Cauchy Tetrahedron Argument applied to Higher Contact Interactions

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Abstract Second gradient theories are nowadays used in many studies in order to describe in detail the layers in the materials where physical properties are sharply varying. Sometimes higher order theories are also evoked. Up to now these models are not based on a construction of stresses similar to the one due to Cauchy, which has been applied only for simple materials. It has been widely recognized that the fundamental assumption by Cauchy that the traction depends only on the normal of the dividing surface cannot be maintained for higher gradient theories. However, this observation did not urge any author, to our knowledge, to revisit the Cauchy construction in order to adapt it to a more general conceptual framework. This is what we do in this paper for a continuum of grade N also called N-th gradient continua. Our construction is very similar to the one due to Cauchy : based on the tetrahedron argument, it does not introduce any argument of a different nature. In particular,, we avoid invoking the principle of virtual work. As one should expect, the balance assumption and the regularity hypotheses have to be adapted to the new framework and tensorial computations become more complex.

Contents

1 Introduction

At the beginning of the 19th century A. L. Cauchy founded Continuum Mechanics by assuming that two parts of a continuum interact by means of surface densities of contact forces concentrated on their dividing surface. Then, assuming that these contact forces depend only on the normal to the dividing surface and that they are balanced by some volume density of force (including inertia), he played with tetrahedrons and proved the existence of the so-called Cauchy stress tensor.

Many authors consider this point of view as the untouchable basis of Continuum Mechanics (see e.g. [48], [25], and the criticism raised in [5] and in [15]). In 1959, W. Noll [33] crystallized this faith by proving that the so-called Cauchy Postulate[1], that is the dependence of contact forces only on the normal of dividing surfaces, is indeed equivalent to the apparently weaker assumption of uniform boundedness of contact forces for all dividing surfaces. The merit of Noll's results consists in pointing out the relationship between tetrahedron argument and measure theory (see e.g. [16]): the drawback is in camouflaging behind a technical hypothesis the physical assumption that the contact forces depend only on the normal.

Many other authors (see e.g. [1], [8], [46], [41], [40], [49], [20], [32], [4], [6], [21], [38], [26]) are accustomed to use the second gradient theory to describe various phenomena in which strong variations of material properties occur. They thus use a theory based on the principle of virtual work and, at least since Toupin [44] [45], they are aware of the fact that the Cauchy Postulate is not valid in this context. Indeed, for instance in [24], it is stated that "*A central consequence of Toupin's work is the observation that Cauchy's hypothesis that the surface traction at a point x on a surface S depend on S through its normal field at x is not valid in a theory involving second gradients of the deformation, because in Toupin's theory the traction depends also on the curvature of S at x*". However, and remarkably enough, almost no effort (except, to our knowledge, the work [11]) has been attempted to adapt that Postulate and the tetrahedron construction in order to encompass theories of higher gradient continua.

All modern mechanicians admit that the introduction of surface contact interactions is the result of some underlying asymptotic process: from discrete atomic networks to continuum models, from microscopically non-homogeneous continua to macroscopic homogenized effective ones, or from thin three-dimensional structures to their lower dimensional limits (i.e. shells, plates, beams). Contact mechanical interactions, i.e. physical entities which act on the kinematics, need not to be forces. They can be for instance couples. It is well known that a distribution of forces, through an asymptotic process, can be transformed into an interaction of a different type, for instance a couple. Remark that the need of considering more general contact interactions is universally accepted in the theory of beams, plates and shells,

[1] We quote here that calling Postulate this hypothesis has been greatly misleading. Actually it is not a fundamental Principle of Mechanics as sometimes believed but a constitutive assumption: nothing comparable, for what concerns generality, for instance to the balance of force, energy or to the principle of virtual work.

where the macro-model has a lower dimension when compared to the corresponding micro-model, whereas it has been regarded as controversial when considering continuum three-dimensional models whose microscopic counterparts are discrete systems of particles moving in a three-dimensional space or again three-dimensional continuum models.

The best way for describing a mechanical interaction, may it be either a force or another type of more complex interaction, is to introduce its power expended on all admissible kinematic fields U. This is a linear form called virtual power. The fact of describing the interactions through their expended virtual power should not be confused with so-called virtual power method. This method popularized by Germain [23] consists in making assumptions on the internal virtual power and deducing properties of contact interactions, in a logical order which reverses the one used by Cauchy [14]. It is remarkable that already Gabrio Piola, in his pioneering works [35], [36], [37] raises several and important theoretical questions about this subject. He understood, by means of a micro-macro identification procedure based on the Principle of Virtual Work, that in general one cannot assume that macroscopic contact interactions reduce simply to contact forces and he introduced continuum models which are much more general than the one studied by Cauchy. Such models have been considered for engineering applications only in the first decade of the 21st century (see [15]).

Many results are available by now (see e.g. [34], [3], [42], [1], [46] or [2]) indicating that it is physically needed and mathematically consistent to consider macroscopic continuum models where contact interactions arise which expend power on high order velocity gradients calculated on dividing surfaces (see also [13]). An essential common ingredient of all systems which, after an homogenization procedure, were found out to present non-Cauchy contact interactions is that they all show highly contrasted physical properties at micro-level (see also [8], [9]). On the purely macroscopic point of view, the necessity of considering such interactions has been proven in some elegant papers [27], [28], [29], [22], [23] when one wants to consider continuum models in which deformation energy may consistently depend on second or higher gradients of displacement. The conceptual framework introduced e.g. by Truesdell and Noll [47] is not suitable for encompassing such models (see e.g. the difficulties arising in [17] and clarified in [11]), [14]. Moreover, the misunderstood range of validity of Noll's theorem persuaded many authors that the dependence of the deformation energy on higher gradients was forbidden by the second principle of thermodynamics (see e.g. [25] and [10]) or that the principles of thermodynamics needed to be modified [17], [31].

The present paper adapts the tetrahedron argument to include a class of continuum models which is much wider than the one originally considered by Cauchy.

To our purposes it is needed to reconsider the assumptions on which Cauchy based his analysis. We explicitly discuss them here because (as it is always true for any theory and model), in order to generalize Cauchy analysis, one has to have a clear understanding of its limits and to be aware of the fact that often its results are used outside of the correct context. In his paper *De la pression ou tension dans un corps solide*, Cauchy wrote ([7]

p. 61 lines 3, 15) : *a small element experiences on its different faces and at each point of them a determined pressure or tension . . . which can depend on the orientation of the surface.* ***This being set****, ...* Therefore he was clearly aware that he was accepting the two assumptions:

($H1$) Contact interactions reduce to surface forces on the boundary,

($H2$) Contact interactions depend on the normal of the boundary.

Later he added (p. 63 lines 16) : *Equilibrium should hold between inertial force and the forces to which are reduced all pressures and tensions exerted on the surfaces, . . .* Therefore he accepted the third assumption

($H3$) Contact interactions are balanced by volume forces.

In his proof he applied the balance of forces to domains with *a volume very small, so that every dimension is an infinitesimal quantity of first order* the mass being *an infinitesimal quantity of the third order* (p. 62 line 9 and p. 64 line 4) and finally he stated that pressure and tension on a small face *experience, by moving from one point to another one on a face, infinitesimal variations of the first order* (p. 62 line 14). Therefore he implicitly accepted the regularity assumptions

($H4$) Contact interactions depend continuously on the position.

The key conceptual advancement performed in the present paper consists in the acceptance of a wider class of contact interactions by weakening assumption ($H1$) (and this cannot be done without getting rid also of ($H2$)). We show, by following as closely as possible the spirit of Cauchy works, how one can indeed consider continuum models in which contact distributions of order larger than one can arise or can concentrate on the geometrical singularities of the dividing surfaces (edges and wedges). We assume that, for a rich enough class of sub-bodies, the contact interactions exerted on a sub-body B is a distribution concentrated on the dividing surface $\partial_2 B$ and on its singularities of dimension one $\partial_1 B$ and of dimension zero $\partial_0 B$. We also assume (and this is an actual restriction on the range of applicability of our theory) that the order of these distributions is uniformly bounded for all possible sub-bodies.

Taking advantage of the fundamental theorem in the theory of distributions due to Laurent Schwartz which states that every distribution having support included in a regular embedded sub-manifold M can be uniquely decomposed as a finite sum of transverse derivatives of extensions of distributions defined on M (see [39]), we write the power of contact interactions by means of the unique representation[2]

$$\mathfrak{S}(B,U) = \sum_{k=0}^{N-1} \int_{\partial_2 B} \mathsf{F}^2_k \mid \nabla^k_\perp U + \sum_{k=0}^{N-2} \int_{\partial_1 B} \mathsf{F}^1_k \mid \nabla^k_\perp U + \sum_{k=0}^{N-3} \int_{\partial_0 B} \mathsf{F}^0_k \mid \nabla^k_\perp U. \tag{1}$$

[2] In the formula (1) the chosen summation bounds may seem restrictive. This is not the case, as one can easily add some extra terms with vanishing densities. We will see later on in this paper the reason for writing the distribution in this form.

The function $\mathfrak{S}$ characterizes the stress state of the continuum which is then said to be in a *stress state of order* N. The fields $\left(\mathsf{F}_k^2, \mathsf{F}_k^1, \mathsf{F}_k^0\right)$ which depend on B and on the position, are dual quantities to the normal gradients $\nabla_\perp^k U$ of the virtual velocity field and are called *the contact* $k+1$ *-forces*. These quantities are, by definition, orthogonal to the shape where they are applied. Thus $\mathsf{F}_k^i \mid \nabla_\perp^k U = \mathsf{F}_k^i \mid \nabla^k U$ and in the sequel it will not be necessary to precise that only the orthogonal part of $\nabla^k U$ is involved.

The kinematics of considered continua may here be very general (e.g. the one specified in [18]). The configuration field may take values in a manifold and the velocity field in its tangent bundle, which can be of any tensorial nature. This tensorial nature is irrelevant in our developments and therefore, for the sake of efficiency, we operate as if the kinematics were described by a real valued function U. Therefore the tensor $\nabla_\perp^k U$ is considered to be of order k, as well as its dual quantities and $\mathsf{F}_k^i \mid \nabla_\perp^k U$ denotes the scalar product of the indicated tensors. It is straightforward, by applying our results component-wise, to extend them to the case where U is a tensor, and in particular to the classical case where U is a vector.

The class of sub-bodies we consider must contain tetrahedrons if we want to follow the trail of Cauchy. Therefore it cannot be limited to domains with smooth boundaries. We thus admit sub-bodies with boundaries (or Cauchy dividing surface) which are piece-wise regular, with normal fields subjected to jumps on a finite set of regular curves eventually concurring into wedges.

One of the points of Cauchy approach which are more often discussed (see e.g. [33] or [19]) is about the assumptions which are needed concerning the dependence of the fields F_k^i upon the sub-body B. We assume that the densities F_k^i depend in a sufficiently regular way on the position and depend on the considered sub-body only in a local way through the shape of the sub-body, a notion which is precisely defined in the following section.

The theory by Cauchy is a particular case of the one we present here: indeed, if we make the extra assumptions that the stress state is of order one and that the contact surface $1-$forces depend on the shape of dividing surfaces only through their normal then we are back to the framework used by Cauchy and our demonstrations and results are identical.

Assuming that the stress state is of order one is indeed a constitutive assumption so deeply rooted in the mind of many authors that it has been very often accepted unconsciously and we emphasize that Noll's theorem [33] cannot be proven without this assumption.

The generalized contact interactions we previously described are not usually considered in the literature. One can find two different reasons for this circumstance. First this is due to the fact that the virtual work is not always the preferred tool for some mechanicians while it gives the conceptual framework in which generalized contact interactions arise naturally. Secondly, it is a fact that many usual materials cannot sustain stress states of order larger than one.

Cauchy's proof of the existence of stress tensor is based on the equilibrium of contact forces with a force which is assumed to be diffuse in the volume. We also need a similar assumption. We assume that *quasi-balance of power*

holds: for every virtual velocity field U, there exists a constant K_U such that,

$$|\mathfrak{S}(B,U)| \leq K_U\,|B|\,. \tag{2}$$

Here $|B|$ denotes the Lebesgue measure of B.

When considering only rigid virtual velocity fields U, one reduces inequality (2) to the quasi-balance of forces which is a weak form of the equilibrium condition used by Cauchy. As remarked in [12], quasi-balance of forces is not sufficient to obtain a description of a stress state of order two or more.

While inequality (2) could seem a very weak assumption, we emphasize that it rules out some possible stress states as for instance those occurring in continua including material surfaces or continua including interfaces with Laplace surface tension.

This paper is organized as follows. In Section 2, we define precisely what we call shape of a body, we precise some technical regularity assumptions, we fix the notation and recall some useful results from differential geometry and tensorial calculus. We also define the boundary operators which will be proven to describe how the shape of the boundary influences the contact interactions.

Section 3 is devoted to the statement of the main result. As this result is based on the tetrahedron argument, we first fix some notation concerning the tetrahedron then we prove the main representation theorem. Due to the complexity of the considered interactions, the proof is much more technical than the original one by Cauchy. The technicalities come also from the fact that we use a non-orthogonal tetrahedron in order to represent not only the surface terms but also the edge and wedge terms. The first version of the theorem is limited to a partial representation result (Theorem 1) which is then extended to a complete one in Theorem 2 owing to a topological lemma and two tensorial lemmas which are postponed to Appendices 8 and 9. It is remarkable that the aforementioned purely topological lemma avoids any need of proving Cauchy action-reaction lemma.

The theorems established in section 3 are only valid for the highest order terms of the stress state and for tetrahedral shapes. We extend the results to more general shapes in section 4 by generalizing Noll's theorem. Finally, in section 5 we show how, using integration by parts on the different elements of the boundary, the previous results can be used to obtain a representation of all terms is the stress state.

2 Notation

2.1 Domains and shapes

In this paper we consider a class of domains D which we call *admissible domains*. These domains have a topological boundary ∂D (*contact surface*) which is piece-wise regular in the following sense : (i) ∂D is a finite union $\cup_{k\in I_2(D)}\mathcal{F}_k$ of two-dimensional C^∞ compact manifolds with border called the *faces* of D; (ii) the union of the borders of these faces is a finite union $\cup_{j\in I_1(D)}\mathcal{L}_j$ of one-dimensional C^∞ compact manifolds with boundary called

the *edges* of D; (iii) the edges are concurring in *wedges* and the set $\partial_0 D := \cup_{\ell \in I_0(D)} \{x_\ell\}$ of wedges is finite. Therefore $I_i(D)$ denotes the set of labels of the elements of dimension i which are parts of the boundary of D.

All internal points of the faces (i.e. those points which do not belong to the border of the faces) are called regular points of the faces. Their set is denoted by $\partial_2 D$. The set of all internal (or regular) points of the edges is denoted by $\partial_1 D$.

On an edge $\mathcal{L}_j$ two faces $\mathcal{F}_k : \ k \in [\mathcal{L}_j]$ concur. Hence $[\mathcal{L}_j]$ denotes the pair of subscripts of the faces concurring there. We denote e_j a unit vector tangent to the edge $\mathcal{L}_j$ and ν_k^j the unit vector orthogonal to the line $\mathcal{L}_j$, tangent to the face $\mathcal{F}_k$ and external to it.

On a wedge $\{x_\ell\}$, a finite number of edges $\mathcal{L}_j : \ j \in [x_\ell]$ concur. Hence $[x_\ell]$ denotes the set of subscripts of the edges concurring there.

As we are interested in the dependence of contact interactions on the shape of the domain, we need to define precisely what we mean by *shape*. We first say that the shape of a domain D at the point x is the same as the shape of the domain D' at the point x' if and only if there exists a neighborhood O_x of x and a translation $t_{x-x'}$ such that

$$t_{x-x'}(x') = x, \quad and \quad t_{x-x'}(D') \cap O_x = D \cap O_x \tag{3}$$

This makes an equivalence relation. Thus we can set :

Definition 1 We call shape of D at the point x the equivalence class with respect to the above defined equivalence relation. We denote this equivalence class by means of the symbol $\widetilde{(D, x)}$.

Note that this notion of shape is local. Note also that, according to the definition, when a surface is rotated its shape changes.

2.2 Regularity assumptions

The shape at x of the domain $D \cap \{y, \ (y - x) \cdot u < 0\}$ depends only on the shape at x of the domain D and on u. We denote it $cut(\widetilde{(D, x)}, u)$.

In all our proofs we will work with shapes belonging to a *prescribed set of shapes* E. This notion is a slight generalization of polyhedral shapes and essentially used in 4. By prescribed set of shapes we mean that there exists a finite sequence (D_i) of admissible domains and a finite sequence (u_j) of unit vectors such that any shape f in E satisfies $f = \widetilde{(D_i, x)}$ or $f = cut(\widetilde{(D_i, x)}, u_j)$ or $f = cut(cut(\widetilde{(D_i, x)}, u_j), u_k)$ for some point x and some i, j, k.[3]

[3] Examples: The shapes of the family of cubes $C_t := [0, t]^3$, for $t \in (0, 1]$ constitute a prescribed set of shapes while the shapes of the family of cubes C'_t image of $[0, 1]^3$ under a rotation of angle t , for $t \in (0, 1]$ around an axis u do not constitute a prescribed set of shapes. Analogously, the shapes of the family of spheres S_t of center 0 and radius t do not constitute a prescribed set of shapes.

A priori, the densities F_q^i in $\mathfrak{S}(D,U)$ depend on the position x and on the domain D. We assume that they depend on the domain only through its shape at point x : $\mathsf{F}_q^i = \mathsf{F}_q^i(x, \widetilde{(D,x)})$.

In the same way as the Cauchy's construction of stress tensor needs the continuity with respect to x of contact forces, our study needs some regularity hypotheses. For any domain D and any unit vectors u, v, we assume the continuity : $\forall \varepsilon > 0$, $\forall \bar{x} \in \partial_i D$, $\exists \eta > 0$ such that $\forall x \in \partial_i D$ satisfying $|| \, x - \bar{x} \, || < \eta$

$$\begin{aligned}
&|| \, \mathsf{F}_q^i(x, \widetilde{(D,x)}) - \mathsf{F}_q^i(\bar{x}, \widetilde{(D,\bar{x})}) \, || < \varepsilon, \\
&|| \, \mathsf{F}_q^i(x, Cut(\widetilde{(D,x)}, u) - \mathsf{F}_q^i(\bar{x}, Cut(\widetilde{(D,\bar{x})}, u) \, || < \varepsilon, \\
&|| \, \mathsf{F}_q^i(x, Cut(Cut(\widetilde{(D,x)}, u), v) - \mathsf{F}_q^i(\bar{x}, Cut(Cut(\widetilde{(D,\bar{x})}, u), v) \, || < \varepsilon.
\end{aligned} \tag{4}$$

and the equicontinuity on a prescribed set of shapes E: $\forall \varepsilon > 0$, $\forall \bar{x}$, $\exists \eta > 0$ such that $\forall x$, $\forall f \in E$,

$$|| \, x - \bar{x} \, || < \eta \Rightarrow || \, \mathsf{F}_q^i(x, f) - \mathsf{F}_q^i(\bar{x}, f) \, || < \varepsilon \tag{5}$$

where $i = 2$, 1 or 0 depending if f is a regular shape, a edge or a wedge shape.

Unlike Noll [33], we do not assume the uniform boundedness of the densities with respect to all possible shapes. Such a strong assumption would kill all possibilities of describing stress states of order larger than one. Instead, our assumptions only imply the uniform boundedness of these densities for shapes in a prescribed set of shapes.

2.3 Differential geometry

When M is a smooth p-dimensional embedded compact manifold in the Euclidean space, we denote $|M|$ the Hausdorff measure of M : if M is a volume, a surface, a line or a discrete system of points $|M|$ denotes respectively its volume, area, length or cardinal number. Integrals over M are integrals with respect to the corresponding Hausdorff measure without explicitly mentioning it.

We denote $\mathit{\Pi}_M$ the orthogonal projector on the tangent subspace of M and $\mathit{\Lambda}_M = Id - \mathit{\Pi}_M$ the projector on its orthogonal subspace.

For any tensor X of order p defined on M, we call *completely orthogonal part* of X the tensor $(X_{\perp M})_{i_1 \ldots . i_p} := X_{j_1 \ldots . j_p} \, (\mathit{\Lambda}_M)_{i_1}^{j_1} \ldots . \, (\mathit{\Lambda}_M)_{i_p}^{j_p}$ and we say that X is completely orthogonal to M if $X = X_{\perp M}$.

We will make use of the following divergence theorem on manifolds with boundary: for any differentiable tensors X and Y (X having an order greater than Y) defined on M we have

$$\int_M (X \cdot \mathit{\Pi}_M) \mid \nabla Y = - \int_M (div_M (X \cdot \mathit{\Pi}_M)) \mid Y + \int_{\partial M} (X \cdot \nu) \mid Y$$

where $div_M(X)$ stands for the standard divergence operator [4] on the manifold M and ν denotes the unit vector orthogonal to ∂M, tangent to M and external to it. The tensorial notation used here is precised in the next section.

2.4 Tensorial analysis

The Einstein convention of summation of repeated indices is used throughout this paper. The subscripts are relative to a three-dimensional basis. They vary in $\{1,2,3\}$ and we adopt the following convention : for $j \in \{1,2,3\}$, $j+1$ denotes the index following j in a circular permutation of $\{1,2,3\}$, similarly $j+2$ denotes the next one...

We use the standard notation $\otimes$ for the tensorial product : $X \otimes Y$ is the $p+q$ tensor defined by

$$(X \otimes Y)_{i_1,i_2,\ldots i_{p+q}} := X_{i_1,i_2,\ldots i_p} Y_{i_{p+1},\ldots i_{p+q}}$$

while the product $\times$ stands for the cross product of two three dimensional vectors. When n is a vector, we also use the less standard notation $n^{\otimes p}$ for denoting the tensor of order p defined by induction by setting $n^{\otimes 1} := n$ and $n^{\otimes p} = n^{\otimes p-1} \otimes n$.

If X and Y are two tensors of order p and q with $p \geq q$, we denote by $X \mid Y$ the $p-q$ tensor defined by

$$(X \mid Y)_{i_1,i_2,\ldots,i_{p-q}} := X_{i_1,i_2,\ldots i_{p-q},j_1,\ldots,j_q} Y_{j_1,j_2,\ldots,j_q}.$$

In particular, when $p=q$, this product coincides with the scalar product of tensors of order p.

We also denote $X \cdot Y$ the $p+q-2$ tensor defined by

$$(X \cdot Y)_{i_1,i_2,\ldots i_{p+q-2}} := X_{i_1,i_2,\ldots i_{p-1},j} Y_{j,i_p,\ldots i_p+q-2}.$$

Note that, when $q=1$, i.e when Y is a vector, the product $X \mid Y$ coincides with $X \cdot Y$.

Given a permutation σ in the symmetric group Σ_p of permutations of $\{1,\ldots,p\}$, the tensor σX is defined by

$$(\sigma X)_{i_1,i_2,\ldots,i_p} := X_{i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{\sigma(p)}}$$

We say that X is completely symmetric if $\sigma X = X$ for every permutation $\sigma \in \Sigma_p$. The application which, to any X, associates

$$\mathcal{K}^p(X) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma X \tag{6}$$

is the orthogonal projection onto the space of completely symmetric tensors and we call completely symmetric part of X its image under this projection. Note that, as the context always prevents any ambiguity, we drop the superscript p in the sequel.

[4] A quick way for defining $div_M(X)$ is to use any smooth extension of X in the vicinity of the manifold and set $div_M(X) := \nabla X \mid \Pi_M$ and to remark thereafter that the result of this operation depends only on the values of X on the manifold.

When M is a smooth p-dimensional embedded compact manifold, we also denote $\mathcal{K}_M(T)$ the completely symmetric part of $T_{\perp M}$:

$$\mathcal{K}_M(T) := \mathcal{K}(T_{\perp M}). \tag{7}$$

2.5 Boundary tensors

For $1 \leq q \leq p$, we define the tensors $\mathcal{P}^p_{M,q}$ and $\mathcal{P}^p_M$ of order $2p$ by [5]

$$(\mathcal{P}^p_{M,q})_{i_1,\dots i_{2p}} := \delta^{i_{p+1}}_{i_1} \dots \delta^{i_{p+q-1}}_{i_{q-1}} (\Pi_M)^{i_{p+q}}_{i_p} (\Lambda_M)^{i_{p+q+1}}_{i_{q+1}} \dots (\Lambda_M)^{i_{2p-1}}_{i_{p-1}} (\Lambda_M)^{i_{2p}}_{i_q} \tag{8}$$

and

$$\mathcal{P}^p_M := \sum_{q=1}^{p} \mathcal{P}^p_{M,q} \tag{9}$$

Now we define the boundary operators in three different ways depending on the shape of the boundary.

Definition 2 At a regular point of a face with external normal n, the boundary operator $\mathcal{O}^p_n$ is the tensor of order $2p-1$:

$$\mathcal{O}^p_n := n^{\otimes(2p-1)}$$

At a regular point of an edge $\mathcal{L}_j$ where concur the faces $\mathcal{F}_k$ ($k \in [\mathcal{L}_j]$) and whose shape f_j is determined by the vectors e_j, n_k, ν^j_k, the boundary operator $\mathcal{O}^p_{f_j}$ is the tensor of order $2p-2$ defined by

$$\left(\mathcal{O}^p_{f_j}\right)_{i_1,\dots,i_{2p-2}} := \sum_{k\in[\mathcal{L}_j]} \left(\nu^j_k\right)_\ell \left(\mathcal{P}^{p-1}_{\mathcal{F}_k}\right)_{i_1,\dots,i_{p-2},\ell,i_{p-1},\dots,i_{2p-3}} \left(n_k\right)_{i_{2p-2}}$$

On a wedge $\hat{x}$ of shape w where concur the edges $\mathcal{L}_j$ ($j \in [\hat{x}]$) with tangent unit vectors e_j pointing outward of $\hat{x}$, the boundary operator $\mathcal{O}_w$ is the tensor of order $2p-3$ defined by

$$(\mathcal{O}^p_w)_{i_1,\dots,i_{2p-3}} := \sum_{j\in[\hat{x}]} \left(e_j\right)_{\ell_1} \left(\mathcal{P}^{p-2}_{\mathcal{L}_j}\right)_{i_1,\dots,i_{p-3},\ell_1,\dots,\ell_{p-1}} \left(\mathcal{O}^p_{f_j}\right)_{\ell_2,\dots,\ell_{p-1},i_{p-2},\dots,i_{2p-3}}$$

Some useful properties of these tensors are, for sake of clarity of our presentation, postponed to Appendix 9.

[5] Note the subtle but important transposition of subscripts i_p and i_q in this formula. Note also that this formula simply reads

$$(\mathcal{P}^p_{M,q})_{i_1,\dots i_{2p}} := \delta^{i_{p+1}}_{i_1} \dots \delta^{i_{2p-1}}_{i_{p-1}} (\Pi_M)^{i_{2p}}_{i_p}$$

when $q = p$ while when $q = p-1$ it reads

$$(\mathcal{P}^p_{M,q})_{i_1,\dots i_{2p}} := \delta^{i_{p+1}}_{i_1} \dots \delta^{i_{2p-2}}_{i_{p-2}} (\Pi_M)^{i_{2p-1}}_{i_p} (\Lambda_M)^{i_{2p}}_{i_{p-1}}$$

3 Cauchy Tetrahedron Argument

3.1 Tetrahedrons

A tetrahedron is the central tool for proving Cauchy representation theorem of contact interactions. This is still true in our case. Thus we fix here some useful notation for geometrical quantities associated to a tetrahedron.

Definition 3 For any point $\hat{x}$, any unit independent vectors n_1, n_2, n_3 (with negative determinant), any unit vector n satisfying for all $j \in \{1,2,3\}$

$$n \cdot n_j < 0 \tag{10}$$

and for any positive real number h we consider the non degenerated tetrahedron

$$\Delta(\hat{x}, n_1, n_2, n_3, n, h) := \{x : (x - \hat{x}) \cdot n_j > 0, (x - \hat{x}) \cdot n < h\}.$$

We denote $\mathcal{F}$ and $\mathcal{F}_j$ the faces of the tetrahedron having respectively for unit outward normals n and n_j. We introduce the vectors $e_j := \frac{n_{j+1} \times n_{j+2}}{\|n_{j+1} \times n_{j+2}\|}$ which are unit tangent vectors to three edges denoted respectively $\mathcal{L}_j$. The point $\hat{x}$ is the vertex of the tetrahedron where these three edges concur.

On each edge $\mathcal{L}_j$, the unit vectors orthogonal to the line $\mathcal{L}_j$, tangent respectively to the faces $\mathcal{F}_{j+1}$ and $\mathcal{F}_{j-1}$ and external to them are $\nu^j_{j+1} := e_j \times n_{j+1}$ and $\nu^j_{j-1} := -e_j \times n_{j-1}$.

The height of the tetrahedron corresponding to the face $\mathcal{F}$ has length h. By projecting each edge vector $|\mathcal{L}_j| e_j$ onto the direction n we get $|\mathcal{L}_j| e_j \cdot n = h$. By projecting $|\mathcal{L}_j| e_j$ onto the normal n_j of $\mathcal{F}_j$, we obtain $h_j = -|\mathcal{L}_j| e_j \cdot n_j$ for the height of the tetrahedron corresponding to the face $\mathcal{F}_j$. The volume of the tetrahedron is

$$|\Delta| = \frac{1}{6} \det(|\mathcal{L}_1| e_1, |\mathcal{L}_2| e_2, |\mathcal{L}_3| e_3) = \frac{d}{6(e_1 \cdot n)(e_2 \cdot n)(e_3 \cdot n)} h^3$$

where $d := \det(e_1, e_2, e_3)$. The areas $|\mathcal{F}_i|$ and $|\mathcal{F}|$ of the faces can then be computed by noticing that $3|\Delta| = |\mathcal{F}|h = |\mathcal{F}_i|h_i$. We get

$$|\mathcal{F}| = \frac{d}{2(e_1 \cdot n)(e_2 \cdot n)(e_3 \cdot n)} h^2$$
$$|\mathcal{F}_i| = \frac{dh^3}{2h_i(e_1 \cdot n)(e_2 \cdot n)(e_3 \cdot n)} = -|\mathcal{F}| \frac{(e_i \cdot n)}{(e_i \cdot n_i)}.$$

Finally we remark that, for any $\bar{x}$ such that $(\bar{x} - \hat{x}) \cdot n = h$,

$$\int_{\mathcal{L}_i} (x - \bar{x}) \cdot n = -\frac{|\mathcal{L}_i|^2}{2} (e_i \cdot n) = -\frac{h^2}{2(e_i \cdot n)} = -\frac{(e_{i+1} \cdot n)(e_{i+2} \cdot n)}{d} |\mathcal{F}|.$$

We need to consider domains which do not change shape under homothety transformations. Among all such domains, the tetrahedron has the primordial advantage of exhibiting the smallest number of shapes. Indeed, it exhibits

fourteen different shapes on its boundary divided in three categories: on the internal points of its four faces the shape is plane and characterized by the normal to the face; on the internal points of its six edges the shape is dihedral and is characterized by the two normals concurring there and the tangent to the edge, for instance the shape f_1 at any internal point of $\mathcal{L}_1$ is characterized by (n_2, n_3, e_1). Finally at any vertex of the tetrahedron the shape is characterized by the triple of normals to the faces concurring in it. For instance the shape w at $\hat{x}$ depends on (n_1, n_2, n_3).

3.2 A Representation Theorem for highest order contact interactions generalizing the Cauchy tetrahedron Theorem.

Let us consider a body occupying the domain D. We assume that it is submitted to a physically admissible stress state $\mathfrak{S}$ of grade N. We prove the following consequence of quasi balance of power.

Theorem 1 *At any point $\bar{x}$ and for any triplet of unit independent vectors (n_1, n_2, n_3) (with negative determinant) there exists a continuous tensor field $\tilde{C}_N$ of order N such that, for any plane shape n satisfying the inequalities (10),*

$$\mathsf{F}^2_{N-1}(\bar{x}, n) = \mathcal{O}^N_n \mid \tilde{C}_N = \left(\tilde{C}_N(\bar{x}) \mid n^{\otimes N}\right) \otimes n^{\otimes N-1}. \tag{11}$$

Moreover

$$\mathsf{F}^1_{N-2}(\bar{x}, f_1) = \mathcal{K}_{\mathcal{L}_1}\left(\mathcal{O}^N_{f_1} \mid \tilde{C}_N\right) \tag{12}$$

and

$$\mathsf{F}^0_{N-3}(\bar{x}, w) = \mathcal{K}\left(\mathcal{O}^N_w \mid \tilde{C}_N\right). \tag{13}$$

Proof For any positive real number ε, we consider the one-parameter family of non degenerated homothetic tetrahedrons $\Delta^\varepsilon = \Delta(\bar{x} - \varepsilon n, n_1, n_2, n_3, n, \varepsilon)$ following Definition 3 and add a superscript ε to all quantities associated to Δ^ε according to the notation stated in section 3.1 We apply the quasi-balance of contact power for this family and for the fixed test field

$$U : x \longrightarrow ((x - \bar{x}) \cdot n)^{N-1} U_0$$

where U_0 is generic in the space $\mathcal{E}$ which describes the kinematics of the continuum. We recall that in whole sequel, the tensorial nature of U_0 is overlooked : without loss of generality U_0 is considered as a scalar quantity. Quasi-balance $|\mathfrak{S}(\Delta^\varepsilon, U)| \leq K_U |\Delta^\varepsilon|$ reads

$$\left| \sum_{k=0}^{N-1} \int_{\partial_2 \Delta^\varepsilon} \mathsf{F}^2_k \mid \nabla^k U + \sum_{k=0}^{N-2} \int_{\partial_1 \Delta^\varepsilon} \mathsf{F}^1_k \mid \nabla^k U + \sum_{k=0}^{N-3} \int_{\partial_0 \Delta^\varepsilon} \mathsf{F}^0_k \mid \nabla^k U \right| \leq K_U |\Delta^\varepsilon| \tag{14}$$

In the sequel we draw conclusions from the fact that $|\mathcal{F}^\varepsilon|^{-1} |\mathfrak{S}(\Delta^\varepsilon, U)|$ has a vanishing limit when ε tends to zero. We start by remarking that the field U together with all its derivatives up to order $N-2$ vanish on the plane

$(x-\bar{x})\cdot n=0$ which include the face $\mathcal{F}^\varepsilon$ and the edges and wedges which border it. Indeed:

$$\nabla^k U=\frac{(N-1)!}{(N-1-k)!}\left((x-\bar{x})\cdot n\right)^{N-1-k}\,U_0\otimes n^{\otimes k}.$$

This last equation shows also that, when ε tends to zero, the asymptotic order of magnitude of $\nabla^k U$ is $O(\varepsilon^{N-1-k})$. Therefore recalling that we assume that the stress state depends regularly on the space variables (4), (5), we get the estimates

$$\begin{aligned}
\left|\int_{\partial_2\Delta^\varepsilon}\mathsf{F}_k^2\mid\nabla^k U\right| &= |\partial_2\Delta^\varepsilon|\,O(\varepsilon^{N-1-k})=O(\varepsilon^{N+1-k}),\\
\left|\int_{\partial_1\Delta^\varepsilon}\mathsf{F}_k^1\mid\nabla^k U\right| &= |\partial_1\Delta^\varepsilon|\,O(\varepsilon^{N-1-k})=O(\varepsilon^{N-k}),\\
\left|\int_{\partial_0\Delta^\varepsilon}\mathsf{F}_k^0\mid\nabla^k U\right| &= O(\varepsilon^{N-1-k}).
\end{aligned}$$

Thus keeping the only terms which are not plainly vanishing in the limit of $|\mathcal{F}^\varepsilon|^{-1}\,|\mathfrak{S}(\Delta^\varepsilon,U)|$ we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}|\mathcal{F}^\varepsilon|^{-1}\Bigg(&\int_{\mathcal{F}^\varepsilon}(N-1)!\,\mathsf{F}_{N-1}^2\mid\left(U_0\otimes n^{\otimes N-1}\right)\\
&+\sum_{i=1}^3\int_{\mathcal{F}_i^\varepsilon}(N-1)!\,\mathsf{F}_{N-1}^2\mid\left(U_0\otimes n^{\otimes N-1}\right)\\
&+\sum_{i=1}^3\int_{L_i^\varepsilon}(N-1)!\,\left((x-\bar{x})\cdot n\right)\mathsf{F}_{N-2}^1\mid\left(U_0\otimes n^{\otimes N-2}\right)\\
&+\frac{(N-1)!\,\varepsilon^2}{2}\mathsf{F}_{N-3}^0(\bar{x}-\varepsilon n,w)\mid\left(U_0\otimes n^{\otimes N-3}\right)\Bigg)=0.
\end{aligned}$$

Making explicit the argument of the functions F and applying the mean value theorem, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}|\mathcal{F}^\varepsilon|^{-1}\Bigg(&|\mathcal{F}^\varepsilon|\,\mathsf{F}_{N-1}^2(x^\varepsilon,n)\mid\left(U_0\otimes n^{\otimes N-1}\right)\\
&+\sum_{i=1}^3|\mathcal{F}_i^\varepsilon|\,\mathsf{F}_{N-1}^2(x_i^\varepsilon,n_i)\mid\left(U_0\otimes\ n^{\otimes N-1}\right)\\
&+\sum_{i=1}^3\left(\mathsf{F}_{N-2}^1\,(\tilde{x}_i^\varepsilon,f_i)\mid\left(U_0\otimes\ n^{\otimes N-2}\right)\int_{\mathcal{L}_i^\varepsilon}\left((x-\bar{x})\cdot n\right)\right)\\
&+\frac{\varepsilon^2}{2}\mathsf{F}_{N-3}^0(\bar{x}-\varepsilon n,w)\mid\left(U_0\otimes\ n^{\otimes N-3}\right)\Bigg)=0.
\end{aligned}$$

for some points $x^\varepsilon \in \mathcal{F}^\varepsilon$, $x_i^\varepsilon \in \mathcal{F}_i^\varepsilon$ and $\tilde{x}_i^\varepsilon \in \mathcal{L}_i^\varepsilon$. Using the geometrical identities we have established in section 3.1, we obtain

$$\lim_{\varepsilon\to 0}\Bigg(\mathsf{F}^2_{N-1}(x^\varepsilon,n)\,|\,\left(U_0\otimes n^{\otimes N-1}\right)-\sum_{i=1}^{3}\frac{(e_i\cdot n)}{(e_i\cdot n_i)}\mathsf{F}^2_{N-1}(x_i^\varepsilon,n_i)\,|\,\left(U_0\otimes n^{\otimes N-1}\right)$$
$$+\sum_{i=1}^{3}\left(-d^{-1}\right)\frac{(e_1\cdot n)\,(e_2\cdot n)\,(e_3\cdot n)}{(e_i\cdot n)}\mathsf{F}^1_{N-2}\left(\tilde{x}_i^\varepsilon,f_i\right)\,|\,\left(U_0\otimes n^{\otimes N-2}\right)$$
$$+d^{-1}\,(e_1\cdot n)\,(e_2\cdot n)\,(e_3\cdot n)\,\mathsf{F}^0_{N-3}(\bar{x}-\varepsilon n,w)\,|\,\left(U_0\otimes n^{\otimes N-3}\right)\Bigg)=0.$$

Passing to the limit is now easy. Using further the arbitrariness of U_0, and introducing the second order tensors $E_i := e_{i+1}\otimes e_{i+2}$, we get

$$\mathsf{F}^2_{N-1}(\bar{x},n)\,|n^{\otimes N-1}-\sum_{i=1}^{3}\frac{(e_i\cdot n)}{(e_i\cdot n_i)}\mathsf{F}^2_{N-1}(\bar{x},n_i)\,|\,n^{\otimes N-1}$$
$$-\sum_{i=1}^{3}\left(d^{-1}\right)\left(E_i\cdot(n\otimes n)\right)\mathsf{F}^1_{N-2}\left(\bar{x},f_i\right)\,|\,n^{\otimes N-2}$$
$$+d^{-1}\,(e_2\cdot n)\,(e_1\cdot n)\,(e_3\cdot n)\,\mathsf{F}^0_{N-3}(\bar{x},w)\,|\,n^{\otimes N-3}=0,$$

or equivalently,

$$\mathsf{F}^2_{N-1}(\bar{x},n)\,|n^{\otimes N-1}-\sum_{i=1}^{3}\frac{1}{(e_i\cdot n_i)}\left(\mathsf{F}^2_{N-1}(\bar{x},n_i)\otimes e_i\right)\,|\,n^{\otimes N}$$
$$-\sum_{i=1}^{3}\left(d^{-1}\right)\left(\mathsf{F}^1_{N-2}\left(\bar{x},f_i\right)\otimes E_i\right)\,|\,n^{\otimes N}$$
$$+d^{-1}\left(\mathsf{F}^0_{N-3}(\bar{x},w)\otimes e_1\otimes e_2\otimes e_3\right)\,|\,n^{\otimes N}=0.$$

Thus, defining the tensor field $\tilde{C}_N(\bar{x})$ of order N by

$$\tilde{C}_N(\bar{x}):=\sum_{i=1}^{3}\frac{1}{(e_i\cdot n_i)}\left(\mathsf{F}^2_{N-1}(\bar{x},n_i)\otimes e_i\right)+\sum_{i=1}^{3}d^{-1}\left(\mathsf{F}^1_{N-2}\left(\bar{x},f_i\right)\otimes E_i\right)$$
$$-d^{-1}\left(\mathsf{F}^0_{N-3}(\bar{x},w)\otimes e_1\otimes e_2\otimes e_3\right),\qquad(15)$$

we obtain $\mathsf{F}^2_{N-1}(\bar{x},n)\,|\,n^{\otimes N-1}=\tilde{C}_N\left(\bar{x}\right)\,|\,n^{\otimes N}$. As $\mathsf{F}^2_{N-1}(\bar{x},n)$ is completely orthogonal to $\mathcal{F}^\varepsilon$, the previous equation implies

$$\mathsf{F}^2_{N-1}(\bar{x},n)=\left(\tilde{C}_N\left(\bar{x}\right)\,|\,n^{\otimes N}\right)\otimes n^{\otimes N-1}.\qquad(16)$$

The first formula is thus obtained.

Let us now compute $\mathcal{K}_{f_1}\left(\mathcal{O}^N_{f_1} \mid (\tilde{C}_N)\right)$, by considering the different terms of formula (15). The term

$$\begin{aligned}\mathcal{O}^N_{f_1} \mid (\mathsf{F}^2_{N-1}(\bar{x}, n_i) \otimes e_i) &= \sum_{k\neq 1} \left(\mathcal{P}^{N-1}_{\mathcal{F}_k} \mid \left(\left(\mathsf{F}^2_{N-1}(\bar{x}, n_i) \otimes e_i\right) \cdot n_k\right)\right) \cdot \nu^1_k \\ &= \sum_{k\neq 1} (e_i \cdot n_k) \left(\mathcal{P}^{N-1}_{\mathcal{F}_k} \mid (\mathsf{F}^2_{N-1}(\bar{x}, n_i)\right) \cdot \nu^1_k\end{aligned}$$

vanishes as $(e_i \cdot n_k) = 0$ if $k \neq i$ and $\mathcal{P}^{N-1}_{\mathcal{F}_k} \mid (\mathsf{F}^2_{N-1}(\bar{x}, n_i)) = 0$ if $k = i$. We have also, using the fact that $(e_2)_{\perp \mathcal{F}_3} = 0$,

$$\begin{aligned}&\mathcal{O}^N_{f_1} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2 \otimes e_3) \\ &\quad= \sum_{k\neq 1} \left(\mathcal{P}^{N-1}_{\mathcal{F}_k} \mid \left((\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2 \otimes e_3) \cdot n_k\right) \cdot \nu^1_k\right) \\ &\quad= (e_3 \cdot n_3) \left(\mathcal{P}^{N-1}_{\mathcal{F}_3} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2)\right) \cdot \nu^1_3 \\ &\quad= (e_3 \cdot n_3) \sum_{q=1}^{N-1} \left(\mathcal{P}^{N-1}_{\mathcal{F}_3,q} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2)\right) \cdot \nu^1_3 \\ &\quad= (e_3 \cdot n_3) \Bigg(\sum_{q=1}^{N-2} \left(\mathcal{P}^{N-1}_{\mathcal{F}_3,q} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2)\right) \cdot \nu^1_3 \\ &\qquad\qquad + \left(\mathcal{P}^{N-1}_{\mathcal{F}_3,N-1} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2)\right) \cdot \nu^1_3 \Bigg) \\ &\quad= (e_3 \cdot n_3) \Bigg(\sum_{q=1}^{N-2} \left(\mathcal{P}^{N-2}_{\mathcal{F}_3,q} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1)\right) \cdot \nu^1_3 \otimes (e_2)_{\perp \mathcal{F}_3} \\ &\qquad\qquad + (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2) \cdot \nu^1_3 \Bigg) \\ &\quad= (e_3 \cdot n_3) \left(\mathcal{P}^{N-1}_{\mathcal{F}_3,N-1} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2)\right) \cdot \nu^1_3 \\ &\quad= (e_3 \cdot n_3)(\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2) \cdot \nu^1_3 \\ &\quad= (e_3 \cdot n_3)(e_2 \cdot \nu^1_3)\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1.\end{aligned}$$

Hence, as $(e_1)_{\perp \mathcal{L}_1} = 0$, the term $\mathcal{K}_{f_1}\left(\mathcal{O}^N_{f_1} \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2 \otimes e_3)\right)$ vanishes. We continue our evaluations:

$$\mathcal{O}^N_{f_1} \mid \left(\mathsf{F}^1_{N-2}(\bar{x}, f_i) \otimes E_i\right) = \sum_{k\neq 1} \left(\mathcal{P}^{N-1}_{\mathcal{F}_k} \mid \left(\left(\mathsf{F}^1_{N-2}(\bar{x}, f_i) \otimes e_{i+1} \otimes e_{i+2}\right) \cdot n_k\right)\right) \cdot \nu^1_k$$

The addends in this sum vanish unless $k = i + 2$. As also $k \neq 1$ the sum vanishes if $i = 2$. Otherwise it reads

$$\mathcal{O}^N_{f_1} \mid (\mathsf{F}^1_{N-2}(\bar{x}, f_i) \otimes E_i)$$

$$
\begin{aligned}
&= (e_{i+2}\cdot n_{i+2})\left(\mathcal{P}^{N-1}_{\mathcal{F}_{i+2}} \mid (\mathsf{F}^1_{N-2}(\bar{x},f_i)\otimes e_{i+1})\right)\cdot \nu^1_{i+2}\\
&= (e_{i+2}\cdot n_{i+2})\left(\sum_{q=1}^{N-2}\tau_q \mathcal{P}^{N-1}_{\mathcal{F}_{i+2},q} \mid (\mathsf{F}^1_{N-2}(\bar{x},f_i)\cdot \nu^1_{i+2}\otimes (e_{i+1})_{\perp\mathcal{F}_{i+2}})\right.\\
&\qquad\qquad \left. + \tau_{N-1}\mathcal{P}^{N-1}_{\mathcal{F}_{i+2},N-1} \mid (\mathsf{F}^1_{N-2}(\bar{x},f_i)\otimes e_{i+1})\cdot \nu^1_{i+2}\right)\\
&= (e_{i+2}\cdot n_{i+2})\left((\mathsf{F}^1_{N-2}(\bar{x},f_i)\otimes e_{i+1})\cdot \nu^1_{i+2}\right)\\
&= (e_{i+2}\cdot n_{i+2})\left(e_{i+1}\cdot \nu^1_{i+2}\right)\left(\mathsf{F}^1_{N-2}(\bar{x},f_i)\right).
\end{aligned}
$$

If $i=3$ the term vanishes, as $e_{i+1}\cdot\nu^1_{i+2} = e_1\cdot\nu^1_2 = 0$. Thus the only non-vanishing term is for $i=1$:

$$
\begin{aligned}
\mathcal{K}_{f_1}\left(\mathcal{O}^N_{f_1}(\mathsf{F}^1_{N-2}(\bar{x},f_1)\otimes E_1)\right) &= (e_3\cdot n_3)\left(e_2\cdot\nu^1_3\right)\mathcal{K}_{f_1}\left(\mathsf{F}^1_{N-2}(\bar{x},f_1)\right)\\
&= (e_3\cdot n_3)\left(e_2\cdot(n_3\times e_1)\right)\mathsf{F}^1_{N-2}(\bar{x},f_1)\\
&= d\mathsf{F}^1_{N-2}(\bar{x},f_1).
\end{aligned}
$$

In this way we have proven formula (12).

We now prove formula (13) by computing $\mathcal{O}^N_w(\tilde{C}_N)$ in a similar way. The term

$$
\begin{aligned}
&\mathcal{O}^N_w(\mathsf{F}^2_{N-1}(\bar{x},n_i)\otimes e_i)\\
&\quad=\sum_{j=1}^{3}\left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \mid \left(\sum_{k\neq j}\left(\mathcal{P}^{N-1}_{\mathcal{F}_k} \mid \left((\mathsf{F}^2_{N-1}(\bar{x},n_i)\otimes e_i)\cdot n_k\right)\cdot\nu^j_k\right)\right)\right)\cdot(-e_j)\\
&\quad=-(e_i\cdot n_i)\sum_{j\neq i}\left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \mid \left(\mathcal{P}^{N-1}_{\mathcal{F}_i} \mid \left(\mathsf{F}^2_{N-1}(\bar{x},n_i)\right)\cdot\nu^j_i\right)\right)\cdot e_j
\end{aligned}
$$

vanishes, as $\mathsf{F}^2_{N-1}(\bar{x},n_i)$ is completely orthogonal to $\mathcal{F}_i$. On the other way, the term

$$
\begin{aligned}
&\mathcal{O}^N_w \mid (\mathsf{F}^1_{N-2}(\bar{x},f_i)\otimes E_i)\\
&=\sum_{j=1}^{3}\left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \mid \left(\sum_{k\neq j}\left(\mathcal{P}^{N-1}_{\mathcal{F}_k} \mid \left((\mathsf{F}^1_{N-2}(\bar{x},f_i)\otimes e_{i+1}\otimes e_{i+2})\cdot n_k\right)\cdot\nu^j_k\right)\right)\right)\cdot(-e_j)\\
&=-(e_{i+2}\cdot n_{i+2})\sum_{j\neq i+2}\left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \mid \left(\mathcal{P}^{N-1}_{\mathcal{F}_{i+2}} \mid \left(\mathsf{F}^1_{N-2}(\bar{x},f_i)\otimes e_{i+1}\right)\cdot\nu^j_{i+2}\right)\cdot e_j\right)\\
&=-(e_{i+2}\cdot n_{i+2})\sum_{j\neq i+2}(e_{i+1}\cdot\nu^j_{i+2})\left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \mid \left(\mathsf{F}^1_{N-2}(\bar{x},f_i)\right)\cdot e_j\right)\\
&=-(e_{i+2}\cdot n_{i+2})(e_{i+1}\cdot\nu^i_{i+2})\left(\mathcal{P}^{N-2}_{\mathcal{L}_i} \mid \left(\mathsf{F}^1_{N-2}(\bar{x},f_i)\right)\cdot e_i\right)
\end{aligned}
$$

also vanishes, as $\mathsf{F}^1_{N-2}(\bar{x}, f_i)$ is completely orthogonal to $\mathcal{L}_i$. Finally we have

$$
\begin{aligned}
&\mathcal{O}^N_w \mid (\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2 \otimes e_3) \\
&= \sum_{j=1}^{3} \left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \middle| \left(\sum_{k \neq j} \left(\mathcal{P}^{N-1}_{\mathcal{F}_k} \middle| \left(\left(\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2 \otimes e_3 \right) \cdot n_k \right) \cdot \nu^j_k \right) \right) \right) \cdot (-e_j) \\
&= -(e_3 \cdot n_3) \sum_{j=1}^{2} \left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \mid \left(\left(\mathcal{P}^{N-1}_{\mathcal{F}_3} \mid \left(\left(\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2 \right) \right) \cdot \nu^j_3 \right) \right) \right) \cdot e_j \\
&= -(e_3 \cdot n_3) \sum_{j=1}^{2} \left(\mathcal{P}^{N-2}_{\mathcal{L}_j} \mid \left(\left(\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \otimes e_2 \right) \cdot \nu^j_3 \right) \right) \cdot e_j \\
&= -(e_3 \cdot n_3)(e_2 \cdot \nu^1_3) \left(\left(\mathcal{P}^{N-2}_{\mathcal{L}_1} \mid \left(\left(\mathsf{F}^0_{N-3}(\bar{x}, w) \otimes e_1 \right) \right) \right) \cdot e_1 \right) \\
&= -(e_3 \cdot n_3)(e_2 \cdot \nu^1_3) \mathsf{F}^0_{N-3}(\bar{x}, w) \\
&= -(e_3 \cdot n_3)(e_2 \cdot (n_3 \times e_1)) \mathsf{F}^0_{N-3}(\bar{x}, w) \\
&= -d\, \mathsf{F}^0_{N-3}(\bar{x}, w),
\end{aligned}
$$

which concludes the proof of the last part of formula (12). □

Formula (11) is a partial representation result. Indeed, as $\tilde{C}_N$ depends on $\bar{x}, e_1, e_2, e_3$ i.e. on $\bar{x}, n_1, n_2, n_3$ but does not depend on n, formula (11) describes the way $\mathsf{F}^2_{N-1}(\bar{x}, n)$ depends on n. However the formula is not valid for any unit vector n in the sphere S_2 but only for those which belong to the cone defined by inequalities (10). This is a common feature of all classical proofs based on Cauchy tetrahedron argument and the results are generally extended by using an action-reaction argument.

On the contrary the formulas (12), (13) are not, at this point, actual representation results. Indeed they are valid only for the particular shapes f_1 and w and the way we introduced $\tilde{C}_N$ explicitly involve $\mathsf{F}^1_{N-2}(\bar{x}, f_1)$ and $\mathsf{F}^0_{N-3}(\bar{x}, w)$.

The next theorem overcomes these difficulties and extends the previous results in an actual representation theorem.

Theorem 2 *There exists a unique continuous completely symmetric tensor field C_N of order N such that, at any point $\bar{x}$ and for any plane shape n,*

$$
\mathsf{F}^2_{N-1}(\bar{x}, n) = \mathcal{O}^N_n \mid C_N = \left(\tilde{C}_N(\bar{x}) \mid n^{\otimes N} \right) \otimes n^{\otimes N-1}. \tag{17}
$$

Moreover, for any edge $\mathcal{L}$ with shape f and any wedge shape with w which coincide locally with the boundary of some non degenerated tetrahedron, we have

$$
\mathsf{F}^1_{N-2}(\bar{x}, f) = \mathcal{K}_{\mathcal{L}} \left(\mathcal{O}^N_f \mid C_N \right) \quad \textit{and} \quad \mathsf{F}^0_{N-3}(\bar{x}, w) = \mathcal{K} \left(\mathcal{O}^N_w \mid C_N \right). \tag{18}
$$

Proof The proof is based on three technical lemmas which are proven in Appendix 8 and 9.

Let us apply Theorem 1 for $\bar{x}$ and any triplet (n_1, n_2, n_3). As $n^{\otimes N}$ is a completely symmetric tensor, it is clear that equation (11) remains valid if we replace $\tilde{C}_N$ by its completely symmetric part C_N. Note that the way we constructed $C_N(\bar{x})$ shows that this tensor depends continuously on $\bar{x}$. We get representation formulas for $\mathsf{F}^2_{N-1}(\bar{x}, n)$ valid in the open cones defined by inequalities (10). Then the purely topological Lemma 1 establishes that this representation actually holds on the whole sphere S_2. It also establishes that, despite the way we defined it, the tensor C_N does not depend on the triplet (e_1, e_2, e_3). Moreover the uniqueness of the representation is assured.

To deal with the edge force representation, we introduce with no loss of generality a tetrahedron such that $f_1 = f$. From Theorem 1 we know that $\mathsf{F}^1_{N-2}(\bar{x}, f) = \mathcal{K}_{\mathcal{L}}\left(\mathcal{O}^N_f \mid \tilde{C}_N\right)$ and Lemma 2 establishes that this identity remains valid when we replace $\tilde{C}_N$ by C_N.

Similarly, we introduce with no loss of generality a tetrahedron such that the wedge shape w coincides with the shape at the vertex $\hat{x}$ of the tetrahedron. We know from Theorem 1 that $\mathsf{F}^0_{N-3}(\bar{x}, w) = \mathcal{K}\left(\mathcal{O}^N_w(\tilde{C}_N)\right)$ and Lemma 3 establishes that this identity remains valid when we replace $\tilde{C}_N$ by C_N. □

4 Extension to more general shapes

We now extend our representation formulas to more general shapes. We prove a theorem analogous to Noll theorem [33] which states that the highest order terms of the stress state depends on the shape of the domain only through the tangent shape.

Our definition of prescribed shapes and assumption (5) play here an essential role.

We restrict ourselves to shapes which are tangent to some shape of a tetrahedron. This is not the more general case : for instance non convex polyhedral domains or vertex where more than three edges concur are not treated. That could be done by playing with union and intersection of domains. We do not do it here for lack of space.

Theorem 3 *If a domain D is tangent at some point $\bar{x} \in \partial_i D$ to a tetrahedron Δ, then*

$$\mathsf{F}^i_{N+1-i}\left(\bar{x}, \widetilde{(D, \bar{x})}\right) = \mathsf{F}^i_{N+1-i}\left(\bar{x}, \widetilde{(\Delta, \bar{x})}\right) \tag{19}$$

Proof We start by considering the case when $\bar{x}$ is a regular point of a face of D ($i = 2$). We denote $\bar{n}$ the outward normal to the boundary at $\bar{x}$. Saying that Δ is tangent to D at $\bar{x}$ simply means that the shape of Δ at $\bar{x}$ is the plane shape $\bar{n}$. Let us temporarily use the orthonormal coordinate system $(\bar{x}, e_1, e_2, e_3)$ (with $e_3 = \bar{n}$) and consider the family of parallelepipeds

$$\square_\varepsilon = [0, \varepsilon] \times [0, \varepsilon] \times [-c\varepsilon^2, c\varepsilon^2].$$

Let us define the sets

$$D_\varepsilon := D \cap \square_\varepsilon, \quad \mathcal{F}_\varepsilon := \partial D \cap \square_\varepsilon \text{ and } \quad \mathcal{F}_- := \{x \in \square_\varepsilon;\ x_3 = -c\varepsilon^2\}$$

As the curvature of S is bounded in a neighborhood of $\bar{x}$, a positive scalar c can be found such that, for ε sufficiently small, $\mathcal{F}_\varepsilon$ does not intersect the face $\mathcal{F}_-$. Clearly, the shapes of the boundary of V_ε for all ε belong to a set of prescribed shapes.

We apply the quasi-balance inequality with the fixed test field $U(x) = (x \cdot \bar{n})^{N-1} U_0$ to the family of domains D_ε. It implies that the limit

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon^{-2} \left(\int_{\mathcal{F}_\varepsilon} \mathsf{F}^2_{N-1} \left(x, \widetilde{(D,x)} \right) |\bar{n}^{\otimes N-1} U_0 + \int_{\mathcal{F}_-} \mathsf{F}^2_{N-1} (x, -\bar{n}) \, |\bar{n}^{\otimes N-1} U_0 \right)$$

vanishes. Indeed (i) all other surface terms in the expression of the stress state are negligible as the areas of the lateral faces of D_ε are of order ε^3, (ii) the edge terms are negligible as the lengths of the edges are of the order ε and the fields $\nabla^q U$ are of order $\varepsilon^{2(N-1-q)}$, (iii) this order of magnitude of the fields $\nabla^q U$ also makes the wedge terms negligible. This leads to

$$\begin{aligned} \mathsf{F}^2_{N-1} \left(\bar{x}, \widetilde{(D,\bar{x})} \right) |\bar{n}^{\otimes N-1} &= -\mathsf{F}^2_{N-1} (\bar{x}, -\bar{n}) \, |\bar{n}^{\otimes N-1} \\ &= -(C_N(\bar{x}) \mid (-\bar{n})^{\otimes N})(-1)^{N-1} \end{aligned}$$

And so

$$\mathsf{F}^2_{N-1} \left(\bar{x}, \widetilde{(D,\bar{x})} \right) = (C_N(\bar{x}) \mid \bar{n}^{\otimes N}) \otimes \bar{n}^{\otimes N-1} = \mathsf{F}^2_{N-1} (\bar{x}, \bar{n}) \, . \tag{20}$$

Now we consider the case when $\bar{x}$ is a regular point of an edge $\mathcal{L}$ of D ($i = 1$). Saying that Δ is tangent to D at $\bar{x}$ means that an edge $\mathcal{L}_1$ of Δ is tangent to $\mathcal{L}$ at $\bar{x}$ and the the two faces $\mathcal{F}_2$, $\mathcal{F}_3$ of the tetrahedron are tangent to the faces of D at $\bar{x}$. We denote $\bar{e}_1$ a unit vector tangent to the line at $\bar{x}$, $\bar{n}_2$, $\bar{n}_3$ the normals of the two faces concurring there. We consider a vector e_3 orthogonal $\bar{e}_1$ and satisfying

$$e_3 \cdot \bar{n}_2 > 0, \quad e_3 \cdot \bar{n}_3 > 0 \tag{21}$$

We temporarily use the orthonormal coordinate system $(\bar{e}_1, e_2, e_3)$ and use in this system of coordinate the same notation for $\square_\varepsilon$. We introduce t_ε the translation of vector $c\,\varepsilon^2 e_3$ and we redefine

$$D_\varepsilon := t_\varepsilon \left(D \cap \square_\varepsilon \right), \quad \mathcal{F}_\varepsilon := t_\varepsilon \left(\partial D \cap \square_\varepsilon \right)$$

and

$$\mathcal{F}_- := t_\varepsilon \left(\{ x \in \square_\varepsilon ; \; x_3 = -c\varepsilon^2 \} \right)$$

(note that $\mathcal{F}_\varepsilon$ is no more a face of the domain but the union of two faces). We moreover introduce the sets

$$\mathcal{L}_\varepsilon := t_\varepsilon \left(\mathcal{L} \cap \square_\varepsilon \right), \quad \mathcal{F}_{2,\varepsilon} := t_\varepsilon \left(\mathcal{F}_2 \cap \square_\varepsilon \right), \quad \mathcal{F}_{3,\varepsilon} := t_\varepsilon \left(\mathcal{F}_3 \cap \square_\varepsilon \right). \tag{22}$$

As the curvature of the edge is bounded in a neighborhood of $\bar{x}$, a positive scalar c can be found such that, for ε sufficiently small, the line $\mathcal{L}_\varepsilon$ does not intersect the face $\mathcal{F}_-$. The shapes of D_ε for all ε still belong to a set of

prescribed shapes. Quasi-balance inequality when applied with the fixed test field $U(x) = ((x-\bar{x})\cdot e_3)^{N-1} U_0$ to the family of domains D_ε implies

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon^{-3} \left(\int_{\mathcal{F}_\varepsilon} \mathsf{F}^2_{N-1}\left(\bar{x}, \widetilde{(D_\varepsilon, \bar{x})}\right) \mid e_3^{\otimes N-1} U_0 + \int_{\mathcal{F}_-} \mathsf{F}^2_{N-1}\left(\bar{x}, -e_3\right) \mid e_3^{\otimes N-1} U_0 \right.$$
$$\left. + \int_{\mathcal{L}_\varepsilon} \mathsf{F}^1_{N-2}\left(\bar{x}, \widetilde{(D_\varepsilon, x)}\right) | e_3^{\otimes N-2} \left((x-\bar{x})\cdot e_3\right) U_0 \right) = 0.$$

Indeed (i) all the other surface terms in the stress state expression are negligible as the areas of the faces $\{x_1 = \pm\varepsilon\}$ are of order ε^4 (ii) the other edge terms are negligible as either their lengths are of the order ε^2 and the field $\nabla^k U$ with $k < N-1$ is of order smaller or equal to ε^2 or they lay in the plane $(x-\bar{x})\cdot e_3 = 0$, (iii) the wedge terms are also negligible as $\nabla^k U$ with $k < N-2$ is of order smaller or equal to ε^4. We get

$$\lim_{\varepsilon \longrightarrow 0} \left(\frac{|\mathcal{F}_{2,\varepsilon}|}{\varepsilon^3} \right) \mathsf{F}^2_{N-1}\left(\bar{x}, \bar{n}_2\right) \mid e_3^{\otimes N-1} + \lim_{\varepsilon \longrightarrow 0} \left(\frac{|\mathcal{F}_{3,\varepsilon}|}{\varepsilon^3} \right) \mathsf{F}^2_{N-1}\left(\bar{x}, \bar{n}_3\right) \mid e_3^{\otimes N-1}$$
$$+ \lim_{\varepsilon \longrightarrow 0} \left(\frac{|\mathcal{F}_-|}{\varepsilon^3} \right) \mathsf{F}^2_{N-1}\left(\bar{x}, -e_3\right) \mid e_3^{\otimes N-1}$$
$$+ \lim_{\varepsilon \longrightarrow 0} \left(\frac{|\mathcal{L}_\varepsilon|\, c\varepsilon^2}{\varepsilon^3} \right) \mathsf{F}^1_{N-2}\left(\bar{x}, \widetilde{(D, \bar{x})}\right) | e_3^{\otimes N-2} = 0.$$

Computing the limits in the previous equality is straightforward. They are clearly proportional to c and depend only on the vectors e_3, $\bar{n}_2$ and $\bar{n}_3$. Therefore the quantity $\mathsf{F}^1_{N-2}\left(\bar{x}, \widetilde{(D, \bar{x})}\right) \mid e_3^{\otimes N-2}$ depends only on these vectors. The vector e_3 is generic in the plane cone defined by (21). We invoke Lemma 1 in the case $d = 2$ to conclude that the plane tensor $\mathsf{F}^1_{N-2}\left(\bar{x}, \widetilde{(D, \bar{x})}\right)$ depends only on $\bar{n}_2$ and $\bar{n}_3$, that is, on the geometry of the tangent tetrahedron.

The case when $\bar{x}$ is a regular point of an edge wedge of D ($i = 0$) is treated in a very similar way. Now three faces of D and three edges are joining at $\bar{x}$. Saying that Δ is tangent to D at $\bar{x}$ means that these geometrical elements are tangent to corresponding elements of the tetrahedron. We denote $\bar{n}_1$, $\bar{n}_2$, $\bar{n}_3$ the normals of the faces concurring there. We consider a vector e_3 satisfying

$$e_3 \cdot \bar{n}_1 > 0, \quad e_3 \cdot \bar{n}_2 > 0, \quad e_3 \cdot \bar{n}_3 > 0 \tag{23}$$

and use notations compatible with the previous ones in an orthonormal coordinate system (e_1, e_2, e_3). Applying the quasi balance of power for the same

test field and with the family of domains defined in (22). We get

$$\sum_{j=1}^{3} \lim_{\varepsilon \longrightarrow 0} \left(\frac{|\mathcal{F}_{j,\varepsilon}|}{\varepsilon^4} \right) \left(\mathsf{F}^2_{N-1}(\bar{x}, \bar{n}_j) \mid e_3^{\otimes N-1} \right)$$
$$+ \lim_{\varepsilon \longrightarrow 0} \left(\frac{|\mathcal{F}_-|}{\varepsilon^4} \right) \left(\mathsf{F}^2_{N-1}(\bar{x}, -e_3) \mid e_3^{\otimes N-1} \right)$$
$$+ \sum_{k=1}^{3} \lim_{\varepsilon \longrightarrow 0} \left(\frac{|\mathcal{L}_{k,\varepsilon}|\, c\varepsilon^2}{2\varepsilon^4} \right) \mathsf{F}^1_{N-2} \left(\bar{x}, \widetilde{(D, \bar{x})} \right) \mid e_3^{\otimes N-2}$$
$$+ \lim_{\varepsilon \longrightarrow 0} \left(\frac{(c\varepsilon^2)^2}{\varepsilon^4} \right) \left(\mathsf{F}^0_{N-3} \left(\bar{x}, \widetilde{(D, \bar{x})} \right) \mid e_3^{\otimes N-3} \right) = 0.$$

Computing the previous limits is again straightforward. They are clearly proportional to c^2 and depend only on the vectors e_3, $\bar{n}_1$, $\bar{n}_2$ and $\bar{n}_3$. Therefore the quantity $\mathsf{F}^0_{N-3} \left(\bar{x}, \widetilde{(D, \bar{x})} \right) \mid e_3^{\otimes N-3}$ depends only on these vectors. As the vector e_3 is generic in the cone defined by (23). We invoke Lemma 1 in the case $d = 3$ to conclude that the plane tensor $\mathsf{F}^0_{N-3} \left(\bar{x}, \widetilde{(D, \bar{x})} \right)$ depends only on $\bar{n}_1$ $\bar{n}_2$ and $\bar{n}_3$, that is, on the geometry of the tangent tetrahedron. □

5 Representation of lower order contact interactions

We still consider a body occupying the domain D and submitted to a physically admissible stress state $\mathfrak{S}$ of grade N. We have established in the previous section that there exists a unique completely symmetric tensor C_N representing the higher order contact interactions F^2_{N-1}, F^1_{N-2} and F^0_{N-3}.

Let us now compute $\int_D C_N \mid \nabla^N U$ by using the divergence theorem :

$$\int_D C_N \mid \nabla^N U = -\int_D div(C_N) \mid \nabla^{N-1} U + \int_{\partial_2 D} (C_N \cdot n) \mid \nabla^{N-1} U$$

The last term can be decomposed using formula (32) and the representation formula (17). It reads

$$\int_{\partial_2 D} \mathsf{F}^2_{N-1} \mid \nabla^{N-1} U + \int_{\partial_2 D} \left(\mathcal{P}^{N-1}_{\partial_2 D} \mid (C_N \cdot n) \right) \mid \nabla^{N-1} U.$$

Owing to (31) we can compute the last term using the surface divergence theorem on each face of D. Setting $\mathsf{G}^2_{N-2} := div_{\partial_2 D} \left(\mathcal{P}^{N-1}_{\partial_2 D} \mid (C_N \cdot n) \right)$, this last term becomes

$$-\int_{\partial_2 D} \mathsf{G}^2_{N-2} \mid \nabla^{N-2} U + \sum_{j \in I_1(D)} \int_{\mathcal{L}_j} \sum_{k \in [\mathcal{L}_j]} \left((\mathcal{P}^{N-1}_{\partial_2 D} \mid (C_N \cdot n)) \cdot \nu^j_k \right) \mid \nabla^{N-2} U$$

or

$$-\int_{\partial_2 D} \mathsf{G}^2_{N-2} \mid \nabla^{N-2} U + \sum_{j \in I_1(D)} \int_{\mathcal{L}_j} \left(\mathcal{O}^N_{f_j} \mid C_N \right) \mid \nabla^{N-2} U.$$

Using formula (32) and the representation formula (18) the last term can be decomposed in

$$\sum_{j\in I_1(D)}\int_{\mathcal{L}_j}\mathsf{F}^1_{N-2}\mid\nabla^{N-2}U+\sum_{j\in I_1(D)}\int_{\mathcal{L}_j}\left(\mathcal{P}^{N-2}_{\mathcal{L}_j}\mid\left(\mathcal{O}^N_{f_j}\mid C_N\right)\right)\mid\nabla^{N-2}U$$

Owing again to (31), we finally apply the divergence theorem along each line $\mathcal{L}j$. Setting on each line $\mathcal{L}_j$, $\mathsf{G}^1_{N-3}:=div_{\mathcal{L}_j}\left(\mathcal{P}^{N-2}_{\mathcal{L}_j}\mid\left(\mathcal{O}^N_{f_j}\mid C_N\right)\right)$, we write the last term as the sum of $-\sum_{j\in I_1(D)}\int_{\mathcal{L}_j}\mathsf{G}^1_{N-3}\mid\nabla^{N-3}U$ and

$$\sum_{\hat{x}\in I_0(D)}\sum_{j\in[\hat{x}]}\left(\left(\mathcal{P}^{N-2}_{\mathcal{L}_j}\mid\left(\mathcal{O}^N_{f_j}\mid C_N\right)\right)\cdot e_j\right)\mid\nabla^{N-3}U$$

i.e., as the sum

$$-\sum_{j\in I_1(D)}\int_{\mathcal{L}_j}\mathsf{G}^1_{N-3}\mid\nabla^{N-3}U+\sum_{\hat{x}\in I_0(D)}\mathsf{F}^0_{N-3}\mid\nabla^{N-3}U.$$

Collecting all these results we obtain the identity

$$\begin{aligned}\int_{\partial_2 D}\mathsf{F}^2_{N-1}\mid\nabla^{N-1}U+\int_{\partial_1 D}\mathsf{F}^1_{N-2}\mid\nabla^{N-2}U+\int_{\partial_0 D}\mathsf{F}^0_{N-3}\mid\nabla^{N-3}U\\ =\int_D C_N\mid\nabla^N U+\int_{\partial_2 D}\mathsf{G}^2_{N-2}\mid\nabla^{N-2}U+\int_{\partial_1 D}\mathsf{G}^1_{N-3}\mid\nabla^{N-3}U\end{aligned}\tag{24}$$

The sum $\int_{\partial_2 D}\mathsf{G}^2_{N-2}\mid\nabla^{N-2}U+\int_{\partial_1 D}\mathsf{G}^1_{N-3}\mid\nabla^{N-3}U$ corresponds to a stress state of order $N-1$ but, as G^2_{N-2} and G^1_{N-3} are not necessarily orthogonal to the shape where they are applied, we have to rewrite it in the canonical form

$$\sum_{k=0}^{N-2}\int_{\partial_2\Delta^\varepsilon}\tilde{\mathsf{F}}^2_k\mid\nabla^k_\perp U+\sum_{k=0}^{N-3}\int_{\partial_1\Delta^\varepsilon}\tilde{\mathsf{F}}^1_k\mid\nabla^k_\perp U+\sum_{k=0}^{N-4}\int_{\partial_0\Delta^\varepsilon}\tilde{\mathsf{F}}^0_k\mid\nabla^k U$$

where the k-forces $\tilde{F}^i_k$ can be made explicit in terms of G^2_{N-2} and G^1_{N-3}. Let us now subtract equality (24) to the quasi-balance inequality (2). As the term $\int_D C_N\mid\nabla^N U$ is itself clearly quasi-balanced, we get that the new stress state

$$\begin{aligned}\mathfrak{S}'(\Delta^\varepsilon,U):=\sum_{k=0}^{N-2}\int_{\partial_2\Delta^\varepsilon}(\mathsf{F}^2_k+\tilde{\mathsf{F}}^2_k)\mid\nabla^k U+\sum_{k=0}^{N-3}\int_{\partial_1\Delta^\varepsilon}(\mathsf{F}^1_k+\tilde{\mathsf{F}}^1_k)\mid\nabla^k U\\ +\sum_{k=0}^{N-4}\int_{\partial_0\Delta^\varepsilon}(\mathsf{F}^0_k+\tilde{\mathsf{F}}^0_k)\mid\nabla^k U\end{aligned}$$

which is of order $N-1$ is also quasi-balanced. The representation theorem can be applied to the higher order terms of this new stress state : there exists a completely symmetric tensor C_{N-1} of order $N-1$ such that

$$\mathsf{F}^2_{N-2} + \tilde{\mathsf{F}}^2_{N-2} = \mathcal{O}^{N-1}_n \mid C_{N-1}, \qquad \mathsf{F}^1_{N-3} + \tilde{\mathsf{F}}^1_{N-3} = \mathcal{K}_{\mathcal{L}} \left(\mathcal{O}^{N-1}_f \mid C_{N-1} \right),$$

$$\mathsf{F}^0_{N-4} + \tilde{\mathsf{F}}^0_{N-4} = \mathcal{K} \left(\mathcal{O}^{N-1}_w \mid C_{N-1} \right).$$

As the interactions G and therefore $\tilde{\mathsf{F}}$ can be made explicit, we have obtained a representation for F^2_{N-2}, F^1_{N-3} and F^0_{N-4}. Clearly this operation can be repeated in order to get a representation of all terms in the stress state through a chain of N completely symmetric tensors with decreasing orders.

We warn the reader that the chain of tensors $(C_1, \ldots, C_N)$ obtained in this way is equivalent but not identical to the one used in [36], [30] [13], this last being more natural when one starts from the principle of virtual work. Indeed if, following Mindlin, we denote $(\overset{1}{\tau}, \ldots, \overset{N}{\tau})$ the last one, we have the relation $\overset{N}{\tau} = C_N$ and, for $1 \leq q < N$, $\overset{q}{\tau} = C_q - div(C_{q+1})$. It happens that, even when following Mindlin, the boundary conditions are written in a more compact form when using $(C_1, \ldots, C_N)$.

We do not try to explicit the representation of all terms. Indeed, such a task needs the introduction of a very heavy notation. We restrict ourselves in section 6 to the description of the representation of a stress state of order one, two or three and let the reader apply the procedure for higher order cases.

6 First, second and third gradient theories

The fact that our results enable us to recover the classical Cauchy theory is an evidence as we have closely followed the path of Cauchy. After quickly checking it, we verify here that they also enable us to recover the now widely used second gradient theory or the third gradient theory described in [30].

6.1 Cauchy first gradient theory

When N=1, the stress state is reduced to

$$\mathfrak{S}(D, U) = \int_{\partial_2 D} \mathsf{F}^2_0 \mid U$$

This corresponds to the Cauchy postulate that contact interactions can be described by to a surface density of forces distributed along the regular part of the boundary. Our theorem reduces to the theorem established by Cauchy. Indeed, as $\mathcal{O}^1_n = n$, our theorem states that there exists a tensor C_1 (the Cauchy stress tensor) of order 1 such that $\mathsf{F}^2_0(x, n) = C_1(x) \cdot n$. The reader should not feel outraged by the fact that the Cauchy stress tensor is of order one. We recall that we have made no assumption on the tensorial nature of U

and considered it in our calculation as a scalar. So are the dual quantities F_0^2. If U is a vector in the physical space, the theorem can be applied component by component and the Cauchy stress tensor becomes a tensor of order two. Note that the complete symmetry stated in our theorem is irrelevant in the case $N = 1$ and that such a symmetry is of quite different nature from the well known symmetry of the Cauchy stress tensor which is a consequence of assumptions (physical nature of the kinematic descriptor and Galilean invariance) out of the scope of our considerations.

6.2 Second gradient theory

When N=2, the stress state reads

$$\mathfrak{S}(D,U) = \int_{\partial_2 D} \mathsf{F}_0^2 \mid U + \mathsf{F}_1^2 \mid \nabla_\perp U + \int_{\partial_1 D} \mathsf{F}_0^1 \mid U.$$

Let us make explicit the boundary operators when $N = 2$. We have $\mathcal{O}_n^2 = n^{\otimes 3}$ and $\mathcal{O}_{f_j}^2 = \sum_{k\in[\mathcal{L}_j]} \nu_k^j \otimes n_k$. Indeed

$$\left(\mathcal{O}_{f_j}^2\right)_{i_1,i_2} := \sum_{k\in[\mathcal{L}_j]} \left(\nu_k^j\right)_\ell \left(\mathcal{P}^1_{\mathcal{F}_k}\right)_{\ell,i_1} \left(n_k\right)_{i_2} = \sum_{k\in[\mathcal{L}_j]} \left(\nu_k^j\right)_\ell \left(\mathit{\Pi}_{\mathcal{F}_k}\right)_{\ell,i_1} \left(n_k\right)_{i_2}$$

Theorem 2 and the procedure described in the previous section establish the existence of tensors C_2 and C_1 such that, at every regular point x of a face with normal n and every regular point y of an edge $\mathcal{L}_j$,

$$\mathsf{F}_1^2(x,n) = (C_2 \mid n^{\otimes 2})\, n, \quad \mathsf{F}_0^1(y,f_j) = (C_2 \mid \sum_{k\in[\mathcal{L}_j]} \nu_k^j \otimes n_k),$$

$$\mathsf{F}_0^2(x,n) = C_1 \cdot n - div_{\partial_2 D}((C_2 \cdot n) \cdot \mathit{\Pi}_{\partial_2 D}).$$

We recover here the expressions stated in [44], [30], [22], [23] or [12].

6.3 Third gradient theory

Let us now consider the case $N = 3$ of third gradient models. The stress state reads

$$\mathfrak{S}(D,U) = \int_{\partial_2 D} \mathsf{F}_0^2 \mid U + \mathsf{F}_1^2 \mid \nabla U + \mathsf{F}_2^2 \mid \nabla^2 U + \int_{\partial_1 D} \mathsf{F}_0^1 \mid U + \mathsf{F}_1^1 \mid \nabla U + \int_{\partial_0 D} \mathsf{F}_0^0 \mid U.$$

We have $\mathcal{O}_n^3 = n^{\otimes 5}$. Moreover definition (8) implies

$$\left(\mathcal{P}^2_{\mathcal{F}_k}\right)_{i_1,\ell,i_2,i_3} = \delta^{i_2}_{i_1} \left(\mathit{\Pi}_{\mathcal{F}_k}\right)^{i_3}_\ell + \left(\mathit{\Pi}_{\mathcal{F}_k}\right)^{i_2}_\ell \left(\mathit{\Lambda}_{\mathcal{F}_k}\right)^{i_3}_{i_1}$$

and

$$\Big(\mathcal{P}^1_{\mathcal{L}_j}\Big)_{\ell_1,\ell_2} = \Big(e_j\Big)_{\ell_1}\Big(e_j\Big)_{\ell_2}.$$

Thus

$$\begin{aligned}\Big(\mathcal{O}^3_{f_j}\Big)_{i_1,i_2,i_3,i_4} &= \sum_{k\in[\mathcal{L}_j]}\Big(\nu^j_k\Big)_\ell\Big(\mathcal{P}^2_{\mathcal{F}_k}\Big)_{i_1,\ell,i_2,i_3}\Big(n_k\Big)_{i_4}\\ &= \sum_{k\in[\mathcal{L}_j]}\delta^{i_2}_{i_1}\Big(\nu^j_k\Big)_{i_3}\Big(n_k\Big)_{i_4} + \Big(n_k\Big)_{i_1}\Big(\nu^j_k\Big)_{i_2}\Big(n_k\Big)_{i_3}\Big(n_k\Big)_{i_4}.\end{aligned}$$

and

$$\begin{aligned}\big(\mathcal{O}^3_w\big)_{i_1,i_2,i_3} &= \sum_{j\in[\hat{x}]}\Big(e_j\Big)_{\ell_1}\Big(\mathcal{P}^1_{\mathcal{L}_j}\Big)_{\ell_1,\ell_2}\Big(\mathcal{O}^3_{f_j}\Big)_{\ell_2,i_1,i_2,i_3}\\ &= \sum_{j\in[\hat{x}]}\Big(e_j\Big)_{\ell_2}\Big(\mathcal{O}^3_{f_j}\Big)_{\ell_2,i_1,i_2,i_3}\\ &= \sum_{j\in[\hat{x}]}\sum_{k\in[\mathcal{L}_j]}\Big(e_j\Big)_{i_1}\Big(\nu^j_k\Big)_{i_2}\Big(n_k\Big)_{i_3}.\end{aligned}$$

Theorem 2 establishes the existence of a completely symmetric tensor C_3 such that, at every regular point x of a face with normal n, every regular point y of an edge $\mathcal{L}_j$ with shape f_j, and every wedge point z with shape w,

$$\mathsf{F}^2_2(x,n) = (C_3 \mid n^{\otimes 3})\, n\otimes n \tag{25}$$

$$\mathsf{F}^1_1(y,f_j) = \sum_{k\in[\mathcal{L}_j]}\Big((C_3 \mid (\nu^j_k\otimes\nu^j_k\otimes n_k))\,\nu^j_k + 2\,(C_3 \mid (\nu^j_k\otimes n_k\otimes n_k))\,n_k\Big) \tag{26}$$

$$\mathsf{F}^0_0(z,w) = \sum_{j\in[\hat{x}]}\sum_{k\in[\mathcal{L}_j]} C_3 \mid (e_j\otimes\nu^j_k\otimes n_k). \tag{27}$$

Lower order terms can again be computed using the procedure described in Section 5. We have

$$\mathcal{P}^2_{\mathcal{F}_k} \mid (C_3\cdot n) = \big(Id + n_k\otimes n_k\big)\cdot(C_3\cdot n)\cdot \mathit{\Pi}_{\mathcal{F}_k}$$

Therefore, on each face $\mathcal{F}_k$, we define G^2_1 as the surface divergence:

$$\mathsf{G}^2_1 = div_{\mathcal{F}_k}\Big(\big(Id + n_k\otimes n_k\big)\cdot(C_3\cdot n)\cdot \mathit{\Pi}_{\mathcal{F}_k}\Big)$$

and, on each line $\mathcal{L}_j$, G^1_0 as the line divergence

$$\mathsf{G}^1_0 = div_{\mathcal{L}_j}\left(\sum_{k\in[\mathcal{L}_j]}\Big(C_3 \mid (n_k\otimes\nu^j_k\otimes e_j)\Big)e_j\right).$$

We then rewrite the stress state $\int_{\partial_2 D} \mathsf{G}_1^2 \mid \nabla U + \int_{\partial_1 D} \mathsf{G}_0^1 \mid U$ in the canonical form $\int_{\partial_2 D} \tilde{\mathsf{F}}_0^2 \mid U + \tilde{\mathsf{F}}_1^2 \mid \nabla U + \int_{\partial_1 D} \tilde{\mathsf{F}}_0^1 \mid U$ by using the surface divergence theorem, setting on each face $\mathcal{F}_k$

$$\tilde{\mathsf{F}}_1^2 = (\mathsf{G}_1^2 \cdot n_k)\, n_k, \qquad \tilde{\mathsf{F}}_0^2 = -div_{\mathcal{F}_k}(\mathsf{G}_1^2 \cdot \mathit{\Pi}_{\mathcal{F}_k})$$

and, on each line $\mathcal{L}_j$,

$$\tilde{\mathsf{F}}_0^1 = \mathsf{G}_0^1 + \sum_{k \in [\mathcal{L}_j]} \mathsf{G}_1^2 \cdot \nu_k^j.$$

As shown in Section 5, the new stress state

$$\tilde{\mathfrak{S}}(D,U) = \int_{\partial_2 D} (\mathsf{F}_0^2 + \tilde{\mathsf{F}}_0^2) \mid U + (\mathsf{F}_1^2 + \tilde{\mathsf{F}}_1^2) \mid \nabla_\perp U + \int_{\partial_1 D} (\mathsf{F}_0^1 + \tilde{\mathsf{F}}_0^1) \mid U$$

is a quasi-balanced stress state of order two. We can apply to it the results obtained in the case $N = 2$: two tensors C_2 and C_1 represent $\mathsf{F}_1^2 + \tilde{\mathsf{F}}_1^2$, $\mathsf{F}_0^1 + \tilde{\mathsf{F}}_0^1$ and $\mathsf{F}_0^2 + \tilde{\mathsf{F}}_0^2$. In [30] Mindlin introduced the operator which, to any tensor X, associates $L_{\partial_2 D}(X) := -div_{\partial_2 D}(X \cdot \mathit{\Pi}_{\partial_2 D})$. Let us write our final results using this operator.

$$\mathsf{F}_1^2(x,n) = \left((C_2 \mid n^{\otimes 2}) + L_{\partial_2 D}\Big(\big(Id + n \otimes n\big) \cdot (C_3 \cdot n) \Big) \cdot n \right) n \tag{28}$$

$$\mathsf{F}_0^2(x,n) = C_1 \cdot n + L_{\partial_2 D}\left(C_2 \cdot n + L_{\partial_2 D}\Big(\big(Id + n \otimes n\big) \cdot (C_3 \cdot n) \Big) \right) \tag{29}$$

$$\begin{aligned} \mathsf{F}_0^1(x,f_j) = (C_2 \mid \sum_{k \in [\mathcal{L}_j]} \nu_k^j \otimes n_k) - div_{\mathcal{L}_j}\left(\Big(C_3 \mid \sum_{k \in [\mathcal{L}_j]} (n_k \otimes \nu_k^j \otimes e_j) \Big) e_j \right) \\ + \sum_{k \in [\mathcal{L}_j]} L_{\mathcal{F}_k}\Big(\big(Id + n_k \otimes n_k\big) \cdot (C_3 \cdot n_k) \Big) \cdot \nu_k^j. \end{aligned} \tag{30}$$

The equations labeled (18a), (18b) and (18c) by Mindlin [30] correspond respectively to our equations (29), (28), (25) while the expressions for the quantities called F and (N_1, N_2, T_1, T_2) and G page 436 in [30] correspond respectively to our equations (30), (26) and (27). However the reader should be aware that, in [30], the decomposition of the plane vector F_1^1 in four components (N_1, N_2, T_1, T_2) is unfounded.

7 Conclusions

We hope that the results stated in this paper will be considered sufficient to give an end to the controversy about the soundness of N-th gradient theories. The method we have presented here is not the simplest way for establishing the relationship between hyperstress tensors and generalized contact interactions. However, as it follows closely Cauchy's path, it should persuade those who consider it as the only physically based one.

In a previous paper [13] the same results were obtained by following the D'Alembertian approach. This approach, first conceived by Lagrange for fluids and by Piola [37] for more general continuous systems consists in postulating the form of internal virtual power (represented by a sequence of hyperstress tensors) and in computing the contact interactions which are compatible with it. As expected the results of both approaches are the same. A careful reader could notice a difference between the boundary operators defined in the present paper and in the reference [13]. Cumbersome tensorial computations can show that they actually give the same representation of contact interactions.

It has to be remarked that we are far from having treated all possible shapes for bodies. For instance, wedges where more (or less !) than three edges concur or reentrant wedges are not treated. The extension to more general bodies could be obtained by considering unions and intersections of simple bodies.

It would also be interesting, at least from a theoretical point of view, to understand what happens in a body where the order of the stress state varies from point to point, eventually being even unbounded.

Finally we have to remark that the presented results cannot encompass stress states for which there are stress concentrations along lower dimensional manifolds, models which are needed if one wants to model for instance a 3D elastic body containing a free moving 2D plate or a fluid containing some unknown interfaces endowed with surface tension. An interesting step in this direction has been performed in [4] when the localization of stress is known. To our knowledge the theoretical tools for attacking the general problem remain to be developed.

8 Appendix A Multilinear Symmetric Operators on the Sphere

Here we prove that an operator which, locally on the sphere, coincides with multilinear completely symmetric operators coincides globally on the sphere with a completely symmetric operator[6].

Lemma 1 *Let φ be a function defined on the unit sphere S_2. Assume that, for any $\bar{n} \in S_2$, there exists an open neighborhood $O_{\bar{n}}$ of $\bar{n}$ in S_2 and a completely symmetric $p-$tensor $\mathcal{C}_{\bar{n}}$, such that, for any $n \in O_{\bar{n}}$,*

$$\varphi(n) = \mathcal{C}_{\bar{n}} \mid n^{\otimes p}.$$

Then there exists a unique completely symmetric $p-$tensor $\mathcal{C}$ such that, for any $n \in S_2$,

$$\varphi(n) = \mathcal{C} \mid n^{\otimes p}.$$

Moreover all tensors $\mathcal{C}_{\bar{n}}$ coincide with $\mathcal{C}$.

[6] It is remarkable that in the particular case of linear operators Lemma 1 allows to skip the classical Cauchy Action-Reaction lemma. Indeed, as soon as the Cauchy representation is proven to be valid in any trihedron, the linearity of $F_2^2(n)$ with respect to n is assured on the whole sphere.

Proof Let us first remark that if two functions $\mathcal{C} \mid n^{\otimes p}$ and $\mathcal{C}' \mid n^{\otimes p}$ coincide on some open subset O of S_2, then $\mathcal{C} = \mathcal{C}'$. Indeed the completely symmetric $p-$tensor $\mathcal{C}'' := \mathcal{C} - \mathcal{C}'$ satisfies $\mathcal{C}'' \mid n^{\otimes p} = 0$ for all $n \in O$ and therefore for all n in the cone $D := \{rn; r \in \mathbb{R},\ n \in O\}$. We use the polarization formula valid for any completely symmetric $p-$tensor (see [43] and references there cited):

$$\begin{aligned}\mathcal{C}'' \mid (u_1 \otimes u_2 \otimes ... \otimes u_p) &= \frac{1}{2^p p!} \sum_{\xi \in \{-1,1\}^p} \xi_1 \xi_2 ... \xi_p \mathcal{C}'' \mid \left(\sum_{i=1}^{p} \xi_i u_i \right)^{\otimes p} \\ &= \frac{1}{2^p p!} \sum_{\xi \in \{-1,1\}^p} (\xi_p)^{2p} \frac{\xi_1}{\xi_p} ... \frac{\xi_p}{\xi_p} \mathcal{C}'' \mid \left(\sum_{i=1}^{p} \frac{\xi_i}{\xi_p} u_i \right)^{\otimes p} \\ &= \frac{1}{2^{p-1} p!} \sum_{\tilde{\xi} \in \{-1,1\}^{p-1}} \tilde{\xi}_1 ... \tilde{\xi}_{p-1} \mathcal{C}'' \mid \left(\sum_{i=1}^{p-1} \tilde{\xi}_i u_i + u_p \right)^{\otimes p}\end{aligned}$$

Let $y \in O$ and $r > 0$ such that $B(y, pr) \subset D$ and let us consider the open subset O' of $\left(\mathbb{R}^3\right)^p$ defined by $O' := (0, 0, ..., 0, y) + B(0, r)^p$. For any $\xi \in \{-1, 1\}^{p-1}$ and any $(u_1, u_2, ..., u_p) \in O'$

$$\sum_{i=1}^{p-1} \xi_i u_i + u_p \in D.$$

As a consequence

$$\mathcal{C}'' \mid \left(\sum_{i=1}^{p-1} \xi_i u_i + u_p \right)^{\otimes p} = 0$$

and from the polarization formula we get $\mathcal{C}'' \mid (u_1 \otimes u_2 \otimes ... \otimes u_p) = 0$. Then the multilinear application $\mathcal{C}''$ vanishes in the open set O' and therefore is identically null.

Now let us fix $\bar{n} \in S_2$ and consider the largest open subset O of S_2 such that $\varphi(n) = \mathcal{C}_{\bar{n}} \mid n^{\otimes p}$ for all $n \in O$. Clearly O is a non empty subset of S_2 as it contains $O_{\bar{n}}$. Let $\tilde{n} \in S_2$ belong to the closure of O. On the non empty open set $O_{\tilde{n}} \cap O$, we have $\varphi(n) = \mathcal{C}_{\bar{n}} \mid n^{\otimes p} = \mathcal{C}_{\tilde{n}} \mid n^{\otimes p}$. Thus $\mathcal{C}_{\bar{n}} = \mathcal{C}_{\tilde{n}}$ and so $\tilde{n} \in O$. The set O is closed. The connectedness of S_2 implies that $O = S_2$ which concludes the proof. □

9 Appendix B. Boundary tensors

This appendix is devoted to the proof of two lemmas which establish that the boundary tensors are completely symmetric with respect to their p last subscripts. We need first to study some properties of the tensors $\mathcal{P}^p_{M,q}$ and $\mathcal{P}^p_M$.

The tensors $\mathcal{P}^p_{M,q}$ have been defined in such a way that, for any tensor X of order p and for any $1 \leq q \leq p$,

$$\left(\mathcal{P}^p_{M,q} \mid X\right) \cdot \mathit{\Pi}_M = \mathcal{P}^p_{M,q} \mid X.$$

Hence, by summation,

$$(\mathcal{P}^p_M \mid X) \cdot \mathit{\Pi}_M = \mathcal{P}^p_M \mid X. \tag{31}$$

Extending definition (8) to the case $p = 0$ by setting

$$(\mathcal{P}^p_{M,0})_{i_1,\ldots i_{2p}} := (\mathit{\Lambda}_M)^{i_{p+1}}_{i_1} \ldots (\mathit{\Lambda}_M)^{i_{2p}}_{i_q}$$

so that $\mathcal{P}^p_{M,0} \mid X := X_{\perp M}$, we obtain, by a simple induction argument that, for any tensor X of order p, any completely symmetric tensor Y of order p and for any $0 \leq r \leq p$,

$$\left(\sum_{q=0}^{r} \left(\mathcal{P}^p_{M,q} \mid X\right)\right) \mid Y = X_{i_1\ldots i_r j_{r+1}\ldots j_p} \mathit{\Lambda}^{j_{r+1}}_{i_{r+1}} \ldots . \mathit{\Lambda}^{j_p}_{i_p} Y_{i_1\ldots . i_p}$$

and, in particular, $\left(\sum_{q=0}^{p} \left(\mathcal{P}^p_{M,q} \mid X\right)\right) \mid Y = X \mid Y$ which can be written

$$(X_{\perp M} + (\mathcal{P}^p_M \mid X)) \mid Y = X \mid Y. \tag{32}$$

The tensor $\mathcal{P}^*_M := \sum_{q=1}^{p} \mathcal{P}^{p*}_{M,q}$ where

$$\left(\mathcal{P}^{p*}_{M,q}\right)_{i_1\ldots . i_{2p}} := \delta^{i_{p+1}}_{i_1} \ldots \delta^{i_{p+q-1}}_{i_{q-1}} (\mathit{\Pi}_M)^{i_{2p}}_{i_q} (\mathit{\Lambda}_M)^{i_{p+q+1}}_{i_{q+1}} \ldots (\mathit{\Lambda}_M)^{i_{2p-1}}_{i_{p-1}} (\mathit{\Lambda}_M)^{i_{p+q}}_{i_p}$$

is the adjoint of $\mathcal{P}^p_M$ in the following sense: for any pair (X, Y) of tensors of order p,

$$(X \mid (\mathcal{P}^p_M \mid Y)) = \left((\mathcal{P}^{p*}_M \mid X) \mid Y\right).$$

When M is a surface with normal n, then $\mathit{\Lambda}_M = n \otimes n$ and thus, for any tensor Y of order $p-1$ and any vector ν tangent to M,

$$\mathcal{P}^{p*}_M \mid (Y \otimes \nu) = \sum_{q=1}^{p} \left(Y \mid n^{\otimes p-q}\right) \otimes \nu \otimes n^{\otimes p-q}. \tag{33}$$

We are now in position to prove the following lemma.

Lemma 2 *The tensor $\mathcal{K}_{\mathcal{L}_j}(\mathcal{O}^p_{f_j} \mid X)$ depends only on the completely symmetric part of X.*

Proof Let us start by noticing that, for any tensors X and Y of order respectively p and $p-2$,

$$\begin{aligned}(\mathcal{O}_{f_j}^{p-1} \mid X) \mid Y &= \left(\sum_{k\in[\mathcal{L}_j]} (\mathcal{P}_{\mathcal{F}_k}^{p-1} \mid (X \cdot n_k)) \cdot \nu_k^j\right) \mid Y \\ &= \sum_{k\in[\mathcal{L}_j]} (\mathcal{P}_{\mathcal{F}_k}^{p-1} \mid (X \cdot n_k)) \mid (Y \otimes \nu_k^j) \\ &= \sum_{k\in[\mathcal{L}_j]} (X \cdot n_k) \mid (\mathcal{P}_{\mathcal{F}_k}^{(p-1)\star} \mid (Y \otimes \nu_k^j)) \\ &= X \mid \left(\sum_{k\in[\mathcal{L}_j]} (\mathcal{P}_{\mathcal{F}_k}^{(p-1)\star} \mid (Y \otimes \nu_k^j)) \otimes n_k\right).\end{aligned}$$

We have to prove that, for any tensors X and Y,

$$\mathcal{K}_{\mathcal{L}_j}(\mathcal{O}_{f_j}^{p-1} \mid X) \mid Y = \mathcal{K}_{\mathcal{L}_j}(\mathcal{O}_{f_j}^{p-1} \mid \mathcal{K}(X)) \mid Y.$$

As the operator $\mathcal{K}_{\mathcal{L}_j}$ is self adjoint, it is equivalent to prove that for any tensor X and any tensor Y completely symmetric and orthogonal to $\mathcal{L}_j$,

$$(\mathcal{O}_{f_j}^{p-1} \mid X) \mid Y = (\mathcal{O}_{f_j}^{p-1} \mid \mathcal{K}(X)) \mid Y.$$

Owing to our preliminary remark and using identity (33), we are reduced to prove that, for any tensor Y completely symmetric and orthogonal to $\mathcal{L}_j$, the tensor

$$\left(\sum_{k\in[\mathcal{L}_j]} (\mathcal{P}_{\mathcal{F}_k}^{(p-1)\star} \mid (Y \otimes \nu_k^j)) \otimes n_k\right) = \sum_{q=1}^{p-1} (Y \mid n_k^{\otimes p-q-1}) \otimes \nu_k^j \otimes n_k^{\otimes p-q}$$

is completely symmetric. To that aim, we check the invariance of this tensor with respect to the transposition of subscripts $(i, i{+}1)$ for all $i \in \{1, \ldots, p{-}1\}$.

All terms in the sum satisfying $q > i+1$ are clearly invariant owing to the complete symmetry of Y. The terms for which $q < i$ are also clearly invariant owing to the complete symmetry of $n_k^{\otimes p-q}$. So, for any $i \in \{1, \ldots p-2\}$, the only two terms in the sum which are not clearly invariant are those for which $q = i+1$ or $q = i$. The sum of these two addends reads

$$(Y \mid n_k^{\otimes p-i-1}) \otimes \nu_k^j \otimes n_k^{\otimes p-i} + (Y \mid n_k^{\otimes p-i-2}) \otimes \nu_k^j \otimes n_k^{\otimes p-i-1}. \tag{34}$$

Using the fact that Y is totally orthogonal to $\mathcal{L}_j$, we have the equality $(Y \mid n_k^{\otimes p-i-2}) \cdot e_j = 0$ so that we can decompose the second of these terms in

$$(Y \mid n_k^{\otimes p-i-1}) \otimes n_k \otimes \nu_k^j \otimes n_k^{\otimes p-i-1} + (Y \mid n_k^{\otimes p-i-2}) \cdot \nu_k^j) \otimes \nu_k^j \otimes \nu_k^j \otimes n_k^{\otimes p-i-1}.$$

Therefore the sum (34) can be rewritten

$$(Y \mid n_k^{\otimes p-i-2}) \cdot \Big(n_k \otimes (\nu_k^j \otimes n_k + n_k \otimes \nu_k^j) + \nu_k^j \otimes \nu_k^j \otimes \nu_k^j\Big) \otimes n_k^{\otimes p-i-1}$$

the invariance of which is now clear.

We still have to deal with the case $i = p-1$. In that case there is only one term $Y \otimes \nu_k^j \otimes n_k$, which is not clearly invariant with respect to the transposition of subscripts $p-1$ and p. Indeed it is not and we have to remind that we deal with a sum over all $k \in [\mathcal{L}_j]$. Actually, the sum

$$\sum_{k\in[\mathcal{L}_j]} \Big(Y \otimes \nu_k^j \otimes n_k\Big) = Y \otimes \left(\sum_{k\in[\mathcal{L}_j]} \nu_k^j \otimes n_k\right)$$

is invariant as the matrix $\sum_{k\in[\mathcal{L}_j]} \nu_k^j \otimes n_k$ is symmetric. Indeed let us compute its skew-symmetric part : up to a factor 2 it reads

$$\sum_{k\in[\mathcal{L}_j]} \Big(\nu_k^j \otimes n_k - n_k \otimes \nu_k^j\Big) = R_{e_j} - R_{e_j} = 0,$$

where R_{e_j} means the rotation of angle $\pi/2$ about the axis e_j and where we have used that the two bases $\Big(e_j, n_k, \nu_k^j\Big)$ for $k \in [\mathcal{L}_j]$ have opposite orientations. The proof is completed as the transpositions of successive subscripts generate all permutations. □

Remark 1 Now let Y be a tensor of order $p-3$, e a unit vector tangent to a line $\mathcal{L}$ and $1 \le i \le p-3$. We remark that $(\mathcal{P}^{p-2}_{\mathcal{L},i} + \mathcal{P}^{p-2}_{\mathcal{L},i+1}) \mid (Y \otimes e)$ is invariant with respect to transposition of the subscripts i and $i+1$.

Indeed as the subscripts different from i and $i+1$ play no role, we can assume, without loss of generality, that $p-3 = i = 1$. In which case, the matrix $e \otimes Y_{\perp\mathcal{L}} + Y \otimes e = e \otimes Y_{\perp\mathcal{L}} + Y_{\perp\mathcal{L}} \otimes e + (Y \cdot e) \otimes e \otimes e$ is clearly symmetric.

This remark is useful to prove the following lemma.

Lemma 3 *The tensor $\mathcal{K}(\mathcal{O}^p_w \mid X)$ depends only on the completely symmetric part of X.*

Proof The proof is similar to the proof of Lemma 2. We start by noticing that, for any tensors X and Y of order respectively p and $p-3$,

$$\begin{aligned}(\mathcal{O}^p_w \mid X)\,|\,Y &= \left(\sum_{j\in[\hat{x}]} \left(\mathcal{P}^{p-2}_{\mathcal{L}_j} \mid \left(\sum_{k\in[\mathcal{L}_j]} \Big(\Big(\mathcal{P}^{p-1}_{\mathcal{F}_k} \mid (X \cdot n_k)\Big) \cdot \nu_k^j\Big)\right)\right) \cdot e_j\right) \mid Y \\ &= X\,|\left(\sum_{j\in[\hat{x}]} \sum_{k\in[\mathcal{L}_j]} (\mathcal{P}^{(p-1)\star}_{\mathcal{F}_k} \mid ((\mathcal{P}^{(p-2)\star}_{\mathcal{L}_j} \mid (Y \otimes e_j)) \otimes \nu_k^j)) \otimes n_k\right).\end{aligned}$$

We have to prove that, for any completely symmetric tensor Y of order $p-3$, the tensor

$$Z := \sum_{j\in[\hat{x}]} \sum_{k\in[\mathcal{L}_j]} (\mathcal{P}^{(p-1)\star}_{\mathcal{F}_k} \mid ((\mathcal{P}^{(p-2)\star}_{\mathcal{L}_j} \mid (Y \otimes e_j)) \otimes \nu_k^j)) \otimes n_k$$

is completely symmetric. Using (33) we get

$$\begin{aligned}(\mathcal{P}^{(p-1)\star}_{\mathcal{F}_k} &|((\mathcal{P}^{(p-2)\star}_{\mathcal{L}_j} \mid (Y \otimes e_j)) \otimes \nu^j_k)) \otimes n_k \\ &= \sum_{q=1}^{p-1}((\mathcal{P}^{(p-2)*}_{\mathcal{L}_j} \mid (Y \otimes e_j)) \mid n_k^{\otimes p-q-1}) \otimes \nu^j_k \otimes n_k^{\otimes p-q} \\ &= \sum_{q=1}^{p-1}\left(\sum_{r=1}^{p-2} \mathcal{P}^{(p-2)\star}_{\mathcal{L}_j,r} \mid ((Y \otimes e_j)) \mid n_k^{\otimes p-q-1}\right) \otimes \nu^j_k \otimes n_k^{\otimes p-q}.\end{aligned}$$

When $q \leq r$, $(\mathcal{P}^{(p-2)\star}_{\mathcal{L}_j,r} \mid (Y \otimes e_j)) \mid n_k^{\otimes p-q-1} = 0$, otherwise, after some computation and using the complete symmetry of Y,

$$(\mathcal{P}^{(p-2)\star}_{\mathcal{L}_j,r} \mid (Y \otimes e_j)) \mid n_k^{\otimes p-q-1} = \mathcal{P}^{(q-1)\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes p-q-1} \otimes e_j)).$$

Thus we get

$$Z = \sum_{j\in[\hat{x}]}\sum_{k\in[\mathcal{L}_j]}\sum_{q=1}^{p-1}\left(\sum_{r=1}^{q-1} \mathcal{P}^{(q-1)\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes p-q-1}) \otimes e_j)\right) \otimes \nu^j_k \otimes n_k^{\otimes p-q} \quad (35)$$

We now check the invariance of Z with respect to all transpositions of subscripts i and $i+1$ for $1 \leq i < p$. This is clear, owing to the complete symmetry of Y, for every term in the sum (35) satisfying $r > i+1$ or $r < i < q-1$. This is also evident if $q < i$. Remark 1 shows that the sum of a pair of terms corresponding to $r = i$ and $r = i+1$ with the same $q > i+1$ is also invariant.

When $i < p-1$ the sum of a pair of terms corresponding to $q = i$ and $q = i+1$ with the same $r < i$ is also symmetric with respect to the considered transposition. Indeed it reads

$$\begin{aligned}\mathcal{P}^{(i-1)\star}_{\mathcal{L}_j,r} &|((Y \mid n_k^{\otimes p-i-1}) \otimes e_j) \otimes \nu^j_k \otimes n_k^{\otimes p-i} \\ &+ \mathcal{P}^{i\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes p-i-2}) \otimes e_j) \otimes \nu^j_k \otimes n_k^{\otimes p-i-1}\end{aligned}$$

But as $r < i$ implies $\mathcal{P}^{i\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes p-i-2}) \otimes e_j) \cdot e_j = 0$, we have

$$\begin{aligned}\mathcal{P}^{i\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes p-i-2}) \otimes e_j) = &\left(\mathcal{P}^{(i-1)\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes p-i-1}) \otimes e_j)\right) \otimes n_k \\ &+ \left(\mathcal{P}^{i\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes p-i-2}) \otimes e_j) \cdot \nu^j_k\right) \otimes \nu^j_k.\end{aligned}$$

So the sum of a pair of terms corresponding to $q = i$ and $q = i+1$ becomes

$$\begin{aligned}&\mathcal{P}^{i\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes n-i-1}) \otimes e_j) \otimes \left(\nu^j_k \otimes n_k + n_k \otimes \nu^j_k\right) \otimes n_k^{\otimes n-i-1} \\ &+ \left(\mathcal{P}^{(i-1)\star}_{\mathcal{L}_j,r} \mid ((Y \mid n_k^{\otimes n-i-2}) \otimes e_j) \cdot \nu^j_k\right) \otimes \nu^j_k \otimes \nu^j_k \otimes n_k^{\otimes n-i-1}\end{aligned}$$

which is now clearly invariant under the considered transposition.

From now on we must consider the whole sum defining Z. Indeed we are left with some terms whose symmetry cannot be established for a fixed pair of subscripts (j,k).

When $i = p-1$ the terms corresponding to $q = i$ and $r < i$ can be treated for a fixed j. Indeed we have already noticed in the proof of Lemma 2 that the matrix $\sum_{k\in[\mathcal{L}_j]} \nu_k^j \otimes n_k$ is symmetric. Thus each sum

$$\sum_{k\in[\mathcal{L}_j]} \left(\mathcal{P}^{(p-2)\star}_{\mathcal{L}_j,r} \mid (Y \otimes e_j)\right) \otimes \nu_k^j \otimes n_k$$

is invariant with respect to the considered transposition.

The only remaining term corresponds to $r = i$, $q = i+1$. It can be treated for a fixed k. Indeed consider the two edges (denoted $j \in [\hat{x},k]$) which concur in the wedge $[\hat{x}]$ and border the face $\mathcal{F}_k$. We have

$$\begin{aligned}\sum_{j\in[\hat{x},k]} \left(\left(\mathcal{P}^{(p-2)\star}_{\mathcal{L}_j,i} \mid (Y \otimes e_j)\right) \mid n_k^{\otimes p-i-2}\right) \otimes \nu_k^j \otimes n_k^{\otimes p-i-1} \\ = \sum_{j\in[\hat{x},k]} \left(Y \mid n_k^{\otimes p-i-2} \otimes e_j \otimes \nu_k^j \otimes n_k^{\otimes p-i-1}\right) \\ = Y \mid n_k^{\otimes p-i-2} \otimes \left(\sum_{j\in[\hat{x},k]} e_j \otimes \nu_k^j\right) \otimes n_k^{\otimes p-i-1}\end{aligned}$$

which is invariant with respect to the considered transposition, as the matrix $\sum_{j\in[\hat{x},k]} e_j \otimes \nu_k^j$ is symmetric (it skew symmetric part is $R_{n_k} - R_{n_k} = 0$). The proof is completed as we have explored all terms in the sum (35). □

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