z]_l(x, t) = \frac{1}{l} \int_t^{t+l} z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (0, T - \varepsilon),$$

$$[z]_{-l}(x, t) = \frac{1}{l} \int_{t-l}^t z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (\varepsilon, T).$$

The idea to use this approximation for  $R$ -solutions can be found in [7]. Recall some properties, given in [23]. Let  $\varepsilon \in (0, T)$ , and  $\varphi_1 \in C_c^\infty(\overline{\Omega} \times [0, T])$ ,  $\varphi_2 \in C_c^\infty(\overline{\Omega} \times (0, T])$  with  $\text{Supp}\varphi_1 \subset \overline{\Omega} \times [0, T - \varepsilon]$ ,  $\text{Supp}\varphi_2 \subset \overline{\Omega} \times [\varepsilon, T]$ . There holds:

- (i) If  $z \in X$ , then  $\varphi_1[z]_l$  and  $\varphi_2[z]_{-l} \in W$ .
- (ii) If  $z \in X$  and  $z_t \in X' + L^1(Q)$ , then, as  $l \rightarrow 0$ ,  $(\varphi_1[z]_l)$  and  $(\varphi_2[z]_{-l})$  converge respectively to  $\varphi_1 z$  and  $\varphi_2 z$  in  $X$ , and *a.e.* in  $Q$ ; and  $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$  converge to  $(\varphi_1 z)_t, (\varphi_2 z)_t$  in  $X' + L^1(Q)$ .
- (iii) If moreover  $z \in L^\infty(Q)$ , then from any sequence  $\{l_n\} \rightarrow 0$ , there exists a subsequence  $\{l_\nu\}$  such that  $\{[z]_{l_\nu}\}, \{[z]_{-l_\nu}\}$  converge to  $z, c_p^Q$ -quasi everywhere in  $Q$ .

Next we recall the approximation used in several articles [8, 12, 11], first introduced in [17].

**Definition 2.6** Let  $k > 0$ , and  $y \in L^\infty(\Omega)$  and  $Y \in X$  such that  $\|y\|_{L^\infty(\Omega)} \leq k$  and  $\|Y\|_{L^\infty(Q)} \leq k$ . For any  $\nu \in \mathbb{N}$ , a **Landes-time approximation**  $\langle Y \rangle_\nu$  of the function  $Y$  is defined as follows:

$$\langle Y \rangle_\nu(x, t) = \nu \int_0^t Y(x, s) e^{\nu(s-t)} ds + e^{-\nu t} z_\nu(x), \quad \forall (x, t) \in Q.$$

where  $\{z_\nu\}$  is a sequence of functions in  $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ , such that  $\|z_\nu\|_{L^\infty(\Omega)} \leq k$ ,  $\{z_\nu\}$  converges to  $y$  *a.e.* in  $\Omega$ , and  $\nu^{-1} \|z_\nu\|_{W_0^{1,p}(\Omega)}^p$  converges to 0.

Therefore, we can verify that  $(\langle Y \rangle_\nu)_t \in X$ ,  $\langle Y \rangle_\nu \in X \cap L^\infty(Q)$ ,  $\|\langle Y \rangle_\nu\|_{\infty, Q} \leq k$  and  $\{\langle Y \rangle_\nu\}$  converges to  $Y$  strongly in  $X$  and *a.e.* in  $Q$ . Moreover,  $\langle Y \rangle_\nu$  satisfies the equation  $(\langle Y \rangle_\nu)_t = \nu(Y - \langle Y \rangle_\nu)$  in the sense of distributions in  $Q$ , and  $\langle Y \rangle_\nu(0) = z_\nu$  in  $\Omega$ . In this paper, we only use the **Landes-time approximation** of the function  $Y = T_k(U)$ , where  $y = T_k(u_0)$ .

## 2.3 First properties

In the sequel we use the following notations: for any function  $J \in W^{1,\infty}(\mathbb{R})$ , nondecreasing with  $J(0) = 0$ , we set

$$\overline{\mathcal{J}}(r) = \int_0^r J(\tau) d\tau, \quad \mathcal{J}(r) = \int_0^r J'(\tau) \tau d\tau. \quad (2.11)$$

It is easy to verify that  $\mathcal{J}(r) \geq 0$ ,

$$\mathcal{J}(r) + \overline{\mathcal{J}}(r) = J(r)r, \quad \text{and} \quad \mathcal{J}(r) - \mathcal{J}(s) \geq s(J(r) - J(s)) \quad \forall r, s \in \mathbb{R}. \quad (2.12)$$

In particular we define, for any  $k > 0$ , and any  $r \in \mathbb{R}$ ,

$$\overline{T}_k(r) = \int_0^r T_k(\tau) d\tau, \quad \mathcal{T}_k(r) = \int_0^r T'_k(\tau) \tau d\tau, \quad (2.13)$$

and we use several times a truncature used in [14]:

$$H_m(r) = \chi_{[-m, m]}(r) + \frac{2m - |s|}{m} \chi_{m < |s| \leq 2m}(r), \quad \overline{H}_m(r) = \int_0^r H_m(\tau) d\tau. \quad (2.14)$$

The next Lemma allows to extend the range of the test functions in (2.2).

**Lemma 2.7** *Let  $u$  be a  $R$ -solution of problem (1.1). Let  $J \in W^{1, \infty}(\mathbb{R})$  be nondecreasing with  $J(0) = 0$ , and  $\overline{J}$  defined by (2.11). Then,*

$$\begin{aligned} & \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla (\xi J(S(U))) + \int_Q S''(U) A(x, t, \nabla u) \cdot \nabla U \xi J(S(U)) \\ & - \int_{\Omega} \xi(0) J(S(u_0)) S(u_0) dx - \int_Q \xi_t \overline{J}(S(U)) \leq \int_Q S'(U) \xi J(S(U)) d\widehat{\mu}_0, \end{aligned} \quad (2.15)$$

for any  $S \in W^{2, \infty}(\mathbb{R})$  such that  $S'$  has compact support on  $\mathbb{R}$  and  $S(0) = 0$ , and for any  $\xi \in C^1(Q) \cap W^{1, \infty}(Q)$ ,  $\xi \geq 0$ .

**Proof.** Let  $\mathcal{J}$  be defined by (2.11). Let  $\zeta \in C_c^1([0, T])$  with values in  $[0, 1]$ , such that  $\zeta_t \leq 0$ , and  $\varphi = \zeta \xi [j(S(U))]_l$ . Clearly,  $\varphi \in X \cap L^\infty(Q)$ ; we choose the pair of functions  $(\varphi, S)$  as test function in (2.2). From the convergence properties of Steklov time-averages, we easily will obtain (2.15) if we prove that

$$\lim_{l \rightarrow 0, \zeta \rightarrow 1} \left( - \int_Q (\zeta \xi [j(S(U))]_l)_t S(U) \geq - \int_Q \xi_t \overline{J}(S(U)) \right).$$

We can write  $- \int_Q (\zeta \xi [j(S(U))]_l)_t S(U) = F + G$ , with

$$F = - \int_Q (\zeta \xi)_t [j(S(U))]_l S(U), \quad G = - \int_Q \zeta \xi S(U) \frac{1}{l} (j(S(U))(x, t + l) - j(S(U))(x, t)).$$

Using (2.12) and integrating by parts we have

$$\begin{aligned} G & \geq - \int_Q \zeta \xi \frac{1}{l} (\mathcal{J}(S(U))(x, t + l) - \mathcal{J}(S(U))(x, t)) = - \int_Q \zeta \xi \frac{\partial}{\partial t} ([\mathcal{J}(S(U))]_l) \\ & = \int_Q (\zeta \xi)_t [\mathcal{J}(S(U))]_l + \int_{\Omega} \zeta(0) \xi(0) [\mathcal{J}(S(U))]_l(0) dx \geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(U))]_l, \end{aligned}$$

since  $\mathcal{J}(S(U)) \geq 0$ . Hence,

$$- \int_Q (\zeta \xi [j(S(U))]_l)_t S(U) \geq \int_Q (\zeta \xi)_t [\mathcal{J}(S(U))]_l + F = \int_Q (\zeta \xi)_t ([\mathcal{J}(S(U))]_l - [J(S(U))]_l S(U)).$$

Otherwise,  $\mathcal{J}(S(U))$  and  $J(S(U)) \in C([0, T]; L^1(\Omega))$ , thus  $\{(\zeta \xi)_t ([\mathcal{J}(S(u))]_l - [J(S(u))]_l S(u))\}$  converges to  $-(\zeta \xi)_t \overline{J}(S(u))$  in  $L^1(Q)$  as  $l \rightarrow 0$ . Therefore,

$$\lim_{l \rightarrow 0, \zeta \rightarrow 1} \left( - \int_Q (\zeta \xi [J(S(U))]_l)_t S(U) \right) \geq \lim_{\zeta \rightarrow 1} \left( - \int_Q (\zeta \xi)_t \overline{J}(S(U)) \right) \geq - \int_Q \xi_t \overline{J}(S(U)),$$



which achieves the proof.  $\blacksquare$

Next we give estimates of the function and its gradient, following the first ones of [11], inspired by the estimates of the elliptic case of [4]. In particular we extend and make more precise the a priori estimates of [22, Proposition 4] given for solutions with smooth data; see also [15, 18].

**Proposition 2.8** *If  $u$  is a  $R$ -solution of problem (1.1), then there exists  $C_1 = C_1(p, \Lambda_1, \Lambda_2)$  such that, for any  $k \geq 1$  and  $\ell \geq 0$ ,*

$$\int_{\ell \leq |U| \leq \ell+k} |\nabla u|^p + \int_{\ell \leq |U| \leq \ell+k} |\nabla U|^p \leq C_1 k M, \quad (2.16)$$

$$\|U\|_{L^\infty((0,T);L^1(\Omega))} \leq C_1(M + |\Omega|), \quad (2.17)$$

where  $M = \|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q} + \|g\|_{p',Q}^{p'} + \|h\|_X^p + \|a\|_{p',Q}^{p'}$ .  
As a consequence, for any  $k \geq 1$ ,

$$\text{meas}\{|U| > k\} \leq C_2 M_1 k^{-pc}, \quad \text{meas}\{|\nabla U| > k\} \leq C_2 M_2 k^{-mc}, \quad (2.18)$$

$$\text{meas}\{|u| > k\} \leq C_2 M_2 k^{-pc}, \quad \text{meas}\{|\nabla u| > k\} \leq C_2 M_2 k^{-mc}, \quad (2.19)$$

where  $C_2 = C_2(N, p, \Lambda_1, \Lambda_2)$ , and  $M_1 = (M + |\Omega|)^{\frac{p}{N}} M$  and  $M_2 = M_1 + M$ .

**Proof.** Set for any  $r \in \mathbb{R}$ , and  $m, k, \ell > 0$ ,

$$T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.$$

For  $m > k + \ell$ , we can choose  $(J, S, \xi) = (T_{k,\ell}, \overline{H_m}, \xi)$  as test functions in (2.15), where  $\overline{H_m}$  is defined in (2.14) and  $\xi \in C^1([0, T])$  with values in  $[0, 1]$ , independent on  $x$ . Since  $T_{k,\ell}(\overline{H_m}(r)) = T_{k,\ell}(r)$  for all  $r \in \mathbb{R}$ , we obtain

$$\begin{aligned} & - \int_{\Omega} \xi(0) T_{k,\ell}(u_0) \overline{H_m}(u_0) dx - \int_Q \xi_t \overline{T_{k,\ell}}(\overline{H_m}(U)) \\ & + \int_{\{\ell \leq |U| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla U - \frac{k}{m} \int_{\{m \leq |U| < 2m\}} \xi A(x, t, \nabla u) \cdot \nabla U \leq \int_Q H_m(U) \xi T_{k,\ell}(U) d\widehat{\mu}_0. \end{aligned}$$

And

$$\int_Q H_m(U) \xi T_{k,\ell}(U) d\widehat{\mu}_0 = \int_Q H_m(U) \xi T_{k,\ell}(U) f + \int_{\{\ell \leq |U| < \ell+k\}} \xi \nabla U \cdot g - \frac{k}{m} \int_{\{m \leq |U| < 2m\}} \xi \nabla U \cdot g.$$

Let  $m \rightarrow \infty$ ; then, for any  $k \geq 1$ , since  $U \in L^1(Q)$  and from (2.3), (2.4), and (2.10), we find

$$- \int_Q \xi_t \overline{T_{k,\ell}}(U) + \int_{\{\ell \leq |U| < \ell+k\}} \xi A(x, t, \nabla u) \cdot \nabla U \leq \int_{\{\ell \leq |U| < \ell+k\}} \xi \nabla U \cdot g + k(\|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q}). \quad (2.20)$$

Next, we take  $\xi \equiv 1$ . We verify that

$$A(x, t, \nabla u) \cdot \nabla U - \nabla U \cdot g \geq \frac{\Lambda_1}{4} (|\nabla u|^p + |\nabla U|^p) - c_1 (|g|^{p'} + |\nabla h|^p + |a|^{p'})$$

for some  $c_1 = c_1(p, \Lambda_1, \Lambda_2) > 0$ . Hence (2.16) follows. Thus, from (2.20) and the Hölder inequality, we get, for any  $\xi \in C^1([0, T])$  with values in  $[0, 1]$ ,

$$- \int_Q \xi_t \overline{T_{k,\ell}}(U) \leq c_2 k M$$

for some  $c_2 = c_2(p, \Lambda_1, \Lambda_2) > 0$ . Thus  $\int_{\Omega} \overline{T_{k,\ell}}(U)(t) dx \leq c_2 k M$ , for a.e.  $t \in (0, T)$ . We deduce (2.17) by taking  $k = 1, \ell = 0$ , since  $\overline{T_{1,0}}(r) = \overline{T_1}(r) \geq |r| - 1$ , for any  $r \in \mathbb{R}$ .

Next, from the Gagliardo-Nirenberg embedding Theorem, see [13, Proposition 3.1], we have

$$\int_Q |T_k(U)|^{\frac{p(N+1)}{N}} \leq c_3 \|U\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} \int_Q |\nabla T_k(U)|^p,$$

where  $c_3 = c_3(N, p)$ . Then, from (2.16) and (2.17), we get, for any  $k \geq 1$ ,

$$\text{meas}\{|U| > k\} \leq k^{-\frac{p(N+1)}{N}} \int_Q |T_k(U)|^{\frac{p(N+1)}{N}} \leq c_3 \|U\|_{L^\infty((0,T);L^1(\Omega))}^{\frac{p}{N}} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(U)|^p \leq c_4 M_1 k^{-p_c},$$

with  $c_4 = c_4(N, p, \Lambda_1, \Lambda_2)$ . We obtain

$$\begin{aligned} \text{meas}\{|\nabla U| > k\} &\leq \frac{1}{k^p} \int_0^{k^p} \text{meas}(\{|\nabla U|^p > s\}) ds \\ &\leq \text{meas}\{|U| > k^{\frac{N}{N+1}}\} + \frac{1}{k^p} \int_0^{k^p} \text{meas}\left(\{|\nabla U|^p > s, |U| \leq k^{\frac{N}{N+1}}\}\right) ds \\ &\leq c_4 M_1 k^{-m_c} + \frac{1}{k^p} \int_{|U| \leq k^{\frac{N}{N+1}}} |\nabla U|^p \leq c_5 M_2 k^{-m_c}, \end{aligned}$$

with  $c_5 = c_5(N, p, \Lambda_1, \Lambda_2)$ . Furthermore, for any  $k \geq 1$ ,

$$\text{meas}\{|h| > k\} + \text{meas}\{|\nabla h| > k\} \leq c_6 k^{-p} \|h\|_X^p,$$

where  $c_6 = c_6(N, p)$ . Therefore, we easily get (2.19). ■

**Remark 2.9** If  $\mu \in L^1(Q)$  and  $a \equiv 0$  in (1.2), then (2.16) holds for all  $k > 0$  and the term  $|\Omega|$  in inequality (2.17) can be removed, where  $M = \|u_0\|_{1,\Omega} + |\mu|(Q)$ . Furthermore, (2.19) is stated as follows:

$$\text{meas}\{|u| > k\} \leq C_2 M^{\frac{p+N}{N}} k^{-p_c}, \quad \text{meas}\{|\nabla u| > k\} \leq C_2 M^{\frac{N+2}{N+1}} k^{-m_c}, \quad \forall k > 0. \quad (2.21)$$

with  $C_2 = C_2(N, p, \Lambda_1, \Lambda_2)$ . To see last inequality, we do in the following way:

$$\begin{aligned} \text{meas}\{|\nabla U| > k\} &\leq \text{meas}\left\{|U| > M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas}\left\{|\nabla U|^p > s, |U| \leq M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} ds \\ &\leq C_2 M^{\frac{N+2}{N+1}} k^{-m_c}. \end{aligned}$$

**Proposition 2.10** Let  $\{\mu_n\} \subset \mathcal{M}_b(Q)$ , and  $\{u_{0,n}\} \subset L^1(\Omega)$ , such that

$$\sup_n |\mu_n|(Q) < \infty, \quad \text{and} \quad \sup_n \|u_{0,n}\|_{1,\Omega} < \infty.$$

Let  $u_n$  be a R-solution of (1.1) with data  $\mu_n = \mu_{n,0} + \mu_{n,s}$  and  $u_{0,n}$ , relative to a decomposition  $(f_n, g_n, h_n)$  of  $\mu_{n,0}$ , and  $U_n = u_n - h_n$ . Assume that  $\{f_n\}$  is bounded in  $L^1(Q)$ ,  $\{g_n\}$  bounded in  $(L^{p'}(Q))^N$  and  $\{h_n\}$  bounded in  $X$ .

Then, up to a subsequence,  $\{U_n\}$  converges a.e. to a function  $U \in L^\infty((0, T); L^1(\Omega))$ , such that  $T_k(U) \in X$  for any  $k > 0$  and  $U \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega))$  for any  $\sigma \in [1, m_c)$ . And

- (i)  $\{U_n\}$  converges to  $U$  strongly in  $L^\sigma(Q)$  for any  $\sigma \in [1, m_c)$ , and  $\sup \|U_n\|_{L^\infty((0,T);L^1(\Omega))} < \infty$ ,
- (ii)  $\sup_{k>0} \sup_n \frac{1}{k+1} \int_Q |\nabla T_k(U_n)|^p < \infty$ ,
- (iii)  $\{T_k(U_n)\}$  converges to  $T_k(U)$  weakly in  $X$ , for any  $k > 0$ ,
- (iv)  $\{A(x, t, \nabla (T_k(U_n) + h_n))\}$  converges to some  $F_k$  weakly in  $(L^{p'}(Q))^N$ .

**Proof.** Take  $S \in W^{2,\infty}(\mathbb{R})$  such that  $S'$  has compact support on  $\mathbb{R}$  and  $S(0) = 0$ . We combine (2.6) with (2.16), and deduce that  $\{S(U_n)_t\}$  is bounded in  $X' + L^1(Q)$  and  $\{S(U_n)\}$  bounded in  $X$ . Hence,  $\{S(U_n)\}$  is relatively compact in  $L^1(Q)$ . On the other hand, we choose  $S = S_k$  such that  $S_k(z) = z$ , if  $|z| < k$  and  $S_k(z) = 2k \operatorname{sign} z$ , if  $|z| > 2k$ . From (2.17), we obtain

$$\begin{aligned} \operatorname{meas} \{|U_n - U_m| > \sigma\} &\leq \operatorname{meas} \{|U_n| > k\} + \operatorname{meas} \{|U_m| > k\} + \operatorname{meas} \{|S_k(U_n) - S_k(U_m)| > \sigma\} \\ &\leq \frac{c}{k} + \operatorname{meas} \{|S_k(U_n) - S_k(U_m)| > \sigma\}, \end{aligned}$$

where  $c$  does not depend of  $n, m$ . Thus, up to a subsequence  $\{u_n\}$  is a Cauchy sequence in measure, and converges *a.e.* in  $Q$  to a function  $u$ . Thus,  $\{T_k(U_n)\}$  converges to  $T_k(U)$  weakly in  $X$ , since  $\sup_n \|T_k(U_n)\|_X < \infty$  for any  $k > 0$ . And  $\{|\nabla (T_k(U_n) + h_n)|^{p-2} \nabla (T_k(U_n) + h_n)\}$  converges to some  $F_k$  weakly in  $(L^{p'}(Q))^N$ . Furthermore, from (2.18),  $\{U_n\}$  strongly converges to  $U$  in  $L^\sigma(Q)$ , for any  $\sigma < p_c$ .  $\blacksquare$

### 3 The convergence theorem

We first recall some properties of the measures, see [22, Lemma 5], [14].

**Proposition 3.1** *Let  $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q)$ , where  $\mu_s^+$  and  $\mu_s^-$  are concentrated, respectively, on two disjoint sets  $E^+$  and  $E^-$  of zero  $c_p^Q$ -capacity. Then, for any  $\delta > 0$ , there exist two compact sets  $K_\delta^+ \subseteq E^+$  and  $K_\delta^- \subseteq E^-$  such that*

$$\mu_s^+(E^+ \setminus K_\delta^+) \leq \delta, \quad \mu_s^-(E^- \setminus K_\delta^-) \leq \delta,$$

and there exist  $\psi_\delta^+, \psi_\delta^- \in C_c^1(Q)$  with values in  $[0, 1]$ , such that  $\psi_\delta^+, \psi_\delta^- = 1$  respectively on  $K_\delta^+, K_\delta^-$ , and  $\operatorname{supp}(\psi_\delta^+) \cap \operatorname{supp}(\psi_\delta^-) = \emptyset$ , and

$$\|\psi_\delta^+\|_X + \|(\psi_\delta^+)_t\|_{X'+L^1(Q)} \leq \delta, \quad \|\psi_\delta^-\|_X + \|(\psi_\delta^-)_t\|_{X'+L^1(Q)} \leq \delta.$$

There exist decompositions  $(\psi_\delta^+)_t = (\psi_\delta^+)_t^1 + (\psi_\delta^+)_t^2$  and  $(\psi_\delta^-)_t = (\psi_\delta^-)_t^1 + (\psi_\delta^-)_t^2$  in  $X' + L^1(Q)$ , such that

$$\left\| (\psi_\delta^+)_t^1 \right\|_{X'} \leq \frac{\delta}{3}, \quad \left\| (\psi_\delta^+)_t^2 \right\|_{1,Q} \leq \frac{\delta}{3}, \quad \left\| (\psi_\delta^-)_t^1 \right\|_{X'} \leq \frac{\delta}{3}, \quad \left\| (\psi_\delta^-)_t^2 \right\|_{1,Q} \leq \frac{\delta}{3}. \quad (3.1)$$

Both  $\{\psi_\delta^+\}$  and  $\{\psi_\delta^-\}$  converge to 0, weak-\* in  $L^\infty(Q)$ , and strongly in  $L^1(Q)$  and up to subsequences, *a.e.* in  $Q$ , as  $\delta$  tends to 0.

Moreover if  $\rho_n$  and  $\eta_n$  are as in Theorem 1.1, we have, for any  $\delta, \delta_1, \delta_2 > 0$ ,

$$\int_Q \psi_\delta^- d\rho_n + \int_Q \psi_\delta^+ d\eta_n = \omega(n, \delta), \quad \int_Q \psi_\delta^- d\mu_s^+ \leq \delta, \quad \int_Q \psi_\delta^+ d\mu_s^- \leq \delta, \quad (3.2)$$

$$\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\rho_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\mu_s^+ \leq \delta_1 + \delta_2, \quad (3.3)$$

$$\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\eta_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\mu_s^- \leq \delta_1 + \delta_2. \quad (3.4)$$

Hereafter, if  $n, \varepsilon, \dots, \nu$  are real numbers, and a function  $\phi$  depends on  $n, \varepsilon, \dots, \nu$  and eventual other parameters  $\alpha, \beta, \dots, \gamma$ , and  $n \rightarrow n_0, \varepsilon \rightarrow \varepsilon_0, \dots, \nu \rightarrow \nu_0$ , we write  $\phi = \omega(n, \varepsilon, \dots, \nu)$ , then this means that, for fixed  $\alpha, \beta, \dots, \gamma$ , there holds  $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} |\phi| = 0$ . In the same way,  $\phi \leq \omega(n, \varepsilon, \delta, \dots, \nu)$  means  $\overline{\lim}_{\nu \rightarrow \nu_0} \dots \overline{\lim}_{\varepsilon \rightarrow \varepsilon_0} \overline{\lim}_{n \rightarrow n_0} \phi \leq 0$ , and  $\phi \geq \omega(n, \varepsilon, \dots, \nu)$  means  $-\phi \leq \omega(n, \varepsilon, \dots, \nu)$ .

**Remark 3.2** *In the sequel we recall a convergence property still used in [14]: If  $\{b_{1,n}\}$  is a sequence in  $L^1(Q)$  converging to  $b_1$  weakly in  $L^1(Q)$  and  $\{b_{2,n}\}$  a bounded sequence in  $L^\infty(Q)$  converging to  $b_2$ , a.e. in  $Q$ , then  $\lim_{n \rightarrow \infty} \int_Q b_{1,n} b_{2,n} = \int_Q b_1 b_2$ .*

Next we prove Theorem 1.1.

**Scheme of the proof.** Let  $\{\mu_n\}, \{u_{0,n}\}$  and  $\{u_n\}$  satisfy the assumptions of Theorem 1.1. Then we can apply Proposition 2.10. Setting  $U_n = u_n - h_n$ , up to subsequences,  $\{u_n\}$  converges a.e. in  $Q$  to some function  $u$ , and  $\{U_n\}$  converges a.e. to  $U = u - h$ , such that  $T_k(U) \in X$  for any  $k > 0$ , and  $U \in L^\sigma((0, T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0, T); L^1(\Omega))$  for every  $\sigma \in [1, m_c)$ . And  $\{U_n\}$  satisfies the conclusions (i) to (iv) of Proposition 2.10. We have

$$\begin{aligned} \mu_n &= (f_n - \operatorname{div} g_n + (h_n)_t) + (\rho_n^1 - \operatorname{div} \rho_n^2) - (\eta_n^1 - \operatorname{div} \eta_n^2) + \rho_{n,s} - \eta_{n,s} \\ &= \mu_{n,0} + (\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-, \end{aligned}$$

where

$$\mu_{n,0} = \lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \quad \text{with } \lambda_{n,0} = f_n - \operatorname{div} g_n + (h_n)_t, \quad \rho_{n,0} = \rho_n^1 - \operatorname{div} \rho_n^2, \quad \eta_{n,0} = \eta_n^1 - \operatorname{div} \eta_n^2. \quad (3.5)$$

Hence

$$\rho_{n,0}, \eta_{n,0} \in \mathcal{M}_b^+(Q) \cap \mathcal{M}_0(Q), \quad \text{and } \rho_n \geq \rho_{n,0}, \quad \eta_n \geq \eta_{n,0}. \quad (3.6)$$

Let  $E^+, E^-$  be the sets where, respectively,  $\mu_s^+$  and  $\mu_s^-$  are concentrated. For any  $\delta_1, \delta_2 > 0$ , let  $\psi_{\delta_1}^+, \psi_{\delta_2}^+$  and  $\psi_{\delta_1}^-, \psi_{\delta_2}^-$  as in Proposition 3.1 and set

$$\Phi_{\delta_1, \delta_2} = \psi_{\delta_1}^+ \psi_{\delta_2}^+ + \psi_{\delta_1}^- \psi_{\delta_2}^-.$$

Suppose that we can prove the two estimates, near  $E$

$$I_1 := \int_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla (U_n - \langle T_k(U) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2), \quad (3.7)$$

and far from  $E$ ,

$$I_2 := \int_{\{|U_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla (U_n - \langle T_k(U) \rangle_\nu) \leq \omega(n, \nu, \delta_1, \delta_2). \quad (3.8)$$

Then it follows that

$$\overline{\lim}_{n, \nu} \int_{\{|U_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (U_n - \langle T_k(U) \rangle_\nu) \leq 0, \quad (3.9)$$

which implies

$$\overline{\lim}_{n \rightarrow \infty} \int_{\{|U_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (U_n - T_k(U)) \leq 0, \quad (3.10)$$

since  $\{(T_k(U))_\nu\}$  converges to  $T_k(U)$  in  $X$ . On the other hand, from the weak convergence of  $\{T_k(U_n)\}$  to  $T_k(U)$  in  $X$ , we verify that

$$\int_{\{|U_n| \leq k\}} A(x, t, \nabla(T_k(U) + h_n)) \cdot \nabla(T_k(U_n) - T_k(U)) = \omega(n).$$

Thus we get

$$\int_{\{|U_n| \leq k\}} (A(x, t, \nabla u_n) - A(x, t, \nabla(T_k(U) + h_n))) \cdot \nabla(u_n - (T_k(U) + h_n)) = \omega(n).$$

Then, it is easy to show that, up to a subsequence,

$$\{\nabla u_n\} \text{ converges to } \nabla u, \quad a.e. \text{ in } Q. \quad (3.11)$$

Therefore,  $\{A(x, t, \nabla u_n)\}$  converges to  $A(x, t, \nabla u)$  weakly in  $(L^{p'}(Q))^N$ ; and from (3.10) we find

$$\overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) \leq \int_Q A(x, t, \nabla u) \nabla T_k(U).$$

Otherwise,  $\{A(x, t, \nabla(T_k(U_n) + h_n))\}$  converges weakly in  $(L^{p'}(Q))^N$  to some  $F_k$ , from Proposition 2.10, and we obtain that  $F_k = A(x, t, \nabla(T_k(U) + h))$ . Hence

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(U_n) + h_n)) \cdot \nabla(T_k(U_n) + h_n) \\ & \leq \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) + \overline{\lim}_{n \rightarrow \infty} \int_Q A(x, t, \nabla(T_k(U_n) + h_n)) \cdot \nabla h_n \\ & \leq \int_Q A(x, t, \nabla(T_k(U) + h)) \cdot \nabla(T_k(U) + h). \end{aligned}$$

As a consequence

$$\{T_k(U_n)\} \text{ converges to } T_k(U), \text{ strongly in } X, \quad \forall k > 0. \quad (3.12)$$

Then to finish the proof we have to check that  $u$  is a solution of (1.1).  $\blacksquare$

In order to prove (3.7) we need a first Lemma, inspired of [14, Lemma 6.1]. It extends the results of [22, Lemma 6 and Lemma 7] relative to sequences of solutions with smooth data:

**Lemma 3.3** *Let  $\psi_{1,\delta}, \psi_{2,\delta} \in C^1(Q)$  be uniformly bounded in  $W^{1,\infty}(Q)$  with values in  $[0, 1]$ , and such that  $\int_Q \psi_{1,\delta} d\mu_s^- \leq \delta$  and  $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$ . Let  $\{u_n\}$  satisfying the assumptions of Theorem 1.1, and  $U_n = u_n - h_n$ . Then*

$$\frac{1}{m} \int_{\{m \leq U_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{\{m \leq U_n < 2m\}} |\nabla U_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (3.13)$$

$$\frac{1}{m} \int_{-2m < U_n \leq -m} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \frac{1}{m} \int_{-2m < U_n \leq -m} |\nabla U_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad (3.14)$$

and for any  $k > 0$ ,

$$\int_{\{m \leq U_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad \int_{\{m \leq U_n < m+k\}} |\nabla U_n|^p \psi_{2,\delta} = \omega(n, m, \delta), \quad (3.15)$$

$$\int_{\{-m-k < U_n \leq -m\}} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta), \quad \int_{\{-m-k < U_n \leq -m\}} |\nabla U_n|^p \psi_{1,\delta} = \omega(n, m, \delta). \quad (3.16)$$

**Proof.** (i) Proof of (3.13), (3.14). Set for any  $r \in \mathbb{R}$  and any  $m, \ell \geq 1$

$$S_{m,\ell}(r) = \int_0^r \left( \frac{-m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,2m+\ell]}(\tau) + \frac{4m + 2h - \tau}{2m + \ell} \chi_{(2m+\ell,4m+2h]}(\tau) \right) d\tau,$$

$$S_m(r) = \int_0^r \left( \frac{-m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,\infty)}(\tau) \right) d\tau.$$

Note that  $S''_{m,\ell} = \chi_{[m,2m]}/m - \chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$ . We choose  $(\xi, J, S) = (\psi_{2,\delta}, T_1, S_{m,\ell})$  as test functions in (2.15) for  $u_n$ , and observe that, from (3.5),

$$\widehat{\mu_{n,0}} = \mu_{n,0} - (h_n)_t = \widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0} = f_n - \operatorname{div} g_n + \rho_{n,0} - \eta_{n,0}. \quad (3.17)$$

Thus we can write  $\sum_{i=1}^6 A_i \leq \sum_{i=7}^{12} A_i$ , where

$$A_1 = - \int_{\Omega} \psi_{2,\delta}(0) T_1(S_{m,\ell}(u_{0,n})) S_{m,\ell}(u_{0,n}) dx, \quad A_2 = - \int_Q (\psi_{2,\delta})_t \overline{T_1}(S_{m,\ell}(U_n)),$$

$$A_3 = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) A(x, t, \nabla u_n) \nabla \psi_{2,\delta}, \quad A_4 = \int_Q (S'_{m,\ell}(U_n))^2 \psi_{2,\delta} T'_1(S_{m,\ell}(U_n)) A(x, t, \nabla u_n) \nabla U_n,$$

$$A_5 = \frac{1}{m} \int_{\{m \leq U_n \leq 2m\}} \psi_{2,\delta} T_1(S_{m,\ell}(U_n)) A(x, t, \nabla u_n) \nabla U_n,$$

$$A_6 = - \frac{1}{2m + \ell} \int_{\{2m+\ell \leq U_n < 2(2m+\ell)\}} \psi_{2,\delta} A(x, t, \nabla u_n) \nabla U_n,$$

$$A_7 = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} f_n, \quad A_8 = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) g_n \cdot \nabla \psi_{2,\delta},$$

$$A_9 = \int_Q (S'_{m,\ell}(U_n))^2 T'_1(S_{m,\ell}(U_n)) \psi_{2,\delta} g_n \cdot \nabla U_n, \quad A_{10} = \frac{1}{m} \int_{m \leq U_n \leq 2m} T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} g_n \cdot \nabla U_n,$$

$$A_{11} = - \frac{1}{2m + \ell} \int_{\{2m+\ell \leq U_n < 2(2m+\ell)\}} \psi_{2,\delta} g_n \cdot \nabla U_n, \quad A_{12} = \int_Q S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} d(\rho_{n,0} - \eta_{n,0}).$$

Since  $\|S_{m,\ell}(u_{0,n})\|_{1,\Omega} \leq \int_{\{m \leq u_{0,n}\}} u_{0,n} dx$ , we find  $A_1 = \omega(\ell, n, m)$ . Otherwise

$$|A_2| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq U_n\}} U_n, \quad |A_3| \leq \|\psi_{2,\delta}\|_{W^{1,\infty}(Q)} \int_{\{m \leq U_n\}} (|a| + \Lambda_2 |\nabla u_n|^{p-1}),$$

which imply  $A_2 = \omega(\ell, n, m)$  and  $A_3 = \omega(\ell, n, m)$ . Using (2.3) for  $u_n$ , we have

$$A_6 = - \int_Q \psi_{2,\delta} d(\rho_{n,s} - \eta_{n,s})^+ + \omega(\ell) = \omega(\ell, n, m, \delta).$$

Hence  $A_6 = \omega(\ell, n, m, \delta)$ , since  $(\rho_{n,s} - \eta_{n,s})^+$  converges to  $\mu_s^+$  as  $n \rightarrow \infty$  in the narrow topology, and  $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$ . We also obtain  $A_{11} = \omega(\ell)$  from (2.10).

Now  $\left\{ S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n)) \right\}_\ell$  converges to  $S'_m(U) T_1(S_m(U))$ ,  $\{S'_{m,\ell}(U_n) T_1(S_{m,\ell}(U_n))\}_n$  converges to  $S'_m(U) T_1(S_m(U))$ ,  $\{S'_m(U) T_1(S_m(U))\}_m$  converges to 0, weak-\* in  $L^\infty(Q)$  and  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$ ,  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ . From Remark 3.2, we obtain

$$\begin{aligned} A_7 &= \int_Q S'_m(U_n) T_1(S_{m,\ell}(U_n)) \psi_{2,\delta} f_n + \omega(\ell) = \int_Q S'_m(U) T_1(S_m(U)) \psi_{2,\delta} f + \omega(\ell, n) = \omega(\ell, n, m), \\ A_8 &= \int_Q S'_m(U_n) T_1(S_{m,\ell}(U_n)) g_n \cdot \nabla \psi_{2,\delta} + \omega(\ell) = \int_Q S'_m(U) T_1(S_m(U)) g \nabla \psi_{2,\delta} + \omega(\ell, n) = \omega(\ell, n, m). \end{aligned}$$

Otherwise,  $A_{12} \leq \int_Q \psi_{2,\delta} d\rho_n$ , and  $\left\{ \int_Q \psi_{2,\delta} d\rho_n \right\}$  converges to  $\int_Q \psi_{2,\delta} d\mu_s^+$ , thus  $A_{12} \leq \omega(\ell, n, m, \delta)$ . Using Holder inequality and the condition (1.2), we have

$$g_n \cdot \nabla U_n - A(x, t, \nabla u_n) \nabla U_n \leq c_1 \left( |g_n|^{p'} + |\nabla h_n|^p + |a|^{p'} \right)$$

with  $c_1 = c_1(p, \Lambda_1, \Lambda_2)$ , which implies

$$A_9 - A_4 \leq c_1 \int_Q (S'_{m,\ell}(U_n))^2 T_1'(S_{m,\ell}(U_n)) \psi_{2,\delta} \left( |g_n|^{p'} + |h_n|^p + |a|^{p'} \right) = \omega(\ell, n, m).$$

Similarly we also show that  $A_{10} - A_5/2 \leq \omega(\ell, n, m)$ . Combining the estimates, we get  $A_5/2 \leq \omega(\ell, n, m, \delta)$ . Using Holder inequality we have

$$A(x, t, \nabla u_n) \nabla U_n \geq \frac{\Lambda_1}{2} |\nabla u_n|^p - c_2 (|a|^{p'} + |\nabla h_n|^p).$$

with  $c_2 = c_2(p, \Lambda_1, \Lambda_2)$ , which implies

$$\frac{1}{m} \int_{\{m \leq U_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{m,\ell}(U_n)) = \omega(\ell, n, m, \delta).$$

Note that for all  $m > 4$ ,  $S_{m,\ell}(r) \geq 1$  for any  $r \in [\frac{3}{2}m, 2m]$ ; hence  $T_1(S_{m,\ell}(r)) = 1$ . So,

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq U_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

Since  $|\nabla U_n|^p \leq 2^{p-1} |\nabla u_n|^p + 2^{p-1} |\nabla h_n|^p$ , there also holds

$$\frac{1}{m} \int_{\{\frac{3}{2}m \leq U_n < 2m\}} |\nabla U_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

We deduce (3.13) by summing on each set  $\{(\frac{4}{3})^i m \leq U_n \leq (\frac{4}{3})^{i+1} m\}$  for  $i = 0, 1, 2$ . Similarly, we can choose  $(\xi, \psi, S) = (\psi_{1,\delta}, T_1, \tilde{S}_{m,\ell})$  as test functions in (2.15) for  $u_n$ , where  $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$ , and we obtain (3.14).

(ii) Proof of (3.15), (3.16). We set, for any  $k, m, \ell \geq 1$ ,

$$S_{k,m,\ell}(r) = \int_0^r \left( T_k(\tau - T_m(\tau)) \chi_{[m, k+m+\ell]} + k \frac{2(k+\ell+m) - \tau}{k+m+\ell} \chi_{(k+m+\ell, 2(k+m+\ell))} \right) d\tau$$

$$S_{k,m}(r) = \int_0^r T_k(\tau - T_m(\tau)) \chi_{[m,\infty)} d\tau.$$

We choose  $(\xi, \psi, S) = (\psi_{2,\delta}, T_1, S_{k,m,\ell})$  as test functions in (2.15) for  $u_n$ . In the same way we also obtain

$$\int_{\{m \leq U_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{k,m,\ell}(U_n)) = \omega(\ell, n, m, \delta).$$

Note that  $T_1(S_{k,m,\ell}(r)) = 1$  for any  $r \geq m+1$ , thus  $\int_{\{m+1 \leq U_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta)$ , which implies (3.15) by changing  $m$  into  $m-1$ . Similarly, we obtain (3.16).  $\blacksquare$

Next we look at the behaviour near  $E$ .

**Lemma 3.4** *Estimate (3.7) holds.*

**Proof.** There holds

$$I_1 = \int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) - \int_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(U) \rangle_\nu.$$

From Proposition 2.10, (iv),  $\{A(x, t, \nabla (T_k(U_n) + h_n)) \cdot \nabla \langle T_k(U) \rangle_\nu\}$  converges weakly in  $L^1(Q)$  to  $F_k \nabla \langle T_k(U) \rangle_\nu$ . And  $\{\chi_{\{|U_n| \leq k\}}\}$  converges to  $\chi_{|U| \leq k}$ , a.e. in  $Q$ , and  $\Phi_{\delta_1, \delta_2}$  converges to 0 a.e. in  $Q$  as  $\delta_1 \rightarrow 0$ , and  $\Phi_{\delta_1, \delta_2}$  takes its values in  $[0, 1]$ . From Remark 3.2, we have

$$\begin{aligned} & \int_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(U) \rangle_\nu = \int_Q \chi_{\{|U_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla (T_k(U_n) + h_n)) \cdot \nabla \langle T_k(U) \rangle_\nu \\ & = \int_Q \chi_{|U| \leq k} \Phi_{\delta_1, \delta_2} F_k \cdot \nabla \langle T_k(U) \rangle_\nu + \omega(n) = \omega(n, \nu, \delta_1). \end{aligned}$$

Therefore, if we prove that

$$\int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(U_n) \leq \omega(n, \delta_1, \delta_2), \quad (3.18)$$

then we deduce (3.7). As noticed in [14, 22], it is precisely for this estimate that we need the double cut  $\psi_{\delta_1}^+ \psi_{\delta_2}^+$ . To do this, we set, for any  $m > k > 0$ , and any  $r \in \mathbb{R}$ ,

$$\hat{S}_{k,m}(r) = \int_0^r (k - T_k(\tau)) H_m(\tau) d\tau,$$

where  $H_m$  is defined at (2.14). Hence  $\text{supp } \hat{S}_{k,m} \subset [-2m, k]$ ; and  $\hat{S}_{k,m}'' = -\chi_{[-k, k]} + \frac{2k}{m} \chi_{[-2m, -m]}$ . We choose  $(\varphi, S) = (\psi_{\delta_1}^+ \psi_{\delta_2}^+, \hat{S}_{k,m})$  as test functions in (2.2). From (3.17), we can write

$$A_1 + A_2 - A_3 + A_4 + A_5 + A_6 = 0,$$

where

$$\begin{aligned} A_1 &= - \int_Q (\psi_{\delta_1}^+ \psi_{\delta_2}^+)_t \hat{S}_{k,m}(U_n), \quad A_2 = \int_Q (k - T_k(U_n)) H_m(U_n) A(x, t, \nabla u_n) \cdot \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+), \\ A_3 &= \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla T_k(U_n), \quad A_4 = \frac{2k}{m} \int_{\{-2m < U_n \leq -m\}} \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \cdot \nabla U_n, \\ A_5 &= - \int_Q (k - T_k(U_n)) H_m(U_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\widehat{\lambda}_{n,0}, \quad A_6 = \int_Q (k - T_k(U_n)) H_m(U_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d(\eta_{m,0} - \rho_{n,0}). \end{aligned}$$



We first estimate  $A_3$ . As in [22, p.585], since  $\{\hat{S}_{k,m}(U_n)\}$  converges to  $\hat{S}_{k,m}(U)$  weakly in  $X$ , and  $\hat{S}_{k,m}(U) \in L^\infty(Q)$ , using (3.1), we find

$$A_1 = - \int_Q (\psi_{\delta_1}^+)_t \psi_{\delta_2}^+ \hat{S}_{k,m}(U) - \int_Q \psi_{\delta_1}^+ (\psi_{\delta_2}^+)_t \hat{S}_{k,m}(U) + \omega(n) = \omega(n, \delta_1).$$

Next consider  $A_2$ . Notice that  $U_n = T_{2m}(U_n)$  on  $\text{supp}(H_m(U_n))$ . From Proposition 2.10, (iv), the sequence  $\{A(x, t, \nabla(T_{2m}(U_n) + h_n)) \cdot \nabla(\psi_{\delta_1}^+ \psi_{\delta_2}^+)\}$  converges to  $F_{2m} \cdot \nabla(\psi_{\delta_1}^+ \psi_{\delta_2}^+)$  weakly in  $L^1(Q)$ . From Remark 3.2 and the convergence of  $\psi_{\delta_1}^+ \psi_{\delta_2}^+$  in  $X$  to 0 as  $\delta_1$  tends to 0, we find

$$A_2 = \int_Q (k - T_k(U)) H_m(U) F_{2m} \cdot \nabla(\psi_{\delta_1}^+ \psi_{\delta_2}^+) + \omega(n) = \omega(n, \delta_1).$$

Then consider  $A_4$ . Then for some  $c_1 = c_1(p, \Lambda_2)$ ,

$$|A_4| \leq c_1 \frac{2k}{m} \int_{\{-2m < U_n \leq -m\}} (|\nabla u_n|^p + |\nabla U_n|^p + |a|^{p'}) \psi_{\delta_1}^+ \psi_{\delta_2}^+.$$

Since  $\psi_{\delta_1}^+$  takes its values in  $[0, 1]$ , from Lemma 3.3, we get in particular  $A_4 = \omega(n, \delta_1, m, \delta_2)$ .

Now we estimate  $A_5$ . The sequence  $\{(k - T_k(U_n)) H_m(U_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+\}$  converges to  $(k - T_k(U)) H_m(U) \psi_{\delta_1}^+ \psi_{\delta_2}^+$ , weakly in  $X$ , and  $\{(k - T_k(U_n)) H_m(U_n)\}$  converges to  $(k - T_k(U)) H_m(U)$ , weak-\* in  $L^\infty(Q)$  and *a.e.* in  $\bar{Q}$ . Otherwise  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$  and  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ . From Remark 3.2 and the convergence of  $\psi_{\delta_1}^+ \psi_{\delta_2}^+$  to 0 in  $X$  and *a.e.* in  $Q$  as  $\delta_1 \rightarrow 0$ , we deduce that

$$A_5 = - \int_Q (k - T_k(U_n)) H_m(U) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\hat{\nu}_0 + \omega(n) = \omega(n, \delta_1),$$

where  $\hat{\nu}_0 = f - \text{div } g$ .

Finally  $A_6 \leq 2k \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\eta_n$ ; using (3.2) we also find  $A_6 \leq \omega(n, \delta_1, m, \delta_2)$ . By addition, since  $A_3$  does not depend on  $m$ , we obtain

$$A_3 = \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla T_k(U_n) \leq \omega(n, \delta_1, \delta_2).$$

Arguing as before with  $(\psi_{\delta_1}^- \psi_{\delta_2}^-, \check{S}_{k,m})$  as test function in (2.2), where  $\check{S}_{k,m}(r) = -\hat{S}_{k,m}(-r)$ , we get in the same way

$$\int_Q \psi_{\delta_1}^- \psi_{\delta_2}^- A(x, t, \nabla u_n) \nabla T_k(U_n) \leq \omega(n, \delta_1, \delta_2).$$

Then, (3.18) holds. ■

Next we look at the behaviour far from  $E$ .

**Lemma 3.5** . *Estimate (3.8) holds.*

**Proof.** Here we estimate  $I_2$ ; we can write

$$I_2 = \int_{\{|U_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \nabla (T_k(U_n) - \langle T_k(U) \rangle_\nu).$$

Following the ideas of [25], used also in [22], we define, for any  $r \in \mathbb{R}$  and  $\ell > 2k > 0$ ,

$$R_{n,\nu,\ell} = T_{\ell+k}(U_n - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U_n - T_k(U_n)).$$

Recall that  $\|\langle T_k(U) \rangle_\nu\|_{\infty, Q} \leq k$ , and observe that

$$R_{n,\nu,\ell} = 2k \operatorname{sign}(U_n) \quad \text{in } \{|U_n| \geq \ell + 2k\}, \quad |R_{n,\nu,\ell}| \leq 4k, \quad R_{n,\nu,\ell} = \omega(n, \nu, \ell) \text{ a.e. in } Q, \quad (3.19)$$

$$\lim_{n \rightarrow \infty} R_{n,\nu,\ell} = T_{\ell+k}(U - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U - T_k(U)), \quad \text{a.e. in } Q, \text{ and weakly in } X. \quad (3.20)$$

Next consider  $\xi_{1,n_1} \in C_c^\infty([0, T])$ ,  $\xi_{2,n_2} \in C_c^\infty((0, T])$  with values in  $[0, 1]$ , such that  $(\xi_{1,n_1})_t \leq 0$  and  $(\xi_{2,n_2})_t \geq 0$ ; and  $\{\xi_{1,n_1}(t)\}$  (resp.  $\{\xi_{2,n_2}(t)\}$ ) converges to 1, for any  $t \in [0, T]$  (resp.  $t \in (0, T]$ ); and moreover, for any  $a \in C([0, T]; L^1(\Omega))$ ,  $\left\{ \int_Q a(\xi_{1,n_1})_t \right\}$  and  $\int_Q a(\xi_{2,n_2})_t$  converge respectively to  $-\int_\Omega a(\cdot, T)dx$  and  $\int_\Omega a(\cdot, 0)dx$ . We set

$$\varphi = \varphi_{n,n_1,n_2,l_1,l_2,\ell} = \xi_{1,n_1}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell+k}(U_n - \langle T_k(U) \rangle_\nu)]_{l_1} - \xi_{2,n_2}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell-k}(U_n - T_k(U_n))]_{-l_2}.$$

We observe that

$$\varphi - (1 - \Phi_{\delta_1,\delta_2})R_{n,\nu,\ell} = \omega(l_1, l_2, n_1, n_2) \quad \text{in norm in } X \text{ and a.e. in } Q. \quad (3.21)$$

We can choose  $(\varphi, S) = (\varphi_{n,n_1,n_2,l_1,l_2,\ell}, \overline{H_m})$  as test functions in (2.7) for  $u_n$ , where  $\overline{H_m}$  is defined at (2.14), with  $m > \ell + 2k$ . We obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,$$

with

$$\begin{aligned} A_1 &= \int_\Omega \varphi(T) \overline{H_m}(U_n(T)) dx, & A_2 &= - \int_\Omega \varphi(0) \overline{H_m}(u_{0,n}) dx, & A_3 &= - \int_Q \varphi_t \overline{H_m}(U_n), \\ A_4 &= \int_Q H_m(U_n) A(x, t, \nabla u_n) \cdot \nabla \varphi, & A_5 &= \int_Q \varphi H'_m(U_n) A(x, t, \nabla u_n) \cdot \nabla U_n, \\ A_6 &= \int_Q H_m(U_n) \varphi d\widehat{\lambda}_{n,0}, & A_7 &= \int_Q H_m(U_n) \varphi d(\rho_{n,0} - \eta_{n,0}). \end{aligned}$$

**Estimate of  $A_4$ .** This term allows to study  $I_2$ . Indeed,  $\{H_m(U_n)\}$  converges to 1, a.e. in  $Q$ ; From (3.21), (3.19) (3.20), we have

$$\begin{aligned} A_4 &= \int_Q (1 - \Phi_{\delta_1,\delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n,\nu,\ell} - \int_Q R_{n,\nu,\ell} A(x, t, \nabla u_n) \cdot \nabla \Phi_{\delta_1,\delta_2} + \omega(l_1, l_2, n_1, n_2, m) \\ &= \int_Q (1 - \Phi_{\delta_1,\delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n,\nu,\ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + \int_{\{|U_n| > k\}} (1 - \Phi_{\delta_1,\delta_2}) A(x, t, \nabla u_n) \cdot \nabla R_{n,\nu,\ell} + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell) \\ &= I_2 + B_1 + B_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell), \end{aligned}$$

where

$$\begin{aligned} B_1 &= \int_{\{|U_n| > k\}} (1 - \Phi_{\delta,\eta}) (\chi_{|U_n - \langle T_k(U) \rangle_\nu| \leq \ell+k} - \chi_{|U_n| - k \leq \ell-k}) A(x, t, \nabla u_n) \cdot \nabla U_n, \\ B_2 &= - \int_{\{|U_n| > k\}} (1 - \Phi_{\delta_1,\delta_2}) \chi_{|U_n - \langle T_k(U) \rangle_\nu| \leq \ell+k} A(x, t, \nabla u_n) \cdot \nabla \langle T_k(U) \rangle_\nu. \end{aligned}$$

Now  $\{A(x, t, \nabla(T_{\ell+2k}(U_n) + h_n)) \cdot \nabla \langle T_k(U) \rangle_\nu\}$  converges to  $F_{\ell+2k} \nabla \langle T_k(U) \rangle_\nu$ , weakly in  $L^1(Q)$ . Otherwise  $\{\chi_{|U_n|>k} \chi_{|U_n - \langle T_k(U) \rangle_\nu| \leq \ell+k}\}$  converges to  $\chi_{|U|>k} \chi_{|U - \langle T_k(U) \rangle_\nu| \leq \ell+k}$ , a.e. in  $Q$ . And  $\{\langle T_k(U) \rangle_\nu\}$  converges to  $T_k(U)$  strongly in  $X$ . From Remark 3.2 we get

$$\begin{aligned} B_2 &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|U|>k} \chi_{|U - \langle T_k(U) \rangle_\nu| \leq \ell+k} F_{\ell+2k} \cdot \nabla \langle T_k(U) \rangle_\nu + \omega(n) \\ &= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|U|>k} \chi_{|U - T_k(U)| \leq \ell+k} F_{\ell+2k} \cdot \nabla T_k(U) + \omega(n, \nu) = \omega(n, \nu), \end{aligned}$$

since  $\nabla T_k(U) \chi_{|U|>k} = 0$ . Besides, we see that, for some  $c_1 = c_1(p, \Lambda_2)$ ,

$$|B_1| \leq c_1 \int_{\{\ell-2k \leq |U_n| < \ell+2k\}} (1 - \Phi_{\delta_1, \delta_2}) (|\nabla u_n|^p + |\nabla U_n|^p + |a|^{p'}) .$$

Using (3.3) and (3.4) and applying (3.15) and (3.16) to  $1 - \Phi_{\delta_1, \delta_2}$ , we obtain, for  $k > 0$ ,

$$\int_{\{m \leq |U_n| < m+4k\}} (|\nabla u_n|^p + |\nabla U_n|^p) (1 - \Phi_{\delta_1, \delta_2}) = \omega(n, m, \delta_1, \delta_2). \quad (3.22)$$

Thus,  $B_1 = \omega(n, \nu, \ell, \delta_1, \delta_2)$ , hence  $B_1 + B_2 = \omega(n, \nu, \ell, \delta_1, \delta_2)$ . Then

$$A_4 = I_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2). \quad (3.23)$$

**Estimate of  $A_5$ .** For  $m > \ell + 2k$ , since  $|\varphi| \leq 2\ell$ , and (3.21) holds, we get, from the dominated convergence Theorem,

$$\begin{aligned} A_5 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} H'_m(U_n) A(x, t, \nabla u_n) \cdot \nabla U_n + \omega(l_1, l_2, n_1, n_2) \\ &= -\frac{2k}{m} \int_{\{m \leq |U_n| < 2m\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla U_n + \omega(l_1, l_2, n_1, n_2); \end{aligned}$$

here, the final equality followed from the relation, since  $m > \ell + 2k$ ,

$$R_{n, \nu, \ell} H'_m(U_n) = -\frac{2k}{m} \chi_{m \leq |U_n| \leq 2m}, \quad a.e. \text{ in } Q. \quad (3.24)$$

Next we go to the limit in  $m$ , by using (2.3), (2.4) for  $u_n$ , with  $\phi = (1 - \Phi_{\delta_1, \delta_2})$ . There holds

$$A_5 = -2k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d((\rho_{n, s} - \eta_{n, s})^+ + (\rho_{n, s} - \eta_{n, s})^-) + \omega(l_1, l_2, n_1, n_2, m).$$

Then, from (3.3) and (3.4), we get  $A_5 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ .

**Estimate of  $A_6$ .** Again, from (3.21),

$$\begin{aligned} A_6 &= \int_Q H_m(U_n) \varphi f_n + \int_Q g_n \cdot \nabla (H_m(U_n) \varphi) \\ &= \int_Q H_m(U_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} f_n + \int_Q g_n \cdot \nabla (H_m(U_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell}) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

Thus we can write  $A_6 = D_1 + D_2 + D_3 + D_4 + \omega(l_1, l_2, n_1, n_2)$ , where

$$\begin{aligned} D_1 &= \int_Q H_m(U_n)(1 - \Phi_{\delta_1, \delta_2})R_{n, \nu, \ell} f_n, & D_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2})R_{n, \nu, \ell} H'_m(U_n) g_n \cdot \nabla U_n, \\ D_3 &= \int_Q H_m(U_n)(1 - \Phi_{\delta_1, \delta_2})g_n \cdot \nabla R_{n, \nu, \ell}, & D_4 &= - \int_Q H_m(U_n)R_{n, \nu, \ell} g_n \cdot \nabla \Phi_{\delta_1, \delta_2}. \end{aligned}$$

Since  $\{f_n\}$  converges to  $f$  weakly in  $L^1(Q)$ , and (3.19)-(3.20) hold, we get, from Remark 3.2,

$$D_1 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) (T_{\ell+k}(U - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U - T_k(U))) f + \omega(m, n) = \omega(m, n, \nu, \ell).$$

We deduce from (2.10) that  $D_2 = \omega(m)$ . Next consider  $D_3$ . Note that  $H_m(U_n) = 1 + \omega(m)$ , and (3.20) holds, and  $\{g_n\}$  converges to  $g$  strongly in  $(L^{p'}(Q))^N$ , and  $\langle T_k(U) \rangle_\nu$  converges to  $T_k(U)$  strongly in  $X$ . Then we obtain successively that

$$\begin{aligned} D_3 &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(U - \langle T_k(U) \rangle_\nu) - T_{\ell-k}(U - T_k(U))) + \omega(m, n) \\ &= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g \cdot \nabla (T_{\ell+k}(U - T_k(U)) - T_{\ell-k}(U - T_k(U))) + \omega(m, n, \nu) \\ &= \omega(m, n, \nu, \ell). \end{aligned}$$

Similarly we also get  $D_4 = \omega(m, n, \nu, \ell)$ . Thus  $A_6 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ .

**Estimate of  $A_7$ .** We have

$$\begin{aligned} |A_7| &= \left| \int_Q S'_m(U_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} d(\rho_{n,0} - \eta_{n,0}) \right| + \omega(l_1, l_2, n_1, n_2) \\ &\leq 4k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d(\rho_n + \eta_n) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

From (3.3) and (3.4) we get  $A_7 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ .

**Estimate of  $A_1 + A_2 + A_3$ .** We set

$$J(r) = T_{\ell-k}(r - T_k(r)), \quad \forall r \in \mathbb{R},$$

and use the notations  $\bar{\mathcal{J}}$  and  $\mathcal{J}$  of (2.11). From the definitions of  $\xi_{1, n_1}, \xi_{1, n_2}$ , we can see that

$$\begin{aligned} A_1 + A_2 &= - \int_\Omega J(U_n(T)) \overline{H_m}(U_n(T)) dx - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu) \overline{H_m}(u_{0,n}) dx + \omega(l_1, l_2, n_1, n_2) \\ &= - \int_\Omega J(U_n(T)) U_n(T) dx - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu) u_{0,n} dx + \omega(l_1, l_2, n_1, n_2, m), \end{aligned} \quad (3.25)$$

where  $z_\nu = \langle T_k(U) \rangle_\nu(0)$ . We can write  $A_3 = F_1 + F_2$ , where

$$\begin{aligned} F_1 &= - \int_Q \left( \xi_{n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(U_n - \langle T_k(U) \rangle_\nu)]_{l_1} \right)_t \overline{H_m}(U_n), \\ F_2 &= \int_Q \left( \xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell-k}(U_n - T_k(U_n))]_{-l_2} \right)_t \overline{H_m}(U_n). \end{aligned}$$

**Estimate of  $F_2$ .** We write  $F_2 = G_1 + G_2 + G_3$ , with

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t \xi_{n_2} [T_{\ell-k}(U_n - T_k(U_n))]_{-l_2} \overline{H_m}(U_n), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2})(\xi_{n_2})_t [T_{\ell-k}(U_n - T_k(U_n))]_{-l_2} \overline{H_m}(U_n), \\ G_3 &= \int_Q \xi_{n_2} (1 - \Phi_{\delta_1, \delta_2}) ([T_{\ell-k}(U_n - T_k(U_n))]_{-l_2})_t \overline{H_m}(U_n). \end{aligned}$$

We find easily that

$$\begin{aligned} G_1 &= - \int_Q (\Phi_{\delta_1, \delta_2})_t J(U_n) U_n + \omega(l_1, l_2, n_1, n_2, m), \\ G_2 &= \int_Q (1 - \Phi_{\delta_1, \delta_2})(\xi_{n_2})_t J(U_n) \overline{H_m}(U_n) + \omega(l_1, l_2) = \int_{\Omega} J(u_{0,n}) u_{0,n} dx + \omega(l_1, l_2, n_1, n_2, m). \end{aligned}$$

Next consider  $G_3$ . Setting  $b = \overline{H_m}(U_n)$ , there holds from (2.13) and (2.12),

$$((J(b))_{-l_2})_t b(\cdot, t) = \frac{b(\cdot, t)}{l_2} (J(b)(\cdot, t) - J(b)(\cdot, t - l_2)).$$

Hence

$$([T_{\ell-k}(U_n - T_k(U_n))]_{-l_2})_t \overline{H_m}(U_n) \geq ([\mathcal{J}(\overline{H_m}(U_n))]_{-l_2})_t = ([\mathcal{J}(U_n)]_{-l_2})_t,$$

since  $\mathcal{J}$  is constant in  $\{|r| \geq m + \ell + 2k\}$ . Integrating by parts in  $G_3$ , we find

$$\begin{aligned} G_3 &\geq \int_Q \xi_{2, n_2} (1 - \Phi_{\delta_1, \delta_2}) ([\mathcal{J}(U_n)]_{-l_2})_t = - \int_Q (\xi_{2, n_2} (1 - \Phi_{\delta_1, \delta_2}))_t [\mathcal{J}(U_n)]_{-l_2} + \int_{\Omega} \xi_{2, n_2}(T) [\mathcal{J}(U_n)]_{-l_2}(T) dx \\ &= - \int_Q (\xi_{2, n_2})_t (1 - \Phi_{\delta_1, \delta_2}) \mathcal{J}(U_n) + \int_Q \xi_{2, n_2} (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(U_n) + \int_{\Omega} \xi_{2, n_2}(T) \mathcal{J}(U_n(T)) dx + \omega(l_1, l_2) \\ &= - \int_{\Omega} \mathcal{J}(u_{0,n}) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{J}(U_n) + \int_{\Omega} \mathcal{J}(U_n(T)) dx + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

Therefore, since  $\mathcal{J}(U_n) - J(U_n)U_n = -\overline{\mathcal{J}}(U_n)$  and  $\overline{\mathcal{J}}(u_{0,n}) = J(u_{0,n})u_{0,n} - \mathcal{J}(u_{0,n})$ , we obtain

$$F_2 \geq \int_{\Omega} \overline{\mathcal{J}}(u_{0,n}) dx - \int_Q (\Phi_{\delta_1, \delta_2})_t \overline{\mathcal{J}}(U_n) + \int_{\Omega} \mathcal{J}(U_n(T)) dx + \omega(l_1, l_2, n_1, n_2, m). \quad (3.26)$$

**Estimate of  $F_1$ .** Since  $m > \ell + 2k$ , there holds  $T_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) = T_{\ell+k}(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu})$  on  $\text{supp} \overline{H_m}(U_n)$ . Hence we can write  $F_1 = L_1 + L_2$ , with

$$\begin{aligned} L_1 &= - \int_Q \left( \xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu})]_{l_1} \right)_t (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu}) \\ L_2 &= - \int_Q \left( \xi_{1, n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k}(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_{\nu})]_{l_1} \right)_t \langle T_k(\overline{H_m}(U)) \rangle_{\nu}. \end{aligned}$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$\begin{aligned}
L_2 &= \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (\langle T_k(\overline{H_m}(U)) \rangle_\nu)_t \\
&\quad + \int_\Omega \xi_{1,n_1} (0) [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (0) \langle T_k(\overline{H_m}(U)) \rangle_\nu (0) dx \\
&= \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) (T_k(U) - \langle T_k(U) \rangle_\nu) + \int_\Omega T_{\ell+k} (u_{0,n} - z_\nu) z_\nu dx + \omega(l_1, l_2, n_1, n_2).
\end{aligned} \tag{3.27}$$

We decompose  $L_1$  into  $L_1 = K_1 + K_2 + K_3$ , where

$$\begin{aligned}
K_1 &= - \int_Q (\xi_{1,n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu) \\
K_2 &= \int_Q \xi_{1,n_1} (\Phi_{\delta_1, \delta_2})_t [T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu) \\
K_3 &= - \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) ([T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1})_t (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu).
\end{aligned}$$

Then we check easily that

$$\begin{aligned}
K_1 &= \int_\Omega T_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) (T) (U_n - \langle T_k(U) \rangle_\nu) (T) dx + \omega(l_1, l_2, n_1, n_2, m), \\
K_2 &= \int_Q (\Phi_{\delta_1, \delta_2})_t T_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) (U_n - \langle T_k(U) \rangle_\nu) + \omega(l_1, l_2, n_1, n_2, m).
\end{aligned}$$

Next consider  $K_3$ . Here we use the function  $\mathcal{T}_k$  defined at (2.13). We set  $b = \overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu$ . Hence from (2.12),

$$\begin{aligned}
([T_{\ell+k}(b)]_{l_1})_t(\cdot, t) &= \frac{b(\cdot, t)}{l_1} (T_{\ell+k}(b)(\cdot, t + l_1) - T_{\ell+k}(b)(\cdot, t)) \\
&\leq \frac{1}{l_1} (\mathcal{T}_{\ell+k}(b)((\cdot, t + l_1)) - \mathcal{T}_{\ell+k}(b)(\cdot, t)) = ([\mathcal{T}_{\ell+k}(b)]_{l_1})_t.
\end{aligned}$$

Thus

$$([T_{\ell+k} (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu)]_{l_1})_t (\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)) \rangle_\nu) \leq ([\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1})_t.$$

Then

$$\begin{aligned}
K_3 &\geq - \int_Q \xi_{1,n_1} (1 - \Phi_{\delta_1, \delta_2}) ([\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1})_t \\
&= \int_Q (\xi_{1,n_1})_t (1 - \Phi_{\delta_1, \delta_2}) [\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1} - \int_Q \xi_{1,n_1} (\Phi_{\delta_1, \delta_2})_t [\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1} \\
&\quad + \int_\Omega \xi_{1,n_1} (0) [\mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu)]_{l_1} (0) dx \\
&= - \int_\Omega \mathcal{T}_{\ell+k} (U_n(T) - \langle T_k(U) \rangle_\nu(T)) dx - \int_Q (\Phi_{\delta_1, \delta_2})_t \mathcal{T}_{\ell+k} (U_n - \langle T_k(U) \rangle_\nu) \\
&\quad + \int_\Omega \mathcal{T}_{\ell+k} (u_{0,n} - z_\nu) dx + \omega(l_1, l_2, n_1, n_2).
\end{aligned}$$

We find by addition, since  $T_{\ell+k}(r) - \mathcal{T}_{\ell+k}(r) = \bar{T}_{\ell+k}(r)$  for any  $r \in \mathbb{R}$ ,

$$\begin{aligned} L_1 &\geq \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) dx + \int_{\Omega} \bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) dx \\ &\quad + \int_Q (\Phi_{\delta_1, \delta_2})_t \bar{T}_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned} \quad (3.28)$$

We deduce from (3.28), (3.27), (3.26),

$$\begin{aligned} A_3 &\geq \int_{\Omega} \bar{J}(u_{0,n}) dx + \int_{\Omega} \mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) dx + \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu}) z_{\nu} dx \\ &\quad + \int_{\Omega} \bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) dx + \int_{\Omega} \mathcal{J}(U_n(T)) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U_n)) \\ &\quad + \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) (T_k(U) - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned} \quad (3.29)$$

Next we add (3.25) and (3.29). Note that  $\mathcal{J}(U_n(T)) - J(U_n(T))U_n(T) = -\bar{J}(U_n(T))$ , and also

$$\mathcal{T}_{\ell+k}(u_{0,n} - z_{\nu}) - T_{\ell+k}(u_{0,n} - z_{\nu})(z_{\nu} - u_{0,n}) = -\bar{T}_{\ell+k}(u_{0,n} - z_{\nu}).$$

Then we find

$$\begin{aligned} A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{J}(u_{0,n}) - \bar{T}_{\ell+k}(u_{0,n} - z_{\nu})) dx + \int_{\Omega} (\bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) - \bar{J}(U_n(T))) dx \\ &\quad + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U_n)) \\ &\quad + \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(U_n - \langle T_k(U) \rangle_{\nu}) (T_k(U) - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m). \end{aligned}$$

Notice that  $\bar{T}_{\ell+k}(r-s) - \bar{J}(r) \geq 0$  for any  $r, s \in \mathbb{R}$  such that  $|s| \leq k$ ; thus

$$\int_{\Omega} (\bar{T}_{\ell+k}(U_n(T) - \langle T_k(U) \rangle_{\nu}(T)) - \bar{J}(U_n(T))) dx \geq 0.$$

And  $\{u_{0,n}\}$  converges to  $u_0$  in  $L^1(\Omega)$  and  $\{U_n\}$  converges to  $U$  in  $L^1(Q)$  from Proposition 2.10. Thus we obtain

$$\begin{aligned} A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{J}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U)) \\ &\quad + \nu \int_Q (1 - \Phi_{\delta_1, \delta_2}) T_{\ell+k}(U - \langle T_k(U) \rangle_{\nu}) (T_k(U) - \langle T_k(U) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m, n). \end{aligned}$$

Moreover  $T_{\ell+k}(r-s)(T_k(r) - s) \geq 0$  for any  $r, s \in \mathbb{R}$  such that  $|s| \leq k$ , hence

$$\begin{aligned} A_1 + A_2 + A_3 &\geq \int_{\Omega} (\bar{J}(u_0) - \bar{T}_{\ell+k}(u_0 - z_{\nu})) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\bar{T}_{\ell+k}(U - \langle T_k(U) \rangle_{\nu}) - \bar{J}(U)) \\ &\quad + \omega(l_1, l_2, n_1, n_2, m, n). \end{aligned}$$

As  $\nu \rightarrow \infty$ ,  $\{z_\nu\}$  converges to  $T_k(u_0)$ , a.e. in  $\Omega$ , thus we get

$$A_1 + A_2 + A_3 \geq \int_{\Omega} (\overline{J}(u_0) - \overline{T}_{\ell+k}(u_0 - T_k(u_0))) dx + \int_Q (\Phi_{\delta_1, \delta_2})_t (\overline{T}_{\ell+k}(U - T_k(U)) - \overline{J}(U)) \\ + \omega(l_1, l_2, n_1, n_2, m, n, \nu).$$

Finally  $|\overline{T}_{\ell+k}(r - T_k(r)) - \overline{J}(r)| \leq 2k|r|\chi_{\{|r| \geq \ell\}}$  for any  $r \in \mathbb{R}$ , thus

$$A_1 + A_2 + A_3 \geq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell).$$

Combining all the estimates, we obtain  $I_2 \leq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ , which implies (3.8), since  $I_2$  does not depend on  $l_1, l_2, n_1, n_2, m, \ell$ .  $\blacksquare$

Next we conclude the proof of Theorem 1.1:

**Lemma 3.6** *The function  $u$  is a  $R$ -solution of (1.1).*

**Proof.** (i) First show that  $u$  satisfies (2.2). Here we proceed as in [22]. Let  $\varphi \in X \cap L^\infty(Q)$  such  $\varphi_t \in X' + L^1(Q)$ ,  $\varphi(\cdot, T) = 0$ , and  $S \in W^{2, \infty}(\mathbb{R})$ , such that  $S'$  has compact support on  $\mathbb{R}$ ,  $S(0) = 0$ . Let  $M > 0$  such that  $\text{supp} S' \subset [-M, M]$ . Taking successively  $(\varphi, S)$  and  $(\varphi\psi_\delta^\pm, S)$  as test functions in (2.2) applied to  $u_n$ , we can write

$$A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7, \quad A_{2, \delta, \pm} + A_{3, \delta, \pm} + A_{4, \delta, \pm} = A_{5, \delta, \pm} + A_{6, \delta, \pm} + A_{7, \delta, \pm},$$

where

$$A_1 = - \int_{\Omega} \varphi(0)S(u_{0,n})dx, \quad A_2 = - \int_Q \varphi_t S(U_n), \quad A_{2, \delta, \pm} = - \int_Q (\varphi\psi_\delta^\pm)_t S(U_n), \\ A_3 = \int_Q S'(U_n)A(x, t, \nabla u_n) \cdot \nabla \varphi, \quad A_{3, \delta, \pm} = \int_Q S'(U_n)A(x, t, \nabla u_n) \cdot \nabla (\varphi\psi_\delta^\pm), \\ A_4 = \int_Q S''(U_n)\varphi A(x, t, \nabla u_n) \cdot \nabla U_n, \quad A_{4, \delta, \pm} = \int_Q S''(U_n)\varphi\psi_\delta^\pm A(x, t, \nabla u_n) \cdot \nabla U_n, \\ A_5 = \int_Q S'(U_n)\varphi d\widehat{\lambda}_{n,0}, \quad A_6 = \int_Q S'(U_n)\varphi d\rho_{n,0}, \quad A_7 = - \int_Q S'(U_n)\varphi d\eta_{n,0}, \\ A_{5, \delta, \pm} = \int_Q S'(U_n)\varphi\psi_\delta^\pm d\widehat{\lambda}_{n,0}, \quad A_{6, \delta, \pm} = \int_Q S'(U_n)\varphi\psi_\delta^\pm d\rho_{n,0}, \quad A_{7, \delta, \pm} = - \int_Q S'(U_n)\varphi\psi_\delta^\pm d\eta_{n,0}.$$

Since  $\{u_{0,n}\}$  converges to  $u_0$  in  $L^1(\Omega)$ , and  $\{S(U_n)\}$  converges to  $S(U)$ , strongly in  $X$  and weak-\* in  $L^\infty(Q)$ , there holds, from (3.2),

$$A_1 = - \int_{\Omega} \varphi(0)S(u_0)dx + \omega(n), \quad A_2 = - \int_Q \varphi_t S(U) + \omega(n), \quad A_{2, \delta, \psi_\delta^\pm} = \omega(n, \delta).$$

Moreover  $T_M(U_n)$  converges to  $T_M(U)$ , then  $T_M(U_n) + h_n$  converges to  $T_k(U) + h$  strongly in  $X$ , thus

$$A_3 = \int_Q S'(U_n)A(x, t, \nabla (T_M(U_n) + h_n)) \cdot \nabla \varphi = \int_Q S'(U)A(x, t, \nabla (T_M(U) + h)) \cdot \nabla \varphi + \omega(n) \\ = \int_Q S'(U)A(x, t, \nabla u) \cdot \nabla \varphi + \omega(n);$$



and

$$\begin{aligned} A_4 &= \int_Q S''(U_n) \varphi A(x, t, \nabla(T_M(U_n) + h_n)) \cdot \nabla T_M(U_n) \\ &= \int_Q S''(U) \varphi A(x, t, \nabla(T_M(U) + h)) \cdot \nabla T_M(U) + \omega(n) = \int_Q S''(U) \varphi A(x, t, \nabla u) \cdot \nabla U + \omega(n). \end{aligned}$$

In the same way, since  $\psi_\delta^\pm$  converges to 0 in  $X$ ,

$$\begin{aligned} A_{3,\delta,\pm} &= \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla(\varphi \psi_\delta^\pm) + \omega(n) = \omega(n, \delta), \\ A_{4,\delta,\pm} &= \int_Q S''(U) \varphi \psi_\delta^\pm A(x, t, \nabla u) \cdot \nabla U + \omega(n) = \omega(n, \delta). \end{aligned}$$

And  $\{g_n\}$  strongly converges to  $g$  in  $(L^{p'}(\Omega))^N$ , thus

$$\begin{aligned} A_5 &= \int_Q S'(U_n) \varphi f_n + \int_Q S'(U_n) g_n \cdot \nabla \varphi + \int_Q S''(U_n) \varphi g_n \cdot \nabla T_M(U_n) \\ &= \int_Q S'(U) \varphi f + \int_Q S'(U) g \cdot \nabla \varphi + \int_Q S''(U) \varphi g \cdot \nabla T_M(U) + \omega(n) \\ &= \int_Q S'(U) \varphi d\widehat{\mu}_0 + \omega(n). \end{aligned}$$

Now  $A_{5,\delta,\pm} = \int_Q S'(U) \varphi \psi_\delta^\pm d\widehat{\lambda}_{n,0} + \omega(n) = \omega(n, \delta)$ . Then  $A_{6,\delta,\pm} + A_{7,\delta,\pm} = \omega(n, \delta)$ . From (3.2) we verify that  $A_{7,\delta,+} = \omega(n, \delta)$  and  $A_{6,\delta,-} = \omega(n, \delta)$ . Moreover, from (3.6) and (3.2), we find

$$|A_6 - A_{6,\delta,+}| \leq \int_Q |S'(U_n) \varphi| (1 - \psi_\delta^+) d\rho_{n,0} \leq \|S\|_{W^{2,\infty}(\mathbb{R})} \|\varphi\|_{L^\infty(Q)} \int_Q (1 - \psi_\delta^+) d\rho_n = \omega(n, \delta).$$

Similarly we also have  $|A_7 - A_{7,\delta,-}| \leq \omega(n, \delta)$ . Hence  $A_6 = \omega(n)$  and  $A_7 = \omega(n)$ . Therefore, we finally obtain (2.2):

$$-\int_\Omega \varphi(0) S(u_0) dx - \int_Q \varphi_t S(U) + \int_Q S'(U) A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(U) \varphi A(x, t, \nabla u) \cdot \nabla U = \int_Q S'(U) \varphi d\widehat{\mu}_0. \quad (3.30)$$

(ii) Next, we prove (2.3) and (2.4). We take  $\varphi \in C_c^\infty(Q)$  and take  $((1 - \psi_\delta^-) \varphi, \overline{H_m})$  as test functions in (3.30), with  $\overline{H_m}$  as in (2.14). We can write  $D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m}$ , where

$$\begin{aligned} D_{1,m} &= -\int_Q ((1 - \psi_\delta^-) \varphi)_t \overline{H_m}(U), & D_{2,m} &= \int_Q H_m(U) A(x, t, \nabla u) \cdot \nabla ((1 - \psi_\delta^-) \varphi), \\ D_{3,m} &= \int_Q H_m(U) (1 - \psi_\delta^-) \varphi d\widehat{\mu}_0, & D_{4,m} &= \frac{1}{m} \int_{m \leq U \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla U, \\ D_{5,m} &= -\frac{1}{m} \int_{-2m \leq U \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u) \cdot \nabla U. \end{aligned} \quad (3.31)$$

Taking the same test functions in (2.2) applied to  $u_n$ , there holds  $D_{1,m}^n + D_{2,m}^n = D_{3,m}^n + D_{4,m}^n + D_{5,m}^n$ , where

$$\begin{aligned} D_{1,m}^n &= - \int_Q ((1 - \psi_\delta^-) \varphi)_t \overline{H_m}(U_n), & D_{2,m}^n &= \int_Q H_m(U_n) A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-) \varphi), \\ D_{3,m}^n &= \int_Q H_m(U_n) (1 - \psi_\delta^-) \varphi d(\widehat{\lambda_{n,0}} + \rho_{n,0} - \eta_{n,0}), & D_{4,m}^n &= \frac{1}{m} \int_{m \leq U \leq 2m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla U_n, \\ D_{5,m}^n &= - \frac{1}{m} \int_{-2m \leq U_n \leq -m} (1 - \psi_\delta^-) \varphi A(x, t, \nabla u_n) \cdot \nabla U_n \end{aligned} \quad (3.32)$$

In (3.32), we go to the limit as  $m \rightarrow \infty$ . Since  $\{\overline{H_m}(U_n)\}$  converges to  $U_n$  and  $\{H_m(U_n)\}$  converges to 1, a.e. in  $Q$ , and  $\{\nabla H_m(U_n)\}$  converges to 0, weakly in  $(L^p(Q))^N$ , we obtain the relation  $D_1^n + D_2^n = D_3^n + D_4^n$ , where

$$\begin{aligned} D_1^n &= - \int_Q ((1 - \psi_\delta^-) \varphi)_t U_n, & D_2^n &= \int_Q A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta^-) \varphi), & D_3^n &= \int_Q (1 - \psi_\delta^-) \varphi d\widehat{\lambda_{n,0}} \\ D_4^n &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_{n,0} - \eta_{n,0}) + \int_Q (1 - \psi_\delta^-) \varphi d((\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-) \\ &= \int_Q (1 - \psi_\delta^-) \varphi d(\rho_n - \eta_n). \end{aligned}$$

Clearly,  $D_{i,m} - D_i^n = \omega(n, m)$  for  $i = 1, 2, 3$ . From Lemma (3.3) and (3.2)-(3.4), we obtain  $D_{5,m} = \omega(n, m, \delta)$ , and

$$\frac{1}{m} \int_{\{m \leq U < 2m\}} \psi_\delta^- \varphi A(x, t, \nabla u) \cdot \nabla U = \omega(n, m, \delta),$$

thus,

$$D_{4,m} = \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U + \omega(n, m, \delta).$$

Since  $\left| \int_Q (1 - \psi_\delta^-) \varphi d\eta_n \right| \leq \|\varphi\|_{L^\infty} \int_Q (1 - \psi_\delta^-) d\eta_n$ , it follows that  $\int_Q (1 - \psi_\delta^-) \varphi d\eta_n = \omega(n, m, \delta)$  from (3.4). And  $\left| \int_Q \psi_\delta^- \varphi d\rho_n \right| \leq \|\varphi\|_{L^\infty} \int_Q \psi_\delta^- d\rho_n$ , thus, from (3.2),  $\int_Q (1 - \psi_\delta^-) \varphi d\rho_n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$ . Then  $D^n = \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta)$ . Therefore by subtraction, we get successively

$$\begin{aligned} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U &= \int_Q \varphi d\mu_s^+ + \omega(n, m, \delta), \\ \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U &= \int_Q \varphi d\mu_s^+, \end{aligned} \quad (3.33)$$

which proves (2.3) when  $\varphi \in C_c^\infty(Q)$ . Next assume only  $\varphi \in C^\infty(\overline{Q})$ . Then

$$\begin{aligned} &\lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla U \\ &= \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi \psi_\delta^+ A(x, t, \nabla u) \cdot \nabla U + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla U \\ &= \int_Q \varphi \psi_\delta^+ d\mu_s^+ + \lim_{m \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi (1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla U = \int_Q \varphi d\mu_s^+ + D, \end{aligned}$$

where

$$D = \int_Q \varphi(1 - \psi_\delta^+) d\mu_s^+ + \lim_{n \rightarrow \infty} \frac{1}{m} \int_{\{m \leq U < 2m\}} \varphi(1 - \psi_\delta^+) A(x, t, \nabla u) \cdot \nabla U = \omega(\delta).$$

Therefore, (3.33) still holds for  $\varphi \in C^\infty(\overline{Q})$ , and we deduce (2.3) by density, and similarly, (2.4). This completes the proof of Theorem 1.1.  $\blacksquare$

## 4 Approximations of measures

Corollary 1.2 is a direct consequence of Theorem 1.1 and the following approximation property:

**Proposition 4.1** *Let  $\mu = \mu_0 + \mu_s \in \mathcal{M}_b^+(Q)$  with  $\mu_0 \in \mathcal{M}_0^+(Q)$  and  $\mu_s \in \mathcal{M}_s^+(Q)$ .*

*(i) Then, we can find a decomposition  $\mu_0 = (f, g, h)$  with  $f \in L^1(Q)$ ,  $g \in (L^{p'}(Q))^N$ ,  $h \in X$  such that*

$$\|f\|_{1,Q} + \|g\|_{p',Q} + \|h\|_X + \mu_s(\Omega) \leq 2\mu(Q) \quad (4.1)$$

*(ii) Furthermore, there exists sequences of measures  $\mu_{0,n} = (f_n, g_n, h_n)$ ,  $\mu_{s,n}$  such that  $f_n, g_n, h_n \in C_c^\infty(Q)$  strongly converge to  $f, g, h$  in  $L^1(Q)$ ,  $(L^{p'}(Q))^N$  and  $X$  respectively, and  $\mu_{s,n} \in (C_c^\infty(Q))^+$  converges to  $\mu_s$  and  $\mu_n := \mu_{0,n} + \mu_{s,n}$  converges to  $\mu$  in the narrow topology, and satisfying  $|\mu_n|(Q) \leq \mu(Q)$ ,*

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X + \mu_{s,n}(Q) \leq 2\mu(Q). \quad (4.2)$$

**Proof.** (i) Step 1. Case where  $\mu$  has a compact support in  $Q$ . By [15], we can find a decomposition  $\mu_0 = (f, g, h)$  with  $f, g, h$  have a compact support in  $Q$ . Let  $\{\varphi_n\}$  be sequence of mollifiers in  $\mathbb{R}^{N+1}$ . Then  $\mu_{0,n} = \varphi_n * \mu_0 \in C_c^\infty(Q)$  for  $n$  large enough. We see that  $\mu_{0,n}(Q) = \mu_0(Q)$  and  $\mu_{0,n}$  admits the decomposition  $\mu_{0,n} = (f_n, g_n, h_n) = (\varphi_n * f, \varphi_n * g, \varphi_n * h)$ . Since  $\{f_n\}, \{g_n\}, \{h_n\}$  strongly converge to  $f, g, h$  in  $L^1(Q), (L^{p'}(Q))^N$  and  $X$  respectively, we have for  $n_0$  large enough,

$$\|f - f_{n_0}\|_{1,Q} + \|g - g_{n_0}\|_{p',Q} + \|h - h_{n_0}\|_{L^p((0,T);W_0^{1,p}(\Omega))} \leq \frac{1}{2}\mu_0(Q).$$

Then we obtain a decomposition  $\mu = (\hat{f}, \hat{g}, \hat{h}) = (\mu_{n_0} + f - f_{n_0}, g - g_{n_0}, h - h_{n_0})$ , such that

$$\|\hat{f}\|_{1,Q} + \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X + \mu_s(Q) \leq \frac{3}{2}\mu(Q) \quad (4.3)$$

Step 2. General case. Let  $\{\theta_n\}$  be a nonnegative, nondecreasing sequence in  $C_c^\infty(Q)$  which converges to 1, a.e. in  $Q$ . Set  $\tilde{\mu}_0 = \theta_0\mu$ , and  $\tilde{\mu}_n = (\theta_n - \theta_{n-1})\mu$ , for any  $n \geq 1$ . Since  $\tilde{\mu}_n = \tilde{\mu}_{0,n} + \tilde{\mu}_{s,n} \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$  has compact support with  $\tilde{\mu}_{0,n} \in \mathcal{M}_0(Q)$ ,  $\tilde{\mu}_{s,n} \in \mathcal{M}_s(Q)$ , by Step 1, we can find a decomposition  $\tilde{\mu}_{0,n} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$  such that

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X + \tilde{\mu}_{s,n}(\Omega) \leq \frac{3}{2}\tilde{\mu}_n(Q).$$

Let  $\bar{f}_n = \sum_{k=0}^n \tilde{f}_k$ ,  $\bar{g}_n = \sum_{k=0}^n \tilde{g}_k$ ,  $\bar{h}_n = \sum_{k=0}^n \tilde{h}_k$  and  $\bar{\mu}_{s,n} = \sum_{k=0}^n \tilde{\mu}_{s,k}$ . Clearly,  $\theta_n\mu_0 = (\bar{f}_n, \bar{g}_n, \bar{h}_n)$ ,  $\theta_n\mu_s = \bar{\mu}_{s,n}$  and  $\{\bar{f}_n\}, \{\bar{g}_n\}, \{\bar{h}_n\}$  and  $\{\bar{\mu}_{s,n}\}$  converge strongly to some  $f, g, h$ , and  $\mu_s$  respectively in  $L^1(Q), (L^{p'}(Q))^N$ ,  $X$  and  $\mathcal{M}_b^+(Q)$ , and

$$\|\bar{f}_n\|_{1,Q} + \|\bar{g}_n\|_{p',Q} + \|\bar{h}_n\|_X + \bar{\mu}_{s,n}(Q) \leq \frac{3}{2}\mu(Q).$$

Therefore,  $\mu_0 = (f, g, h)$ , and (4.1) holds.

(ii) We take a sequence  $\{m_n\}$  in  $\mathbb{N}$  such that  $f_n = \varphi_{m_n} * \bar{f}_n$ ,  $g_n = \varphi_{m_n} * \bar{g}_n$ ,  $h_n = \varphi_{m_n} * \bar{h}_n$ ,  $\varphi_{m_n} * \bar{\mu}_{s,n} \in (C_c^\infty(Q))^+$ ,  $\int_Q \varphi_{m_n} * \bar{\mu}_{s,n} dxdt = \bar{\mu}_{s,n}(Q)$  and

$$\|f_n - \bar{f}_n\|_{1,Q} + \|g_n - \bar{g}_n\|_{p',Q} + \|h_n - \bar{h}_n\|_X \leq \frac{1}{n+2} \mu(Q).$$

Let  $\mu_{0,n} = \varphi_{m_n} * (\theta_n \mu_0) = (f_n, g_n, h_n)$ ,  $\mu_{s,n} = \varphi_{m_n} * \bar{\mu}_{s,n}$  and  $\mu_n = \mu_{0,n} + \mu_{s,n}$ . Therefore,  $\{f_n\}, \{g_n\}, \{h_n\}$  strongly converge to  $f, g, h$  in  $L^1(Q), (L^{p'}(Q))^N$  and  $X$  respectively. And (4.2) holds. Furthermore,  $\{\mu_{s,n}\}, \{\mu_n\}$  converge to  $\mu_s, \mu$  in the weak topology of measures, and  $\mu_{s,n}(Q) = \int_Q \theta_n d\mu_s, \mu_n(Q) = \int_Q \theta_n d\mu$  converges to  $\mu_s(Q), \mu(Q)$ , thus  $\{\mu_{s,n}\}, \{\mu_n\}$  converges to  $\mu_s, \mu$  in the narrow topology and  $|\mu_n|(Q) \leq \mu(Q)$ . ■

Observe that part (i) of Proposition 4.1 was used in [22], even if there was no explicit proof. Otherwise part (ii) is a *key point* for finding applications to the stability Theorem. Note also a very useful consequence for approximations by *nondecreasing* sequences:

**Proposition 4.2** *Let  $\mu \in \mathcal{M}_b^+(Q)$  and  $\varepsilon > 0$ . Let  $\{\mu_n\}$  be a nondecreasing sequence in  $\mathcal{M}_b^+(Q)$  converging to  $\mu$  in  $\mathcal{M}_b(Q)$ . Then, there exist  $f_n, f \in L^1(Q)$ ,  $g_n, g \in (L^{p'}(Q))^N$  and  $h_n, h \in X$ ,  $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$  such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \quad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and  $\{f_n\}, \{g_n\}, \{h_n\}$  strongly converge to  $f, g, h$  in  $L^1(Q), (L^{p'}(Q))^N$  and  $X$  respectively, and  $\{\mu_{n,s}\}$  converges to  $\mu_s$  (strongly) in  $\mathcal{M}_b(Q)$  and

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X + \mu_{n,s}(\Omega) \leq 2\mu(Q). \quad (4.4)$$

**Proof.** Since  $\{\mu_n\}$  is nondecreasing, then  $\{\mu_{n,0}\}, \{\mu_{n,s}\}$  are nondecreasing too. Clearly,  $\|\mu - \mu_n\|_{\mathcal{M}_b(Q)} = \|\mu_0 - \mu_{n,0}\|_{\mathcal{M}_b(Q)} + \|\mu_s - \mu_{n,s}\|_{\mathcal{M}_b(Q)}$ . Hence,  $\{\mu_{n,s}\}$  converges to  $\mu_s$  and  $\{\mu_{n,0}\}$  converges to  $\mu_0$  (strongly) in  $\mathcal{M}_b(Q)$ . Set  $\tilde{\mu}_{0,0} = \mu_{0,0}$ , and  $\tilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$  for any  $n \geq 1$ . By Proposition 4.1, (i), we can find  $\tilde{f}_n \in L^1(Q)$ ,  $\tilde{g}_n \in (L^{p'}(Q))^N$  and  $\tilde{h}_n \in X$  such that  $\tilde{\mu}_{n,0} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$  and

$$\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq 2\tilde{\mu}_{n,0}(Q)$$

Let  $f_n = \sum_{k=0}^n \tilde{f}_k$ ,  $G_n = \sum_{k=0}^n \tilde{g}_k$  and  $h_n = \sum_{k=0}^n \tilde{h}_k$ . Clearly,  $\mu_{n,0} = (f_n, g_n, h_n)$  and the convergence properties hold with (4.4), since

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X \leq 2\mu_0(Q). \quad \blacksquare$$

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