The computation of wavelet-Galerkin three-term connection coefficients on a bounded domain

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Abstract

Computation of triple product integrals involving Daubechies scaling functions may be necessary when using the wavelet-Galerkin method to solve differential equations involving nonlinearities or parameters with field variable dependence. Numerical algorithms for determining these triple product integrals, known as three-term connection coefficients, exist but tend to suffer from ill-conditioning. A more stable numerical solution algorithm is presented herein and shown to be both accurate and robust.

Keywords

three-term connection coefficient — wavelet-Galerkin method — triple product integral — numerical method

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1. Introduction

The use of Daubechies wavelet families as Galerkin basis functions for solving differential equations is of growing interest [1, 2]. These oscillatory functions have compact support which allow sparse representation of complex responses on unbounded, bounded or periodic domains [3, 4, 5, 6, 7, 8, 9, 10, 11]. For the discrete orthogonal wavelet-Galerkin method, Daubechies scaling functions are commonly used as the functional basis [5, 6, 11]. The Galerkin formulation for equations containing nonlinearities requiring the product of the field variable with itself or its derivative require the integration of a scaling function triple product. The Daubechies scaling functions cannot be defined explicitly making analytic integration intractable, and their fractal nature (i.e. discontinuities which are independent of scale) make numerical integration error prone [12].

Innovative work by Chen *et al.* [6, 13], Latto *et al.* [8], and Romine *et al.* [11] provide algorithms to compute the exact solution to the three-term connection coefficients on a bounded domain. In each of these references the authors solve for the connection coefficients using a set of rank deficient scaling equations defined recursively using the two-scale definition of the scaling functions. The rank deficiency is filled by replacing a corresponding number of equations with theoretically independent moment equations, allowing determination of a unique set of connection coefficients. Due to the numerical error introduced during implementation of these algorithm, the scaling equations and moment equations are generally no longer independent. This has been found to lead to ill-conditioning of the system and calculation of erroneous connection coefficients. A novel algorithm is presented herein which can account for this numerical error by solving for a set of connection coefficients which satisfy all the constraining equations in a least-squares sense.

In Section 2 a brief review of Daubechies wavelet notation is included and key references which contain derivations of some necessary parameters are cited. Section 3 details the proposed method of computing the three-term connection coefficients, included an example calculation. The results are compared with existing coefficients found in the literature to validate the method. Conclusions are presented in Section 4.

2. Daubechies Wavelet Notation

The Daubechies scaling function is defined by a set of L filter coefficients $p_\ell : \ell \in [0, L-1]$, where L is an even integer. The fundamental two-scale equation is defined as

$$\phi(x) = \sum_{\ell=0}^{L-1} p_\ell \phi(2x - \ell) \tag{1}$$

where $\phi(x)$ is the scaling function with fundamental support over the finite intervals $[0, L-1]$. The filter coefficients p_ℓ are derived by imposing a number of constraints given by Daubechies [7].

It is useful to define the n^{th} derivative of the scaling function as $\phi^{(n)}(x)$, where

$$\phi^{(n)}(x) = \frac{\mathrm{d}^n \phi}{\mathrm{d}x^n}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\phi^{(n-1)}(x), \quad \phi^{(0)}(x) = \phi(x). \tag{2}$$

By amalgamating Eqs. (1) and (2) and accounting for the conditions in Ref [7], it is possible to write [6]

$$\phi^{(n)}(x) = 2^n \sum_{\ell=0}^{L-1} p_\ell \phi^{(n)}(2x - \ell), \quad n = 0, 1, \ldots, L/2 - 1. \tag{3}$$

It is also useful to define the inner product of the scaling function and its derivative over a bounded interval

$$\Gamma_k^n(x) = \int_0^x \phi(y)\phi^{(n)}(y-k)\mathrm{d}y. \tag{4}$$

The solutions $\Gamma_k^n(x)$ are known as the two-term connection coefficients [6]; these coefficients are required in the next section when determining three-term connection coefficients. One algorithm for computing these two-term connection coefficients was derived by Chen *et al.* [6], with corrections presented by Zhang *et al.* [13].

3. Evaluation of the three-term connection coefficients

The three-term connection coefficients over a bounded domain are defined as follows [6]

$$\Omega_{j,k}^{m,n}(x) = \int_0^x \phi(y)\phi^{(m)}(y-j)\phi^{(n)}(y-k)\mathrm{d}y \tag{5}$$

for $0 \le m, n \le (L/2 - 1)$ and $j, k, m, n, x \in \mathbf{Z}$, with the following properties

$$\Omega_{j,k}^{m,n}(x) = 0 \qquad \text{for } |j|, |k|, \text{ or } |j-k| \ge L-1 \tag{6}$$

$$\Omega_{j,k}^{m,n}(x) = 0 \qquad \text{for } x-j, x-k, \text{ or } x \le 0 \tag{7}$$

$$\Omega_{j,k}^{m,n}(x) = \Omega_{j,k}^{m,n}(L-1) \qquad \text{for } x-j, x-k, \text{ or } x \ge L-1. \tag{8}$$

Substituting the two-scale relations (1) and (3) into Eq. (5) and performing a change of variable gives

$$\Omega_{j,k}^{m,n}(x) = 2^{m+n-1} \sum_{i_a=0}^{L-1} \sum_{i_b=0}^{L-1} \sum_{i_c=0}^{L-1} p_{i_a} p_{i_b} p_{i_c} \Omega_{2j+i_b-i_a, 2k+i_c-i_a}^{m,n}(2x - i_a). \tag{9}$$

Accounting for the constraints provided by Eqs. (6) to (8), the scaling equations (9) can be written in matrix form as

$$2^{(1-m-n)}\tilde{\mathbf{\Omega}}^{m,n}(x) = \mathbf{S}\tilde{\mathbf{\Omega}}^{m,n}(x) \tag{10}$$

for $x = 1, 2, \ldots, L-1$, where $\mathbf{S}$ has entries compiled from summing the relevant triple products $p_{i_a} p_{i_b} p_{i_c}$ as defined in Eq. (9). This implies the connection coefficient vector $\tilde{\mathbf{\Omega}}^{m,n}(x)$ belongs to the eigenspace corresponding to the eigenvalue $2^{(1-m-n)}$ from Eq. (10). The connection coefficient vector is of the form [6, 13]

$$\tilde{\mathbf{\Omega}}^{m,n}(x) = \left[\mathbf{\Omega}^{m,n}(1), \mathbf{\Omega}^{m,n}(2), \cdots, \mathbf{\Omega}^{m,n}(L-1)\right]^T \tag{11}$$

$$\mathbf{\Omega}^{m,n}(x) = \begin{cases} \left[\mathbf{\Omega}^{m,n}_{x-L+2}(x), \mathbf{\Omega}^{m,n}_{x-L+3}(x), \ldots, \mathbf{\Omega}^{m,n}_{x-1}(x)\right]^T & \text{for } x = 1, 2, \cdots, L-2 \\ \left[\mathbf{\Omega}^{m,n}_{2-L}(x), \mathbf{\Omega}^{m,n}_{3-L}(x), \ldots, \mathbf{\Omega}^{m,n}_{L-2}(x)\right]^T & \text{for } x = L-1 \end{cases} \tag{12}$$

$$\mathbf{\Omega}^{m,n}_{j}(x) = \begin{cases} \left[\Omega^{m,n}_{j,x-L+2}(x), \Omega^{m,n}_{j,x-L+3}(x), \cdots, \Omega^{m,n}_{j,x-1}(x)\right]^T & \text{for } x = 1, 2, \ldots, L-2 \\ \left[\Omega^{m,n}_{j,\nu}(x), \Omega^{m,n}_{j,\nu+1}(x), \cdots, \Omega^{m,n}_{j,\mu}(x)\right]^T & \text{for } x = L-1 \end{cases} \tag{13}$$

where $\nu = \max(j+2-L, 2-L)$, $\mu = \min(j+L-2, L-2)$. The vector $\tilde{\mathbf{\Omega}}^{m,n}(x)$ contains $(L-2)^3$ unknowns for $x \in [1, L-2]$ and $3L^2 - 9L + 7$ unknowns for $x = L-1$. It can be shown the matrix $\mathbf{S}$ has q eigenvalues equal to $2^{(1-m-n)}$, where [6]

$$q = (m+n+1) + \begin{cases} \displaystyle\sum_{i=1}^{m+n} i & \text{if } m+n \leq L/2 \\ \displaystyle\sum_{i=L/2+1}^{m+n} \left(\frac{3L}{2} - 2i\right) + \frac{(L+2)L}{8} & \text{if } L/2 < m+n \leq L-2. \end{cases} \tag{14}$$

The eigenvectors corresponding to these q eigenvalues describe the solution space of the scaling equations (10) for a given m and n. In fact, since the scaling equations depend only on the summation $(m+n)$ and not the specific derivatives, this set of eigenvectors gives the scaling equation solution space for all three-term connection coefficients whose derivatives sum to $(m+n)$ [8].

The unique solution for derivatives m and n is found by considering the set of moment equations which are derived in Ref [7]; the derivation can be found in Refs. [6, 13]:

$$\sum_k k^n \Omega^{m,n}_{j,k}(x) = n!\, \Gamma^m_j(x) \tag{15}$$

$$\sum_j j^m \Omega^{m,n}_{j,k}(x) = m!\, \Gamma^n_k(x). \tag{16}$$

Thus $\tilde{\mathbf{\Omega}}^{m,n}(x)$ is uniquely described by the intersection of the scaling equation solution space with that of the moment equations. This implies the solution must be a linear combination of the q eigenvectors; the participation factors can be computed from the moment equations as detailed below.

3.1 Example calculation

Consider the specific case of $m = 0, n = 1$. Eq. (14) states $q = 3$ eigenvectors describe the solution space to the scaling equations, thus

$$\tilde{\mathbf{\Omega}}^{m,n}(x) = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} \tag{17}$$

where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the eigenvectors corresponding to the eigenvalue $2^{(1-m-n)} = 1$ from Eq. (10); the constants c_1, c_2 and c_3 are the respective participation factors to be determined. Substituting Eq. (17) into the moment equations results in

$$c_1 \sum_k k^n \mathbf{u} + c_2 \sum_k k^n \mathbf{v} + c_3 \sum_k k^n \mathbf{w} = n!\, \Gamma^m_j(x) \qquad \forall j \in [2-L, L-2] \tag{18}$$

$$c_1 \sum_j j^m \mathbf{u} + c_2 \sum_j j^m \mathbf{v} + c_3 \sum_j j^m \mathbf{w} = m!\, \Gamma^n_k(x) \qquad \forall k \in [2-L, L-2] \tag{19}$$

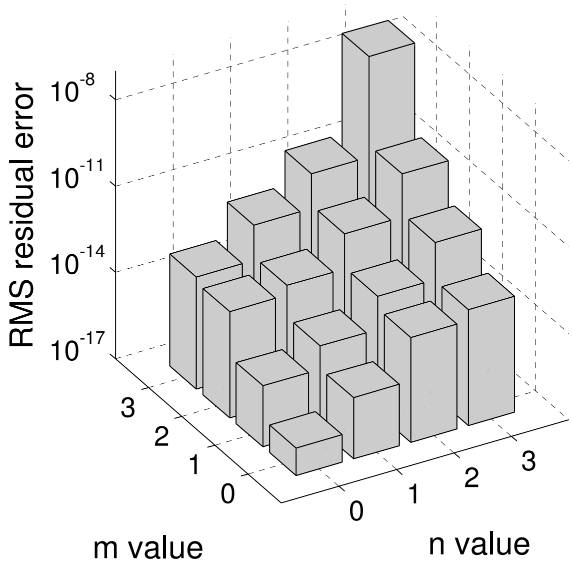

Figure 1. RMS of the residual error from all moment equations

for $x = 1, 2, \ldots, L-1$. Note there are only $g = 3/2L^2 - 7/2L + 2$ non-trivial equations described by each (18) and (19) due to the constraints of Eqs. (6) to (8).

These moment equations can be written in matrix form as

$$\mathbf{M} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \mathbf{b} \tag{20}$$

where $\mathbf{M}$ is a rectangular matrix of size $2g \times 3$, comprised of the summation terms on the lefthand side of Eqs. (18) and (19), and $\mathbf{b}$ is a vector of length $2g$ composed of the respective righthand side terms. The participation factors are determined using a pseudoinverse

$$\mathbf{c} = (\mathbf{M}^T \mathbf{M})^{-1} \mathbf{M}^T \mathbf{b}. \tag{21}$$

This gives a robust "best fit" in a least-squares sense, which allows for any dependencies between the scaling and moment equations resulting from accumulated numerical error. The unique solution of the three-term connection coefficients can thus be found by substituting the participation factors found in Eq. (21) into Eq. (17). The algorithm is analogous for different values of m and n.

The accuracy of the computed $\tilde{\mathbf{\Omega}}^{m,n}(x)$ vector is quantified by first calculating the residual of each moment equation given in Eqs. (15) and (16); as $\tilde{\mathbf{\Omega}}^{m,n}(x)$ is a linear combination of the eigenvectors from Eq. (10), the scaling equations are automatically satisfied. To allow meaningful comparison of different derivative combinations, the residuals are normalized by the $L2$-norm of the righthand-side of the moment equation. The RMS value of this normalized residual vector is then calculated to give a scalar measure of the absolute error for a given n and m combination. This error measure for Daubechies scaling functions ($L = 8$) at all allowable derivative combinations is provided in Fig. 1. As shown, the error in $\tilde{\mathbf{\Omega}}^{m,n}(x)$ grows with higher derivatives, but even at $n, m = 3$ the error norm remains relatively low. As a comparison, the three-term connection coefficients for $L = 6, m = 0, n = 1$ published in Chen *et al.* [6] result in an RMS error norm of 1.55e-10, whereas for the current algorithm it is 1.82e-16. The three-term connection coefficients are tabulated in the Appendix for comparison.

4. Conclusions

Computation of the three-term connection coefficients is necessary when using the wavelet-Galerkin method to solve differential equations involving nonlinearities or parameters with variable dependence. Algorithms currently exist to solve for these coefficients but they have been found to suffer from ill-conditioning which can result in erroneous results. The current investigation introduces a novel solution algorithm which appears to be more numerically robust and comparatively more accurate than previously published algorithms.

A. Appendix

Table 1 lists the 125 three-term connection coefficients for $L = 6$, $m = 0$, $n = 1$, $\tilde{\mathbf{\Omega}}^{0,1}(x)$, computed using the algorithm described above.

Table 1. Three-term connection coefficients for $L = 6$, $m = 0$, $n = 1$

x	j	k	$\Omega^{0,1}_{j,k}(x)$	x	j	k	$\Omega^{0,1}_{j,k}(x)$	x	j	k	$\Omega^{0,1}_{j,k}(x)$
1	-3	-3	-7.15591411E-04	3	1	1	3.17107026E-01	5	-1	-2	8.74622766E-02
1	-3	-2	1.35502680E-03	3	1	2	-1.33459578E-01	5	-1	-1	-2.48445901E-01
1	-3	-1	7.87118799E-06	3	2	-1	5.29897817E-03	5	-1	0	3.13587753E-01
1	-3	0	-6.14851011E-04	3	2	0	-1.15661098E-02	5	-1	1	-1.25272546E-01
1	-2	-3	5.29897817E-03	3	2	1	2.54926579E-02	5	-1	2	-6.17732216E-03
1	-2	-2	-1.15661098E-02	3	2	2	-1.93594670E-02	5	-1	3	3.29019694E-05
1	-2	-1	2.54926579E-02	4	0	0	2.53068293E-08	5	0	-4	2.45926496E-06
1	-2	0	-1.93594670E-02	4	0	1	4.96892129E-01	5	0	-3	2.37636417E-03
1	-1	-3	-2.24262187E-02	4	0	2	3.29019100E-02	5	0	-2	9.35483714E-02
1	-1	-2	6.06841495E-02	4	0	3	1.45424640E-03	5	0	-1	-6.27175506E-01
1	-1	-1	-1.75635286E-01	4	1	0	-2.48445211E-01	5	0	0	3.61020792E-15
1	-1	0	1.37823004E-01	4	1	1	3.13595282E-01	5	0	1	4.96891801E-01
1	0	-3	8.23609549E-03	4	1	2	-1.25290082E-01	5	0	2	3.29026887E-02
1	0	-2	1.96354133E-01	4	1	3	-6.13510292E-03	5	0	3	1.45229718E-03
1	0	-1	-9.14074187E-01	4	2	0	-1.64544140E-02	5	0	4	1.52412730E-06
1	0	0	7.09481500E-01	4	2	1	3.77788872E-02	5	1	-3	-4.45648096E-04
2	-2	-2	-1.64544140E-02	4	2	2	-4.67007112E-02	5	1	-2	-2.07415149E-02
2	-2	-1	3.77788872E-02	4	2	3	2.03167757E-02	5	1	-1	8.74622766E-02
2	-2	0	-4.67007112E-02	4	3	0	-7.15591411E-04	5	1	0	-2.48445901E-01
2	-2	1	2.03167757E-02	4	3	1	1.35502680E-03	5	1	1	3.13587753E-01
2	-1	-2	8.73555499E-02	4	3	2	7.87118799E-06	5	1	2	-1.25272546E-01
2	-1	-1	-2.49815835E-01	4	3	3	-6.14851011E-04	5	1	3	-6.17732216E-03
2	-1	0	3.17107026E-01	5	-4	-4	-7.62063649E-07	5	1	4	3.29019694E-05
2	-1	1	-1.33459578E-01	5	-4	-3	-4.46399112E-07	5	2	-2	1.33940712E-04
2	0	-2	9.40317902E-02	5	-4	-2	1.94935669E-06	5	2	-1	4.92552163E-03
2	0	-1	-6.20031558E-01	5	-4	-1	4.88738554E-07	5	2	0	-1.64513443E-02
2	0	0	-1.91465368E-02	5	-4	0	-1.22963248E-06	5	2	1	3.78102693E-02
2	0	1	5.42767481E-01	5	-3	-4	-3.24555703E-05	5	2	2	-4.67741857E-02
2	1	-2	-2.24262187E-02	5	-3	-3	-7.26148591E-04	5	2	3	2.04916885E-02
2	1	-1	6.06841495E-02	5	-3	-2	1.25180052E-03	5	2	4	-1.35890068E-04
2	1	0	-1.75635286E-01	5	-3	-1	2.49826364E-04	5	3	-1	-3.24555703E-05
2	1	1	1.37823004E-01	5	-3	0	-1.18818208E-03	5	3	0	-7.26148591E-04
3	-1	-1	-2.48445211E-01	5	-3	1	4.45159357E-04	5	3	1	1.25180052E-03
3	-1	0	3.13595282E-01	5	-2	-4	1.33940712E-04	5	3	2	2.49826364E-04
3	-1	1	-1.25290082E-01	5	-2	-3	4.92552163E-03	5	3	3	-1.18818208E-03
3	-1	2	-6.13510292E-03	5	-2	-2	-1.64513443E-02	5	3	4	4.45159357E-04
3	0	-1	-6.27152133E-01	5	-2	-1	3.78102693E-02	5	4	0	-7.62063649E-07
3	0	0	2.88213076E-04	5	-2	0	-4.67741857E-02	5	4	1	-4.46399112E-07
3	0	1	4.96129851E-01	5	-2	1	2.04916885E-02	5	4	2	1.94935669E-06
3	0	2	3.48068749E-02	5	-2	2	-1.35890068E-04	5	4	3	4.88738554E-07
3	1	-1	8.73555499E-02	5	-1	-4	-4.45648096E-04	5	4	4	-1.22963248E-06
3	1	0	-2.49815835E-01	5	-1	-3	-2.07415149E-02				

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