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Observer synthesis for a class of MIMO non triangular nonlinear systems

M. Triki^{1,2}, M. Farza¹, M. M'Saad¹, T. Maatoug², B. Dahhou^{3,4}

Abstract—A high gain observer is proposed for a class of nonlinear systems involving some uncertainties. In the absence of the uncertainties, the considered class of systems becomes a canonical form that characterizes a class of uniformly observable systems and the observation error exponentially converges to zero. In the presence of uncertainties, it is shown that the observation error can be made as small as desired by appropriately specifying the design parameter of the observer gain. Two main contributions are worth to be emphasized: the first one is related to the structure of the considered class of systems which does not assume a complete triangular structure. That is, each block may contain nonlinearities which depend on the whole state. The second one lies in the simplicity of the observer gain design since its calibration is reduced to the choice of a single design parameter. More specifically, it involves a design function that has to satisfy a mild condition which is given. Different expressions of such a function are proposed. Of particular interest, it is shown that high gain observers and sliding mode like observers can be derived by considering particular expressions of the design function. An example with simulation results is given for illustration purposes.

keyword : Nonlinear systems. High gain observers. Sliding mode observers. MIMO systems.

I. INTRODUCTION

The need to study the observer design problem for nonlinear dynamical systems is, from a control point of view, well understood by now. The list of references herein covers part of the recent works done in that area ([11], [12], [13], [17], [10], [8], [16], [14], [3], [1], [6], [7], [15], [4], [9]). In this paper, one aims at designing a high gain observer for a class of nonlinear uncertain systems satisfying some regularity assumptions. The general framework of this observer design is based on the contributions given in [2], [6], [9]. The gain of the proposed observer is issued from the resolution of a constant Lyapunov algebraic equation and it is explicitly given. Its tuning is achieved through the choice of a single parameter whatever the dimension of the considered system is.

More specifically, one exhibits a state observer for nonlinear systems which are diffeomorphic to:

$$\begin{cases} \dot{x} = Ax + \varphi(u, x) + \bar{\varepsilon}(t) \\ y = Cx \end{cases} \quad (1)$$

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where the state $x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix} \in \mathbb{R}^n$, with $x^k =$

$\begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}$, $x_i^k \in \mathbb{R}^{p_k}$, $i = 1, \dots, \lambda_k$, $k = 1, \dots, q$, $\sum_{k=1}^q n_k = \sum_{k=1}^q p_k \lambda_k = n$ with $p_k \geq 1$ and $\lambda_k \geq 2$; the

output $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{pmatrix} \in \mathbb{R}^p$ with $y_k \in \mathbb{R}^{p_k}$, $k =$

$1, \dots, q$ and $\sum_{k=1}^q p_k = p$; $A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_q \end{bmatrix}$, $A_k =$

$\begin{bmatrix} 0 & I_{p_k} & & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & I_{p_k} \\ 0 & \dots & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_q \end{bmatrix}$,

$C_k = [I_{p_k} \ 0 \ \dots \ 0]$ and the nonlinear function

field $\varphi(u, x) = \begin{pmatrix} \varphi^1(u, x) \\ \varphi^2(u, x) \\ \vdots \\ \varphi^q(u, x) \end{pmatrix} \in \mathbb{R}^n$; $\varphi^k(u, x) =$

$\begin{pmatrix} \varphi_1^k(u, x) \\ \varphi_2^k(u, x) \\ \vdots \\ \varphi_{\lambda_k}^k(u, x) \end{pmatrix} \in \mathbb{R}^{n_k}$ where for $k = 1, \dots, q$, the element

$\varphi_i^k(u, x) \in \mathbb{R}^{p_k}$ has the structural dependence on the states:

- for $1 \leq i \leq \lambda_k - 1$:

$$\varphi_i^k(u, x) = \varphi_i^k(u, x^1, x^2, \dots, x^{k-1}, x_1^k, x_2^k, \dots, x_i^k, x_1^{k+1}, x_1^{k+2}, \dots, x_1^q) \quad (2)$$

- for $i = \lambda_k$:

$$\varphi_{\lambda_k}^k(u, x) = \varphi_{\lambda_k}^k(u, x^1, x^2, \dots, x^q) \quad (3)$$

The function $\bar{\varepsilon}(t)$ is a completely unknown function. Many components of this function are zero since its structure can be described as follows:

$$\bar{\varepsilon}(u, x) = \begin{pmatrix} \bar{\varepsilon}^1(t) \\ \bar{\varepsilon}^2(t) \\ \vdots \\ \bar{\varepsilon}^q(t) \end{pmatrix} \in \mathbb{R}^n \text{ with}$$

- $\bar{\varepsilon}^k \equiv 0$ for $k = 2, \dots, q$.
- $\bar{\varepsilon}_i^1 \equiv 0$ for $i = 1, \dots, \lambda_1 - 1$.
- $\bar{\varepsilon}_{\lambda_1}^1 \equiv \varepsilon(t)$ where $\varepsilon(t)$ is an unknown bounded function i.e.

$$\exists \eta > 0; \forall t \geq 0; \|\varepsilon(t)\| \leq \eta \quad (4)$$

Notice that in the absence of uncertainties, the class of considered systems generalizes that considered in [15] in two directions. Firstly, the output x_1^k of each sub-block k is not a scalar as in [15] but belongs to \mathbb{R}^{p_k} . Secondly, unlike in [15] where the nonlinearity intervening in the last equation of each sub-block, namely $\varphi_{\lambda_k}^k(u, x)$, assumes the same triangular state dependence as the previous variables of the same sub-block, i.e. $\varphi_{\lambda_k}^k(u, x)$ satisfies (2) with $i = \lambda_k$, a more general state dependence (3) is assumed in this paper. Moreover, it has been also shown in [9] that in the case where no uncertainty is considered, the class of systems the authors considered is diffeomorphic to system (1) with $q = 1$ which means that in [9], all the output belong to the same block. In the case where the uncertainties are considered, system (1) includes the class of systems considered in ([4]). In fact, many systems involving unknown inputs can be put under form (1) and this allows to design nonlinear observers that simultaneously estimate the state variables as well as the unknown inputs as shown in [5].

This paper is organized as follows. In section 2, the observer design is given and a full convergence analysis is detailed. It is shown that the observation error exponentially converges to zero in the absence of uncertainties, and can be made as small as desired in the presence of uncertainties. The latter result is achieved by appropriately specifying the design parameter involved in the observer gain. In section 3, different expressions of the observer design function are proposed giving rise to different observers. Simulation results are given in section 4 to illustrate the effectiveness of the proposed observer design.

II. OBSERVER DESIGN

As generally used in the high gain observer methodology, one assumes that system (1) satisfies the following Lipschitz assumption:

Assumption 1: $\varphi(u, x)$ is a globally Lipschitz nonlinear function with respect to x uniformly to u .

Before giving a candidate observer, one introduces the following notations:

- Let $\Delta_k(\theta)$ be the diagonal matrix defined by:

$$\Delta_k(\theta) = \text{diag} \left(I_{p_k}, \frac{1}{\theta^{\delta_k}} I_{p_k}, \dots, \frac{1}{\theta^{\delta_k(\lambda_k-1)}} I_{p_k} \right) \quad (5)$$

where $\theta > 0$ is a real number and one defines δ_k which indicates the power of θ as follows:

$$\begin{cases} \delta_k = 2^{q-k} \left(\prod_{i=k+1}^q (\lambda_i - 1) \right) \\ \quad \text{for } k = 1, \dots, q-1 \\ \delta_q = 1 \end{cases} \quad (6)$$

Notice that for any $k = 1, \dots, q-1$, one has

$$\frac{\delta_k}{2} = (\lambda_{k+1} - 1) \delta_{k+1} \quad (7)$$

Since $\lambda_k \geq 2$, the δ_q 's constitute a decreasing sequence of positive real numbers, i.e.

$$\delta_1 \geq \delta_2 \geq \dots \geq \delta_q > 0 \quad (8)$$

It is easy to show that the following identities hold:

$$\begin{aligned} \bullet \Delta_k(\theta) A_k \Delta_k^{-1}(\theta) &= \theta^{\delta_k} A_k \\ \bullet \Delta_k^{-1}(\theta) C_k &= C_k \end{aligned} \quad (9)$$

- Let S_k be the unique solution of the algebraic Lyapunov equation

$$S_k + A_k^T S_k + S_k A_k = C_k^T C_k \quad (10)$$

where A_k and C_k are defined in system (1). It can be shown that the explicit solution of (10) is symmetric positive definite for every $\theta > 0$ and in particular, one has

$$S_k^{-1} C_k^T = (C_{n_k}^1 I_{p_k}, \dots, C_{n_k}^{n_k} I_{p_k})^T$$

- $\forall \xi^k \in \mathbb{R}^{p_k}$, let $K_k(\xi^k) \in \mathbb{R}^{p_k}$ be a vector of smooth functions satisfying:

$$\xi^{kT} K_k(\xi^k) \geq \frac{1}{2} \xi^{kT} \xi^k \quad (11)$$

A candidate observer for system (1) is described by the following dynamical system:

$$\dot{\hat{x}}^k = A_k \hat{x}^k + \varphi^k(u, \hat{x}) - \theta^{\delta_k} \Delta_k^{-1}(\theta) S_k^{-1} C_k^T K_k(C_k e^k) \quad (12)$$

for $k = 1, \dots, q$ with

$$\bullet \hat{x} = \begin{pmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \vdots \\ \hat{x}^q \end{pmatrix} \in \mathbb{R}^n, \hat{x}^k = \begin{pmatrix} \hat{x}_1^k \\ \hat{x}_2^k \\ \vdots \\ \hat{x}_{\lambda_k}^k \end{pmatrix} \in \mathbb{R}^{n_k}, \hat{x}_i^k \in \mathbb{R}^{p_k}, i = 1, \dots, \lambda_k, k = 1, \dots, q, \sum_{k=1}^q n_k = n.$$

- $\hat{x}_1^k = x_1^k$ (output injection) for $k = 1, \dots, q$.
- $\hat{x}_i^k = \hat{x}_i^k$ if $i \neq 1$.
- $e^k = \hat{x}^k - x^k$.
- The function K_k 's satisfy condition (11).
- u and y are known inputs and outputs of system (1).

Indeed, one can establish the following fundamental result.

Theorem 2.1: Assume that system (1) satisfies assumption (1), then:

$\forall M > 0; \exists \theta_0 > 0; \forall \theta \geq \theta_0; \exists \lambda_\theta > 0; \exists \mu_\theta > 0; \exists \beta_\theta > 0$ such that for $k \in \{1, \dots, q\}$: $\|\hat{x}^k(t) - x^k(t)\| \leq \lambda_\theta e^{-\mu_\theta t} \|\hat{x}(0) - x(0)\| + \beta_\theta \eta$ for every admissible control

u s.t. $Essup\|u(t)\| \leq M$ where η is the upper bound of $\|\varepsilon\|$ given in (4). Moreover, λ_θ is polynomial in θ , $\lim_{\theta \rightarrow \infty} \mu_\theta = +\infty$ and $\lim_{\theta \rightarrow \infty} \beta_\theta = 0$.

Proof of Theorem (2.1): set the estimation error $e(t) = \hat{x}(t) - x(t)$ and let $e^k(t)$ be the k 'th subcomponent of $e(t)$. For writing convenience and as long as there is no ambiguity, one shall omit the time t for each variable. One has:

$$\begin{aligned} \dot{e}^k &= A_k e^k + \varphi^k(u, \hat{x}) - \varphi^k(u, x) \\ &- \theta^{\delta_k} \Delta_k^{-1}(\theta) S_k^{-1} C_k^T K_k (C_k e^k) - \bar{\varepsilon}^k \end{aligned} \quad (13)$$

where u is an admissible control such that $\|u\|_\infty \leq M$, $M > 0$ is a given constant. For $k = 1, \dots, q$, set

$$\bar{e}^k = \Delta_k(\theta) e^k \quad (14)$$

From equation (13) and using identities (9), one gets:

$$\begin{aligned} \dot{\bar{e}}^k &= \theta^{\delta_k} A_k \bar{e}^k - \theta^{\delta_k} S_k^{-1} C_k^T K_k (C_k \bar{e}^k) \\ &+ \Delta_k(\theta) (\varphi^k(u, \hat{x}) - \varphi^k(u, x)) - \Delta_k(\theta) \bar{\varepsilon}^k \end{aligned} \quad (15)$$

Set

$$V_k(\bar{e}^k) = \theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \bar{e}^k \quad (16)$$

where the σ_1^k 's are some reals that shall be specified later and let $V(\bar{e}) = \sum_{k=1}^q V_k(\bar{e}^k)$ where $S = \text{diag}(S_1, \dots, S_q)$, be the candidate Lyapunov function. One has:

$$\begin{aligned} \dot{V}_k &= 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \dot{\bar{e}}^k \\ &= 2\theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} S_k A_k \bar{e}^k - 2\theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} C_k^T K_k (C_k \bar{e}^k) \\ &+ 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) (\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &- 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) \bar{\varepsilon}^k \end{aligned}$$

Using the algebraic Lyapunov equation (10), one gets:

$$\begin{aligned} \dot{V}_k &= -\theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} S_k \bar{e}^k + \theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} C_k^T C_k \bar{e}^k \\ &- 2\theta^{\delta_k - 2\sigma_1^k} (C_k \bar{e}^k)^T K_k (C_k \bar{e}^k) \\ &+ 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) (\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &- 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) \bar{\varepsilon}^k \end{aligned}$$

Using property (11) satisfied by the function K_k , one obtains:

$$\begin{aligned} \dot{V}_k &\leq -\theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} S_k \bar{e}^k + \theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} C_k^T C_k \bar{e}^k \\ &- 2\theta^{\delta_k - 2\sigma_1^k} (C_k \bar{e}^k)^T C_k \bar{e}^k \\ &+ 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) (\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &- 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) \bar{\varepsilon}^k \\ &= -\theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} S_k \bar{e}^k - \theta^{\delta_k - 2\sigma_1^k} \bar{e}^{kT} C_k^T C_k \bar{e}^k \\ &+ 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) (\varphi^k(u, \hat{x}) - \varphi^k(u, x)) \\ &- 2\theta^{-2\sigma_1^k} \bar{e}^{kT} S_k \Delta_k(\theta) \bar{\varepsilon}^k \\ &\leq -\theta^{\delta_k} V_k \\ &+ 2\theta^{-2\sigma_1^k} \|S_k \bar{e}^k\| \|\Delta_k(\theta) (\varphi^k(u, \hat{x}) - \varphi^k(u, x))\| \\ &+ 2\theta^{-2\sigma_1^k} \|S_k \bar{e}^k\| \|\Delta_k(\theta) \bar{\varepsilon}^k\| \\ &\leq -\theta^{\delta_k} V_k + \\ &+ 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \sum_{i=1}^{\lambda_k} \frac{1}{\theta^{\sigma_i^k}} \|(\varphi_i^k(u, \hat{x}) - \varphi_i^k(u, x))\| \\ &+ 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\alpha_{k,1}\eta}{\theta^{\sigma_1^k + (\lambda_1 - 1)\delta_1}} \end{aligned}$$

where λ_{max}^k is the maximum eigenvalue of S_k , $\sigma_i^k = \sigma_1^k + (i-1)\delta_k$ and $\alpha_{k,1}$ is the Kronecker symbol, i.e $\alpha_{k,1} = 1$ if $k=1$ and 0 otherwise.

Therefore,

$$\begin{aligned} \dot{V}_k &\leq -\theta^{\delta_k} V_k \\ &+ 2\rho_k \sqrt{\lambda_{max}^k} \sqrt{V_k} \sum_{i=1}^{\lambda_k} \sum_{l=1}^q \sum_{j=1}^{\lambda_l} \chi_{l,j}^{k,i} \theta^{\sigma_j^l - \sigma_i^k} \theta^{-\sigma_1^l} \|\bar{e}_j^l\| \\ &+ 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\alpha_{k,1}\eta}{\theta^{\sigma_1^k + (\lambda_1 - 1)\delta_1}} \end{aligned}$$

where $\rho_k = \sup\left\{ \left\| \frac{\partial \varphi_i^k}{\partial x_j^l}(u, x) \right\| ; x \in \mathbb{R}^n \text{ and } \|u\|_\infty \leq M \right\}$ and

$\chi_{l,j}^{k,i} = 0$ if $\frac{\partial \varphi_i^k}{\partial x_j^l}(u, x) \equiv 0$, $\chi_{l,j}^{k,i} = 1$ otherwise.

Now, one has

$$\begin{aligned} \dot{V}_k &\leq -\theta^{\delta_k} V_k \\ &+ 2\rho_k \sqrt{\lambda_{max}^k} \sqrt{V_k} \sum_{i=1}^{\lambda_k} \sum_{l=1}^q \sum_{j=1}^{\lambda_l} \chi_{l,j}^{k,i} \theta^{\sigma_j^l - \sigma_i^k} \theta^{-\sigma_1^l} \|\bar{e}_j^l\| \\ &+ 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\alpha_{k,1}\eta}{\theta^{\sigma_1^k + (\lambda_1 - 1)\delta_1}} \\ &\leq -\theta^{\delta_k} V_k \\ &+ 2\rho_k \sqrt{\lambda_{max}^k} \sqrt{V_k} \sum_{i=1}^{\lambda_k} \sum_{l=1}^q \sum_{j=1}^{\lambda_l} \chi_{l,j}^{k,i} \theta^{\sigma_j^l - \sigma_i^k} \frac{\sqrt{V_l}}{\sqrt{\lambda_{min}^l}} \\ &+ 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\alpha_{k,1}\eta}{\theta^{\sigma_1^k + (\lambda_1 - 1)\delta_1}} \end{aligned}$$

where λ_{min}^l is the minimum eigenvalue of S_l .
Thus,

$$\begin{aligned} \dot{V}_k &\leq -(\sqrt{\theta^{\delta_k} V_k})^2 + 2\rho_k \mu_S \sqrt{\theta^{\delta_k} V_k} \\ &\quad \sum_{i=1}^{\lambda_k} \sum_{l=1}^q \sum_{j=1}^{\lambda_l} \chi_{l,j}^{k,i} \theta^{\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2}} \sqrt{\theta^{\delta_l} V_l} \\ &\quad + 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\alpha_{k,1}\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \end{aligned}$$

$$\text{where } \mu_S = \sqrt{\frac{\lambda_{max}(S)}{\lambda_{min}(S)}}.$$

Please notice the last term appearing in the right hand side of the last inequality is to be dropped for $k \neq 1$. As a result, the real σ_1^k appearing in the denominator of this term is replaced by σ_1^1 .

The remaining of the proof is as follows. One firstly assumes that the reals σ_1^k , $k = 1, \dots, q$ are chosen such that the following conditions are satisfied:

$$0 = \sigma_1^1 > \sigma_1^2 > \dots > \sigma_1^q \quad (17)$$

$$\text{if } \chi_{l,j}^{k,i} = 1 \text{ then } \sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \leq -\frac{1}{2} \quad (18)$$

$$\sigma_1^k + (\lambda_k - 1)\delta_k \text{ is constant for } k = 1, \dots, q \quad (19)$$

Then, one shall show that such a choice is possible and provides a set of σ_1^k 's satisfying such conditions.

Now, suppose that condition (18) holds and assume that $\theta \geq 1$. Then, one gets:

$$\begin{aligned} \dot{V}_k &\leq -(\sqrt{\theta^{\delta_k} V_k})^2 + 2\rho_k \mu_S \sqrt{\theta^{\delta_k} V_k} \sum_{i=1}^{\lambda_k} \sum_{l=1}^q \sum_{j=1}^{\lambda_l} \theta^{-\frac{1}{2}} \sqrt{\theta^{\delta_l} V_l} \\ &\quad \sqrt{V_k(\bar{e}^k)} \leq \exp\left(\frac{-\theta(1 - 2n^2 \rho \mu_S \theta^{-\frac{1}{2}})}{2} t\right) \sqrt{V(\bar{e}(0))} \\ &\quad + 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\alpha_{k,1}\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \\ &\leq -(\sqrt{\theta^{\delta_k} V_k})^2 + 2\lambda_k \rho_k \mu_S \theta^{-\frac{1}{2}} \sqrt{\theta^{\delta_k} V_k} \sum_{l=1}^q \sum_{j=1}^{\lambda_l} \sqrt{\theta^{\delta_l} V_l} \\ &\quad + 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\alpha_{k,1}\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \end{aligned}$$

Now, set $V_k^* = \theta^{\delta_k} V_k$ for $k = 1, \dots, q$ and $V^* = \sum_{k=1}^q V_k^*$.

Notice that

$$\theta V \leq V^* \leq \theta^{\delta_1} V \quad (20)$$

Then,

$$\begin{aligned} \dot{V}_k &\leq -V_k^* + 2\lambda_k \rho_k \mu_S \theta^{-\frac{1}{2}} \sqrt{V_k^*} \sum_{l=1}^q \sum_{j=1}^{\lambda_l} \sqrt{V_l^*} \\ &\quad + 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \\ &\leq -V_k^* + 2\lambda_k \rho_k \mu_S \theta^{-\frac{1}{2}} \sqrt{V_k^*} \sqrt{V^*} \\ &\quad + 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \\ &\leq -V_k^* + 2\lambda_k \rho_k \mu_S \theta^{-\frac{1}{2}} V^* \\ &\quad + 2\sqrt{\lambda_{max}^k} \sqrt{V_k} \frac{\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \end{aligned}$$

Hence,

$$\begin{aligned} \dot{V} &\leq -(1 - 2n^2 \rho \mu_S \theta^{-\frac{1}{2}}) V^* \\ &\quad + 2\sqrt{\lambda_{max}} \sqrt{V} \frac{\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \end{aligned}$$

where $\rho = \max\{\rho_k, 1 \leq k \leq q\}$ and $\lambda_{max} = \max\{\lambda_{max}^k, 1 \leq k \leq q\}$.

Using (20), one gets

$$\dot{V} \leq -\theta(1 - 2n^2 \rho \mu_S \theta^{-\frac{1}{2}}) V + 2\sqrt{\lambda_{max}} \sqrt{V} \frac{\eta}{\theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}}$$

This leads to:

$$\begin{aligned} \sqrt{V(\bar{e}(t))} &\leq \exp\left(\frac{-\theta(1 - 2n^2 \rho \mu_S \theta^{-\frac{1}{2}})}{2} t\right) \sqrt{V(\bar{e}(0))} \\ &\quad + \frac{2\sqrt{\lambda_{max}} \eta}{\theta(1 - 2n^2 \rho \mu_S \theta^{-\frac{1}{2}}) \theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \quad (21) \end{aligned}$$

and therefore one obtains for $k = 1, \dots, q$:

$$\begin{aligned} \sqrt{V_k(\bar{e}^k)} &\leq \exp\left(\frac{-\theta(1 - 2n^2 \rho \mu_S \theta^{-\frac{1}{2}})}{2} t\right) \sqrt{V(\bar{e}(0))} \\ &\quad + \frac{2\sqrt{\lambda_{max}} \eta}{\theta(1 - 2n^2 \rho \mu_S \theta^{-\frac{1}{2}}) \theta^{\sigma_1^1 + (\lambda_1 - 1)\delta_1}} \quad (22) \end{aligned}$$

On one hand, from the definition of V and V_k (equation (16)), one has

$$\theta^{\sigma_1^k} \frac{\sqrt{V_k(\bar{e}^k)}}{\sqrt{\lambda_{max}^k}} \leq \|\bar{e}^k\| \leq \theta^{\sigma_1^k} \frac{\sqrt{V_k(\bar{e}^k)}}{\sqrt{\lambda_{min}^k}} \quad (23)$$

On another hand, from the definition of \bar{e}^k (equation (14)), one has:

$$\|\bar{e}^k\| \leq \|e^k\| \leq \theta^{(\lambda_k - 1)\delta_k} \|\bar{e}^k\| \quad (24)$$

Finally, from the definition of V_k and according to (17) and (5), one has

$$(\sqrt{V(\bar{e})} \leq \sqrt{\lambda_{max}} \|\bar{e}\| \leq \sqrt{\lambda_{max}} \|e\| \quad (25)$$

Combining (22), (23) and (24), one gets:

$$\begin{aligned}
\|e^k\| &\leq \theta^{\sigma_1^k + (\lambda_k - 1)\delta_k} \frac{\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}} \\
&\quad \exp\left(\frac{-\theta(1 - 2n^2\rho\mu_S\theta^{-\frac{1}{2}})_t}{2}\right) \|e(0)\| \\
&+ \frac{2\sqrt{\lambda_{max}}\eta}{\sqrt{\lambda_{min}}} \frac{\theta^{\sigma_1^k + (\lambda_k - 1)\delta_k}}{\theta(1 - 2n^2\rho\mu_S\theta^{-\frac{1}{2}})\theta^{\sigma_1^k + (\lambda_1 - 1)\delta_1}} \\
&= \theta^{\sigma_1^k + (\lambda_k - 1)\delta_k} \frac{\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}} \\
&\quad \exp\left(\frac{-\theta(1 - 2n^2\rho\mu_S\theta^{-\varepsilon})_t}{2}\right) \|e(0)\| \\
&+ \frac{2\sqrt{\lambda_{max}}\eta}{\sqrt{\lambda_{min}}} \frac{\theta^{-\sigma_{\lambda_1}^1 + \sigma_{\lambda_k}^k}}{\theta(1 - 2n^2\rho\mu_S\theta^{-\frac{1}{2}})} \\
&= \theta^{(\lambda_1 - 1)\delta_1} \frac{\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}} \\
&\quad \exp\left(\frac{-\theta(1 - 2n^2\rho\mu_S\theta^{-\varepsilon})_t}{2}\right) \|e(0)\| \\
&+ \frac{2\sqrt{\lambda_{max}}\eta}{\sqrt{\lambda_{min}}} \frac{1}{\theta(1 - 2n^2\rho\mu_S\theta^{-\frac{1}{2}})} \quad (26)
\end{aligned}$$

The last equality comes from property (19) satisfied by the σ_1^k 's. The parameters λ_θ , μ_θ and β_θ of the theorem are hence:

$$\begin{aligned}
\lambda_\theta &= \theta^{(\lambda_1 - 1)\delta_1} \frac{\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}}, \quad \mu_\theta = \frac{-\theta(1 - 2n^2\rho\mu_S\theta^{-\frac{1}{2}})}{2} \\
\beta_\theta &= \frac{2\sqrt{\lambda_{max}}}{\sqrt{\lambda_{min}}} \frac{1}{\theta(1 - 2n^2\rho\mu_S\theta^{-\frac{1}{2}})}
\end{aligned}$$

To end the proof of the theorem, one shall exhibit a set of reals σ_1^k satisfying conditions (17), (18) and (19).

Before providing the set of σ_1^k 's, one firstly notes that according to the state dependence given by (2) and (3), the case where $\chi_{l,j}^{k,i} = 1$, $k, l \in \{1, \dots, q\}$ and $j \in \{2, \dots, \lambda_l\}$, occurs if and only if one of the following three situations is met:

- $k > l$ and i takes any value in $\{1, \dots, \lambda_k\}$.
- $k = l$ and $i \in \{1, \dots, \lambda_k\}$ with $i \geq j$.
- $k < l$ and $i = \lambda_k$.

Now, for $k = 1, \dots, q$ and $i = 1, \dots, \lambda_k$, let us define the σ_i^k 's as follows:

$$\begin{aligned}
\sigma_i^k &= \sigma_1^k + (i - 1)\delta_k \\
\text{where } \sigma_1^k &= -(\lambda_k - 1)\delta_k + (\lambda_1 - 1)\delta_1 \quad (27)
\end{aligned}$$

and the δ_k 's are given by (6). According to (27), one has:

$$\begin{aligned}
\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} &= \sigma_1^l + (j - 1)\delta_l - \sigma_1^k - (i - 1)\delta_k \\
&\quad - \frac{\delta_k}{2} - \frac{\delta_l}{2} \\
&= -(\lambda_l - 1)\delta_l + (j - 1)\delta_l \\
&\quad + (\lambda_k - 1)\delta_k - (i - 1)\delta_k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \\
&= (\lambda_k - i - \frac{1}{2})\delta_k + (j - \lambda_l - \frac{1}{2})\delta_l \quad (28)
\end{aligned}$$

Let us now check condition (18) by considering the three cases listed above.

- $k > l$ and i takes any value in $\{1, \dots, \lambda_k\}$:

Please notice that since $k > l$, one has $k \geq 2$. Equality (28) becomes:

$$\begin{aligned}
\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} &\leq (\lambda_k - \frac{3}{2})\delta_k - \frac{\delta_l}{2} \\
&= (\lambda_k - 1)\delta_k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \\
&\leq (\lambda_k - 1)\delta_k - \frac{\delta_k}{2} - \frac{\delta_{k-1}}{2} \quad (29)
\end{aligned}$$

The last inequality results from (8).

Using (7), one obtains

$$\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} \leq -\frac{\delta_k}{2} \leq -\frac{1}{2} \quad (30)$$

- $k = l$ and $i \in \{1, \dots, \lambda_k\}$ with $i \geq j$:

Equation (28) specializes as follows:

$$\begin{aligned}
\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} &= (j - i - 1)\delta_k \\
&\leq -\delta_k \leq -\delta_q \leq -\frac{1}{2}
\end{aligned}$$

- $k < l$ and $i = \lambda_k$:

Equation (28) becomes:

$$\begin{aligned}
\sigma_j^l - \sigma_i^k - \frac{\delta_k}{2} - \frac{\delta_l}{2} &= -\frac{\delta_k}{2} + (j - \lambda_l - \frac{1}{2})\delta_l \\
&\leq -\frac{\delta_k}{2} - \frac{\delta_l}{2} \leq -\frac{1}{2}
\end{aligned}$$

To end the proof of the theorem, one has to check property (17). It is easy to check from (27) that $\sigma_1^1 = 0$. Thus, it remains to show that

$$\sigma_1^{k+1} - \sigma_1^k > 0 \quad \text{for } k = 1, \dots, q - 1 \quad (31)$$

On one hand, one has:

$$\sigma_1^k = -(\lambda_k - 1)\delta_k + (\lambda_1 - 1)\delta_1$$

On another hand, one has:

$$\begin{aligned}
\sigma_1^{k+1} &= -(\lambda_{k+1} - 1)\delta_{k+1} + (\lambda_1 - 1)\delta_1 \\
&= -\frac{\delta_k}{2} + (\lambda_1 - 1)\delta_1
\end{aligned}$$

This leads to

$$\sigma_1^{k+1} - \sigma_1^k = -\frac{\delta_k}{2} + (\lambda_k - 1)\delta_k = (\lambda_k - \frac{3}{2})\delta_k > 0$$

III. SOME PARTICULAR DESIGN FUNCTIONS

Some particular expressions of the design function K_k that satisfy condition (11) shall be given. It will be shown that each of these expressions yields a well known observer. Indeed, one has:

- The usual high gain design function given by

$$K_k(e_1^k) = \rho e_1^k \quad (32)$$

where ρ is a positive scalar satisfying $\rho \geq \frac{1}{2}$. Notice that the function K_k is bounded as soon as e_1^k is.

- The design function involved in the actual sliding mode framework

$$K_k(e_1^k) = \rho \operatorname{sign}(e_1^k) \quad (33)$$

where ρ is a positive scalar and 'sign' is the usual signum function. It is worth mentioning that the required property (11) holds in the case of bounded input bounded state systems for relatively high values of ρ . However, this design function induces a chattering phenomena which is by no means suitable in practical situations.

- The design functions that are commonly used in the sliding mode practice, namely

$$K_k(e_1^k) = \rho \tanh(k_o e_1^k) = \rho \tanh(k_o e_1^k) \quad (34)$$

where \tanh denotes the hyperbolic tangent function and ρ and k_o are positive scalars. One can easily show that the design function (34) satisfies the property (11) for relatively high values of k_o . More particularly, recall that one has $\lim_{k_o \rightarrow +\infty} \tanh(k_o e_1^k) = \operatorname{sign}(e_1^k)$.

IV. EXAMPLE

Consider the following dynamical system

$$\begin{cases} \dot{x}_1 = x_2 - \tanh(x_1) \\ \dot{x}_2 = x_2^2 + \varepsilon(t) - x_4 - \sin(x_5) \\ \dot{x}_3 = x_4 - x_2^2 - \tanh(x_3) \\ \dot{x}_4 = x_5 + x_2/(1+x_2^2) + \cos(x_4) \\ \dot{x}_5 = -x_2 * \sin(x_2) - x_3^2 \\ y = (x_1 \ x_3)^T \end{cases} \quad (35)$$

where the uncertainty term $\varepsilon(t)$ is given by $\varepsilon(t) = 2\cos(t)$. It is clear that system (35) is under form (1) with $\lambda_1 = 2$ and $\lambda_2 = 3$. An observer of the form (12) can be hence designed for the estimation of the state without any prior knowledge on the expression of the uncertainties. Please notice that one has $\delta_1 = 2(\lambda_2 - 1) = 4$ according to (6).

Simulation results corresponding to the non measured state variables are given in figure 1. This clearly shows the ability of the observer in providing accurate state estimates as the curves of x_2 and \hat{x}_2 become quickly superimposed in spite of different corresponding initial conditions. Indeed, these simulations have been carried out using the following initial conditions $x_i(0) = 1$ for the model and $\hat{x}_1 = \hat{x}_3 = 1$, $\hat{x}_i = -1$ for observer. The value of the design parameter θ was set to 8.

V. CONCLUSION

A high gain observer for a large class of nonlinear MIMO systems has been proposed in this paper. Unlike previous works related to high gain observer design, a complete triangular structure is not assumed. Moreover, the gain of the proposed observer involves a well defined design function. The latter provides a unified framework for the high gain observer design, namely classical high gain observers and several versions of sliding mode like observers are obtained by considering particular expressions of the design function.

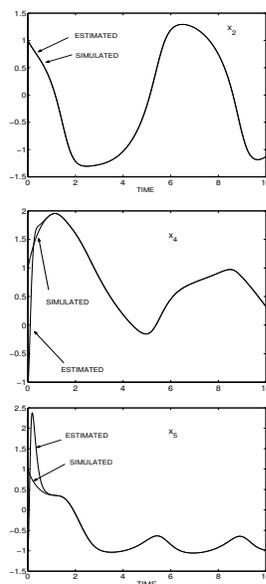


Fig. 1. Simulation results for x_i , $i = 2, 4, 5$

REFERENCES

- [1] M. Arcak and P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica*, 37:1923–1930, 2001.
- [2] G. Bornard and H. Hammouri. A graph approach to uniform observability of nonlinear multi output systems. In *Proc. of the 41st IEEE Conference on Decision and Control*, Las Vegas, Nevada, USA, 2002.
- [3] X. Fan and M. Arcak. Observer design for systems with multivariable monotone nonlinearities. *Systems & Control letters*, 50:319–330, 2003.
- [4] M. Farza, M. M'Saad, and L. Rossignol. Observer design based on triangular form generated by injective map. *Automatica*, 40:135–143, 2004.
- [5] M. Farza, M. Triki, T. Maatoug, M. M'Saad, and B. Dahhou. Unknown inputs observers for a class of nonlinear systems. In *Proc. of the 10th int. Conf. of Sciences & Techniques of Automatic*, Hammamet, Tunisia, 2009.
- [6] J.P. Gauthier, H. Hammouri, and S. Othman. A simple observer for nonlinear systems - application to bioreactors. *IEEE Trans. on Aut. Control*, 37:875–880, 1992.
- [7] J.P. Gauthier and I.A.K. Kupka. Observability and observers for nonlinear systems. *SIAM J. Control. Optim.*, 32:975–994, 1994.
- [8] M. Guay. Observer linearization by output-dependent time-scale transformations. *IEEE Transactions on Automatic Control*, 47:1730–1735, 2002.
- [9] H. Hammouri and M. Farza. Nonlinear observers for locally uniformly observable systems. *ESAIM J. on Control, Optimisation and Calculus of Variations*, 9:353–370, 2003.
- [10] M. Hou and A. C. Pugh. Observer with linear error dynamics for nonlinear multi-output systems. *Syst. Contr. Lett.*, 37:1–9, 1999.
- [11] R. E. Kalman and R. S. Bucy. New results in linear filtering and prediction theory. *J. Basic Eng.*, pages 95–108, 1961.
- [12] A. J. Krener and A. Isidori. Linearization by output injection and nonlinear observers. *Syst. Contr. Lett.*, 3:47–52, 1983.
- [13] A. J. Krener and W. Respondek. Nonlinear observers with linearizable error dynamics. *SIAM J. Contr. Optim.*, 23:197–216, 1985.
- [14] R. Rajamani. Observers for Lipschitz Nonlinear Systems. *IEEE Transactions on Automatic Control*, 43(3):397–401, 1998.
- [15] H. Shim, Y.I.Son, and J.H.Seo. Semi-global observer for multi-output nonlinear systems. *Syst. Contr. Lett.*, 42:233–244, 2001.
- [16] I. Souleiman, A. Glumineau, and G. Schreier. Direct transformation of nonlinear systems into state affine MISO form and nonlinear observers design. *IEEE Trans. on Automat. Contr.*, 48:2191–2196, 2003.
- [17] X.H. Xia and W.B. Gao. Nonlinear observer design by observer error linearization. *SIAM J.Control Optim.*, 27(1):199–216, 1989.