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REALIZING COMPACTLY GENERATED PSEUDO-GROUPS OF DIMENSION ONE

ABSTRACT. Many compactly generated pseudo-groups of local transformations on 1-manifolds are realizable as the transverse dynamic of a foliation of codimension 1 on a compact manifold of dimension 3 or 4.

Ga'el Meigniez ¹

After C. Ehresmann, [2], given a foliation \mathcal{F} of codimension q on a compact manifold M , its transverse dynamics is represented by its *holonomy pseudo-group* of local transformations on any exhaustive transversal T . The inverse problem has been raised by A. Haefliger: given a pseudo-group of local transformation of some manifold of dimension q , realize it, if possible, as the dynamic of some foliation of codimension q on some compact manifold. The difficulty here lies in the compactity. More precisely, Haefliger discovered a necessary condition: the pseudo-group must be *compactly generated* [5][7]. He asked if this condition is sufficient.

The present paper intends to study the case $q = 1$.

A counterexample is known: there exists a compactly generated pseudo-group of local transformations of the line, which is not realizable. It contains a *paradoxical Reeb component*: a full subpseudo-group equivalent to the holonomy of a Reeb component, but whose boundary orbit has some complicated isotropy group on the exterior side [8].

The object of the present paper is, on the contrary, to give a *positive* answer to Haefliger's question for many pseudo-groups of dimension one.

Recall that a codimension-1 foliation is (topologically) *taut* if through every point there passes a transverse loop, or a transverse path with extremities on ∂M (we refer e.g. to [1] for elements on foliations). Equivalently, the foliation has no *dead end component*. These notions are easily translated for pseudo-groups: one has the notions of tautness and of dead end components for a pseudo-group of dimension 1 (see paragraph 1.3 below). It turns out that, to realize a given compactly generated pseudo-group of dimension 1, the extra necessary and/or sufficient conditions that we find, bear on the isotropy groups of the closed orbits bounding the dead end components, if any.

We also pay attention to the dimension of the realization. Of course, a pseudo-group which is realized by some foliation \mathcal{F} on some manifold M , is also realized by the pullback of \mathcal{F} into $M \times \mathbf{S}^1$. One can ask to realize a

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pseudo-group, if possible, in the smallest possible dimension. The dynamics of the foliations on surfaces being very constraint, the dimension 3 will be in general the first candidate.

There is a well-known constraint specific to dimension 3 in the nontaut case. Namely, remember that for elementary Euler characteristic reasons, in every compact foliated 3-manifold which is not taut, every leaf bounding a dead end component is a 2-torus, an annulus, a Klein bottle, or a M'obius strip (S. Goodman) [1]. This phenomenon has a counterpart in the holonomy pseudo-group: for every orbit bounding a dead end component, its isotropy group is commutative of rank at most two.

A few precisions must be given before the results.

Equivalence: the pseudo-groups must be considered up to an equivalence called *Haefliger equivalence*. Given two different exhaustive transversals for a same foliated manifold, the two holonomy pseudo-groups are Haefliger-equivalent [5][6][7]. A foliated manifold is said to *realize* a pseudo-group G if its holonomy pseudo-group on any exhaustive transversal is Haefliger-equivalent to G .

Differentiability: The given pseudo-group being of class C^r , $0 \leq r \leq \infty$, the realizing foliation will be $C^{\infty,r}$, that is, globally C^r and tangentially smooth [1]. We also consider the pseudo-groups of class PL : the realizing foliations will be $C^{\infty,PL}$.

Orientation: for simplicity, we consider only the *orientable* pseudo-groups of dimension one, corresponding to the transversely orientable foliations of codimension one. However, we don't always ask the realizing foliations to be tangentially orientable, nor, equivalently, the ambient manifolds to be orientable.

Boundaries: by a "foliated manifold", we understand a manifold M with a smooth boundary (maybe empty), endowed with a foliation \mathcal{F} such that each connected component of ∂M is either a leaf of \mathcal{F} or transverse to \mathcal{F} . So, ∂M splits into a tangential boundary $\partial_{\parallel} M$, which is seen in the holonomy pseudo-group, and a transverse boundary $\partial_{\perp} M$, which is not. However, in the realization problem, the choice of allowing a transverse boundary or not, only affects the dimension of the realization. For, if G is realized by some foliation \mathcal{F} on some manifold M , with some transverse boundary components, then it is also realized, without transverse boundary components, by the pullback of \mathcal{F} in a manifold of one more dimension, namely:

$$(M \times \mathbf{S}^1) \cup_{(\partial_{\perp} M \times \mathbf{S}^1)} (\partial_{\perp} M \times \mathbf{D}^2)$$

THEOREM A. *Every orientable pseudo-group of dimension 1 which is compactly generated and taut, is realized by some foliated compact orientable 3-manifold, without transverse boundary.*

Essentially, our method is that the pseudo-group is first easily realized as the dynamic of a Morse-singular foliation on a compact 3-manifold. The singularities are of Morse indices 1 and 2, in equal number. Then, thanks

to tautness, from every singularity of index 2 there is a positively transverse path to some (distant) singularity of index 1 . Thanks to some geometric manifestation of compact generation, the pair is cancelled, *not* in Morse's way, but rather by the means of an elementary surgery of index 2 performed on the manifold, without changing the dynamic of the foliation.

We also get a *characterization* of the dynamics of all foliations, taut or not, on compact 3-manifolds. It turns out that there is an interplay between the conditions of orientability and boundaries.

THEOREM B. *Let G be an orientable pseudo-group of dimension 1 . Then, the three following properties are equivalent.*

- (1) *G is realizable by some foliated compact 3-manifold (orientable, possibly with a transverse boundary).*
- (2) *G is realizable by some foliated compact 3-manifold (without transverse boundary, possibly nonorientable).*
- (3) *G is compactly generated; and for every orbit of G in the boundary of every dead end component, its isotropy group is commutative of rank at most 2 .*

As a basic but fundamental example, the pseudo-group of local transformations of the real line generated by two homotheties $t \mapsto \lambda t$, $t \mapsto \mu t$, with $\lambda, \mu > 0$ and $\log \mu / \log \lambda \notin \mathbf{Q}$, verifies the conditions of theorem B, and is *not* realizable by any foliated compact orientable 3-manifold without boundary — see [ref]. We know no simple, necessary and sufficient conditions for realizing a pseudo-group on an orientable, compact 3-manifold without transverse boundary.

More generally, skipping the condition of rank at most 2:

THEOREM C. *Let G be an orientable pseudo-group of dimension 1 which is compactly generated and such that, for every orbit in the boundary of every dead end component, its isotropy group is commutative. Then, G is realizable by some foliated compact orientable 4-manifold, without transverse boundary.*

COROLLARY 0.1. *Every orientable pseudo-group of dimension 1 and of class PL which is compactly generated, is realizable (in dimension 4).*

There remain several open questions between these positive results and the negative result of [8].

Regarding the isotropy groups of the orbits bounding the dead end components, the zoology of the groups that always allow a realization in high dimension, remains obscure (and may be intractable).

Consider a pseudo-group G of dimension 1 which is compactly generated. If G is real analytic, is it necessarily realizable? If G is realizable, is it necessarily realizable in dimension 4 ?

Also, beyond the realization problem itself, one can ask for a more universal property. Call G *universally realizable* if there is a system of foliated

compact manifolds, each realizing G , and of foliation-preserving embeddings, whose inductive limit is a Haefliger classifying space for G . One can prove (not tackled in the present paper) that every compactly generated pseudo-group of dimension one and class PL is universally realizable. What if we change PL for "taut"? for "real analytic"?

The problem is that the method of the present paper, essentially the cancellation of a pair of distant singularities of indices 1 and $n - 1$ in a Morse-singular foliation on a n -manifold by an elementary surgery without changing the dynamic, is very specific to these indices, and we know no equivalent e.g. for a pair of singularities of index 2 in ambient dimension 4.

1. PRELIMINARIES ON PSEUDO-GROUPS

In the two first paragraphs 1.1 and 1.2, we recall concepts and facts about Haefliger equivalence and compact generation, in a form that fits our purposes. The material here is essentially due to Haefliger. In the third paragraph 1.3, we translate into the frame of pseudo-groups of dimension 1, the notion of topological tautness, which is classical in the frame of foliations of codimension 1.

In \mathbf{R}^n , one writes \mathbf{D}^n the compact unit ball; \mathbf{S}^{n-1} its boundary; * the basepoint $(1, 0, \dots, 0) \in \mathbf{S}^{n-1}$; and dx_n the foliation $x_n = \text{constant}$. "Smooth" means C^∞ .

1.1. Pseudo-groups and Haefliger equivalences. An arbitrary differentiability class is understood. Let T, T' be manifolds of the same dimension, not necessarily compact. Smooth boundaries are allowed.

A *local transformation* from T to T' is a diffeomorphism γ between two nonempty, topologically open subsets $Dom(\gamma) \subset T$, $Im(\gamma) \subset T'$. Note that the boundary is necessarily invariant :

$$Dom(\gamma) \cap \partial T = \gamma^{-1}(\partial T')$$

Given also a local transformation γ' from T' to T'' , the compose $\gamma'\gamma$ is defined whenever $Im(\gamma)$ meets $Dom(\gamma')$ (an inclusion is not necessary), and one has :

$$Dom(\gamma'\gamma) = \gamma^{-1}(Dom(\gamma'))$$

Given two sets of local transformations A, B , as usual, AB denotes the set of the composes of all composable pairs $\alpha\beta$, where $\alpha \in A$ and $\beta \in B$. Also, 1_U denotes the identity map of the set U .

DEFINITION 1.1. [10] *A pseudo-group on a manifold T is a set G of local self-transformations of T such that :*

- (1) *For every nonempty, topologically open $U \subset T$, the identity map 1_U belongs to G ;*
- (2) *$GG = G^{-1} = G$;*

- (3) For every local self-transformation γ of T , if $\text{Dom}(\gamma)$ admits an open cover (U_i) such that every restriction $\gamma|_{U_i}$ belongs to G , then γ belongs to G .

Then, by (1) and (2), G is also stable by restrictions: if γ belongs to G and if $U \subset \text{Dom}(\gamma)$ is nonempty open, then $\gamma|_U$ belongs to G .

Example 1. Every set S of local self-transformations of T is contained in a smallest pseudo-group $\langle S \rangle$ containing S , called the pseudo-group *generated* by S . A local transformation γ of T belongs to $\langle S \rangle$ if and only if, in a neighborhood of every point in its domain, γ splits as a compose $\sigma_\ell \dots \sigma_1$, with $\ell \geq 0$ and $\sigma_1, \dots, \sigma_\ell \in (S \cup S^{-1})$.

Example 2. Given a pseudo-group (G, T) , and a nonempty open subset $U \subset T$, one has on U a *restricted* pseudo-group $G|_U := 1_U G 1_U$: the set of the elements of G whose domains and images are both contained in U .

Example 3. More generally, given a pseudo-group (G, T) , a manifold T' , and a set F of local transformations from T' to T , one has on T' a *pullback* pseudo-group $F^*(G) := \langle F^{-1} G F \rangle$.

Under a pseudo-group (G, T) , every point $t \in T$ has :

- (1) An *orbit* $G(t)$: the set of the images $\gamma(t)$ through the local transformations $\gamma \in G$ defined at t ;
- (2) An *isotropy group* G_t : the group of the germs at t of the local transformations $\gamma \in G$ defined at t and fixing t .

Call an open subset $T' \subset T$ *exhaustive* if T' meets every orbit. Call the pseudo-group G *cocompact* if T admits a relatively compact exhaustive open subset. Call the pseudo-group G *connected* if every two points of T are linked by a finite sequence of points of T , of which every two consecutive ones lie in the same orbit or in the same connected component of T . Obviously, every pseudo-group splits as a disjoint sum of connected ones.

Let (M, \mathcal{F}) be a manifold foliated in codimension q . A smooth boundary is allowed, in which case each connected component of ∂M must be tangent to \mathcal{F} or transverse to \mathcal{F} . One writes $\partial_{\parallel} M$ the union of the tangential components. By a *transversal*, one means a q -manifold T immersed into M transversely to \mathcal{F} , not necessarily compact, and such that $\partial T = T \cap \partial_{\parallel} M$. One calls T *exhaustive* (or *total*) if it meets every leaf.

DEFINITION 1.2. [2] *The holonomy pseudo-group $\text{Hol}(\mathcal{F}, T)$ of a foliation \mathcal{F} on an exhaustive transversal T is the pseudo-group generated by the local transformations γ of T for which there exists a map*

$$f_\gamma : [0, 1] \times \text{Dom}(\gamma) \rightarrow M$$

such that :

- $f_\gamma \pitchfork \mathcal{F}$ and $f_\gamma^* \mathcal{F}$ is the slice foliation on $[0, 1] \times \text{Dom}(\gamma)$, whose leaves are the $[0, 1] \times t$'s ($t \in \text{Dom}(\gamma)$);
- $f_\gamma(0, t) = t$ and $f_\gamma(1, t) = \gamma(t)$, for every $t \in \text{Dom}(\gamma)$.

This holonomy pseudo-group does represent the dynamic of the foliation: there is a one-to-one correspondence $L \mapsto L \cap T$ between the leaves of \mathcal{F} and the orbits of $Hol(\mathcal{F}, T)$; a topologically closed orbit corresponds to a topologically closed leaf; the isotropy group of $Hol(\mathcal{F}, T)$ at any point is isomorphic with the holonomy group of the corresponding leaf; etc.

DEFINITION 1.3. [4] *A Haefliger equivalence between two pseudo-groups (T_i, G_i) ($i = 0, 1$) is a pseudo-group G on the disjoint union $T_0 \sqcup T_1$, such that $G|_{T_i} = G_i$ ($i = 0, 1$) and that every orbit of G meets both T_0 and T_1 .*

Example 1. The two holonomy pseudo-groups of a same foliation on two exhaustive transversals are Haefliger equivalent.

Example 2. The restriction of a pseudo-group (G, T) to any exhaustive open subset of T is Haefliger-equivalent to (G, T) .

Example 3. More generally, let (G, T) be a pseudo-group, and let F be a set of local transformations from T' to T . Assume that:

- (1) $FF^{-1} \subset G$;
- (2) $\cup_{\phi \in F} Dom(\phi) = T'$;
- (3) $\cup_{\phi \in F} Im(\phi)$ is G -exhaustive in T .

Then, the pseudo-group $\langle F \cup G \rangle$ on $T \sqcup T'$ is a Haefliger equivalence between (G, T) and $(F^*(G), T')$.

The Haefliger equivalence is actually an equivalence relation between pseudo-groups. Given two Haefliger equivalences: G between (T_0, G_0) and (T_1, G_1) , and G' between (T_1, G_1) and (T_2, G_2) , one forms the pseudo-group $\langle G \cup G' \rangle$ on $T_0 \sqcup T_1 \sqcup T_2$. Then, $\langle G \cup G' \rangle|(T_0 \sqcup T_2)$ is a Haefliger equivalence between (T_0, G_0) and (T_2, G_2) .

Every Haefliger equivalence induces a one-to-one correspondence between the orbit spaces T_i/G_i ($i = 0, 1$). A closed orbit corresponds to a closed orbit. The isotropy groups at points on corresponding orbits are isomorphic.

1.2. Compact generation. Let (T, G) be a pseudo-group. We say that $\gamma \in G$ is (G) -*extendable* if there exists some $\bar{\gamma} \in G$ such that $Dom(\gamma)$ is contained *and relatively compact* in $Dom(\bar{\gamma})$, and that $\gamma = \bar{\gamma}|_{Dom(\gamma)}$. The compose of two extendable elements is also extendable. The inverse of an extendable element is also extendable.

DEFINITION 1.4. (Haefliger)[5] *A pseudo-group (T, G) is compactly generated if there are an exhaustive, relatively compact, open subset $T' \subset T$, and finitely many elements of $G|_{T'}$ which are G -extendable, and which generate $G|_{T'}$.*

PROPOSITION 1.5. (Haefliger)[5][7] *Compact generation is invariant by Haefliger equivalence.*

PROPOSITION 1.6. (Haefliger)[5][7] *The holonomy pseudo-group of every foliated compact manifold is compactly generated.*

We shall also use the following fact, which amounts to say that the choice of T' is arbitrary.

LEMMA 1.7. [5][7] *Let (T, G) be a compactly generated pseudo-group, and $T'' \subset T$ be any exhaustive, relatively compact, open subset. Then there are finitely many elements of $G|T''$ that are extendable in G , and that generate $G|T''$.*

Note: pseudo-groups vs. groupoids. Despite some appearances, the above definition of compact generation is relevant; in particular because it is preserved through Haefliger equivalences. Recently, N. Raimbaud has shown that compact generation has a somewhat more natural generalization in the frame of topological groupoids. Write $\Gamma(T)$ the topological groupoid of the germs of local transformations of T . Let G be any pseudo-group on T . Then, the set Γ of germs $[g]_t$, for all $g \in G$ and all $t \in \text{Dom}(g)$, is in $\Gamma(T)$ an open subgroupoid whose space of objects is the all of T . It is easily verified that one gets this way a bijection between the set of pseudo-groups on T and the set of open subgroupoids in $\Gamma(T)$ whose space of objects is T . The pseudo-group G is compactly generated if and only if the topological groupoid Γ contains an exhaustive, relatively compact, open subset, which generates a full subgroupoid [9].

1.3. Tautness for pseudo-groups of dimension 1. We now consider a pseudo-group (G, T) of dimension 1, that is, $\dim T = 1$, and oriented, that is, T is oriented and G is orientation-preserving. From now on, all pseudo-groups will be understood of dimension 1 and oriented.

By a *positive arc* $[t, t']$ of origin t and extremity t' , we mean an orientation-preserving embedding of the interval $[0, 1]$ into T sending 0 to t and 1 to t' .

A *positive chain* is a finite sequence of positive arcs, such that the extremity of each (but the last) lies on the same orbit as the origin of the next. A *positive loop* is a positive chain such that the extremity of the last arc lies on the same orbit at the origin of the first.

DEFINITION 1.8. *A oriented pseudo-group (G, T) of dimension 1 is taut if every point of T lies either on a positive chain whose origin and extremity belong to ∂T , or on a positive loop.*

PROPOSITION 1.9. *Let (G, T) be a cocompact, oriented pseudo-group of dimension 1. Then, (G, T) is taut if and only if it is Haefliger-equivalent to some pseudo-group (G', T') s.t. T' is a finite disjoint union of compact intervals and circles.*

Proof. One first easily verifies that tautness is invariant by Haefliger equivalence. "If" follows.

Conversely, given a taut cocompact pseudo-group (T, G) , by cocompacity there is a finite family C of positive chains, each being a loop or having

extremities on ∂T , such that every orbit of G meets on at least one of them.

Consider one of these chains $c = ([t_i, t'_i])$ ($0 \leq i \leq \ell(c)$) which is not a loop: its origin t_0 and extremity $t'_{\ell(c)}$ lie on ∂T . For every $1 \leq i \leq \ell(c)$, one has $t_i = g_i(t'_{i-1})$ for some $g_i \in G$ whose domain and image are small. Let

$$U_0 := [t_0, t'_0] \cup \text{Dom}(g_1)$$

$$U_{\ell(c)} := \text{Im}(g_{\ell(c)}) \cup [t_{\ell(c)}, t'_{\ell(c)}]$$

and for each $1 \leq i \leq \ell(c) - 1$, let

$$U_i := \text{Im}(g_i) \cup [t_i, t'_i] \cup \text{Dom}(g_{i+1})$$

One makes an abstract copy U'_i of each U_i . Write $f_{c,i} : U'_i \rightarrow U_i$ the identity. These abstract copies are glued together by means of the g_i 's into a single compact segment T'_c . Thus, T'_c has an atlas of maps which are local transformations $f_{c,i}$ ($0 \leq i \leq \ell(c)$) from T'_c onto T , such that every change of map $g_i = f_i f_{i-1}^{-1}$ belongs to G . The images of the f_i 's cover the chain c .

In the same way, for every $c \in C$ which is a loop, one makes a circle T'_c together with an atlas $f_{c,i}$ ($0 \leq i \leq \ell(c)$) of maps which are local transformations from T'_c to T , such that every change of map belongs to G . The images of the maps cover the chain c .

Let T' be the disjoint union of the T'_c 's, for $c \in C$. By the example 3 above after the definition 1.3, G is Haefliger-equivalent to the pseudo-group $F^*(G)$ of local transformations of T' . \square

In case (G, T) is connected, one can be more precise (left as an exercise):

PROPOSITION 1.10. *Let (G, T) be a connected, cocompact, oriented pseudo-group of dimension 1. Then, (G, T) is taut if and only if it is Haefliger-equivalent to some pseudo-group (G', T') s.t. T' is either a finite disjoint union of compact intervals, or a single circle.*

Our last lemma has no relation to tautness. For a compactly generated pseudo-group of dimension one, one can give a more precise form to the generating system defining compact generation:

LEMMA 1.11. *Let (T, G) be a compactly generated pseudo-group of dimension 1. Then :*

- (1) *There is a G -exhaustive, open, relatively compact $T' \subset T$ which has finitely many connected components;*
- (2) *For every T' as above, $G|T'$ admits a finite set of G -extendable generators whose domains and images are intervals.*

Proof. (1) The pseudo-group G being compactly generated, is in particular cocompact: there is a compact $K \subset T$ meeting every orbit. Being compact, K meets only finitely many connected components T_i of T . For each i , let $T'_i \subset T_i$ be relatively compact, open, connected, and contain $K \cap T_i$. Then,

$T' := \cup_i T'_i$ is G -exhaustive, open, relatively compact, and has finitely many connected components.

(2) By the lemma 1.7, $G|T'$ admits a finite set (g_i) ($i = 1, \dots, p$) of G -extendable generators. For each $1 \leq i \leq p$, let \bar{g}_i be a G -extension of g_i . Let $U_i \subset \text{Dom}(\bar{g}_i)$ be open, relatively compact, contain $\text{Dom}(g_i)$, and have finitely many connected components. Then, $U_i \cap T'$ has finitely many connected components. Each of these components is either an interval or a circle. In the second case, we cover this circle by two open intervals. We get a cover of $U_i \cap T'$ by a finite family (I_j) ($j \in J_i$) of intervals open and relatively compact in $\text{Dom}(\bar{g}_i)$. The finite family $(\bar{g}_i|I_j)$ ($1 \leq i \leq p, j \in J_i$) is G -extendable and generates $G|T'$. \square

2. PROOF OF THEOREM A

We are given a taut, compactly generated pseudo-group (G, T) of dimension 1 and class C^r , $0 \leq r \leq \infty$, or PL . We have to realize G as the holonomy pseudo-group of some foliated compact 3-manifold. By proposition 1.9, we can assume that T is compact: a finite disjoint union of compact intervals and circles. By lemma 1.11 applied to $T' = T$, the pseudo-group G admits a finite system g_1, \dots, g_p of G -extendable generators whose domains and images are intervals.

The proof uses Morse-singular foliations. It would be natural to define them as the Haefliger structures whose singularities are quadratic, but this would lead to irrelevant technicalities. A simpler concept will do.

DEFINITION 2.1. *A Morse foliation \mathcal{F} on a smooth n -manifold M is a foliation of codimension one and class $C^{\infty, r}$ on the complement of finitely many singular points, such that on some open neighborhood of each, \mathcal{F} is conjugate to the level hypersurfaces of some nondegenerate quadratic form on some neighborhood of 0 in \mathbf{R}^n . The conjugation must be C^0 ; it must be smooth except maybe at the singular point.*

We write $\text{Sing}(\mathcal{F}) \subset M$ the finite set of singularities. Note that \mathcal{F} is smooth on some neighborhood of $\text{Sing}(\mathcal{F})$, minus $\text{Sing}(\mathcal{F})$. The holonomy pseudo-group of \mathcal{F} is defined, on any exhaustive transversal disjoint from the singularities, as the holonomy pseudo-group of the regular foliation $\mathcal{F}|(M \setminus \text{Sing}(\mathcal{F}))$.

We shall first realize (G, T) as the holonomy pseudo-group of a Morse foliation on a compact 3-manifold. Then, compact generation will allow us to surgerize this manifold and regularize the foliation, without changing its transverse structure.

To fix ideas, in a first time we assume that T is without boundary: that is, a finite disjoint union of circles.

Let $M_0 := \mathbf{S}^2 \times T$ and let \mathcal{F}_0 be the foliation of M_0 by 2-spheres: its holonomy pseudo-group on the exhaustive transversal $* \times T$ is the trivial pseudo-group. Write $\text{pr}_2 : M_0 \rightarrow T$ the second projection.

For every $1 \leq i \leq p$, write $(u_i, u'_i) \subset T$ the open interval that is the domain of g_i , and write (v_i, v'_i) the image of g_i . Fix some extension $\bar{g}_i \in G$.

Choose two embeddings $e_i : \mathbf{D}^3 \rightarrow M_0$ and $f_i : \mathbf{D}^3 \rightarrow M_0$ s.t.

- (1) $e_i(\mathbf{D}^3)$ and $f_i(\mathbf{D}^3)$ are disjoint from each other and from $T \times *$;
- (2) $\text{pr}_2(e_i(\mathbf{D}^3)) = [u_i, u'_i]$ and $\text{pr}_2(f_i(\mathbf{D}^3)) = [v_i, v'_i]$;
- (3) $e_i^* \mathcal{F}_0 = f_i^* \mathcal{F}_0$ is the trivial foliation $dx_3|_{\mathbf{D}^3}$;
- (4) $\text{pr}_2 \circ f_i = \bar{g}_i \circ \text{pr}_2 \circ e_i$.

We perform on M_0 an elementary surgery of index 1 by cutting the interiors of $e_i(\mathbf{D}^3)$ and of $f_i(\mathbf{D}^3)$, and by pasting their boundary 2-spheres. The points $e_i(x)$ and $f_i(x)$ are pasted, for every $x \in \partial \mathbf{D}^3$.

We perform such a surgery on M_0 for every $1 \leq i \leq p$, choosing of course the embeddings e_i, f_i two by two disjoint. Let M_1 be the surgerized manifold.

Obviously, \mathcal{F}_0 induces on M_1 a Morse foliation \mathcal{F}_1 , with $2p$ singularities, one at every point $s_i := e_i(0, 0, -1) = f_i(0, 0, -1)$, of Morse index 1; and one at every point $s'_i := e_i(0, 0, +1) = f_i(0, 0, +1)$, of Morse index 2. It is easy and standard to endow M_1 with a smooth structure, such that \mathcal{F}_1 is of class $C^{\infty, r}$, and smooth in a neighborhood of the singularities, minus the singularities.

By (4), the holonomy of \mathcal{F}_1 on the transversal $T \times * \cong T$ is generated by the local transformations g_i . That is, it coincides with G .

Up to now, we have not used fully the fact that G is compactly generated. Now, we point a consequence of this fact, which is actually its geometric translation.

Consider in general some Morse foliation \mathcal{X} on some 3-manifold X , and some singularity s of index 1. On some neighborhood of s , the Morse foliation \mathcal{X} admits the first integral $Q := -x_0^2 + x_1^2 + x_2^2$ in some continuous local coordinates x_0, x_1, x_2 , smooth except maybe at the singularity. The two components of the singular cone at s , namely $Q^{-1}(0) \cap \{x_0 < 0\}$ and $Q^{-1}(0) \cap \{x_0 > 0\}$, may either belong to the same leaf of the regular foliation $\mathcal{X}|_{(X \setminus \text{Sing}(\mathcal{X}))}$, or not. If they do, then there is a loop $\lambda : [0, 1] \rightarrow X$ s.t.

- $\lambda(0) = \lambda(1) = s$;
- λ is tangential to \mathcal{X} ;
- $\lambda(t) \notin \text{Sing}(\mathcal{X})$ for every $0 < t < 1$;
- $x_0(\lambda(t)) \leq 0$ (resp. ≥ 0) for every t close enough to 0 (resp. 1).

Such a loop has a holonomy germ $h(\lambda)$ on the pseudo-transversal arc $x_0 = x_1 = 0, x_2 \geq 0$. This is the germ at 0 of some homeomorphism of the nonnegative half-line.

DEFINITION 2.2. *If moreover the holonomy $h(\lambda)$ is the identity, then we call λ a Levitt loop for \mathcal{X} at s .*

In the same way, at every singularity of index 2, the Morse foliation \mathcal{X} admits the first integral $Q' := x_0^2 - x_1^2 - x_2^2$ in some local coordinates $x_0,$

x_1, x_2 . The notion of a Levitt loop is defined symmetrically by reversing the transverse orientation of \mathcal{X} .

LEMMA 2.3. *The Morse foliation \mathcal{F}_1 admits a Levitt loop at every singularity.*

Proof. Consider e.g. a singularity s_i of index 1. In M_0 , let a be a path from the point $(u_i, *)$ to the point $e_i(0, 0, -1)$ in the sphere $u_i \times \mathbf{S}^2$; and let b be a path from $(v_i, *)$ to $f_i(0, 0, -1)$ in the sphere $v_i \times \mathbf{S}^2$. Then, in M_1 , the path ab^{-1} is tangential to \mathcal{F}_1 and passes through s_i . Obviously, the holonomy $h(ab^{-1})$ of \mathcal{F}_1 on $T \times * \cong T$ along this path is well-defined on the right-hand side of u_i . That is, $h(ab^{-1})$ is a germ of homeomorphism of T from some interval $[u_i, u_i + \epsilon) \subset T$ onto some interval $[v_i, v_i + \eta) \subset T$. By properties (2) through (4) above,

$$h(ab^{-1}) = g_i|_{[u_i, u_i + \epsilon)}$$

On the other hand, recall that $\bar{g}_i \in G$. Since G is the holonomy pseudo-group of \mathcal{F}_1 on $T \times *$, there is in $M \setminus \text{Sing}(\mathcal{F}_1)$, a path c from u_i to v_i , tangential to \mathcal{F}_1 , and whose holonomy on $T \times *$ is the germ of \bar{g}_i at u_i . Then, $\lambda := a^{-1}cb$ is a Levitt loop at s_i . \square

To simplify the argument in the rest of the construction, it is convenient (although in fact not necessary) that \mathcal{F}_1 admit at each singularity a *simple* Levitt loop. We can get this extra property as follows. Let s be a singularity of \mathcal{F}_1 , let λ be a Levitt loop for \mathcal{F}_1 at s , and let L be the leaf singular at s , containing λ . After a generic perturbation of λ in L , the loop λ is immersed and self-transverse in L . Let x be a self-intersection point of λ . Since \mathcal{F}_1 is taut, there passes through x an embedded transverse circle $C \subset M_1$, disjoint from $T \times *$. We surgerize M_1 , cutting a small tubular neighborhood $N \cong \mathbf{D}^2 \times \mathbf{S}^1$ of C in M_1 , in which \mathcal{F}_1 is the foliation by the $\mathbf{D}^2 \times t$'s; and we glue $\Sigma \times \mathbf{S}^1$, where Σ is the compact connected orientable surface of genus 1 bounded by one circle, foliated by the $\Sigma \times t$'s. The holonomy pseudo-group of the foliation \mathcal{F}_1 on $T \times *$ is not changed. After the surgery, there is at s a Levitt loop with one less self-intersection. Of the two pieces of λ that passed through x , now one passes in the new handle and is disjoint from the other.

After a finite number of such surgeries, for every $1 \leq i \leq p$, the Morse foliation \mathcal{F}_1 admits at the singularity s_i (resp. s'_i) a simple Levitt loop λ_i (resp. λ'_i).

Fix some $1 \leq i \leq p$. We shall somewhat cancel the pair of singularities s_i and s'_i of \mathcal{F}_1 , at the price of a surgery on M_1 , without changing the transverse structure of \mathcal{F}_1 .

First, we use fully the fact that G is taut: there is a path $p_i : [0, 1] \rightarrow M_1$ from $p_i(0) = s'_i$ to $p_i(1) = s_i$, and positively transverse to \mathcal{F}_1 except at its endpoints.

The geometry is as follows (figure 1). Let $Q(x_1, x_2, x_3)$ be a quadratic form of Morse index 1 w.r.t. some local system of coordinates at s_i , which is a local first integral for \mathcal{F}_1 . Then, p_i arrives at s_i by one of the two

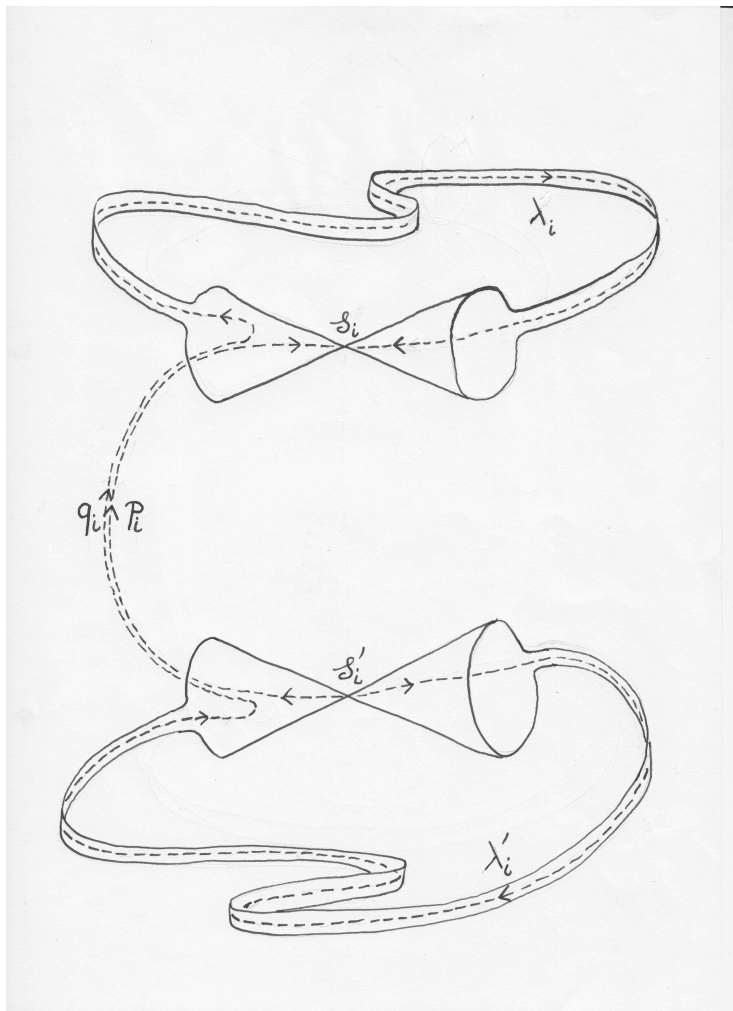


FIGURE 1

components of the cone $Q < 0$. Reversing if necessary the orientation of λ_i , one can arrange that λ_i quits s_i in the boundary of the same half cone. Symmetrically, let $Q'(x_1, x_2, x_3)$ be a quadratic form of Morse index 2 w.r.t. some local system of coordinates at s'_i , which is a local first integral for \mathcal{F}_1 . Then, p_i quits s'_i by one of the two components of the cone $Q' > 0$. Reversing if necessary the orientation of λ'_i , one can arrange that λ'_i arrives at s'_i in the boundary of the same half cone.

We shall surgerize M_1 , and modify \mathcal{F}_1 , in an arbitrarily small neighborhood of $\lambda'_i \cup p_i \cup \lambda_i$, to cancel the singularities s_i, s'_i , without changing the holonomy pseudo-group of the foliation.

To this aim, the composed path $\lambda'_i p_i \lambda_i$ (that is, λ'_i followed by p_i followed by λ_i) is homotoped, relatively to its endpoints, into some path q_i also

positively transverse to \mathcal{F}_1 , except at its endpoints $q_i(0) = s'_i$ and $q_i(1) = s_i$. The homotopy consists in pushing the two tangential Levitt loops to some nearby, positively transverse paths, and in rounding the two corners; it is C^0 -small.

Notice that p_i and q_i arrive at s_i by the two opposed components of the cone $Q < 0$. Symmetrically, p_i and q_i quit s'_i by the two opposed components of the cone $Q' > 0$.

By construction, q_i is \mathcal{F}_1 -equivalent to p_i , that is, the diffeomorphism $p_i(t) \mapsto q_i(t)$ belongs to the holonomy pseudo-group of \mathcal{F}_1 on the union of the two transversal open arcs $p_i \cup q_i \setminus \{s_i, s'_i\}$. Indeed, this diffeomorphism coincides with the holonomy of the loop λ_i (resp. λ'_i) for t close to 1 (resp. 0); and with the identity for the other values of t .

After a small, generic perturbation of p_i and q_i relative to their endpoints, we arrange that p_i and q_i are two embeddings of the interval into M_1 ; that they are disjoint, but at their endpoints; and also that they are disjoint from p_j, q_j for every $j \neq i$, and also disjoint from the transversal $T \times *$. After a C^r -small perturbation of \mathcal{F}_1 in some small neighborhood of p_i and q_i , relative to some small neighborhoods of their endpoints, \mathcal{F} is smooth in some neighborhood of $p_i \cup q_i$.

Now, we shall perform on M_1 an elementary surgery of index 2 along every embedded circle $p_i \cup q_i$ ($1 \leq i \leq p$) (figure 2). That is, we cut some small tubular neighborhood $N_i \cong \mathbf{S}^1 \times \mathbf{D}^2$ of $p_i \cup q_i$, and we paste $\mathbf{D}^2 \times \mathbf{S}^1$ (here the choice of the framing is irrelevant). We shall obtain a closed 3-manifold M . We shall, for a convenient choice of the N_i 's, extend the foliation $\mathcal{F}_1|(M_1 \setminus \cup_i N_i)$ to M , as a (regular) foliation \mathcal{F} , still admitting $T \times *$ as an exhaustive transversal, and whose holonomy pseudo-group on $T \times *$ will still be G .

To this end, first notice that, by definition 2.2, and since λ_i (resp. λ'_i) is a simple loop, there is some small open neighborhood U_i (resp. U'_i) of λ_i (resp. λ'_i) in M_1 , such that the foliation \mathcal{F}_1 admits in $U_i \setminus s_i$ (resp. $U'_i \setminus s'_i$) a first integral F_i (resp. F'_i) whose level sets are connected. Precisely, for every $t < F_i(s_i)$ (resp. $t > F'_i(s'_i)$), the level set $F_i^{-1}(t)$ (resp. $F'_i{}^{-1}(t)$) is an open disk. For every $t > F_i(s_i)$ (resp. $t < F'_i(s'_i)$), the level set $F_i^{-1}(t)$ (resp. $F'_i{}^{-1}(t)$) is the connected orientable open surface of genus one with one end.

Choose a compact 3-ball $B_i \subset U_i$ containing s_i and such that $F_i|_{B_i}$ is topologically conjugate to a quadratic form Q of signature $-++$, with three different eigenvalues, on the unit ball. Choose a compact 3-ball $B'_i \subset U'_i$ containing s'_i and such that $F'_i|_{B'_i}$ is topologically conjugate to a quadratic form Q' of signature $--+$, with three different eigenvalues, on the unit ball. Choose some tubular neighborhood S_i of the circle $p_i \cup q_i$, so thin that $S_i \cap \partial B_i$ (resp. $S_i \cap \partial B'_i$) is contained in the cone $Q < 0$ (resp. $Q' > 0$), and such that $\mathcal{F}_1|_{S_i}$ is a foliation by disks, except on the intersections of S_i

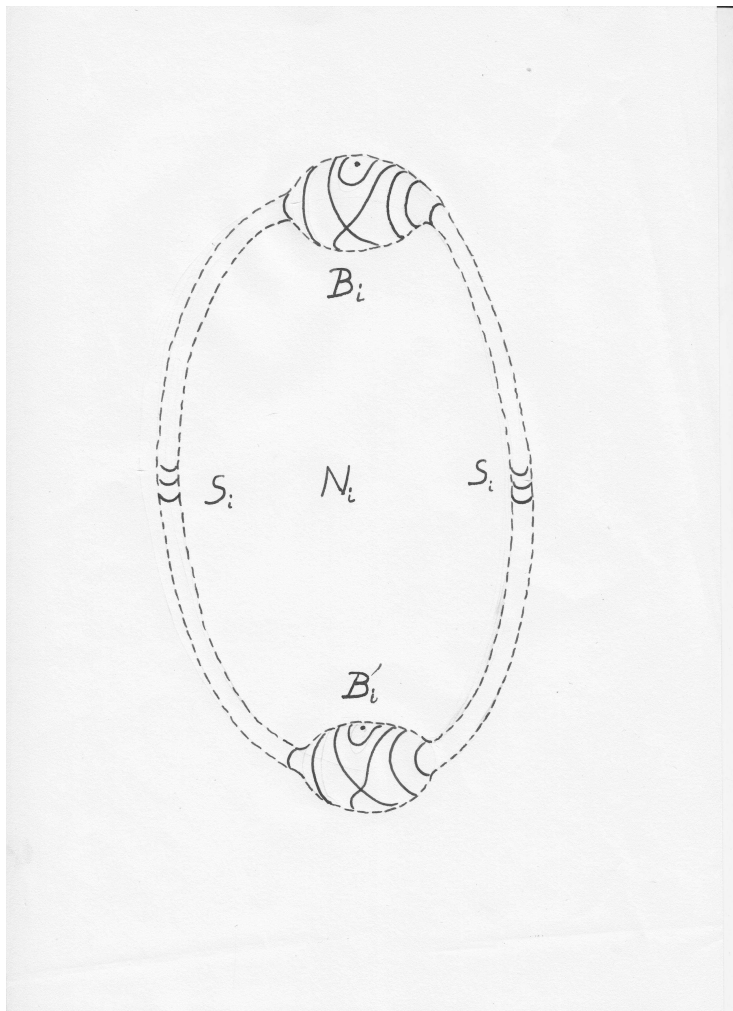


FIGURE 2

with $Q \geq 0$ and with $Q' \leq 0$. Define $N_i := B_i \cup B'_i \cup S_i$. We can arrange that N_i is a smooth solid torus.

Then, after reparametrizing the values of F_i and of F'_i , they obviously extend to a function F''_i on $N_i \setminus \{s_i, s'_i\}$ as follows.

- (1) F''_i is a first integral for \mathcal{F}_1 on $N_i \setminus \{s_i, s'_i\}$;
- (2) F''_i coincides with F_i on B_i and with F'_i on B'_i ;
- (3) $F''_i|_{\partial N_i}$ has exactly eight Morse critical points: two minima and two critical points of index 1 on $\partial B'_i$, two critical points of index 1 and two maxima on ∂B_i ;
- (4) The values of F''_i at these critical points are respectively $-2, -2, -1, -1, 1, 1, 2, 2$;

- (5) The sign of the tangency between F_i'' and ∂N_i at each critical point is as follows: the descending gradient of F_i'' exits N_i at the four critical points on $\partial B_i'$, and enters N_i at the four critical points on ∂B_i ;
- (6) One has $F_i''(p_i(u)) = F_i''(q_i(u))$ for every $u \in [0, 1]$.

On the other hand, in the handle $H_i := \mathbf{D}^2 \times \mathbf{S}^1$, one has the function $h := x_2(1 + y_1^2)$, where $\mathbf{D}^2 \subset \mathbf{R}^2$ (resp. $\mathbf{S}^1 \subset \mathbf{R}^2$) is defined by $x_1^2 + x_2^2 \leq 1$ (resp. $y_1^2 + y_2^2 = 1$). By (3), (4) and elementary Morse theory, $h|_{\partial H_i}$ is smoothly conjugate to $F_i''|_{\partial N_i}$. We attach H_i to $M \setminus \text{Int}(N_i)$ so that the functions F_i'' and h coincide on $\partial N_i \cong \partial H_i$. We extend \mathcal{F}_1 inside H_i as the foliation defined by h . By (5), the sign of the tangency between h and ∂H_i at each singularity is the same as the sign of the tangency between F_i'' and ∂N_i . So, the resulting foliation is regular.

Having done this for every pair of singularity $s_i, s_i', i = 1, \dots, p$, we get a regular foliation \mathcal{F} on a closed 3-manifold M .

We claim that \mathcal{F} admits $T \times *$ as an exhaustive transversal, and has the same holonomy pseudo-group G as \mathcal{F}_1 on $T \times *$. Obviously, \mathcal{F} has no leaf contained in any H_i . So, the claim amounts to verify the following. Let $\gamma : [0, 1] \rightarrow M_1$ (resp. M) be a path tangential to \mathcal{F}_1 (resp. \mathcal{F}) and whose endpoints belong to $M_1 \setminus \cup_i \text{Int}(N_i)$. Then, there is a path $\gamma' : [0, 1] \rightarrow M$ (resp. M_1) tangential to \mathcal{F} (resp. \mathcal{F}_1) with the same endpoints, and such that the holonomy of \mathcal{F}_1 (resp. \mathcal{F}) along γ is the same as the holonomy of \mathcal{F} (resp. \mathcal{F}_1) along γ' .

We can assume that γ is contained in some N_i (resp. H_i), with endpoints on $\partial N_i = \partial H_i$. Let $t := F_i''(\gamma(0)) = h(\gamma(0)) = F_i''(\gamma(1)) = h(\gamma(1))$.

First, consider the case where γ is contained in N_i and tangential to \mathcal{F}_1 . In the first subcase where $F_i''(s_i') < t < F_i''(s_i)$, the level set $F_i''^{-1}(t)$ is the disjoint union of two disks, so γ has the same endpoints as some path γ' contained in $\partial F_i''^{-1}(t)$, and we are done. In the second subcase $t \geq F_i''(s_i)$, consider the level set $F_i^{-1}(t) \subset U_i$. Obviously, the intersection of this level set with $U_i \setminus \text{Int}(B_i)$ is connected (a pair of pants). So, γ has the same endpoints as some path γ' contained in this intersection, and we are done. (If the endpoints of γ do not lie on the same connected component of the boundary of the annulus $F_i^{-1}(t) \cap B_i$, then the path γ' will be close to the Levitt loop λ_i). The third and last subcase $t \leq F_i''(s_i')$ is symmetric to the second.

Now, consider the second case where γ is contained in H_i and tangential to \mathcal{F} . In the first subcase where $F_i''(s_i') < t < F_i''(s_i)$, the level set $F_i''^{-1}(t)$ is the disjoint union of two disks centered respectively at $p_i(u)$ and $q_i(u)$ for some $0 < u < 1$ (by (6)). If both endpoints of γ belong to the same disk, then they are also the endpoints of some path γ' contained in $\partial F_i''^{-1}(t)$, and we are done. If the endpoints of γ don't belong to the same disk, then recall that the diffeomorphism $p_i(u) \mapsto q_i(u)$ between the transversals p_i and q_i belongs to the holonomy pseudo-group of \mathcal{F}_1 on $p_i \cup q_i$. By (6), we

are done. In the subcases $t \geq F_i''(s_i)$ and $t \leq F_i''(s'_i)$, the argument is the same as in the first case.

Theorem A is proved in the case of a pseudo-group (G, T) without boundary.

Now, let us prove theorem A for a taut, compactly generated pseudo-group (G, T) such that T has a boundary. One can assume that (G, T) is connected. Thus, one is reduced to the case where T is a finite disjoint union of compact intervals (proposition 1.10).

The construction is much as in the case without boundary. We stress the few differences.

We start from the manifold $M_0 := \mathbf{S}^2 \times T$. For some of the generators g_i , their domains and images meet the boundary, i.e. they are semi-open intervals. Consider for example a g_i whose domain meets the positive boundary $\partial_+ T$ (the boundary points where the tangent vectors which are positive w.r.t. the orientation of T , exit from T). That is, $Dom(g_i) = (u_i, u'_i]$ and $Im(g_i) = (v_i, v'_i]$ and $Dom(g_i) \cap \partial T = u'_i$ and $Im(g_i) \cap \partial T = v'_i$.

Such a generator will be introduced in the holonomy of the foliation by performing, somewhat, a *half* elementary surgery of index 1 on the manifold M_0 . Put for every n :

$$2^{-1}\mathbf{D}^n := \{(x_1, \dots, x_n) \in \mathbf{R}^n / x_1^2 + \dots + x_n^2 \leq 1, x_n \leq 0\}$$

Its boundary splits as the union of \mathbf{D}^{n-1} (the subset defined in $2^{-1}\mathbf{D}^n$ by $x_n = 0$) and $2^{-1}\mathbf{S}^{n-1}$ (the subset defined in $2^{-1}\mathbf{D}^n$ by $x_1^2 + \dots + x_n^2 = 1$). Fix some extension $\bar{g}_i \in G$.

Choose two embeddings $e_i : 2^{-1}\mathbf{D}^3 \rightarrow M_0$ and $f_i : 2^{-1}\mathbf{D}^3 \rightarrow M_0$ s.t.

- (1) $e_i^{-1}(\partial M_0) = f_i^{-1}(\partial M_0) = \mathbf{D}^2$;
- (2) $e_i(2^{-1}\mathbf{D}^3)$ and $f_i(2^{-1}\mathbf{D}^3)$ are disjoint from each other and from $T \times *$;
- (3) $\text{pr}_2(e_i(2^{-1}\mathbf{D}^3)) = [u_i, u'_i]$ and $\text{pr}_2(f_i(2^{-1}\mathbf{D}^3)) = [v_i, v'_i]$;
- (4) $e_i^* \mathcal{F}_0 = f_i^* \mathcal{F}_0$ is the trivial foliation dx_3 on $2^{-1}\mathbf{D}^3$;
- (5) $\text{pr}_2 \circ f_i = \bar{g}_i \circ \text{pr}_2 \circ e_i$.

We perform on M_0 a surgery by cutting $e_i(\mathbf{D}^3 \setminus 2^{-1}\mathbf{S}^2)$ and $f_i(\mathbf{D}^3 \setminus 2^{-1}\mathbf{S}^2)$ and by pasting $e_i(2^{-1}\mathbf{S}^2)$ with $f_i(2^{-1}\mathbf{S}^2)$. The points $e_i(x)$ and $f_i(x)$ are pasted, for every $x \in 2^{-1}\mathbf{S}^2$.

This surgery produces a single singularity $s_i := e_i(0, 0, -1) = f_i(0, 0, -1)$, of Morse index 1.

The case of a generator g_i whose domain meets $\partial_- T$ is of course symmetric.

After performing a surgery for every generator, we get a surgerized compact manifold M_1 , and a Morse foliation \mathcal{F}_1 induced by \mathcal{F}_0 on M_1 , with some singularities of indices 1 and 2. The boundary of M_1 is the disjoint union of two closed connected surfaces $\partial_- M_1, \partial_+ M_1$, both tangential to \mathcal{F}_1 .

At every point of $\partial_- M_1$ (resp. $\partial_+ M_1$), the tangent vectors positively transverse to \mathcal{F}_1 enter into (resp. exit from) M_1 . The holonomy pseudo-group of \mathcal{F}_1 on $T \times *$ coincides with G .

These singularities are eliminated one after the other (*not* by pairs). Let us eliminate e.g. a singularity s_i of index 1.

On the one hand, by tautness, there is a path p_i , positively transverse to \mathcal{F}_1 but at s_i , from $p_i(0) \in \partial_- M_1$ to $p_i(1) = s_i$.

On the other hand, by compact generation, \mathcal{F}_1 admits a Levitt loop λ_i at s_i . We can arrange that λ_i is a simple loop: if it has a transverse self-intersection x , then, by tautness, through x there passes an arc A embedded in M_1 , positively transverse to \mathcal{F}_1 , and whose endpoints lie on ∂M_1 . We surgerize M_1 along A , cutting a small tubular neighborhood $\cong \mathbf{D}^2 \times [0, 1]$ and pasting $\Sigma \times [0, 1]$ (recall that Σ is the disk endowed with a handle). The holonomy pseudo-group of the foliation is not changed. After the surgery, s_i admits a Levitt loop with one less self-intersection.

The composed path $p_i \lambda_i$ is homotoped to a path q_i positively transverse to \mathcal{F}_1 , arriving at s_i through the component of the cone $Q < 0$ opposed to that of p_i ; and q_i is \mathcal{F}_1 -equivalent to p_i . During the homotopy, the extremity endpoint s_i is fixed, but the origin endpoint moves in $\partial_- M_1$. One arranges that $p_i \cap q_i = s_i$.

The singularity s_i is eliminated by, somewhat, a *half* elementary surgery of index 2 along the arc $p_i \cup q_i$: one cuts a small tubular neighborhood of this arc, $N_i \cong [0, 1] \times \mathbf{D}^2$, such that $N_i \cap \partial M_1 \cong \{0, 1\} \times \mathbf{D}^2$; and one pastes $2^{-1}\mathbf{D}^2 \times \mathbf{S}^1$ foliated by the restricted function $h|(2^{-1}\mathbf{D}^2 \times \mathbf{S}^1)$. The details are just like in the case without boundary.

3. PROOF OF THEOREMS B AND C

3.1. Examples: realizing the homothety pseudo-groups. First, we discuss the realization of some elementary but fundamental examples: the homothety pseudo-groups. They constitute the most simple nontaut, compactly generated pseudo-groups.

Given some positive real numbers $\lambda_1, \dots, \lambda_r$, let $G(\lambda_1, \dots, \lambda_r)$ be the pseudo-group of local transformations of the real line generated by the homotheties $t \mapsto \lambda_1 t, \dots, t \mapsto \lambda_r t$. We assume that the family $\log \lambda_1, \dots, \log \lambda_r$ is of linear rank r over \mathbf{Q} .

For $r = 1$, the pseudo-group $G(\lambda_1)$ has three obvious realizations of interest. The first is on the annulus $A := \mathbf{S}^1 \times [0, 1]$. The compact leaf is $\mathbf{S}^1 \times (1/2)$; the other leaves are transverse to ∂A and spiral towards $\mathbf{S}^1 \times (1/2)$. The second realization is on the Klein bottle K . The closed leaf is a circle that splits K into two M'obius strips, in each of which the foliation is a twisted planar Reeb component. The third realization is on $\partial(A \times \mathbf{D}^2) \cong \mathbf{S}^2 \times \mathbf{S}^1$. The compact leaf is a 2-torus, and splits $\mathbf{S}^2 \times \mathbf{S}^1$ into two Reeb components.

On the contrary, $G(\lambda_1)$ is *not* realizable on T^2 . Indeed, the foliation would be transversely oriented and have a single compact leaf, whose linear holonomy would be nontrivial, a contradiction.

The case $r = 2$ is analogous. One realizes $G(\lambda_1, \lambda_2)$ on $V := T^2 \times [0, 1]$ by a foliation $\mathcal{F}(\lambda_1, \lambda_2)$ transverse to both boundary tori, where its trace is the linear irrational foliation $dx \log \lambda_1 + dy \log \lambda_2 = 0$. The torus $T^2 \times (1/2)$ is a compact leaf; the other leaves spiral towards it. One has also a second realization by a foliation on $K \times \mathbf{S}^1$: one first pulls $\mathcal{F}(\lambda_1^2, \lambda_2)$ back into $T^3 = \partial(V \times [0, 1])$, and then quotients by the involution $(x, y, t, i) \mapsto (-x, y, t, 1-i)$ of T^3 .

Notice that $G(\lambda_1, \lambda_2)$ is *not* realizable by any foliation \mathcal{F} on any closed, orientable 3-manifold M . For, by contradiction, \mathcal{F} would have a unique compact leaf L diffeomorphic to T^2 , along which M would split into two compact 3-manifolds M' , M'' . On $\mathbf{R} \setminus 0$, the differential 1-form dt/t is invariant by $G(\lambda_1, \lambda_2)$. There would correspond on $M \setminus L$ a nonsingular closed 1-form ω of rank $r = 2$, such that $\mathcal{F}|(M \setminus L) = \ker \omega$. In $H^1(M'; \mathbf{R})$, the de Rham cohomology class $[\omega]$ decomposes as $(\log \lambda_1)e_1 + (\log \lambda_2)e_2$, with $e_1, e_2 \in H^1(M'; \mathbf{Z})$. The restriction $[\omega]|L \in H^1(L; \mathbf{R})$ is of rank 2, being the class of the linear holonomy of \mathcal{F} along L . Thus, $e_1|L$ and $e_2|L$ are not \mathbf{Q} -colinear in $H^1(L; \mathbf{Q})$. Since L is a 2-torus, $(e_1|L) \wedge (e_2|L) \neq 0$ in $H^2(L; \mathbf{Z})$. In other words, $e_1 \wedge e_2 \in H^2(M'; \mathbf{Z})$ is nonnull on the fundamental class of $\partial M'$. This contradicts Stokes theorem, M' being an orientable compact 3-manifold.

For *every* $r \geq 2$, the pseudo-group $G(\lambda_1, \dots, \lambda_r)$ is realizable on a closed orientable 4-manifold. Indeed, in a first place, for each $2 \leq i \leq r$, just as above, realize $G(\lambda_1, \lambda_i)$ by a foliation $\mathcal{F}(\lambda_1, \lambda_i)$ on $V := T^2 \times [0, 1]$. So, $G(\lambda_1, \lambda_i)$ is also realized by the pullback \mathcal{F}_i of $\mathcal{F}(\lambda_1, \lambda_i)$ in the 4-manifold

$$M_i := \partial(V \times \mathbf{D}^2) \cong T^2 \times \mathbf{S}^2$$

The compact leaf L_i of \mathcal{F}_i is the 3-torus $T^2 \times \mathbf{S}^1$. For each $i = 3, \dots, r$, in L_2 and in L_i , we pick some embedded circle $C_i \subset L_2$ (resp. $C'_i \subset L_i$) parallel to the first circle factor: the holonomy of \mathcal{F}_2 (resp. \mathcal{F}_i) along C_i (resp. C'_i) is the germ of $t \mapsto \lambda_1 t$ at 0. We arrange that C_3, \dots, C_r are two by two disjoint. The loop C_i (resp. C'_i) has in M_2 (resp. M_i) a small tubular neighborhood N_i (resp. N'_i) $\cong \mathbf{D}^3 \times \mathbf{S}^1$, on the boundary of which \mathcal{F}_2 (resp. \mathcal{F}_i) traces a foliation composed of two Reeb components, realizing $G(\lambda_1)$. We cut from M_2, \dots, M_r the interiors of $N_3, \dots, N_r, N'_3, \dots, N'_r$. We paste every ∂N_i with $\partial N'_i$, such that $\mathcal{F}_2|_{\partial N_i}$ matches $\mathcal{F}_i|_{\partial N'_i}$. We get a closed orientable 4-manifold with a foliation realizing $G(\lambda_1, \dots, \lambda_r)$.

The realization of pseudo-groups of homotheties *with boundary* is much alike: let $2^{-1}G(\lambda_1, \dots, \lambda_r)$ be the pseudo-group of local transformations of the *half*-line $\mathbf{R}_{\geq 0}$ generated by some family of homotheties $t \mapsto \lambda_1 t, \dots, t \mapsto \lambda_r t$, of rank r . Each of the above realizations of $G(\lambda_1, \dots, \lambda_r)$ splits along its unique compact leaf into two realizations of $2^{-1}G(\lambda_1, \dots, \lambda_r)$.

3.2. Novikov decomposition for pseudo-groups, and hinges. Let (G, T) be a compactly generated pseudo-group of dimension 1 .

We consider the closed orbits (the orbits topologically closed in T).

LEMMA 3.1. *The union of the closed orbits is topologically closed in T .*

Proof. We know no proof for this fact in the pseudo-group frame. To prove it, we realize the pseudo-group, as in section 2, by a Morse foliation \mathcal{F} on a compact manifold M . Since the homology of $M \setminus \text{Sing}(\mathcal{F})$ is of finite rank, Haefliger’s argument [3] applies and shows that the union of the closed leaves is closed. \square

We call a closed orbit *isolated* (resp. *left isolated*) (resp. *right isolated*) if it admits some neighborhood (resp. left neighborhood) (resp. right neighborhood) in T which meets no other closed orbit.

By a *component* of (G, T) , one means a submanifold $T' \subset T$ of dimension 1, topologically closed in T , and saturated for G .

By an *I-bundle* (resp. a \mathbf{S}^1 -*bundle*) we mean the pseudo-group generated by a finite number r of *global* diffeomorphisms on the compact interval (resp. circle). It is of course realized on some compact 3-manifold fibred over some closed surface (“suspension”). Every pseudo-group Haefliger-equivalent to an *I-bundle* (resp. a \mathbf{S}^1 -bundle) is also called an *I-bundle* (resp. a \mathbf{S}^1 -bundle). The smallest possible r is the *rank* of the *I-bundle* (resp. \mathbf{S}^1 -bundle).

Any closed orbit whose isotropy group has infinitely many fix points bounds an *I-bundle*. Precisely,

LEMMA 3.2. *Let $G(t) \subset T$ be a closed orbit, and let h_1, \dots, h_r be elements of G whose germs at t generate the isotropy group G_t . Assume that h_1, \dots, h_r admit a sequence (t_n) of common fix points other than t , decreasing (resp. increasing) to t . Put $I_n := [t, t_n]$ (resp. $[t_n, t]$).*

Then, for every n large enough,

- *The restricted pseudo-group $(G|_{I_n, I_n})$ is generated by $h_1|_{I_n}, \dots, h_r|_{I_n}$;*
- *The G -saturation of I_n is an *I-bundle* component of (G, T) , Haefliger-equivalent to $(G|_{I_n, I_n})$.*

Proof. One first reduces oneself to the case where $G(t) = \{t\}$, as follows. Let $U := T \setminus G(t) \cup t$. By Baire’s theorem, every closed orbit is discrete. So, U is open in T and meets every orbit. We change (G, T) for $(G|_U, U)$, which is also compactly generated by proposition 1.5.

So, we assume that $G(t) = \{t\}$.

Since G is compactly generated, one has a topologically open, relatively compact $T' \subset T$ meeting every G -orbit (in particular $t \in T'$) such that $G|_{T'}$ admits a system of generators g_1, \dots, g_p which are G -extendable. Let $\bar{g}_1, \dots, \bar{g}_p \in G$ be some extensions.

If t lies in the topological boundary of $\text{Dom}(g_i)$ w.r.t. T , then we can avoid this by changing g_i for $\bar{g}_i|(\text{Dom}(g_i) \cup (t - \epsilon, t + \epsilon'))$, where $(t - \epsilon, t + \epsilon')$

is relatively compact in $Dom(\bar{g}_i) \cap T'$. The like holds for $Im(g_i)$. Thus, after permuting the generators, for some $0 \leq q \leq p$, the point t belongs to the domains and to the images of g_1, \dots, g_q ; but t does not belong, nor is adherent, to the domains nor to the images of g_{q+1}, \dots, g_p .

Also, restricting the domain of each h_i , we arrange that $h_i \in G|T'$ and that h_i is G -extendable. Then, we can add the family (h_i) to the family of generators (g_i) . So, we can assume that $r \leq q$ and that $g_1 = h_1, \dots, g_r = h_r$.

Then, for every $r + 1 \leq i \leq q$, the generator g_i coincides with some compose of g_1, \dots, g_r on some small compact neighborhood N_i of t . We can change g_i for $g_i|(Dom(g_i) \setminus N_i)$. Finally, we have obtained a family of generators g_1, \dots, g_p for $G|T'$, such that t belongs to the domains and to the images of $g_1 = h_1, \dots, g_r = h_r$; but that t does not belong, nor is adherent, to the domains nor to the images of g_{r+1}, \dots, g_p .

For every n large enough, I_n is contained in T' and in the domains of g_1, \dots, g_r ; and I_n is invariant by $g_1 = h_1, \dots, g_r = h_r$; and I_n is disjoint from the supports of g_{r+1}, \dots, g_p . Thus, I_n is saturated for G , and $G|I_n$ is generated by $g_1|I_n, \dots, g_r|I_n$. The interval I_n is an I -bundle component of G . \square

We call an orbit *essential* (w.r.t. (G, T)) if it meets no transverse positive loop and no transverse positive chain whose both endpoints lie on ∂T . Every essential orbit is closed (obviously). In the union of the closed orbits, the union of the essential orbits is topologically closed (obviously) and open (by lemma 3.2).

We call an I -bundle component of G *essential* (w.r.t. (G, T)) if its boundary orbits are essential w.r.t. (G, T) . Then, every closed orbit interior to this I -bundle is also essential w.r.t. (G, T) .

The ‘‘Novikov decomposition’’ is well-known for foliations on compact manifolds. Every compact connected manifold endowed with a transversely orientable foliation of codimension one, either is a \mathbf{S}^1 -bundle, or splits along finitely many compact leaves bounding some dead end components, into compact components, such that each component is an I -bundle, or its interior is topologically taut. For compactly generated pseudo-groups, one has an analogous decomposition (exercise):

PROPOSITION 3.3. *(Novikov decomposition) Let (G, T) be a connected, compactly generated, orientable pseudo-group of dimension 1. Assume that (G, T) is not a \mathbf{S}^1 -bundle.*

Then, T splits, along finitely many essential orbits, into finitely many components T_i , such that for each i :

(a) the component $(G|T_i, T_i)$ is an essential I -bundle,

or

(b) the interior of the component, $(G|Int(T_i), Int(T_i))$, is taut.

Novikov decompositions are functorial with respect to Haefliger equivalences: given a Haefliger equivalence between two pseudo-groups, to every

Novikov decomposition of the one corresponds naturally a Novikov decomposition of the other.

We shall not use this decomposition under this form, nor prove it in general. We need, under the hypotheses of theorems B and C, the more precise form 3.7 below.

From now on, we assume moreover that in the orientable, compactly generated, 1-dimensionnal pseudo-group (G, T) , every essential orbit is commutative, that is, its isotropy group is commutative. By the *essential rank* of (G, T) , we mean the supremum of the ranks of the isotropy groups of the essential orbits.

The proof of theorems B and C somewhat consists in realizing independently every component of some Novikov decomposition, and pasting these realizations together. The interior of every component falling to (b) is realized on a closed 3-manifold, thanks to theorem A and to the following

LEMMA 3.4. *Let (G, T) be a compactly generated pseudo-group of dimension 1 . Let $G(t_0) \subset T$ be an isolated closed orbit, whose isotropy group is commutative.*

Then, the subpseudo-group $G|(T \setminus G(t_0))$ is also compactly generated.

Proof. We treat the case where the orbit $G(t_0)$ is contained in $\partial_- T$. Of course, the case where it is contained in $\partial_+ T$ is symmetric; and the case where it is contained in $Int(T)$ is much alike.

Since G is compactly generated, one has a topologically open, relatively compact $T' \subset T$ meeting every G -orbit such that $G|T'$ admits a system of generators g_1, \dots, g_p which are G -extendable.

Just as in the proof of lemma 3.2, one can arrange that $G(t_0) = \{t_0\}$ (in particular, $t_0 \in T'$); and that, for some $0 \leq r \leq p$, the point t_0 belongs to the domains and to the images of g_1, \dots, g_r ; and that t_0 does not belong, nor is adherent, to the domains nor to the images of g_{r+1}, \dots, g_p .

Since t_0 is isolated as a closed orbit of G , one has $r \geq 1$. We can arrange moreover, to simplify notations, that the family (g_i) is symmetric: the inverse of every g_i is some g_j .

The isotropy group of t_0 being commutative, there is a $u_0 > t_0$ so close to t_0 that

- (1) For every $r+1 \leq i \leq p$, the interval $[t_0, u_0]$ does not meet $Dom(g_i)$;
- (2) For every $1 \leq i, j \leq r$ and every $t \in [t_0, u_0]$, one has $t \in Dom(g_i)$ and $g_i(t) \in Dom(g_j)$ and $g_i g_j(t) = g_j g_i(t)$.

Put $T'' := T' \setminus [t_0, u_0]$ and $G_0 := G|(T \setminus t_0)$. We shall show that every orbit of G_0 meets T'' , and that the pseudo-group $G_0|T''$ is generated by $g_1|T'' , \dots, g_p|T''$. Every $g_i|T''$ being G_0 -extendable, it will follow that G_0 is compactly generated.

To this end, define by induction two sequences $u_n \in [t_0, u_0]$ and $1 \leq i(n) \leq r$, such that $u_{n+1} := g_{i(n)}(u_n)$ is the minimum of $g_1(u_n) , \dots, g_r(u_n)$. Because t_0 is isolated as a closed orbit of G , there is no common

fixed point for g_1, \dots, g_r in the interval $(t_0, u_0]$. Thus, (u_n) decreases to t_0 . Also, for every $n \geq 0$:

$$g_{i(n)}^{-1}((u_{n+1}, u_0]) \subset (u_n, u_0] \cup T'' \quad (*)$$

In particular, every orbit of G_0 meets T'' .

Consider the germ $[g]_t$ of some $g \in G_0$ at some point $t \in \text{Dom}(g)$ such that $t \in T''$ and $g(t) \in T''$. Since the g_i 's generate G/T' , this germ can be decomposed as a word w in the germs of the generators:

$$[g]_t = [g_{j(\ell)}]_{t(\ell-1)} \cdots [g_{j(1)}]_{t(0)}$$

where $1 \leq j(1), \dots, j(\ell) \leq p$, where $t(0) = t$, and where for every $0 \leq k \leq \ell$ one has $t(k) := g_{j(k)} \circ \cdots \circ g_{j(1)}(t) \in T'$.

We call the finite sequence $t(0), \dots, t(\ell)$ the *trace* of w . We have to prove that $[g]_t$ admits also a second such decomposition, whose trace is moreover contained in T'' .

We make a double induction: on the smallest integer $n \geq 0$ such that the trace of w is disjoint from $[t_0, u_n]$, and, if $n \geq 1$, on the number of k 's for which $t_k \in (u_n, u_{n-1}]$.

Assume that $n \geq 1$. Let $1 \leq k \leq \ell - 1$ be an index for which $t_k \in (u_n, u_{n-1}]$. Consider the word

$$w' := g_{j(\ell)} \cdots g_{j(k+2)} g_{i(n-1)} g_{j(k+1)} g_{j(k)} g_{i(n-1)}^{-1} g_{j(k-1)} \cdots g_{j(1)}$$

By the property (2) above applied at the point t_k to the pair $g_{i(n-1)}^{-1}, g_{j(k+1)}$ and to the pair $g_{i(n-1)}^{-1}, g_{j(k)}^{-1}$, the compose w' is defined at t , and w' has the same germ at t as w .

The trace of w' at t is the same as the trace of w , except that $t(k)$ has been changed for the three points $g_{i(n-1)}^{-1}(t(k-1))$, $g_{i(n-1)}^{-1}(t(k))$ and $g_{i(n-1)}^{-1}(t(k+1))$. By (*), none of the three lies in $[t_0, u_n]$. The induction is complete. \square

The pasting of the realizations of the Novikov components will be a little delicate. The following notion allows us to take in account, with every commutative closed orbit, its isotropy group; and with every commutative I -bundle, the holonomy of its boundary orbits on the exterior side.

DEFINITION 3.5. *We call an oriented pseudo-group (Γ, Ω) of dimension 1 a hinge if Ω is an interval, either open, or compact, or semi-open; and if there exist a Γ -invariant compact interval $[a, b] \subset \Omega$, with $a \leq b$, and a system of generators $\gamma_1, \dots, \gamma_r$ for Γ , s.t.*

- (1) *The domains and the images of $\gamma_1, \dots, \gamma_r$ are intervals containing $[a, b]$;*
- (2) *For every $\gamma, \eta \in \Gamma$, one has $\gamma\eta = \eta\gamma$ and $\gamma^{-1}\eta = \eta\gamma^{-1}$ and $\gamma^{-1}\eta^{-1} = \eta^{-1}\gamma^{-1}$ wherever both composes are defined;*
- (3) *Every orbit of Γ meets every neighborhood of $[a, b]$ in Ω .*

We call $[a, b]$ the *core*. The hinge is *degenerate* if $a = b$, in which case $a = b$ is a closed orbit of (Γ, Ω) . The hinge is *nondegenerate* if $a < b$, in which case $[a, b]$ is an I -bundle component of (Γ, Ω) . The smallest possible r is the *rank* of (Γ, Ω) . Notice that by (3), in the nondegenerate (resp. degenerate) case, the boundary $\partial\Omega$ of Ω as a manifold is contained in $\{a, b\}$ (resp. is empty).

Every hinge is easily realized:

LEMMA 3.6. *Let (Γ, Ω) be a hinge of rank $r \geq 1$. Assume that $\partial\Omega = \emptyset$ (resp. $\partial\Omega = \{a\}$ or $\{b\}$) (resp. $\partial\Omega = \{a, b\}$). Then, (Γ, Ω) is realized:*

- (1) *If $r \leq 2$, on $T^2 \times [0, 1]$, with two (resp. one) (resp. zero) transverse boundary components;*
- (2) *If $r \leq 2$, on $K \times \mathbf{S}^1$ (resp. $M \times \mathbf{S}^1$) (resp. $A \times \mathbf{S}^1$), where K (resp. M) (resp. A) is the Klein bottle (resp. the compact M'obius strip) (resp. the compact annulus), without transverse boundary component;*
- (3) *For every r , on some orientable compact 4-manifold, without transverse boundary component.*

Proof. The realization is just like in the particular case of the homothety pseudo-groups, seen above at paragraph 3.1. Assume for example that $\partial\Omega = \emptyset$ and that $r = 2$, and let us realize (Γ, Ω) on $T^2 \times [0, 1]$ (1).

The suspension of γ_1 and γ_2 over T^2 provides, in $T^2 \times \Omega$, a foliation \mathcal{F} on some open neighborhood U of $T^2 \times [a, b]$. By property (3), γ_1 and γ_2 have no common fix point outside $[a, b]$. Consequently, one has an embedding of $T^2 \times [0, 1]$ into U containing $T^2 \times [a, b]$ in its interior, and meeting every leaf of \mathcal{F} , and such that $T^2 \times 0$ and $T^2 \times 1$ are embedded transversely to \mathcal{F} . It is easily verified that $\mathcal{F}|(T^2 \times [0, 1])$ realizes (Γ, Ω) . \square

PROPOSITION 3.7. *Let (G, T) be a compactly generated, oriented pseudo-group of dimension 1, in which every essential orbit is commutative.*

Then, after a Haefliger equivalence, T splits as a disjoint union

$$T = T_0 \sqcup \Omega_1 \sqcup \cdots \sqcup \Omega_\ell$$

such that

- (1) *T_0 is a finite disjoint union of circles and compact intervals;*
- (2) *Each Ω_k ($1 \leq k \leq \ell$) is the domain of a hinge $\Gamma_k \subset G$ whose rank is at most the essential rank of (G, T) ;*
- (3) *Each core $[a_k, b_k] \subset \Omega_k$ is G -saturated;*
- (4) *For every $t \in \Omega_k \setminus [a_k, b_k]$ ($1 \leq k \leq \ell$), the orbit $G(t)$ meets T_0 .*

We begin to prove proposition 3.7.

By a *subpseudo-group* in (G, T) , we mean a pseudo-group (Γ, Ω) such that $\Omega \subset T$ is topologically open, and that $\Gamma \subset G|_\Omega$.

DEFINITION 3.8. *Let $(\Gamma, \Omega) \subset (G, T)$ be a hinge subpseudo-group. Let $[a, b] \subset \Omega$ be its core.*

(a) *Assume that (Γ, Ω) is degenerate ($a = b$). We call the hinge subpseudo-group faithful if $G(a)$ is closed in T and if $\Gamma_a = G_a$ (isotropy groups).*

(b) Assume that (Γ, Ω) is nondegenerate ($a \neq b$). We call the hinge subpseudo-group faithful if the G -saturation of $[a, b]$ is a component of G , if $\Gamma|[a, b] = G|[a, b]$, if $\Gamma_a = G_a$, and if $\Gamma_b = G_b$.

In case (b), the G -saturation of $[a, b]$ is necessarily an I -bundle component of G .

The notion of subpseudo-group is *not* functorial with respect to the Haefliger equivalences. The following notion solves this difficulty.

DEFINITION 3.9. Given two pseudo-groups (G, T) and (Γ, Ω) , an extension of (G, T) by (Γ, Ω) is a pseudo-group \bar{G} on the disjoint union $\bar{T} := T \sqcup \Omega$ s.t.

- T is exhaustive for \bar{G} ;
- $G = \bar{G}|T$;
- $\Gamma \subset \bar{G}$.

In particular, (\bar{G}, \bar{T}) is Haefliger-equivalent to (G, T) , and (Γ, Ω) is a subpseudo-group of (\bar{G}, \bar{T}) .

For example, given two pseudo-groups (G, T) , (Γ, Ω) and given a Haefliger equivalence $(\bar{\Gamma}, \Omega \sqcup \Omega_0)$ between (Γ, Ω) and some subpseudo-group $(\Gamma_0, \Omega_0) \subset (G, T)$, one has a natural extension of (G, T) by (Γ, Ω) : namely, \bar{G} is the pseudo-group on $T \sqcup \Omega$ generated by $G \cup \bar{\Gamma}$.

An extension (\bar{G}, \bar{T}) of a pseudo-group (G, T) by a hinge (Γ, Ω) is called *faithful* if $(\Gamma, \Omega) \subset (\bar{G}, \bar{T})$ is faithful; *essential* if its core is an essential orbit or an essential I -bundle in (\bar{G}, \bar{T}) .

LEMMA 3.10. For every $t \in T$ such that $G(t)$ is essential, there is an essential faithful extension (\bar{G}, \bar{T}) of (G, T) by a hinge (Γ, Ω) s.t.:

- The rank of the hinge (Γ, Ω) is at most the essential rank of (G, T) ;
- The core of (Γ, Ω) meets $\bar{G}(t)$;
- The core of (Γ, Ω) meets also every essential orbit of \bar{G} close enough to $\bar{G}(t)$.

Proof. First case: $G(t)$ is not contained in any I -bundle component of (G, T) of rank 0.

In this case, we shall actually find a faithful hinge subpseudo-group in (G, T) whose core meets $G(t)$ and every neighboring essential orbit.

Let $r := \text{rank}(G_t)$ and choose $h_1, \dots, h_r \in G$ such that their germs at t are a basis of G_t . Let Ω be a small interval containing t , topologically open in T , and contained in the intersection of the domains and of the images of h_1, \dots, h_r . Put $\gamma_i := h_i|(\Omega \cap h_i^{-1}(\Omega))$ ($i = 1, \dots, r$) and $\Gamma := \langle \gamma_1, \dots, \gamma_r \rangle$. For Ω small enough, the properties (1) and (2) of definition 3.5 are fulfilled.

First subcase: $G(t)$ is isolated. Put $a = b := t$. For Ω small enough, by lemma 3.2, h_1, \dots, h_r have no common fix point in Ω . In consequence, for every $t' \in \Omega \setminus \{t\}$, there is an i for which one of the four following properties holds: $t' \in \text{Dom}(\gamma_i)$ and $t < \gamma_i(t') < t'$, or $t' \in \text{Dom}(\gamma_i^{-1})$ and $t < \gamma_i^{-1}(t') < t'$, or $t' \in \text{Dom}(\gamma_i)$ and $t' < \gamma_i(t') < t$, or $t' \in \text{Dom}(\gamma_i^{-1})$ and $t' < \gamma_i^{-1}(t') < t$. The property (3) of definition 3.5 follows.

Second subcase: $G(t)$ is not isolated from either side. In that subcase, by lemma 3.2, we can shorten Ω to arrange that moreover none of the endpoints of Ω is a fix point common to h_1, \dots, h_r . Let a and b be the smallest and the largest fix points common to h_1, \dots, h_r in Ω . Then, $a < t < b$. For every $t' \in \Omega \setminus [a, b]$, there is an i for which one of the four following properties holds: $t' \in \text{Dom}(\gamma_i)$ and $b < \gamma_i(t') < t'$, or $t' \in \text{Dom}(\gamma_i^{-1})$ and $b < \gamma_i^{-1}(t') < t'$, or $t' \in \text{Dom}(\gamma_i)$ and $t' < \gamma_i(t') < a$, or $t' \in \text{Dom}(\gamma_i^{-1})$ and $t' < \gamma_i^{-1}(t') < a$. The property (3) of definition 3.5 follows.

Third subcase: $G(t)$ is isolated from exactly one side. The argument is similar to the two first subcases.

Second case: $G(t)$ is contained in a I -bundle component $C \subset T$ of rank 0. That is, C is a 1-manifold, topologically closed in T , and $G|_C$ is Haefliger-equivalent to the trivial pseudo-group on the interval $[0, 1] \subset \mathbf{R}$. In other words, one has an orientation-preserving etale map $f : C \rightarrow [0, 1]$; and the Haefliger equivalence is nothing but the pseudo-group on the disjoint union $C \sqcup [0, 1]$ generated by the set of the local sections of f . The boundary ∂C is made of of two orbits $\partial_- C = G(t_0)$ and $\partial_+ C = G(t_1)$. We can assume that C is maximal among the I -bundle components of rank 0. Assume also, to fix ideas, that C is interior to T (the other cases being alike and simpler). Thus, the isotropy group of G at t_0 (resp. t_1) is nontrivial on the left (resp. right).

Pick some small open interval $(u_0, v_0) \subset T$ containing t_0 and whose intersection with C is $[t_0, v_0)$; and pick some small open interval $(u_1, v_1) \subset T$ containing t_1 and whose intersection with C is $(u_1, t_1]$. Take the intervals so small that $f(v_0) < f(u_1)$. One extends $f|[t_0, v_0)$ into a diffeomorphism f_0 from the interval (u_0, v_0) onto the interval $(-\infty, f(v_0))$. The choice among the extensions is arbitrary. Similarly, one extends $f|(u_1, t_1]$ into a diffeomorphism f_1 from the interval (u_1, v_1) onto the interval $(f(u_1), +\infty)$. Let T' be the disjoint union $T \sqcup \mathbf{R}$. Let G' be the pseudo-group on T' generated by G , f , f_0 , and f_1 . Obviously, T is exhaustive in (G', T') , and $G = G'|_T$, and $G'|_{[0, 1]}$ is the trivial pseudo-group on $[0, 1]$, and the orbit $G'(t)$ meets $[0, 1]$ at $f(t)$. Let $r := \max(\text{rank}(G'_0), \text{rank}(G'_1))$. One immediately makes $h_1, \dots, h_r \in G'|\mathbf{R}$ whose domains and images contain $[0, 1]$, which are the identity on $(0, 1)$, whose germs at 0 generate G'_0 , and whose germs at 1 generate G'_1 . Let $\Omega \subset \mathbf{R}$ be an open interval containing $[0, 1]$, and contained in the intersection of the domains and of the images of h_1, \dots, h_r . For Ω small enough, the property (2) of definition 3.5 is fulfilled. By lemma 3.2, we can moreover shorten Ω to arrange that none of its endpoints is a fix point common to h_1, \dots, h_r . Put $\gamma_i := h_i|(\Omega \cap h_i^{-1}(\Omega))$ ($i = 1, \dots, r$) and $\Gamma := \langle \gamma_1, \dots, \gamma_r \rangle$. Let a and b be the smallest and the largest fix points common to h_1, \dots, h_r in Ω . The property (3) of definition 3.5 is fulfilled. The pseudo-group $(G'|(T \sqcup \Omega), \Omega)$ is a faithful extension of (G, T) by the hinge (Γ, Ω) . \square

Proof. of proposition 3.7. The pseudo-group (G, T) being cocompact, and the union of the essential leaves being topologically closed in T , one has a compact $K \subset T$ whose G -saturation coincides with this union. By lemma 3.10, every point of K has a neighborhood in K whose orbits meet the core of the hinge after one essential, faithful hinge extension, whose rank is at most the essential rank of (G, T) . One extracts a finite subcover. There corresponds a finite sequence of essential faithful extensions by hinges (Γ_k, Ω_k) ($1 \leq k \leq \ell$), whose ranks are at most the essential rank of (G, T) . Let (\bar{G}, \bar{T}) be the resulting global extension of (G, T) ; let $[a_k, b_k]$ be the core of (Γ_k, Ω_k) ; and let $C_k \subset \bar{T}$ be the \bar{G} -saturation of $[a_k, b_k]$. It is easy to arrange that C_1, \dots, C_ℓ are two by two disjoint. A closed orbit of \bar{G} is contained in $C_1 \cup \dots \cup C_\ell$ iff it is essential. Consequently, the pseudo-group $(\bar{G}|U = G|U, U)$ is taut, where

$$U := T \setminus ((C_1 \cup \dots \cup C_\ell) \cap T)$$

Also, the topological closure \bar{U} of U in T being a component of (G, T) , the restricted pseudo-group $(G|\bar{U}, \bar{U})$ is compactly generated. By lemma 3.4, $(G|U, U)$ is also compactly generated. By proposition 1.9, $(G|U, U)$ is Haefliger-equivalent to some pseudo-group (G_0, T_0) on a finite disjoint union T_0 of compact intervals and circles. By the example that follows the definition 3.9 above, we get an extension (\tilde{G}, \tilde{T}) of (\bar{G}, \bar{T}) by (G_0, T_0) . One has $\tilde{T} = \bar{T} \sqcup T_0$. Let

$$\tilde{T}' := T_0 \sqcup \Omega_1 \cdots \sqcup \Omega_\ell \subset \tilde{T}$$

By construction, \tilde{T}' is exhaustive in (\tilde{G}, \tilde{T}) . We change (G, T) for $(\tilde{G}|\tilde{T}', \tilde{T}')$. The properties of proposition 3.7 are fulfilled. \square

3.3. End of the proofs of theorems B and C. Let, as before, (G, T) be an oriented, compactly generated pseudo-group of dimension 1, in which every essential orbit is commutative. Our task is to realize (G, T) , in dimension 3 if possible, and 4 if not.

Without loss of generality, (G, T) is under the form described by proposition 3.7. We shall first realize separately $(G|T_0, T_0)$, (Γ_1, Ω_1) , \dots , $(\Gamma_\ell, \Omega_\ell)$; and then surgerize along some loops in the realizations, transverse to the foliations. It is convenient to begin with introducing these loops into the pseudo-group.

For each k , if $a_k \notin \partial_- \Omega_k$ (resp. $b_k \notin \partial_+ \Omega_k$), write Ω_k^- (resp. Ω_k^+) the connected component of $\Omega_k \setminus [a_k, b_k]$ on the left of a_k (resp. on the right of b_k).

LEMMA 3.11. *a) In case $a_k \notin \partial_- \Omega_k$, there exist in Ω_k^- two points $a'_k < a''_k < a_k$, and $\phi_k \in \Gamma_k$, s.t.*

- i) The interval (a'_k, a''_k) is exhaustive for $\Gamma_k|\Omega_k^-$;*
- ii) $[a'_k, a''_k] \subset \text{Dom}(\psi_k) \cap \text{Im}(\psi_k)$;*
- iii) $\phi_k(t) > t$ for every $t \in [a'_k, a''_k]$;*
- iv) $\phi_k(a'_k) < a''_k$.*

- b) Symmetrically, in case $b_k \notin \partial_+\Omega_k$, there exist in Ω_k^+ two points $b_k < b'_k < b''_k$, and $\psi_k \in \Gamma_k$, s.t.
- i) The interval (b'_k, b''_k) is exhaustive for $\Gamma_k|\Omega_k^+$;
 - ii) $[b'_k, b''_k] \subset \text{Dom}(\psi_k) \cap \text{Im}(\psi_k)$;
 - iii) $\psi_k(t) > t$ for every $t \in [b'_k, b''_k]$;
 - iv) $\psi_k(b'_k) < b''_k$.

Proof. of a). Recall $\gamma_1, \dots, \gamma_r$ of definition 3.5. Choose $a'_k < a_k$, so close to a_k that it belongs to the domain and to the image of γ_i , for every $1 \leq i \leq r$. Let $\gamma_j^{\epsilon_j}(a'_k)$ be the maximum of the values $\gamma_i(a'_k), \gamma_i^{-1}(a'_k)$ ($1 \leq i \leq r$). Put $\phi_k := \gamma_j^{\epsilon_j}$. Choose a''_k in the interval $(\gamma_j^{\epsilon_j}(a'_k), a_k)$, so close to $\gamma_j^{\epsilon_j}(a'_k)$ that iii) holds. The properties i), ii) and iv) are obvious. \square

For each $k = 1, \dots, \ell$, it follows from ii), iii) and iv) that, in case $a_k \notin \partial_-\Omega_k$ (resp. $b_k \notin \partial_+\Omega_k$), the subpseudo-group of (Γ_k, Ω_k) generated by $\phi_k|(a'_k, a''_k)$ (resp. $\psi_k|(b'_k, b''_k)$) is Haefliger-equivalent to the trivial pseudo-group on the circle. In case $\partial\Omega_k = \emptyset$ (resp. $\{a_k\}$) (resp. $\{b_k\}$) (resp. $\{a_k, b_k\}$), by the example following the definition 3.9, we get an extension $(\hat{\Gamma}_k, \hat{\Omega}_k)$ of the hinge (Γ_k, Ω_k) by the trivial pseudo-group on the disjoint union of two circles $S_k^- \sqcup S_k^+$ (resp. one circle S_k^+) (resp. one circle S_k^-) (resp. \emptyset).

In other words, we have an extension (\hat{G}, \hat{T}) of (G, T) by the trivial pseudo-group on the disjoint union S of all the S_k^\pm 's ($1 \leq k \leq \ell$). In particular, $\hat{T} = T \sqcup S$. Write $\hat{T}_0 := T_0 \sqcup S \subset \hat{T}$ and $\hat{G}_0 := \hat{G}|\hat{T}_0$. Also write

$$A := [a_1, b_1] \cup \dots \cup [a_\ell, b_\ell]$$

LEMMA 3.12. *The pseudo-group \hat{G} on \hat{T} is generated by \hat{G}_0 , by $\hat{\Gamma}_1, \dots$, and by $\hat{\Gamma}_\ell$.*

Proof. We have to verify that the germ $[g]_t$ of every $g \in \hat{G}$ at every $t \in \text{Dom}(g)$, is generated by \hat{G}_0 and the $\hat{\Gamma}_k$'s.

If $t \in \Omega_k \setminus [a_k, b_k]$, then (lemma 3.11, i)) there is some $\gamma \in \Gamma_k$ s.t. $\gamma(t) \in (a'_k, a''_k)$ or $\gamma(t) \in (b'_k, b''_k)$, and thus some $\hat{\gamma} \in \hat{\Gamma}_k$ s.t. $\hat{\gamma}(t) \in S_k^\pm$. We are thus reduced to the case $t \in \hat{T}_0 \cup A$. Symmetrically, one can assume also that $g(t) \in \hat{T}_0 \cup A$.

By proposition 3.7, (3), either $t, g(t) \in \hat{T}_0$ (and thus $[g]_t \in \hat{G}_0$) or $t, g(t) \in [a_k, b_k]$ for some $1 \leq k \leq \ell$. In that second case, the extension of (G, T) by (Γ_k, Ω_k) being faithful, $g \in \Gamma_k$. \square

Proof. that (3) implies (1) in theorem B. Start from a pseudo-group (\hat{G}, \hat{T}) as in lemma 3.12, Haefliger-equivalent to (G, T) .

On the one hand, the restriction $\hat{G}|\hat{T} \setminus \text{Int}(A)$, being a component of (\hat{G}, \hat{T}) , is also compactly generated. By lemma 3.4, the restriction of \hat{G} to $\hat{T} \setminus \text{Int}(A)$ is compactly generated. Since $\hat{T}_0 \subset \hat{T} \setminus \text{Int}(A)$ is exhaustive, (\hat{G}_0, \hat{T}_0) is compactly generated. This pseudo-group is also taut, \hat{T}_0 being

a disjoint union of circles and compact intervals. By theorem A, (\hat{G}_0, \hat{T}_0) is realized by a foliated compact 3-manifold (M_0, \mathcal{F}_0) , without transverse boundary. More precisely, from the proof of theorem A, \hat{T}_0 is embedded into M_0 as an exhaustive transversal to \mathcal{F}_0 , and \hat{G}_0 is the holonomy pseudo-group of \mathcal{F}_0 on \hat{T}_0 . One takes off from M_0 a small open tubular neighborhood N_0 of S , s.t. $\mathcal{F}_0|_{\partial N_0}$ is the trivial foliation by 2-spheres.

On the other hand, for each $k = 1, \dots, \ell$, one realizes (Γ_k, Ω_k) by a foliation \mathcal{F}_k on $M_k := T^2 \times [0, 1]$ (lemma 3.6). Obviously, \mathcal{F}_k admits transverse loops corresponding to S_k^\pm , in the sense that $\hat{\Omega}_k$ embeds into M_k as an exhaustive transversal to \mathcal{F}_k , and $\hat{\Gamma}_k$ is the holonomy pseudo-group of \mathcal{F}_k on $\hat{\Omega}_k$. One takes off from M_k a small open tubular neighborhood N_k of S_k^\pm , s.t. $\mathcal{F}_k|_{\partial N_k}$ is the trivial foliation by 2-spheres.

One pastes $\sqcup_{1 \leq k \leq \ell} \partial N_k \cong \mathbf{S}^2 \times S$ with $\partial N_0 \cong \mathbf{S}^2 \times S$, w.r.t. the identity of S . One gets a foliation \mathcal{F} on

$$M_0 \cup_{\mathbf{S}^2 \times S} (M_1 \sqcup \dots \sqcup M_\ell)$$

whose holonomy on the exhaustive transversal T coincides with G , by lemma 3.12. \square

Proof. that (3) implies (2) in theorem B. The same as for (3) implies (1), but instead of $T^2 \times [0, 1]$, we use $K \times \mathbf{S}^1$, where K is the Klein bottle. \square

Proof. of theorem C. The same as for (3) implies (1) in theorem B, but instead of the foliated 3-manifold (M_0, \mathcal{F}_0) , we use the foliated 4-manifold $(M_0 \times \mathbf{S}^1, pr_1^*(\mathcal{F}_0))$; and instead of $T^2 \times [0, 1]$, we use a 4-dimensional realization of (Γ_k, Ω_k) (lemma 3.6). \square

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