A link at infinity for minimal surfaces in R^4

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Abstract

We look at complete minimal surfaces of finite total curvature in $\mathbb{R}^4$. Similarly to the case of complex curves in $\mathbb{C}^2$ we introduce their *link at infinity*; we derive the *writhe number at infinity* which gives a formula for the total normal curvature of the surface. The knowledge of the link at infinity can sometimes help us determine if a surface has self-intersection and we illustrate this idea by looking at genus zero surfaces of small total curvature.

1 Introduction - Sketch of the paper

The study of complete minimal surfaces of finite total curvature in $\mathbb{R}^4$ has been initiated by a paper of Chern and Osserman ([Ch-Os]): they show that the Gauss map giving the data of the oriented tangent planes can be seen as a holomorphic map into a quadric in $\mathbb{C}P^2$. This quadric is actually the product of two projective lines and the Gauss map splits into two meromorphic functions. These meromorphic functions would be the starting point for the twistor representation of minimal surfaces by Eells and Salamon ([E-S]).
In the 1980's Ossermann and other authors wrote a series of papers (see for example [Ho-Os1], [Ho-Os2] and [Mo-Os]) pursuing the investigation of the Gauss map. In $\mathbb{R}^3$, the planes are characterized among minimal surfaces as having a constant Gauss map. Similarly, complex curves in $\mathbb{R}^4$ have one of the two meromorphic Gauss maps equal a constant.
Much research has been done about the Gauss map, using tools of complex analysis as it gives us good information about the minimal surface. However it cannot really help us determined when an immersed minimal surface is actually embedded and it is this problem that we would like to address here.

We start by recalling the definitions of the Gauss maps via the quadric and also the Eells-Salamon approach. This material is classical and well-known but we felt it was useful to present in the same paper both definitions and to give a concrete way of going from one to the other. We then recall the curvature formulae derived from these maps.
For an embedded minimal surface we define the link at infinity, which is the intersection of the surface with a sphere of very large radius in $\mathbb{R}^4$. We give a formula relating this link to the total normal curvature of the surface and derive some restrictions on the asymptotic behaviour of the surface. These give us a necessary condition for a degenerate minimal surface to be embedded.
Finally we look at minimal surfaces of small total curvature. If the curvature is -4π, we are able to classify all complete embedded non holomorphic ones. We get some partial information for curvatures -6π and -8π.

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2 The Gauss maps and the curvatures

In this section, Σ is a complete embedded minimal surface in $\mathbb{R}^4$ of finite total curvature; its Gauss map maps a point p of Σ to the tangent plane $T_p\Sigma$ inside the Grassmannian of oriented 2-planes in $\mathbb{R}^4$.

2.1 The Grassmannian of oriented 2-planes in $\mathbb{R}^4$

There are two equivalent ways of describing this Grassmannian which give rise to two different definitions of the Gauss map. They are both classical and well-understood; we recall them both and describe a concrete correspondence between them.
For more details we refer the reader to [Ch-Os], [E-S] and [Mo-Os], [Ho-Os1] and [Ho-Os2].

2.1.1 Complex structures

Here is the definition of the Grassmannian which is used the twistor approach to minimal surfaces.
We let G_2^+ the Grassmannian of oriented 2-planes in $\mathbb{R}^4$: it splits into a product

$$G_2^+ = \mathbb{S}(\Lambda^+(\mathbb{R}^4)) \times \mathbb{S}(\Lambda^-(\mathbb{R}^4))$$

where $\mathbb{S}$ denotes the unit sphere and $\Lambda^+(\mathbb{R}^4)$ (resp. $\Lambda^-(\mathbb{R}^4)$) denotes the subset of 2-vectors which are $+1$ (resp. -1) -eigenvectors for the Hodge operator $* : \Lambda^2(\mathbb{R}^4) \longrightarrow \Lambda^2(\mathbb{R}^4)$.
If P is an oriented 2-plane in $\mathbb{R}^4$, we write it as $\epsilon_1 \wedge \epsilon_2$ where (ϵ_1, ϵ_2) is a positive orthonormal basis on P and we split $\epsilon_1 \wedge \epsilon_2$ as $\epsilon_1 \wedge \epsilon_2 = \frac{1}{\sqrt{2}}(J_+ + J_-)$ with

$$J_+(P) = \frac{1}{\sqrt{2}}[\epsilon_1 \wedge \epsilon_2 + \star(\epsilon_1 \wedge \epsilon_2)] \in \mathbb{S}(\Lambda^+(\mathbb{R}^4)) \tag{1}$$

$$J_-(P) = \frac{1}{\sqrt{2}}[\epsilon_1 \wedge \epsilon_2 - \star(\epsilon_1 \wedge \epsilon_2)] \in \mathbb{S}(\Lambda^-(\mathbb{R}^4)) \tag{2}$$

The space $\mathbb{S}(\Lambda^+(\mathbb{R}^4))$ (resp. $\mathbb{S}(\Lambda^-(\mathbb{R}^4))$) is the space of parallel complex structures on $\mathbb{R}^4$ which are compatible with (resp. reverse) the orientation on $\mathbb{R}^4$. We view J_+ and J_- in (1) and (2) as a complex structure by setting

$$J_+(\epsilon_1) = \epsilon_2 \qquad J_-(\epsilon_1) = \epsilon_2 \tag{3}$$

Then the plane P is a $J_+(P)$- and $J_-(P)$-complex line.

2.1.2 The Grassmannian as a quadric in $\mathbb{C}P^3$

We now present the definition most commonly used by authors working on minimal surfaces in Euclidean spaces.
We fix a positive orthonormal basis (e_1, e_2, e_3, e_4) of $\mathbb{R}^4$ and we extend it to a basis of the complexified space $\mathbb{R}^4 \otimes \mathbb{C} = \mathbb{C}^4$. We denote by z_i the corresponding complex coordinates in $\mathbb{C}^4$ and we define the quadric

$$Q_2 = \{[z_0, ..., z_3] \in \mathbb{C}P^3 / \sum_{i=0}^{3} z_i^2 = 0\}$$

We consider again the plane P generated by ϵ_1, ϵ_2 and we map it to the class in $\mathbb{C}P^3$ of the vector

$$[\epsilon_1 - i\epsilon_2] \in Q_2 \tag{4}$$

We now recall the Segre isomorphism between Q_2 and $\mathbb{C}P^1 \times \mathbb{C}P^1$; it is given by the following two maps:

$$g_+([\phi_1, \phi_2, \phi_3, \phi_4]) = \frac{\phi_3 + i\phi_4}{\phi_1 - i\phi_2} = \frac{\phi_1 + i\phi_2}{-\phi_3 + i\phi_4} \tag{5}$$

$$g_-([\phi_1, \phi_2, \phi_3, \phi_4]) = \frac{-\phi_3 + i\phi_4}{\phi_1 - i\phi_2} = \frac{\phi_1 + i\phi_2}{\phi_3 + i\phi_4} \tag{6}$$

We now explain that these two different ways of describing an oriented 2-plane P as the data of two elements in two 2-spheres are equivalent. We do it for g_+ and J_+; it works the same for g_- and J_-.
We derive from the basis $(e_1, ..., e_4)$ of $\mathbb{R}^4$ a basis of $\mathbb{S}(\Lambda^+(\mathbb{R}^4))$ given by

$$J_0 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4)$$

$$J_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_4 \wedge e_2)$$

$$J_2 = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3)$$

Suppose $J_+(P) = \alpha J_0 + \beta J_1 + \gamma J_2$. We denote by Π the stereographic projection w.r.t. the point J_0 to the plane generated by J_1 and J_2; we have

$$\Pi(J_+(P)) = \frac{\beta}{1-\alpha} J_1 + \frac{\gamma}{1-\alpha} J_2 \tag{7}$$

Convention 1. *We identify the plane generated by J_1 and J_2 with the complex plane, with J_2 (resp. J_1) identified with 1 (resp. i).*

Following Convention 1, we rewrite (7) as

$$\Pi(J_+(P)) = \frac{\gamma + i\beta}{1-\alpha} \tag{8}$$

On the other hand, we identify $\mathbb{C}P^1$ with $\mathbb{C} \cup \{\infty\}$ by

$$[z_1, z_2] \mapsto \frac{z_1}{z_2} \tag{9}$$

Proposition 1. *Under the above identifications, if P is an oriented 2-plane,*

$$g_+(P) = \Pi(J_+(P))$$

Proof. We first check

Lemma 1. *If $J \in \mathbb{S}(\Lambda^+(\mathbb{R}^4))$, the planes of G_2^+ which are J-complex lines form a complex line in G_2^+.*

Proof. We first prove Lemma 1 if $J = J_0$. A J_0-complex plane is generated by two vectors

$$\epsilon_1 = ae_1 + be_2 + ce_3 + de_4 \qquad \epsilon_2 = -be_1 + ae_2 - de_3 + ce_4 \tag{10}$$

Then

$$\epsilon_1 - i\epsilon_2 = (\lambda, -i\lambda, \mu, -i\mu) \tag{11}$$

where $\lambda = a + ib$ and $\mu = c + id$. It is clear that (11) describes a line L in Q_2.
A general J is given by $J = B^{-1}J_0B$ for some $B \in SO(4)$. For a unit vector u, we have

$$u - iJu = B^{-1}(Bu - iJ_0u)$$

hence J belongs to the line $B^{-1}L$. □

It follows from Lemma 1 that it is enough to prove Prop. 1 if P is generated by $e_1, J_+(P)e_1$. Then

$$e_1 - iJ_+(P)e_1 = (1, -i\alpha, -i\beta, -i\gamma)$$

hence

$$g_+(P) = \frac{i\beta + \gamma}{1 - \alpha} = \Pi(J_+(P))$$

□

2.2 The Gauss map: notations

Let Σ be a Riemann surface and $F : \Sigma \longrightarrow \mathbb{R}^4$ be an immersion. If $p \in \Sigma$, the Gauss map $\Gamma(p) \in G_2^+$ of F at p is the oriented tangent plane to $dF(T_p\Sigma)$. Namely, if $z = x + iy$ is a local holomorphic coordinate on Σ around p, we can write

$$\Gamma(p) = [\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial x}] \in Q_2 \tag{12}$$

If F is minimal, then $\Gamma : \Sigma \longrightarrow Q_2$ is holomorphic.
Using the notations of (1) and (2) we define

$$\gamma_+(p) = J_+(\Gamma(p)) \qquad \gamma_-(p) = J_-(\Gamma(p)) \tag{13}$$

If F is minimal and we use Convention 1, the maps γ_+ and γ_+ are holomorphic.

2.3 Curvatures of the tangent and normal bundle

If Σ is a surface immersed in $\mathbb{R}^4$, we use the Gauss maps (13) to compute the curvatures of the tangent bundle $T\Sigma$ and normal bundle $N\Sigma$. We have ([Ch-T],[Vi]):

$$\frac{1}{2}\|\nabla\gamma_+\|^2 = -K^T - K^N \qquad \frac{1}{2}\|\nabla\gamma_-\|^2 = -K^T + K^N \tag{14}$$

If Σ is minimal,

$$|K^N| \leq -K^T \tag{15}$$

the equality being attained at points where Σ is superminimal.

3 Complete minimal surfaces of finite total curvature

In this section Σ is a Riemann surface and $F : \Sigma \longrightarrow \mathbb{R}^4$ is a conformal harmonic map such that $F(\Sigma)$ is a complete minimal surface in $\mathbb{R}^4$. We recall some basic properties (see [Ch-Os]).
There exists a compact Riemann surface $\hat{\Sigma}$ without boundary and a finite number of points $p_1, ..., p_d$ in $\hat{\Sigma}$ such that

$$\Sigma = \hat{\Sigma}\backslash\{p_1, ..., p_d\}$$

and the Gauss map Γ extends to a holomorphic map

$$\hat{\Gamma} : \hat{\Sigma} \longrightarrow Q_2.$$

Moreover there exist meromorphic differentials $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ on Σ such that, for every $k = 1, ..., 4$, the corresponding i-th coordinate of F can be written as

$$x_k = \int \alpha_k \tag{16}$$

We assume that for R large enough, $F(\Sigma) \cap (\mathbb{R}^4\backslash\mathbb{S}(0, R))$ is a finite union of annuli. These annuli are called *ends* of Σ and these ends correspond to the

p_k's.
Given a p_k, we call the plane $P_k = \hat{\Gamma}(p_k)$ the tangent plane to Σ at infinity for the corresponding end. Using the expression (16), we parametrize the end as follows

Proposition 2. *Let $D(p_k, \epsilon)$ be the disk in $\hat{\Sigma}$ centered at p_k and of radius ϵ. For $\epsilon > 0$ small enough, we reparametrize $D(p_k, \epsilon)\backslash\{p_k\}$ as $\{z \in \mathbb{C}/|z| > R\}$ for some $R > 0$. The restriction of F to the end can be written as*

$$\{z/|z| > R\} \longrightarrow \mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$$

$$z \mapsto (Re(z^N) + o(|z^N|), Im(z^N) + o(|z^N|), o(|z^N|), o(|z^N|)) \qquad (17)$$

where the first complex coordinate on $\mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$ generates P_k.

REMARK. The condition that $F(\Sigma)$ is complete is essential for Prop. 2:

Exemple 1. *The following minimal surface has finite total curvature:*

$$\mathbb{C} \longrightarrow \mathbb{C}^2$$

$$z \mapsto (e^z z, e^z z^2).$$

If $g(\hat{\Sigma}) = 0$ we also also derive from (16),

Proposition 3. *If $m_1, ..., m_d$ are points in $\mathbb{C}$ and F is a complete minimal immersion of $\mathbb{C}\backslash\{m_1, ..., m_d\}$ in $\mathbb{R}^4$ of finite total curvature, it can be written as*

$$F : \mathbb{C} \longrightarrow \mathbb{C}^2$$

$$F : z \mapsto (f_1(z) + \bar{f}_2(z), f_3(z) + \bar{f}_4(z))$$

where f_1, f_2, f_3, f_4 are meromorphic functions verifying

$$f_1'(z) f_2'(z) + f_3'(z) f_4'(z) = 0 \qquad (18)$$

If $\Sigma = \mathbb{C}$, the f_i's are polynomials.

The identity (18) follows from the fact that F is conformal (cf. [M-W]).

3.1 Homology computations

The maps γ_+ and γ_- also extend to finite degree maps

$$\hat{\gamma}_+ : \hat{\Sigma} \longrightarrow \mathbb{S}(\Lambda^+(\mathbb{R}^4)), \quad \hat{\gamma}_- : \hat{\Sigma} \longrightarrow \mathbb{S}(\Lambda^-(\mathbb{R}^4)).$$

We denote by d_+ (resp. d_-) the degree of $\hat{\gamma}_+$ (resp. $\hat{\gamma}_-$) and we express the $d_\pm$'s in terms of the homology class of $\hat{\Gamma}(\hat{\Sigma})$ in G_2^+, see ([Ho-Os2]).
We let S_+ (resp. S_-) be the class in $H_2(G_2^+, \mathbb{Z})$ of $\mathbb{S}(\Lambda^+(\mathbb{R}^4)) \times \{x\}$ (resp. $\{x\} \times \mathbb{S}(\Lambda^-(\mathbb{R}^4))$), where x is an element of $\mathbb{S}(\Lambda^-(\mathbb{R}^4))$ (resp. $\mathbb{S}(\Lambda^+(\mathbb{R}^4))$. The homology class of $\hat{\Gamma}(\hat{\Sigma})$ verifies

$$[\hat{\Gamma}(\hat{\Sigma})] = d_+ S_+ + d_- S_- \tag{19}$$

We have

$$-\int_\Sigma K^T = 2\pi(d_+ + d_-) \qquad -\int_\Sigma K^N = 2\pi(d_+ - d_-) \tag{20}$$

3.2 Total curvature of the tangent bundle

The Gauss-Bonnet formula together and Th. A of [Shi] yield

Proposition 4. *If d is the number of ends of Σ and $\chi(\Sigma)$ is its Euler characteristic, we have*

$$\frac{1}{2\pi}\int_\Sigma K^T = -\sum_{i=1}^{b} N_i + \chi(\Sigma) = 2 - \sum_{i=1}^{b}(1 + N_i) - 2g(\Sigma) \tag{21}$$

where the N_i's are as the N in (17).

4 The normal bundle

In this section we assume that Σ is embedded.

4.1 The knots and link at infinity

We define a link at infinity similarly to what is done for complex curves (cf. [N-R]):

Theorem 1. *There exists a $R_0 > 0$ such that for $R > R_0$, $L_R = \mathbb{S}(0,R) \cap \Sigma$ is a link; and its link type does not depend on $R > R_0$.*

Proof. We go back to Prop. 2 and we consider for each end i, the knot $K_i(R) = \mathbb{S}(0,R) \cap F(D(p_i,\epsilon)\backslash\{p_i\})$; it follows from the expression of (17) that for R large enough, $K_i(R)$ is transverse to $\mathbb{S}(0,R)$ and for $R < R'$, $K_i(R)$ is isotopic to $K_i(R')$. Also, since the $D(p_i,\epsilon)$'s are disjoint for ϵ small enough, $K_i(R)$ and $K_j(R)$ are disjoint if $i \neq j$.
It follows that the L_R's are all well-defined and isotopic for R large enough. □

4.2 The writhe at infinity

Let X be a vector in $\mathbb{R}^4$ which does not belong to any of the P_i's (the tangent planes at infinity). We derive from Prop. 2 that the projection of X to $\mathbb{S}(0,R)$ is not tangent to any of the $K_i(R)$'s if R is large enough. Hence, if we push slightly $K_i(R)$ in the direction of the projection of X along $\mathbb{S}(0,R)$, we get another knot $\hat{K}_i(R)$ which is disjoint from $K_i(R)$. Moreover we can take each $\hat{K}_i(R)$ close enough to $K_i(R)$ so that $\hat{K}_i(R)$ and $\hat{K}_j(R)$ are disjoint if $i \neq j$. Thus the $\hat{K}_i$'s put together form a link $\hat{L}(R)$ and the linking number $lk(L(R), \hat{L}(R))$ is well defined.

4.3 Integral formulae for the normal curvature

Proposition 5. *Let Σ be a complete minimal surface of finite total curvature embedded in $\mathbb{R}^4$. For a large enough positive real number R, we define $lk(L(R), \hat{L}(R))$ as in §4.2. Then, for R large enough, the curvature of the normal bundle is*

$$\frac{1}{2\pi}\int_\Sigma K^N = lk(L(R), \hat{L}(R)) \tag{22}$$

the equality being attained if and only if Σ is holomorphic for a parallel complex structure on $\mathbb{R}^4$.

Proof. We let X^N be the projection of X to the normal bundle $N\Sigma$. We let J be the complex structure (compatible with the metric and orientation) on $N\Sigma$ and we apply Stokes'theorem to the form

$$\omega = -\frac{1}{\|X^N\|^2} < \nabla X^N, JX^N > .$$

We have $d\omega = K^N dA$, where dA is the area element on Σ.

$$\frac{1}{2\pi}\int_{\Sigma\cap B(0,R)} K^N = \frac{1}{2\pi}\int_{\Sigma\cap\partial B(0,R)} \omega + \text{number of zeroes of } X^N \text{ on } \Sigma\cap B(0,R).$$

Lemma 2.

$$\lim_{R\longrightarrow\infty}\int_{\Sigma\cap\partial B(0,R)} \omega = 0$$

Proof. It is enough to consider one end p_1; we denote P_1 the oriented tangent plane at infinity for this end viewed as a 2-vector and by J_+ the corresponding complex structure (i.e. $\hat{\gamma}_+(p_1)$).
Denoting by

$$\star : \Lambda^3(\mathbb{R}) \longrightarrow \Lambda^1(\mathbb{R})$$

the Hodge operator, we let the reader check that

$$X^N = -J_+(\star(X \wedge TF(\Sigma))) \tag{23}$$

where $TF(\Sigma)$ is the tangent plane. It follows that, in order to bound ω, we need to bound $\|\nabla\gamma_+\|$ and $\|\nabla\gamma_-\|$. We achieve ths by putting together (5), (6) and (17) to derive the existence of two complex numbers a and b,

$$\gamma_+(z) = \frac{a}{\bar{z}} + o(\frac{1}{|z|}) \qquad \gamma_-(z) = \frac{b}{z} + o(\frac{1}{|z|}) \tag{24}$$

We conclude by saying that interpreting the number of zeroes of X^N as the number of intersection points inside $B(0,R)$ between $F(\Sigma)$ and a surface obtained by pushing $F(\Sigma)$ slightly in the direction of X^N. □

If we focus on a single end P_1, we proceed as in [Vi2] and view the knot at infinity K_1 as a braid in $\mathbb{S}(0,R)$; or equivalently as a braid in the cylinder $\mathbb{S}^1 \times Q$ where Q is a plane containing X (as in [S-V]). The linking number $lk(K(R), \hat{K}(R))$ can be interpreted as the algebraic length $e(K(R))$ (cf. [Be]) of this braid. We derive from Prop. 5

Corollary 1. *Under the assumptions of Prop. 5, and for R large enough,*
1) If Σ has a single end with knot at infinity

$$\frac{1}{2\pi}\int_{\Sigma} K^N = e(K) \tag{25}$$

2) If Σ has several ends and their tangent planes at infinity P_i are all transverse,

$$\frac{1}{2\pi}\int_\Sigma K^N = \sum_i e(K_i(R)) + 2\sum_{i,j,i\neq j}\sigma(i,j)N_iN_jlk(K_i(R),k_j(R)) \qquad (26)$$

where $\sigma(i,j)$ is 1 (resp. -1) if P_i and P_j intersect positively (resp. negatively) and the N_i's are as in (17).

REMARK. We point out the similarity with the case of a local branch point ([Vi2]): in this case as well, the date of the normal bundle is given by the algebraic length of a braid.

5 Estimates for a complete minimal surface with a single end

In this section, Σ is a complete minimal surface in $\mathbb{R}^4$ of finite total curvature with a single end. We let g be the genus of Σ and K be its knot at infinity. The integral formulae for the tangent and normal curvatures together with the inequality (15) between these curvatures enable us to derive some estimates.

Proposition 6. *Under the assumptions of Cor. 1 1), we have*

$$|e(K)| \leq N - 1 + 2g \qquad (27)$$

the equality being attained if and only if Σ is holomorphic for a parallel complex structure on $\mathbb{R}^4$.

REMARK. The inequality (27) is just Rudolph's slice-Bennequin inequality ([Ru]).

5.1 Computations inside G_2^+

We now consider $\tilde{\Sigma} = \hat{\Gamma}(\hat{\Sigma})$ which is a complex curve in G_2^+. We derive from (20) and (25) that the homology class $[\tilde{\Sigma}]$ verifies

$$[\tilde{\Sigma}].[\tilde{\Sigma}] =$$

$$(d_+S_+ + d_-S_-).(d_+S_+ + d_-S_-) = 2d_+d_- = \frac{1}{2}[(2g+N-1)^2 - e(K)^2] \quad (28)$$

$$< c_1(G_2^+), \tilde{\Sigma} >= 2(d_+ + d_-) = 2(2g+N-1) \quad (29)$$

We can now write the adjunction formula for $\tilde{\Sigma}$ ([G-H]):

$$c_1(T\tilde{\Sigma}) + c_1(N\tilde{\Sigma}) = 2 - 2g + [\tilde{\Sigma}].[\tilde{\Sigma}] + \sum_s m_s =< c_1(G_2^+), \tilde{\Sigma} >$$

where the s's run through the singular points of $\tilde{\Sigma}$ and the m_s are negative numbers. It follows from (28) and (29) that

$$4 - 4g + (2g+N-1)^2 - e(K)^2 \geq 2(2g+N-1).$$

We derive

Proposition 7. *Let Σ be a complete properly embedded in $\mathbb{R}^4$ minimal surface of finite total curvature which is not holomorphic for any parallel complex structure on $\mathbb{R}^4$. Then*

$$e(K)^2 \leq (2g+N-3)^2 - 4g.$$

Equality is attained if the map $\hat{\Gamma} : \hat{\Sigma} \longrightarrow G_2^+$ is an embedding.

5.2 The knot at infinity

We now go back to the expression (2) of the end and focus on the second component; we assume that there exists an integer p, with $0 \leq p < N$ and two complex numbers A and B such that the end is parametrized

$$z \mapsto (z^N + o(|z|^N), Az^p + B\bar{z}^p + o(|z|^p)) \quad (30)$$

We distinguish two cases in (30).
1st case: If $|A| \neq |B|$ in (30), the knot at infinity is the (N,q) torus knot; hence $|e(K)| = (N-1)p$. Hence

Proposition 8. *If $|A| \neq |B|$ in (30),*

$$g(\Sigma) \geq \frac{(N-1)(p-1)}{2} \quad (31)$$

Equality occurs in (30) occurs if Σ is holomorphic for a parallel complex structure on $\mathbb{R}^4$.

Proposition 9. *If $|A| = |B|$ in (30), then*
i) $N - p \leq d_+ \leq p - 1 + 2g, \quad N - p \leq d_- \leq p - 1 + 2g$
ii) $|e| \leq 2p - N - 1 + 2g$

Proof. Both $\hat{\gamma}_+$ and $\hat{\gamma}_-$ have a branch point of order $N-p$ at infinity; since d_+ and d_- are the degrees of these maps, we derive $N-p \leq d_-$ and $N-p \leq d_+$. We derive the other inequalities for the degrees (we write it for d_+, the same proof works for d_-):

$$d_+ = d_+ + d_- - d_- = N - 1 + 2g - d_- \leq N - 1 + 2g - (N - p) = p - 1 + 2g.$$

The inequality ii) follows immediately from i). □

6 Planar degenerate minimal surfaces

We consider here 1-degenerate minimal surfaces (we will drop the 1 from now on): by definition their image under the Gauss map sits inside a hyperplane of $\mathbb{C}P^3$. We refer the reader to [Ho-Os1] for a detailed exposition; unlike [Ho-Os1] we only consider planar degenerate minimal surfaces. We rewrite one of their results

Proposition 10. *([Ho-Os1], Lemma 4.5) Let $F : \mathbb{C} \longrightarrow \mathbb{R}^4$ be a degenerate minimal surface. Then there exists an orthonormal basis of $\mathbb{R}^4$ w.r.t. which we can write F as*

$$z \mapsto (P(z) + \bar{\lambda}\bar{P}(z), u(z) + \bar{v}(z)) \tag{32}$$

where P, u and v are holomorphic functions such that

$$\lambda P'(z)^2 + u'(z)v'(z) = 0 \tag{33}$$

If moreover we assume $F(\Sigma)$ to be of finite total curvature and complete, the functions P', u' and v' are polynomials.

Proposition 11. *Let $F : \mathbb{C} \longrightarrow \mathbb{R}^4$ be a degenerate minimal map as inProp. 10. We denote by P_0 the plane generated by the first two coordonates in (32) and let P_T be the tangent plane at infinity to $F(\mathbb{C})$. If F is an embedding, then P_0 and P_T are not transverse planes.*

Proof. If $|\lambda| = 1$, then $F(\mathbb{C})$ is a minimal surface inside an Euclidean 3-space of $\mathbb{R}^4$, hence it is a 2-plane. Thus we assume, without loss of generality that $|\lambda| < 1$ and that P_0 and P_T are transverse planes. So we can split $\mathbb{R}^4$ into a product $P_0 \times P_T$ w.r.t. which the end is parametrized

$$z \mapsto (Az^N + B\bar{Z}^N + o(|z|^N), z^q + o(|z|^q)) \tag{34}$$

with $q > N$ and $|A| \neq |B|$. The knot at infinity is a torus knot and Σ is not embedded (cf. Prop. 8). □

Here is an example where the planes are transverse

Exemple 2. *The following map is an immersed degenerate minimal surface*

$$H : z \mapsto (2z^N + \bar{z}^N, -\frac{2N^2}{2N-1}z^{2N-1} + \bar{z}) \tag{35}$$

It has $(N-1)N$ transverse double points, all positive; the braid at infinity K is a $(2N-1, N)$ torus knot and its algebraic length is $e(K) = (2N-2)N$.

Proof. It follows from (18) that H is minimal; to check that the braid at infinity is a $(2N-1, N)$ torus knot, we rewrite the first two coordinates as $(3Re(z^N), Im(z^N))$.
A double point of H is the data of a $\nu \neq 1$, with $\nu^N = 1$ and two complex numbers z_1, z_2 with $z_2 = \nu z_1$ and such that $H(z_1) = H(z_2)$. We write the second component of (35) and derive

$$-\frac{2N^2}{2N-1}z_1^{2N-1} + \bar{z_1} = -\frac{2N^2}{2N-1}\bar{\nu}z_1^{2N-1} + \bar{\nu}\bar{z_1}$$

After simplifying by $\bar{\nu} - 1$, we derive

$$-\frac{2N^2}{2N-1}z_1^{2N-1} = \bar{z}_1 \tag{36}$$

Equation (36) has $2N$ solutions. If we go through all the ν's, we count twice every different value of $\{z_1, z_2\}$ (we get the same double point for ν and for $\bar{\nu}$): in total this gives us $N(N-1)$ double points: this number coincides with $\frac{1}{2}e(K)$, hence we know that all these points are all positive. □

7 Minimal surfaces of small total curvature

In this section we restrict ourselves to planar minimal surfaces as in Cor. 3 and we show how the link at infinity can help us determine when the minimal surface is embedded.

7.1 Embedded minimal surfaces of total curvature -4π

[Ho-Os1] show that if $F : \Sigma \longrightarrow \mathbb{R}^4$ is minimal of total curvature is -4π, then Σ is either $\mathbb{C}$ or $\mathbb{C}\backslash\{0\}$; in both these cases, they give a general formula for the coordinates of F. We investigate when the surface is embedded.
NB. In this section, what we mean by holomorphic is holomorphic for some parallel complex structure J on $\mathbb{R}^4$.

Proposition 12. *A non holomorphic minimal map F from $\mathbb{C}$ and of total curvature -4π can always be written as*

$$F : z \mapsto (\frac{z^3}{3} - a^2 z - \bar{\beta}^2 \bar{z}, \beta \frac{z^2}{2} + \bar{\beta} \frac{\bar{z}^2}{2} + \beta a z - \bar{\beta} \bar{a} \bar{z}) \tag{37}$$

for a, β complex numbers with $\beta \neq 0$.
It is an embedding if and only if

$$\frac{\bar{\beta}}{\beta} \neq \frac{a^2}{\bar{a}^2} \tag{38}$$

If (38) is not true, then $F(\mathbb{C})$ has codimension one self-intersections.

Proof. In [Ho-Os1] we also find a general form for immersed surfaces of curvature -4π which is equivalent to (37). Nevertheless we prove (37) here, so as to fit in with our notations.
We consider the f_i's as in Cor. 3. We assume that f_1 has the highest degree; after a rotation in the first component of $\mathbb{C}^2$, we can assume

$$f_1' = (z-b)(z-c) \qquad f_2' = \lambda$$

By replacing z by $z - \frac{b+c}{2}$, we can rewrite

$$f_1' = (z-a)(z+a) \qquad f_2' = \lambda \tag{39}$$

Without loss of generality, we can assume

$$f_3' = \alpha(z+a) \qquad f_4' = \beta(z-a) \tag{40}$$

We have $\alpha\beta + \lambda = 0$; since the knot at infinity is not a torus knot, we have $|\alpha| = |\beta| = \sqrt{|\lambda|}$. We put $\alpha = Re^{i\gamma_1}$, $\beta = Re^{i\gamma_2}$, after multiplying the second coordinate in $\mathbb{C}^2$ by $e^{i(\frac{\gamma_2-\gamma_1}{2})}$, we assume that

$$\beta = \alpha \tag{41}$$

hence

$$\lambda = -\beta^2 \tag{42}$$

We derive in passing that $\beta \neq 0$.
We let z_1, z_2 be two different numbers such that

$$F(z_1) = F(z_2) \tag{43}$$

We introduce

$$X = z_1 - z_2, \quad Y = z_1 + z_2$$

and we point out that $XY = z_1^2 - z_2^2$; this enables us to rewrite the second component in $\mathbb{C}^2$ of (43)

$$\frac{1}{2}\beta XY + \frac{1}{2}\bar{\beta}\bar{X}\bar{Y} + a\beta X - \bar{a}\bar{\beta}\bar{X} = 0 \tag{44}$$

We notice that the sum of the first two terms (resp. of the last two terms) of (44) is real (resp. imaginary), hence we can rewrite (44) as the following two equalities

$$a\beta X - \bar{a}\bar{\beta}\bar{X} = 0 \ \text{ that is } \quad \bar{X} = \frac{a\beta}{\bar{a}\bar{\beta}}X \tag{45}$$

$$\beta XY + \bar{\beta}\bar{X}\bar{Y} = 0 \tag{46}$$

If we plug (45) into (46), we get

$$\bar{Y} = -\frac{\bar{a}}{a}Y \tag{47}$$

We now let the reader check that

$$z_1^3 - z_2^3 = \frac{X}{4}(3Y^2 + X^2)$$

which enables us to rewrite the first component of $F(z_1) = F(z_2)$ as

$$\frac{X}{12}(3Y^2 + X^2) - a^2X - \bar{\beta}^2\bar{X} = 0. \tag{48}$$

We plug (45) into (48), simplify by X and derive

$$\frac{1}{12}(3Y^2 + X^2) - a^2 - \frac{a}{\bar{a}}|\beta|^2 = 0 \tag{49}$$

We derive from (45) and (47) that

$$X^2 = \frac{\bar{a}\bar{\beta}}{a\beta}|X|^2 \qquad Y^2 = -\frac{a}{\bar{a}}|Y|^2 \tag{50}$$

We let $a = |a|e^{iu}$ and rewrite (49) using (50)

$$\frac{1}{12}(-3e^{2iu}|Y|^2 + \frac{\bar{a}\bar{\beta}}{a\beta}|X|^2) - |a|^2e^{2iu} - e^{2iu}|\beta|^2 = 0 \tag{51}$$

The equation (51) has a solution if an only if

$$\frac{\bar{a}\bar{\beta}}{a\beta} = e^{2iu} = \frac{a}{\bar{a}} \tag{52}$$

We recognize the inverse of (38). If (38) is verified, we can rewrite (53) as

$$-3|Y|^2 + |X|^2 = 12(|a|^2 + |\beta|^2) \tag{53}$$

Thus $|X|$ and $|Y|$ belong to a hyperbola $\mathcal{H}$ in $\mathbb{R}^2$: for every point in $\mathcal{H}$, the equations yield four values of the type $((X,Y)$, $(-X,Y)$, $(X,-Y)$ and $(-X,-Y))$. They correspond in turn to two double points of F (which are different except if $Y = 0$). □

The following minimal surface has also curvature -4π:

Exemple 3. *The image of the map*

$$\mathbb{C} \longrightarrow \mathbb{R}^4$$

$$z \mapsto (z + \bar{z}^3, z^2 + \frac{3}{4}\bar{z}^2)$$

is an immersed minimal surface with two transverse double points. Its knot at infinity is the $(2,3)$ *torus knot and it is not holomorphic for any parallel complex structure.*

Proposition 13. *A non holomorphic minimal immersion of total curvature -4π from $\mathbb{C}\backslash\{0\}$ can always be put in the form*

$$F : z \mapsto (az + b\bar{z} + \frac{c}{z}, \alpha \ln z + \alpha \ln \bar{z} + \beta z - \bar{\beta}\bar{z}) \tag{54}$$

where α is real and $\alpha \neq 0$, $b \neq 0$

$$a\bar{b} = \beta^2, c\bar{b} = \alpha^2 \tag{55}$$

It is always an embedding.

Proof. We derive the expression (54) as [Ho-Os1]; we get the identities (55) by using (18).
Let z_1 and z_2 be two different complex numbers such that

$$F(z_1) = F(z_2) \tag{56}$$

We derive from the second component of (56) that $\ln(|z_1|) = \ln(|z_2|)$, hence

$$|z_1| = |z_2| \tag{57}$$

Hence we write $z_1 = \rho e^{i\theta_1}$ and $z_2 = \rho e^{i\theta_2}$.
We now let $\beta = |\beta|e^{i\gamma}$ and rewrite $Im(\beta z_1) = Im(\beta z_2)$ as

$$\sin(\gamma + \theta_1) = \sin(\gamma + \theta_2) \tag{58}$$

hence

$$\gamma + \theta_1 = (2n + 1)\pi - (\gamma + \theta_2)$$

for some integer n; hence

$$z_2 = \eta \bar{z}_1 \tag{59}$$

where

$$\eta = -e^{-2i\gamma} = -\frac{\bar{\beta}}{\beta} \tag{60}$$

We plug (59) into the first component of (56) and get

$$az_1 + b\bar{z}_1 + \frac{c}{z_1} = a\eta\bar{z}_1 + b\bar{\eta}z_1 + \frac{c\bar{\eta}}{\bar{z}_1}$$

After multiplying by $z_1\bar{z}_1$, we derive

$$z_1[a|z_1|^2 - b|z_1|^2\bar{\eta} - c\bar{\eta}] = \bar{z}_1[a\eta|z_1|^2 - b|z_1|^2 - c] \tag{61}$$

$$= \eta\bar{z}_1[a|z_1|^2 - b|z_1|^2\bar{\eta} - c\bar{\eta}] \tag{62}$$

If $a|z_1|^2 - b|z_1|^2\bar{\eta} - c\bar{\eta} \neq 0$, we derive that $\eta\bar{z}_1 = z_1$; in turn this implies that $z_1 = z_2$ (cf. (59). Hence $a|z_1|^2 - b|z_1|^2\bar{\eta} - c\bar{\eta} = 0$ and

$$|z_1|^2 = \frac{c\bar{\eta}}{a - \bar{\eta}b} \tag{63}$$

$$= \frac{c}{a\eta - b} = \frac{\alpha^2}{\bar{b}[a\eta - b]} = \frac{\alpha^2}{\bar{b}a(-\frac{\bar{\beta}}{\beta}) - \bar{b}b} = \frac{\alpha^2}{-\bar{\beta}\beta - \bar{b}b}.$$

Since α is real, this is impossible. □

The curvature -4π is the only case where we are able to get a complete classification of embedded surfaces. For larger total curvature, we only get a couple of partial results which we present now.

7.2 Total curvature -6π

We refer the reader to [F-M] or [Bu-Zi] (among many other possible references) for material about concordant knots and slice knots. The following should be clear:

Proposition 14. *Let $F : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{R}^4$ be a minimal embedding such that $F(\Sigma)$ is complete and of total finite curvature. Then the two knots at infinity are concordant.*

We derive

Proposition 15. *Let $F : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{R}^4$ be a minimal surface of total curvature -6π. If F is embedded and not holomorphic, then the two tangent planes at infinity are not transverse.*

Proof. We have $d_+ + d_- = 3$. Since F is not holomorphic neither of d_+ or d_- is zero and we derive

$$|d_+ - d_-| = 1 \tag{64}$$

We denote by K_1 (resp. K_2) the knot at infinity in the neighbourhood of 0 (resp. infinity). Without loss of generality, we assume that F is equivalent to z^2 (resp. $\frac{1}{z}$) near infinity (resp. near 0). It follows that K_1 is trivial and K_2 is a knot represented by a braid with 2 strings. This braid is a σ^k for some integer k; if $k > 1$, then K_2 is a torus knot and if $k = \pm 1$, then K_2

is trivial. The knots K_1 and K_2 are concordant, hence K_2 is slice: thus it cannot be a torus knot and $e(K_2) = 1$.
We denote by P_1 (resp. P_2) the tangent plane at infinity in the neighbourhood of 0 (resp. infinity) and we let X be a vector in $\mathbb{R}^4$ which does not belong to either P_1 or P_2. Using Cor. 1, we derive

$$|d_+ - d_-| = |e(K_1) + e(K_2) \pm 4| = |\pm 1 + 0 + \pm 4| \geq 3$$

which contradicts (64). □

By contrast, the reader can check using (18) that

Exemple 4. *The following map from $\mathbb{C}\backslash\{0\}$ to $\mathbb{R}^4$ is minimal*

$$z \mapsto (z^2 + \ln z + \ln \bar{z}, 2z - \bar{z} + \frac{1}{2\bar{z}}) \tag{65}$$

The tangent planes at infinity in Prop. 4 are transverse which implies that the surface has self-intersections.

7.3 Total curvature -8π

In the previous cases, we have found obstructions to embeddedness by considering the writhe number of the knot. We present here a situation where it is the topology of the knot at infinity that yields the obstruction. First we state the obvious

Proposition 16. *Let $F : \mathbb{C} \longrightarrow \mathbb{R}^4$ be a minimal map such that $F(\Sigma)$ is complete, embedded and has finite total curvature. Then the knot at infinity is slice.*

We recall

Theorem 2. *([F-M]) The Alexander polynomial of a slice knot must be of the form $p(t)p(\frac{1}{t})$ for some integral polynomial $p(t)$.*

We derive from Prop. 16 and Th. 2

Proposition 17. *Let Σ be a complete minimal surface of genus 0 with a single end and of total curvature -8π given by*

$$z \mapsto (z^5 + P(z) + \bar{Q}(z), Az^4 + B\bar{z}^4 + Cz^3 + D\bar{z}^3 + o(|z|^3)) \tag{66}$$

where $|A| = |B|$*, P and Q are holomorphic polynomials of degree smaller than* 5.
For a generic $(C, D) \in \mathbb{C}^2$*, the surface is not embedded.*

REMARK. The condition $|A| = |B|$ is necessary for Σ to be an embedding.

Proof. If $A = B = 0$, we need $|C| = |D|$, otherwise the knot at infinity would be the $(3, 5)$-torus knot.
We now assume that $A \neq 0$. Possible after a change of coordinates, the end is parametrized by

$$re^{i\theta} \mapsto (r^5 e^{5i\theta} + o(r^5), r^4 \cos 4\theta + o(r^4), r^3 \cos(3\theta + \alpha) + o(r^3)) \tag{67}$$

For a generic C, D, i.e. a generic α, the truncated function

$$e^{i\theta} \mapsto (e^{5i\theta}, \cos 4\theta, \cos(3\theta + \alpha)) \tag{68}$$

is injective hence (68) is enough to define the knot at infinity.
We recognize get knots similar to the ones studied in [So-Vi]. In that paper, we investigated branch points of minimal surfaces in $\mathbb{R}^4$; if a disk around such a branch point p is embedded, we intersect it with a small sphere in $\mathbb{R}^4$ centered at p. We called them *minimal knots*; the simplest ones, which we called *simple minimal knots* are knots in the cylinder given by

$$e^{i\theta} \mapsto (e^{Ni\theta}, \cos p\theta, \cos(q\theta + \alpha)) \tag{69}$$

where N, p, q are integers, with $q > N$, $p > N$ and $(N, q) = (N, p) = 1$. Despite the fact that in [So-Vi] N is smaller than the other two integers and here it is larger, some facts from that paper apply to the knot (68): in particular, up to mirror symmetry, the knot type of (68) does not depend on the phase α.
We use the formulae in [So-Vi] to compute a representation of one of the knots (68) as a braid with 5 strings and derive

$$\beta = \sigma_4 \sigma_2^{-1} \sigma_1 \sigma_3^{-1} \sigma_2^{-1} \sigma_4 \sigma_3 \sigma_1^{-1} \sigma_2 \sigma_4^{-1} \sigma_1^{-1} \sigma_3 \tag{70}$$

To get the Alexander polynomial of β, we use the software [B-F] developped by Andrew Bartholomew and Roger Fenn and we derive for the Alexander polynomial $A(t)$ of β:

$$A(t) = t^2 - 2t + 3 - \frac{2}{t} + 1 \tag{71}$$

It is clear that $A(t)$ does not verify the property of Th. 2, hence the knot represented by β is not slice. This, together with Prop. 16 concludes the proof.

□

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