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Bounded-error identification for closed-loop systems

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Abstract

This paper presents a scheme for the identification of a system which operates in closed-loop and in the presence of bounded output disturbances. Two algorithms are proposed to solve this identification problem. The first algorithm is an Optimal Bounding Ellipsoid (OBE) type algorithm. This first algorithm is analyzed and sufficient conditions for stability and convergence are established. Relaxation of these conditions leads to a second identification algorithm. The implementation of that second algorithm is realized in an iterative scheme. A numerical example is provided to show the efficiency of the scheme.

Key words: Closed-loop identification; Bounded noise; Linear systems

1 Introduction

1.1 The considered identification problem

This paper is devoted to the study of a Set Membership Identification (SMI) algorithm for a dynamic SISO system operating in the presence of feedback. Here the system is assumed to be parameterized by a discrete-time transfer function $G^*(q)$ such that the closed-loop

behavior of the system satisfies
$$\begin{cases} y_t = G^*(q)u_t + w_t \\ u_t = r_t - C(q)y_t \end{cases},$$

it follows

$$y_t = \frac{G^*(q)}{1 + G^*(q)C(q)}r_t + v_t \quad (1)$$

with $v_t = \frac{1}{1 + G^*(q)C(q)}w_t$. $C(q)$ is the linear controller (supposed to be known) and r_t an exogenous input signal. The sequence w_t is not observable but is known to be bounded in the ℓ_1 norm: $|w_t| \leq \delta_w$. Through the closed-loop w_t produces the bounded sequence v_t such that

$$|v_t| \leq \delta_v \quad (2)$$

It represents noise measurements, state disturbances or modeling inaccuracies brought back on the output of the closed loop.

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This closed-loop SMI problem occurs when open-loop experiment is prohibited or has no meaning (safety, stability, economical reasons, efficiency of operation, etc.) and when the diversity of the components on w_t is such that its probability density function is unknown.

1.2 Prior work

The identification of closed-loop systems has received much interest for the last decades (see e.g. [12] and [1]) and three specific groups of methods can be distinguished: (1-) The direct approaches in which the identification is performed as in an usual open-loop context ([7], [6] and references therein), (2-) the indirect approaches which are mainly based on an analysis of the control system sensitivity function using the system output and an external excitation input (see [19], [20], [12], [13]) and (3-) the joint input-output approaches which use the system input-output behavior together with an external excitation input (see [21], [14]). These methods aim at providing an unbiased model of the plant in the stochastic noise assumption. If the only information about the noise is its instantaneous bound, these methods are not able to efficiently identify the system.

SMI methods are the identification methods introduced to deal with system identification when the noise is assumed to be unknown but bounded. Here we consider noise bounded in the ℓ_1 norm. Unlike to the other identification approaches, which provide an estimate, SMI methods propose the estimation of a feasible parameter set i.e. a model set compatible with all the available information. There are two main possible structures for

the design of this feasible parameter set: a polytope or an ellipsoid. In this paper we shall investigate a particular type of ellipsoidal algorithms: the Optimal Bounding Ellipsoid (OBE) type algorithms. The reason is that their computational complexity is low and they are appropriated to handle the identification problem in presence of bounded disturbances. Some contributions have been presented in [11], [9], [2], [18] and [17].

In the above methods very few of them are devoted to the direct identification problem expressed as $|y_t - \hat{G}(q)u_t| \leq \delta_w$ with $\hat{G}(q)$ an IIR filter and δ_w fixed in advance. Among them, some are only suitable for the identification of stable systems ([8], [3], [4], [10], [17]) and others have a high computational complexity ([4], [5]). Above all, none of them ensures the estimation of a model which stabilizes the closed-loop, this is however an essential elementary property.

In this paper, to get around these difficulties, we consider the indirect identification problem expressed as

$\left| y_t - \frac{\hat{G}(q)}{1 + \hat{G}(q)C(q)} r_t \right| \leq \delta_v$ with δ_v fixed in advance. In the above challenging problem, the number of alternatives is very limited. One alternative is to use a SMI algorithm in an indirect two steps approach: 1– the transfer function $\frac{G^*(q)}{1 + G^*(q)C(q)}$ between r_t and y_t is identified, 2– $G^*(q)$ is retrieved from the identified transfer function under the condition that the controller is linear and known. This approach leads however to a high order model and the use of a model reduction step would probably not maintain the property $\left| y_t - \frac{\hat{G}(q)}{1 + \hat{G}(q)C(q)} r_t \right| \leq \delta_v$. This paper consists in the development of a new alternative which alleviates some of the issues of the previous methods.

1.3 Contributions of this paper

The first key idea in our development is the proposition of a first algorithm using an OBE type algorithm together with the closed-loop Output Error (CLOE) parametrization introduced in [15]. Such a parametrization is not linear in the parameter vector. This non-linear effect impacts the stability analysis and a main contribution is the establishment of stability and convergence conditions of the algorithm. The second key idea in our development is the relaxation of the previous stability conditions via a second identification algorithm. This leads to the estimation of a model such that $\left| y_t - \frac{G(q)}{1 + G(q)C(q)} r_t \right| \leq \delta_v$ without over-parametrization. The current paper completes the work presented in [16]. The paper is organized as follows: the identification problem is formulated in Section 2. In Section 3, two identification algorithms are presented. The first one is described and analyzed in detail in subsections 3.1 and 3.2, the second one is introduced in subsection 3.3. The proposed algorithms have been tested on a numerical application, results are given in section 4. Section 5 concludes the paper. Appendices contain most of the

proofs.

2 Problem formulation

Consider the transfer function $G^*(q)$ parameterized as

$$G^*(q) = q^{-d} \frac{B^*(q)}{A^*(q)} \quad (3)$$

with $\begin{cases} B^*(q) = b_0^* + b_1^* q^{-1} + \dots + b_{n_b}^* q^{-n_b} \\ A^*(q) = 1 + a_1^* q^{-1} + \dots + a_{n_a}^* q^{-n_a} \end{cases}$. q^{-1} is the delay operator, d is the delay, n_a and n_b the degrees respectively $A^*(q)$ and $B^*(q)$. Let us denote $\theta^* \in \mathbb{R}^n$ the parameter vector with $n = n_a + n_b + 1$ the number of parameters: $\theta^{*T} = (\dots a_i^* \dots b_i^* \dots)$. Making use of the CLOE parametrization, $y_t = \frac{G^*(q)}{1 + G^*(q)C(q)} r_t + v_t$ can be re-expressed as $y_t = \hat{y}_t + v_t$ where \hat{y}_t is determined by $\hat{y}_t = \phi_t^T \theta^*$ with $\phi_t = (\dots -\hat{y}_{t-i} \dots \hat{u}_{t-d-i} \dots)$ and $\hat{u}_{t-d-i} = r_{t-d-i} - C(q)\hat{y}_{t-d-i}$.

Objective: Given the degrees n_a and n_b , the aim of this paper is to present an identification scheme in order to find an estimate $\hat{\theta}$ for θ^* . The transfer function $\hat{G}(q)$ parameterized by $\hat{\theta}$ must satisfy

$$\left| y_t - \frac{\hat{G}(q)}{1 + \hat{G}(q)C(q)} r_t \right| \leq \delta_v \quad (4)$$

This must be done by using the available data $\{r_t, y_t\}$, the knowledge of the controller $C(q) = \frac{R(q)}{S(q)}$ and the upper bound δ_v .

The estimate for θ^* at the instant t is denoted $\hat{\theta}_t$. For this current time t , \hat{y}_t is replaced by its a priori and a posteriori estimates $\begin{cases} \hat{y}_{t/t-1} = \hat{\phi}_t^T \hat{\theta}_{t-1} \\ \hat{y}_{t/t} = \hat{\phi}_t^T \hat{\theta}_t \end{cases}$. The pseudo

linear regression vector ϕ_t is substituted by $\hat{\phi}_t$ which is simply obtained by replacing the unknown component \hat{y}_{t-i} by its a posteriori estimate $\hat{y}_{t-i/t-i}$ and \hat{u}_{t-d-i} by its a posteriori estimate $\hat{u}_{t-d-i/t-d-i}$:

$$\hat{\phi}_t^T = (\dots -\hat{y}_{t-i/t-i} \dots \hat{u}_{t-d-i/t-d-i} \dots)$$

with $\hat{u}_{t-d-i/t-d-i} = r_{t-d-i} - C(q)\hat{y}_{t-d-i/t-d-i}$. The a priori and a posteriori prediction errors are derived from the previous definitions in the following form:

$\begin{cases} \epsilon_{t/t-1} = y_t - \hat{y}_{t/t-1} \\ \epsilon_{t/t} = y_t - \hat{y}_{t/t} \end{cases}$. Let us notice that the a posteriori

ori prediction error $\epsilon_{t/t}$ can be easily expressed as:

$$\epsilon_{t/t} = \frac{S(q)}{A^*(q)S(q) + q^{-d}B^*(q)R(q)} \hat{\phi}_t^T \tilde{\theta}_t + v_t \quad (5)$$

where $\tilde{\theta}_t = \theta^* - \hat{\theta}_t$ denotes the parameter error vector.

3 Identification algorithms and analysis

3.1 The CLOE-OBE (closed-loop Output Error - OBE) algorithm.

From (1) and (2) the parameter vector θ^* belongs to the set defined by $\bigcap_i^t \mathcal{S}_i$ with $\mathcal{S}_t = \{\theta \in \mathbb{R}^n, |y_t - \phi_t^T \theta| \leq \delta_v\}$. The first OBE algorithm to be presented builds on that property in the sense that its aim is to find a parameter vector $\hat{\theta}_t$ center of an ellipsoid \mathcal{E}_t such that $\mathcal{E}_t \supset \bigcap_i^t \hat{\mathcal{S}}_i$ where $\hat{\mathcal{S}}_t$ is the observation set defined by $\hat{\mathcal{S}}_t = \{\theta \in \mathbb{R}^n, |y_t - \hat{\phi}_t^T \theta| \leq \delta\}$. δ is a user defined bound which has to be specified taking into account the bound δ_v . Given $(y_t, \hat{\phi}_t)$, $\hat{\mathcal{S}}_t$ is the set of all possible θ which are consistent with the chosen bound δ . An important property of this observation set is given in the following theorem. In this theorem $\|\cdot\|_1$ is the l_1 induced norm.

Theorem 1 Consider a parameter vector $\hat{\theta}_t$ such that $\hat{\theta}_t \in \hat{\mathcal{S}}_t$. Assume that $G^*(q)$ and δ are such that:

$$\left\| 1 - \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} \right\|_1 < 1 \quad (6)$$

$$\delta \geq \frac{\left\| \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} \right\|_1}{1 - \left\| 1 - \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} \right\|_1} \delta_v \quad (7)$$

Then

$$\theta^* \in \hat{\mathcal{S}}_t \quad (8)$$

■

This theorem states that the ability to find the true parameter vector inside $\hat{\mathcal{S}}_t$ depends on one condition on $G^*(q)$ and one condition on δ . From (7) the choice on δ depends on the known controller $C(q)$, on the known bound δ_v but also on the unknown polynomials $A^*(q)$ and $B^*(q)$. In subsection 3.3 a filter will be introduced so as to relax these hard conditions.

Remark 2 The previous conditions depend on the design of the controller. In particular, condition (6) implies the asymptotic stability of the controller used during the identification step. There is no stability condition on $G^*(q)$, consequently the proposed approach is suitable both for stable systems and unstable systems.

The first proposed algorithm is named CLOE-OBE. It will be shown in Theorem 3 that it provides a parameter vector $\hat{\theta}_t$ center of an ellipsoid $\mathcal{E}_t \supset \bigcap_i^t \hat{\mathcal{S}}_i$. This algorithm corresponds to a modified weighted recursive least square algorithm. Its update equations are given in table 1 below.

CLOE-OBE algorithm

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \Gamma_t \epsilon_{t/t-1} \quad (9)$$

$$\begin{cases} \Gamma_t = \frac{P_{t-1} \hat{\phi}_t \sigma_t}{\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t} \\ P_t^{-1} = \lambda P_{t-1}^{-1} + \hat{\phi}_t \sigma_t \hat{\phi}_t^T \\ \epsilon_{t/t-1} = y_t - \hat{\phi}_t^T \hat{\theta}_{t-1} \end{cases} \quad (10)$$

$$\sigma_t = \begin{cases} \frac{\lambda}{\hat{\phi}_t^T P_{t-1} \hat{\phi}_t} \left(\left| \frac{\epsilon_{t/t-1}}{\delta} \right| - 1 \right) & \text{if } (|\epsilon_{t/t-1}| > \delta) \text{ and } (\hat{\phi}_t^T P_{t-1} \hat{\phi}_t > 0) \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Table 1

Algorithm 1: CLOE-OBE algorithm

The two weighting terms are λ and σ_t . $0 < \lambda \leq 1$ is the forgetting factor fixed by the user to weight the past information. σ_t is a switching flag which stops the updating of $\hat{\theta}_t$ if the a priori prediction error is below δ or if the arrival data are meaningless (i.e. $\hat{\phi}_t^T P_{t-1} \hat{\phi}_t = 0$). An important point to note is that, the a posteriori prediction error $\epsilon_{t/t}$ can be written as:

$$\epsilon_{t/t} = \frac{\lambda}{\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t} \epsilon_{t/t-1} \quad (12)$$

Using the value of σ_t for $\hat{\phi}_t^T P_{t-1} \hat{\phi}_t > 0$ and $|\epsilon_{t/t-1}| > \delta$ yields $|\epsilon_{t/t}| = \delta$. This clearly shows that the CLOE-OBE algorithm ensures the following key property:

$$\forall t \text{ such that } \hat{\phi}_t^T P_{t-1} \hat{\phi}_t > 0 ; |\epsilon_{t/t}| \leq \delta \quad (13)$$

Consequently the a posteriori prediction error $\epsilon_{t/t}$ is bounded by δ which implies that $\hat{\theta}_t \in \hat{\mathcal{S}}_t$.

3.2 Stability and convergence properties

This subsection is devoted to analyzing the properties of the CLOE-OBE algorithm. The first properties provide some geometrical interpretations, to this end let us define for each time $t-1$ the ellipsoid \mathcal{E}_{t-1} : $\mathcal{E}_{t-1} = \{\theta \in \mathbb{R}^n, (\theta - \hat{\theta}_{t-1})^T P_{t-1}^{-1} (\theta - \hat{\theta}_{t-1}) \leq \rho_{t-1}\}$ with ρ_{t-1} a scalar. From above we have $\hat{\theta}_t \in \hat{\mathcal{S}}_t$. The following Theorem 3 builds the ellipsoid \mathcal{E}_t so as to ensure \mathcal{E}_t to be a bounding ellipsoid of $(\hat{\mathcal{S}}_t \cap \mathcal{E}_{t-1})$. It is shown that provided \mathcal{E}_{t-1} is a sufficiently large ellipsoid such that $\theta^* \in \mathcal{E}_{t-1}$ then \mathcal{E}_t contains θ^* too.

Theorem 3 Consider the class of systems defined in section 2 and the CLOE-OBE algorithm given by (9), (10) and (11). Assume (6), (7) and

$$\theta^* \in \mathcal{E}_{t-1} \quad (14)$$

then

- An outer bounding ellipsoid of $(\hat{\mathcal{S}}_t \cap \mathcal{E}_{t-1})$ is given by the ellipsoid \mathcal{E}_t with ρ_t computed as follows:

$$\rho_t = \lambda \rho_{t-1} + \sigma_t \delta^2 - \lambda \sigma_t \frac{\epsilon_{t/t-1}^2}{\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t} \quad (15)$$

- \mathcal{E}_t is such that $\theta^* \in \mathcal{E}_t$. ■

The size of the ellipsoid \mathcal{E}_t is related to the eigenvalues of P_t and the scalar ρ_t . In this theorem the scalar ρ_t is computed so as to ensure \mathcal{E}_t to be a bounding ellipsoid of $\bigcap_i^t \hat{\mathcal{S}}_i$. By carefully analyzing (15) it can be shown that σ_t minimizes ρ_t . The size of the ellipsoid \mathcal{E}_t is indirectly influenced by the value chosen for the upper bound δ in the sense that an overestimation of this bound generates a larger ellipsoid. An underestimation of the bound is in contradiction with the condition (7) which may generate some difficulties in estimation.

The following Theorem 4 focuses on the center $\hat{\theta}_t$ of the ellipsoid \mathcal{E}_t . It is shown that under some conditions $\hat{\theta}_t$ converges in a neighborhood of θ^* .

Theorem 4 Consider the class of systems defined in section 2 and the CLOE-OBE algorithm given by (9), (10) and (11). Assume (6), (7) then for all initial conditions

- $$\left| \tilde{\theta}_t \right|^2 \leq \gamma_1 \left| \tilde{\theta}_0 \right|^2 \quad (16)$$

with $\gamma_1 = \frac{\mu_{\max}(P_0^{-1})}{\mu_{\min}(P_0^{-1})}$, where $\mu_{\max}(P_0^{-1})$ and $\mu_{\min}(P_0^{-1})$ are respectively the maximum and the minimum eigenvalues of P_0^{-1} .

Furthermore, if $\sigma_i \neq 0$ on an interval $[t; t - o_e + 1]$ and if $\{\hat{\phi}_i\}$ is a persistently exciting sequence of order $o_e \geq n$, i.e there exist $\alpha > 0$ and $\beta > 0$ such that

$$\alpha I_n \leq \sum_{i=0}^{o_e-1} \hat{\phi}_{t-i} \sigma_{t-i} \hat{\phi}_{t-i}^T \leq \beta I_n \quad (17)$$

Then the following properties hold:

- $$\left| \tilde{\theta}_t \right|^2 \leq \gamma_2 \lambda^t \left| \tilde{\theta}_0 \right|^2 \quad (18)$$

$$\text{with } \gamma_2 = \begin{cases} \frac{\mu_{\max}(P_0^{-1})}{\alpha} \left(\frac{\lambda^{-o_e-1}}{\lambda^{-1}-1} \right) & \text{if } \lambda < 1 \\ \frac{\mu_{\max}(P_0^{-1})}{\alpha} & \text{if } \lambda = 1 \end{cases}$$

•

$$\lim_{t \rightarrow \infty} |\epsilon_{t/t-1}| \leq \delta \quad (19)$$

■

Remark 5 In this result, $\left| \tilde{\theta}_t \right|^2$ exponentially decreases as long as $\sigma_t \neq 0$ and the persistent excitation condition holds. The adaptation is frozen once $|\epsilon_{t/t-1}| \leq \delta$.

3.3 The F-CLOE-OBE (Filtered - CLOE-OBE) algorithm.

The above algorithm suffers from conditions (6) and (7): they limit the scope of application of the algorithm (condition (6)) and they limit the choice on δ (condition (7)). To remove them an adaptation filter is introduced here. This adaptation filter must be designed and implemented in such a way to compensate the effect of $\frac{S(q)}{A^*(q)S(q) + q^{-d}B^*(q)R(q)}$ in (5) without increasing the noise level. Denote the adaptation filter by $F(q)$. Let first define the a priori and a posteriori adaptation errors as

$$\begin{cases} \eta_{t/t-1} = \epsilon_{t/t-1} + (F(q) - 1) \epsilon_{t/t} \\ \eta_{t/t} = F(q) \epsilon_{t/t} \end{cases} \quad \text{These definitions allow us to propose a filtered algorithm by simply substituting in (9), (10) and (11):}$$

• $\epsilon_{t/t-1}$ by $\eta_{t/t-1}$ and $\epsilon_{t/t}$ by $\eta_{t/t}$;

- y_t by y_t^F such that $y_t^F = \frac{1}{F(q)} y_t$ and $\hat{\phi}_t$ by $\hat{\phi}_t^F$ such that $\hat{\phi}_t^F = \frac{1}{F(q)} \hat{\phi}_t$.

Taking into account these adjustments, the proposed F-CLOE-OBE algorithm is given in table 2. It can be established that $\eta_{t/t} = \frac{\lambda}{\lambda + \hat{\phi}_t^F T P_{t-1} \hat{\phi}_t^F \sigma_t} \eta_{t/t-1}$ and consequently a similar relation to (13) holds: $\forall t$ such that $\hat{\phi}_t^F T P_{t-1} \hat{\phi}_t^F > 0$ we have $|\eta_{t/t}| \leq \delta$, highlighting that δ is now a bound on the magnitude of the a posteriori adaptation error $\eta_{t/t}$.

After some straightforward lines of calculus it can be shown that the a posteriori adaptation error satisfies

$$\eta_{t/t} = F(q) \frac{S(q)}{A^*(q)S(q) + q^{-d}B^*(q)R(q)} \hat{\phi}_t^F T \tilde{\theta}_t + v_t$$

Using this last equation, the following Theorem 6 presents an analysis of the F-CLOE-OBE algorithm.

Theorem 6 Consider the class of systems defined in section 2 and the F-CLOE-OBE algorithm given by (20), (21) and (22). Assume that $G^*(q)$, $F(q)$ and δ are such that

$$\left\| 1 - \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} \frac{1}{F(q)} \right\|_1 < 1 \quad (23)$$

F-CLOE-OBE algorithm

$$\hat{\theta}_t = \hat{\theta}_{t-1} + \Gamma_t \eta_{t/t-1} \quad (20)$$

$$\begin{cases} \Gamma_t = \frac{P_{t-1} \widehat{\phi}_{t-1}^T \sigma_t}{\lambda + \widehat{\phi}_{t-1}^T P_{t-1} \widehat{\phi}_{t-1} \sigma_t} \\ P_t^{-1} = \lambda P_{t-1}^{-1} + \widehat{\phi}_{t-1}^T \sigma_t \widehat{\phi}_{t-1} \\ \epsilon_{t/t-1} = y_t^F - \widehat{\phi}_{t-1}^T \hat{\theta}_{t-1} \end{cases} \quad (21)$$

$$\sigma_t = \begin{cases} \frac{\lambda}{\widehat{\phi}_{t-1}^T P_{t-1} \widehat{\phi}_{t-1}} (|\frac{\eta_{t/t-1}}{\delta}| - 1) \\ \text{if } (|\eta_{t/t-1}| > \delta) \text{ and } \left(\widehat{\phi}_{t-1}^T P_{t-1} \widehat{\phi}_{t-1} > 0 \right) \\ 0 \text{ otherwise} \end{cases} \quad (22)$$

Table 2

Algorithm 2: F-CLOE-OBE algorithm

$$\delta \geq \frac{\left\| \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} \frac{1}{F(q)} \right\|_1}{1 - \left\| 1 - \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} \frac{1}{F(q)} \right\|_1} \delta_v \quad (24)$$

then for all initial conditions

•

$$|\tilde{\theta}_t|^2 \leq \gamma_1 |\tilde{\theta}_0|^2 \quad (25)$$

Furthermore, if $\sigma_i \neq 0$ on an interval $[t; t - o_e + 1]$ and if $\{\widehat{\phi}_i^F\}$ is a persistently exciting sequence of order $o_e \geq n$, then the following properties hold:

•

$$|\tilde{\theta}_t|^2 \leq \gamma_2 \lambda^t |\tilde{\theta}_0|^2 \quad (26)$$

•

$$\lim_{t \rightarrow \infty} |\eta_{t/t-1}| \leq \delta \quad (27)$$

■

The proof is similar to that of Theorem 4. From a geometrical point of view, by considering the following Filtered-observation set \widehat{S}_t^F :

$$\widehat{S}_t^F = \left\{ \theta \in \mathbb{R}^n, |y_t^F - \widehat{\phi}_t^T \theta + (F(q) - 1)\epsilon_{t/t}| \leq \delta \right\}$$

we get Theorem 7, similar to Theorem 3.

Theorem 7 Consider the class of systems defined in section 2 and the F-CLOE-OBE algorithm given by (20), (21) and (22). Assume (23) and (24) in Theorem 6 hold then

$$\theta^* \in \widehat{S}_t^F \quad (28)$$

Moreover if

$$\theta^* \in \mathcal{E}_{t-1} \quad (29)$$

then

- An outer bounding ellipsoid of $(\widehat{S}_t^F \cap \mathcal{E}_{t-1})$ is given by the ellipsoid \mathcal{E}_t with ρ_t computed as follows:

$$\rho_t = \lambda \rho_{t-1} + \sigma_t \delta^2 - \lambda \sigma_t \frac{\eta_{t/t-1}^2}{\lambda + \widehat{\phi}_{t-1}^T P_{t-1} \widehat{\phi}_{t-1} \sigma_t} \quad (30)$$

- \mathcal{E}_t is such that

$$\theta^* \in \mathcal{E}_t \quad (31)$$

■

The proof is similar to that of Theorem 3. The ideal filter is obviously $F(q) = \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)}$. Thus, conditions (23) and (24) are much milder than conditions (6) and (7) if a reasonable estimated model is available. From the previous considerations on (23) and (24), the following implementation iterative scheme is proposed:

- (1) Choose a high bound δ and apply the CLOE-OBE algorithm to get $\widehat{G}(q)$;
- (2) Given this first estimation, design the filter $F(q) = \frac{\widehat{A}(q)S(q) + q^{-d}\widehat{B}(q)R(q)}{S(q)}$ and choose a lower δ ;
- (3) Apply the F-CLOE-OBE algorithm to get a new $\widehat{G}(q)$;
- (4) Repeat steps 2 and 3 until convergence of step 3 and $\delta = \delta_v$ or until a finite number of iterations.

The first step is an initialization step: an initial estimate of the model is necessary so as to implement the F-CLOE-OBE algorithm. Then at each iteration a temporary model is used to generate the adaptation filter. It is difficult to make a general discussion on the behavior of this iterative scheme, global convergence has not been proven so far. A smooth transition between two iterations may be introduced by using a smooth decreasing bound δ : at iteration i , choose $\delta = \delta^{(i)}$ with $\delta^{(i)}$ given by the formula:

$$\delta^{(i)} = \left(\delta^{(ini)} - \delta^{(fin)} \right) e^{-i/\mu} + \delta^{(fin)} \quad (32)$$

with $\delta^{(ini)} > \delta_v$ and $\delta^{(fin)} = \delta_v$. In our experiments this iterative scheme works well with $\mu = 1$.

Note that if $F(q) \simeq \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)}$ it is possible to choose $\delta = \delta_v$. In that case, from (27) we have $\lim_{t \rightarrow \infty} \hat{\theta}_t = \hat{\theta}$ where $\hat{\theta}$ is such that $|\eta_t| \leq \delta_v$ with $\eta_t = F(q)\epsilon_t = F(q) \left(y_t^F - \widehat{\phi}_t^T \hat{\theta} \right)$ that's to say $\eta_t = y_t - \widehat{\phi}_t^T \hat{\theta}$.

To summarize we see that the contribution of the filter $F(q)$ is twofold: first it relaxes conditions (6) and (7), second it allows the estimation of a $\widehat{G}(q)$ such that:

$$\left| y_t - \frac{\widehat{G}(q)}{1 + \widehat{G}(q)C(q)} r_t \right| \leq \delta_v \quad (33)$$

That is what we expected. This is coherent with the objective stated in section 2 by equation (4).

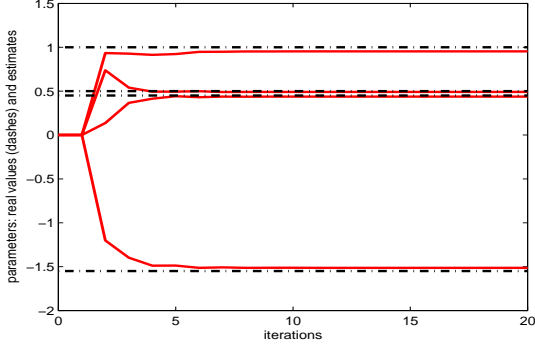


Fig. 1. Convergence of parameters.

4 Simulation results

In this section, different simulation results are reported so as to illustrate performance of the proposed iterative scheme. The system and controller are described by $G^*(q) = \frac{q^{-1}(1+0.5q^{-1})}{1-1.55q^{-1}+0.45q^{-2}}$ and $C(q) = \frac{0.9092-1.3404q^{-1}+0.5464q^{-2}}{1-1.1q^{-1}+0.1q^{-2}}$. Note that the system is unstable and condition (6) in Theorem 1 is violated. The iterative scheme has been applied in different situations. For each of them the default value for the number of iterations has been fixed to 20 (as in MATLAB ©), the forgetting factor λ has been chosen equal to 1 and δ_v has been adjusted so as to have a signal to noise ratio equal to 10dB on the output of the closed-loop.

4.1 First simulation experiment

In a first experiment, we focused on performance of the iterative scheme with respect to the excitation signal and the choice on δ . Two excitation signals of length $N = 2000$ have been used:

- $r_t = r_t^{(rbs)}$ a random binary sequence ;
- $r_t = r_t^{(sin)}$ a sum of 10 sinusoids uniformly distributed over $[0.01; \pi] \text{rads}^{-1}$.

The bounded noise w_t was defined by $w_t = \delta_w \frac{1}{2}(e_t + b_t)$ with e_t a white noise uniformly distributed in $[-1; 1]$ and b_t a binary periodic square wave with period 2 samples.

In a first time the applied exogenous excitation signal was the random binary sequence $r_t^{(rbs)}$. In the iterative scheme we have chosen a decreasing bound as described by (32) with $\delta^{(ini)} = 5 * \delta_v$. The convergence of $\hat{\theta}$ at the end of each iteration towards θ^* is shown in Fig. 1. These results suggest that even if condition (6) is violated, this does not affect the convergence of the algorithm, the adaptation filter relaxes this condition. Fig. 2 presents bounds $\pm\delta_v$ and the output error $y_t - \frac{\hat{G}(q)}{1+\hat{G}(q)C(q)}r_t$ obtained with the final model. This figure illustrates the fact that the final model satisfies (33).

In a second time a Monte Carlo simulation with 100 runs

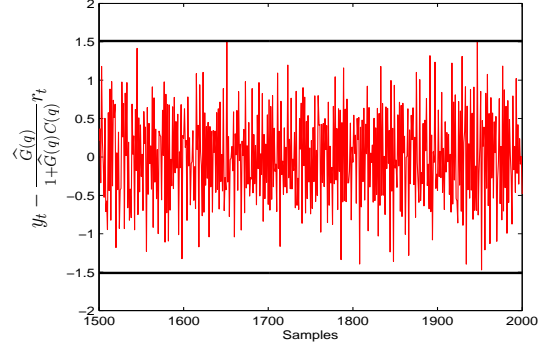


Fig. 2. Bounds $\pm\delta_v$ and the closed-loop output error

$$y_t - \frac{\hat{G}(q)}{1+\hat{G}(q)C(q)}r_t.$$

(with different realizations of the noise) has been carried out in order to study the behavior of F-CLOE-OBE algorithm with respect to the excitation signal and an over-evaluation of δ_v . Fig. 3 presents the mean value of $\|\hat{\theta}\|$ as a function of δ/δ_v . It reveals that the size of the parameter error vector is related to the choice of δ : an over-evaluation of δ_v leads to a highest error. This figure also shows that a poor excitation signal degrades the estimation accuracy, even if the estimated model satisfies (33).

4.2 Second simulation experiment

In a second experiment, the objective was to compare the iterative scheme with a similar closed-loop identification algorithm: the F-CLOE method (see [15]). Here three noise sources have been considered:

- $w_t = w_t^{(1)} = \delta_w (\frac{9}{10}e_t + \frac{1}{10}b_t)$
- $w_t = w_t^{(2)} = \delta_w (\frac{5}{10}e_t + \frac{5}{10}b_t)$
- $w_t = w_t^{(3)} = \delta_w (\frac{1}{10}e_t + \frac{9}{10}b_t)$

with e_t and b_t defined as previously. These three noise sources produce three noises v_t with three different distributions as depicted on Fig. 4.

Each algorithm has been implemented in an iterative scheme with the same number of iterations. A Monte Carlo simulation with 100 runs has been carried out with the three noise sources and with the two previous excitation signals. Table 3 presents the mean value of $\|\hat{\theta}\|$. From the results given, it appears that performance of the F-CLOE-OBE algorithm are inferior to performance of the F-CLOE algorithm when the noise samples are distributed over the entire interval $[-\delta_v; \delta_v]$ (first noise distribution). This is not surprising because the F-CLOE-OBE algorithm freezes the parameters adaptation if $|\epsilon_{t/t-1}| \leq \delta$ while the F-CLOE algorithm never stops adapting. However, performance of the F-CLOE-OBE algorithm become better when the noise samples are close to the bounds δ_v and $-\delta_v$ (third noise distribution). The F-CLOE algorithm is not suitable to a such noise distribution. Also note that the F-CLOE-OBE al-

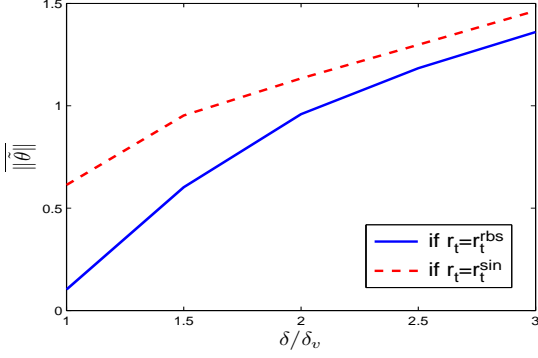


Fig. 3. $\|\tilde{\theta}\|$ as a function of δ/δ_v .

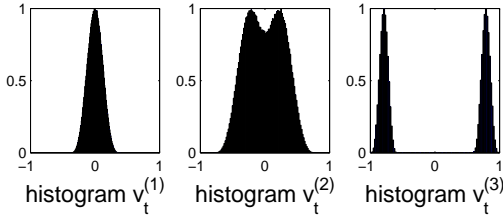


Fig. 4. Histograms of the three noise v_t .

noise		$r_t = r_t^{(rbs)}$	$r_t = r_t^{(sin)}$
$w_t = w_t^{(1)}$	F-CLOE:	0.03	0.08
	F-CLOE-OBE :	0.16	0.72
$w_t = w_t^{(2)}$	F-CLOE:	0.02	0.80
	F-CLOE-OBE :	0.11	0.61
$w_t = w_t^{(3)}$	F-CLOE:	0.08	1.86
	F-CLOE-OBE :	0.08	0.09

Table 3
Mean value of $\|\tilde{\theta}\|$ as a function of the noise source and the excitation signal.

gorithm seems less sensitive than F-CLOE algorithm to a poor excitation signal.

5 Conclusion

In this paper, we have presented a scheme for the identification of a system operating in closed-loop and with bounded noise on the output. We have posed the problem in terms of an OBE type algorithm and we have obtained an identification iterative scheme with a low computational complexity. The study was based on two identification algorithms, for each of them sufficient conditions for stability and convergence have been demonstrated. The relationship between the two algorithms is the introduction of an adaptation filter used to relax some binding conditions. With regard to other SMI methods, the proposed scheme estimates an IIR filter, it can be applied both to stable systems and to unstable

systems and it ensures the estimation of a model stabilizing the closed-loop.

A Proof of Theorem 1

If $\theta^* \in \hat{\mathcal{S}}_t$ then it means that $|y_t - \hat{\phi}_t^T \theta^*| \leq \delta$. This inequality is true if $|y_t - \hat{\phi}_t^T \hat{\theta}_t - \hat{\phi}_t^T (\theta^* - \hat{\theta}_t)| \leq \delta$ that's to say $|\epsilon_{t/t} - b_t| \leq \delta$ with $b_t = \hat{\phi}_t^T \tilde{\theta}_t$. It has been said that $\epsilon_{t/t} = \frac{S(q)}{A^*(q)S(q) + q^{-d}B^*(q)R(q)} b_t + v_t$, thus b_t is such that $b_t = \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} (\epsilon_{t/t} - v_t)$. From the triangular inequality, the last inequality becomes:

$$\left| \left(1 - \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} \right) \epsilon_{t/t} \right| + \left| \frac{A^*(q)S(q) + q^{-d}B^*(q)R(q)}{S(q)} v_t \right| \leq \delta \quad (\text{A.1})$$

$\hat{\theta}_t$ is estimated in such a way that $\hat{\theta}_t \in \bigcap_i^t \hat{\mathcal{S}}_i$ thus $\hat{\theta}_t$ belongs to $\hat{\mathcal{S}}_t$ and then $|\epsilon_{t/t}| \leq \delta$. Moreover if condition (6) is supposed to be true and v_t is such that $|v_t| \leq \delta_v$ then one obtains condition (7).

B Proof of Theorem 3

• Let θ such that $\theta \in \mathcal{E}_{t-1}$ and $\theta \in \hat{\mathcal{S}}_t$ then we have $(\theta - \hat{\theta}_{t-1})^T P_{t-1}^{-1} (\theta - \hat{\theta}_{t-1}) \leq \rho_{t-1}$ and $|y_t - \hat{\phi}_t^T \theta| \leq \delta$. If $\theta \in (\hat{\mathcal{S}}_t \cap \mathcal{E}_{t-1})$ it comes:

$$\lambda(\theta - \hat{\theta}_{t-1})^T P_{t-1}^{-1} (\theta - \hat{\theta}_{t-1}) + \sigma_t \left(y_t - \hat{\phi}_t^T \theta \right)^2 \leq \lambda \rho_{t-1} + \sigma_t \delta^2 \quad (\text{B.1})$$

It can be shown that the left term in (B.1) becomes $(\theta - \hat{\theta}_t)^T P_t^{-1} (\theta - \hat{\theta}_t) + \frac{\sigma_t}{\lambda} \left(\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t \right) \epsilon_{t/t}^2$ and together with (12), (B.1) gives $(\theta - \hat{\theta}_t)^T P_t^{-1} (\theta - \hat{\theta}_t) \leq \lambda \rho_{t-1} + \sigma_t \delta^2 - \lambda \sigma_t \frac{\epsilon_{t/t}^2}{\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t}$. This corresponds to the ellipsoid \mathcal{E}_t .

• From Theorem 1 we have $\theta^* \in \hat{\mathcal{S}}_t$. Together with (14) it gives: $\theta^* \in (\hat{\mathcal{S}}_t \cap \mathcal{E}_{t-1}) \subseteq \mathcal{E}_t$.

C Proof of Theorem 4

• Consider the Lyapounov function $V_t = \tilde{\theta}_t^T P_t^{-1} \tilde{\theta}_t$. From (9), (10) and (12) we have $\tilde{\theta}_{t-1} = \tilde{\theta}_t + P_{t-1} \hat{\phi}_t \frac{\sigma_t}{\lambda} \epsilon_{t/t}$ then V_{t-1} can be rewritten as $V_{t-1} = \frac{1}{\lambda} V_t - \frac{1}{\lambda} \tilde{\theta}_t^T \hat{\phi}_t \sigma_t \hat{\phi}_t^T \tilde{\theta}_t + \frac{2}{\lambda} \tilde{\theta}_t^T \hat{\phi}_t \sigma_t \epsilon_{t/t} + \frac{\sigma_t^2}{\lambda^2} \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \epsilon_{t/t}^2$. This gives $V_t = \lambda V_{t-1} + q_t$ with $q_t = \tilde{\theta}_t^T \hat{\phi}_t \sigma_t \hat{\phi}_t^T \tilde{\theta}_t - 2 \tilde{\theta}_t^T \hat{\phi}_t \sigma_t \epsilon_{t/t} - \frac{\sigma_t^2}{\lambda} \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t^2 \epsilon_{t/t}^2$. If $\sigma_t = 0$, then $q_t = 0$ and $V_t = \lambda V_{t-1}$. Let consider the case where $\sigma_t \neq 0$. q_t can be rewritten as follows: $q_t = \sigma_t \left(\epsilon_{t/t} - b_t \right)^2 - \frac{1}{\lambda} \sigma_t \left(\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t \right) \epsilon_{t/t}^2$. It comes from (12) $q_t = \sigma_t \left(\epsilon_{t/t} - b_t \right)^2 - \lambda \frac{\sigma_t \epsilon_{t/t}^2}{\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t}$. From (11) we have: $\lambda + \hat{\phi}_t^T P_{t-1} \hat{\phi}_t \sigma_t = \lambda \left| \frac{\epsilon_{t/t-1}}{\delta} \right|$, together with q_t this

gives: $q_t = \sigma_t \left(\epsilon_{t/t} - b_t \right)^2 - \epsilon_{t/t-1}^2 \sigma_t \left| \frac{\delta}{\epsilon_{t/t-1}} \right|$. In this part one considers the case $\sigma_t \neq 0$, so $\left| \frac{\delta}{\epsilon_{t/t-1}} \right| < 1$. Thus $q_t \leq 0$ if $(\epsilon_{t/t} - b_t)^2 \leq \delta^2$. In the proof of Theorem 1 it is shown that this inequality is true provided that conditions (6) and (7) hold. This ensures $q_t \leq 0$ and then $V_t \leq \lambda V_{t-1}$. • Henceforth we know that in each cases ($\sigma_t = 0$ or $\sigma_t \neq 0$) one has $V_t \leq \lambda V_{t-1}$ (if conditions (6) and (7) are satisfied). This gives $\tilde{\theta}_t^T P_t^{-1} \tilde{\theta}_t \leq \lambda^t \tilde{\theta}_0^T P_0^{-1} \tilde{\theta}_0$. We have $P_t^{-1} = \lambda P_{t-1}^{-1} + \hat{\phi}_t \sigma_t \hat{\phi}_t^T$ and $\sigma(t) \geq 0$ so it follows that $P_t^{-1} \geq \lambda^t P_0^{-1}$ and consequently $\tilde{\theta}_t^T \lambda^t P_0^{-1} \tilde{\theta}_t \leq \lambda^t \tilde{\theta}_0^T P_0^{-1} \tilde{\theta}_0$ thus $\mu_{\min}(P_0^{-1}) \left| \tilde{\theta}_t \right|^2 \leq \mu_{\max}(P_0^{-1}) \left| \tilde{\theta}_0 \right|^2$. This gives property (16).

• In order to establish (18) and (19) let consider again the following relation $P_t^{-1} = \lambda P_{t-1}^{-1} + \hat{\phi}_t \sigma_t \hat{\phi}_t^T$. After several iterations between $t - o_e$ and t it comes $P_t^{-1} \geq \sum_{i=0}^{o_e-1} \lambda^i \hat{\phi}_{t-i} \sigma_{t-i} \hat{\phi}_{t-i}^T$. It can be shown that $P_t^{-1} \geq \nu(\lambda) \sum_{i=0}^{o_e-1} \hat{\phi}_{t-i} \sigma_{t-i} \hat{\phi}_{t-i}^T$ with

$$\nu(\lambda) = \begin{cases} 1 & \text{if } \lambda = 1 \\ \frac{1-\lambda^{-1}}{1-\lambda^{-o_e}} & \text{if } \lambda < 1 \end{cases}. \text{ On the interval } [t; t - o_e + 1]$$

it is assumed that $\sigma_i \neq 0$ and that the sequence $\{\hat{\phi}_i\}$ is persistent exciting of order $o_e \geq n$, then from (17) this gives $P_t^{-1} \geq \nu(\lambda) \alpha I_n$. It has been established that $\tilde{\theta}_t^T P_t^{-1} \tilde{\theta}_t \leq \lambda^t \tilde{\theta}_0^T P_0^{-1} \tilde{\theta}_0$ it follows $\nu(\lambda) \alpha \left| \tilde{\theta}_t \right|^2 \leq \lambda^t \mu_{\max}(P_0^{-1}) \left| \tilde{\theta}_0 \right|^2$. This gives (18) with $\gamma_2 = \frac{\mu_{\max}(P_0^{-1})}{\nu(\lambda) \alpha}$. From (18), as long as the a priori prediction error is such that $|\epsilon_{t/t-1}| \leq \delta$, the parameter vector is updated and $\left| \tilde{\theta}_t \right|$ decreases exponentially. This yields to the convergence of $\hat{\theta}_t$ in a neighborhood of θ^* , conclusion (19) follows.

References

- [1] J. C. Agüero, G.C. Goodwin, and P.M.J. Van den Hof. A virtual closed-loop method for closed-loop identification. *Automatica*, 47(3):1626–1637, 2011.
- [2] C. Canudas-De-Wit and J. Carrillo. A modified EW-RLS algorithm for systems with bounded noise. *Automatica*, 26(3):599–606, 1990.
- [3] V. Cerone. Feasible parameter set for linear models with bounded errors in all variables. *Automatica*, 29(6):1551–1555, 1993.
- [4] V. Cerone. Parameter bounds for ARMAX models from records with bounded errors in variables. *International Journal of Control*, 57(1):225–235, 1993.
- [5] V. Cerone, D. Piga, and D. Regruto. Set-membership error-in-variables identification through convex relaxation techniques. *Transactions Automatic Control*, 57(2):517–522, 2012.
- [6] A. Chiuso. Asymptotic variance of closed-loop subspace identification methods. *IEEE Transactions on Automatic Control*, 51(8):1299–1314, 2006.
- [7] A. Chiuso and G. Picci. Consistency analysis of some closed-loop subspace identification methods. *Automatica*, 41(3):377–391, 2005.
- [8] T. Clement and S. Gentil. Recursive membership set estimation for output error models. *Mathematics and Computers in Simulation*, 32(5):505–513, 1990.
- [9] S. Dasgupta and Y.F. Huang. Asymptotically convergent modified recursive least square with data-dependent updating and forgetting factor for systems with bounded noise. *IEEE Transactions on Information Theory*, 33(3):383–392, 1987.
- [10] G. Ferreres and M. M'Saad. Estimation of output error models in the presence of unknown but bounded disturbances. *International Journal of Adaptive Control and Signal Processing*, 11(2):115–140, 1997.
- [11] E. Fogel and Y.F. Huang. On the value of information in system identification - bounded noise case. *Automatica*, 18(2):229–238, 1982.
- [12] U. Forssell and L. Ljung. Closed-loop identification revisited. *Automatica*, 35(7):1215–1241, 1999.
- [13] M. Gilson and P.M.J. Van Den Hof. Instrumental variable methods for closed-loop system identification. *Automatica*, 41(2):241–249, 2005.
- [14] T. Katayama and H. Tanaka. An approach to closed-loop subspace identification by orthogonal decomposition. *Automatica*, 43(9):1623–1630, 2007.
- [15] I.D. Landau and A. Karimi. Recursive algorithms for identification in closed-loop: a unified approach and evaluation. *Automatica*, 33(8):1499–1523, 1997.
- [16] M. Poulliquen, O. Gehan, E. Pigeon, and M. Frikel. Closed-loop output error identification with bounded disturbances. In *16th IFAC Symposium on System Identification-Brussels*, 2012.
- [17] M. Poulliquen, E. Pigeon, and O. Gehan. Output error identification for multi-input multi-output systems with bounded disturbances. In *50th IEEE Conference on Decision and Control-Orlando*, 2011.
- [18] G. Tan, C. Wen, and Y.C. Soh. Identification for systems with bounded noise. *IEEE Transactions on Automatic Control*, 42(7):998–1001, 1997.
- [19] P.M.J. Van Den Hof and R.J.P. Schrama. An indirect method for transfer function estimation from closed-loop data. *Automatica*, 29(6):1523–1527, 1993.
- [20] P. Van Overschee and B. De Moor. Closed-loop subspace system identification. *Proceedings 36th Control and Decision Conference, San Diego*, 1997.
- [21] M. Verhaegen. Application of a subspace model identification technique to identify lti systems operating in closed-loop. *Automatica*, 29(4):1027–1040, 1993.