A wave equation including leptons and quarks for the standard model of quantum physics in Clifford Algebra

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August 27, 2014

Abstract

A wave equation with mass term is studied for all particles and antiparticles of the first generation: electron and its neutrino, positron and antineutrino, quarks u and d with three states of color and antiquarks $\overline{u}$ and $\overline{d}$. This wave equation is form invariant under the Cl_3^* group generalizing the relativistic invariance. It is gauge invariant under the $U(1) \times SU(2) \times SU(3)$ group of the standard model of quantum physics. The wave is a function of space and time with value in the Clifford algebra $Cl_{1,5}$. All features of the standard model, charge conjugation, color, left waves, Lagrangian formalism, are linked to the geometry of this extended space-time.

Keywords: invariance group, Dirac equation, electromagnetism, weak interactions, strong interactions, Clifford algebras

Introduction

We use here all notations of "New insights in the standard model of quantum physics in Clifford Algebra" [10]. The wave equation for all particles of the first generation is a generalization of the wave equation obtained in 6.7 for

the electron and its neutrino. This wave equation has obtained a proper mass term compatible with the gauge invariance in [9]. It is a generalization of the homogeneous nonlinear Dirac equation for the electron alone [1] [2] [3] [4] [5] [6] [7]. The link with the usual presentation of the standard model is made by the left and right Weyl spinors used for waves of each particle. These right and left waves are parts of the wave with value in $Cl_{1,5}$.

We used previously the same algebra $Cl_{5,1} = Cl_{1,5}$. It is the same algebra, and this explains very well why sub-algebras $Cl_{1,3}$ and $Cl_{3,1}$ have been equally used to describe relativistic physics [12] [13]. But the signature of the scalar product cannot be free, this scalar product being linked to the gravitation in the general relativity. It happens that vectors of $Cl_{1,5}$ are pseudo-vectors of $Cl_{5,1}$ and more generally that n-vectors of $Cl_{1,5}$ are $(6-n)$-vectors of $Cl_{5,1}$. The generalization of the wave equation for electron-neutrino is simpler if we use $Cl_{1,5}$. This is the first indication that the signature $+----$ is the true one. We explain in Appendix A how the reverse in $Cl_{1,5}$ is linked to the reverse in $Cl_{1,3}$, a necessary condition to get the wave equation of all particles of the first generation.

We have noticed, for the electron alone firstly (See [6] 2.4), next for electron+neutrino [9] the double link existing between the wave equation and the Lagrangian density: It is well known that the wave equation may be obtained from the Lagrangian density by the variational calculus. The new link is that the real part of the invariant wave equation is simply $\mathcal{L} = 0$. The Lagrangian formalism is then necessary, being a consequence of the wave equation. Next we have extended the double link to electro-weak interactions in the leptonic case (electron + neutrino). Now we are extending the double link to the gauge group of the standard model. The Lagrangian density must then be the real part of the invariant wave equation.

Moreover we generalized the non-linear homogeneous wave equation of the electron, and we got a wave equation with mass term [9], form invariant under the $Cl_3^* = GL(2,\mathbb{C})$ group and gauge invariant under the $U(1) \times SU(2)$ gauge group of electro-weak interactions. Our aim is to explain how this may be extended to a wave equation with mass term, both form invariant under Cl_3^* and gauge invariant under the $U(1) \times SU(2) \times SU(3)$ gauge group of the standard model, including both electro-weak and strong interactions.

1 From the lepton case to the full wave

The standard model adds to the leptons (electron e and its neutrino n) in the "first generation" two quarks u and d with three states each. Weak interactions acting only on left waves of quarks (and right waves of antiquarks) we read the

wave of all fermions of the first generation as follows:

$$\Psi = \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix} \; ; \quad \Psi_l = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix} = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_{\overline{n}}\sigma_1 & \widehat{\phi}_{\overline{e}}\sigma_1 \end{pmatrix} \tag{1.1}$$

$$\Psi_r = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{ur} & \widehat{\phi}_{dr} \end{pmatrix} = \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{\overline{u}r}\sigma_1 & \widehat{\phi}_{\overline{d}r}\sigma_1 \end{pmatrix}; \; \Psi_g = \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{ug} & \widehat{\phi}_{dg} \end{pmatrix} = \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{\overline{u}g}\sigma_1 & \widehat{\phi}_{\overline{d}g}\sigma_1 \end{pmatrix}$$

$$\Psi_b = \begin{pmatrix} \phi_{db} & \phi_{ub} \\ \widehat{\phi}_{ub} & \widehat{\phi}_{db} \end{pmatrix} = \begin{pmatrix} \phi_{db} & \phi_{ub} \\ \widehat{\phi}_{\overline{u}b}\sigma_1 & \widehat{\phi}_{\overline{d}b}\sigma_1 \end{pmatrix} \tag{1.2}$$

The electro-weak theory [14] needs three spinorial waves in the electron-neutrino case: the right ξ_e and the left η_e of the electron and the left spinor η_n of the electronic neutrino. The form invariance of the Dirac theory imposes to use ϕ_e for the electron and ϕ_n for its neutrino satisfying

$$\phi_e = \sqrt{2}\begin{pmatrix} \xi_{1e} & -\eta^*_{2e} \\ \xi_{2e} & \eta^*_{1e} \end{pmatrix} = \sqrt{2}(\xi_e \quad -i\sigma_2\eta^*_e) \; ; \quad \phi_n = \sqrt{2}\begin{pmatrix} 0 & -\eta^*_{2n} \\ 0 & \eta^*_{1n} \end{pmatrix} \tag{1.3}$$

$$\widehat{\phi}_e = \sqrt{2}\begin{pmatrix} \eta_{1e} & -\xi^*_{2e} \\ \eta_{2e} & \xi^*_{1e} \end{pmatrix} = \sqrt{2}(\eta_e \quad -i\sigma_2\xi^*_e) \; ; \quad \widehat{\phi}_n = \sqrt{2}\begin{pmatrix} \eta_{1n} & 0 \\ \eta_{2n} & 0 \end{pmatrix} = \sqrt{2}(\eta_n \quad 0). \tag{1.4}$$

Waves ϕ_e and ϕ_n are functions of space and time with value into the Clifford algebra Cl_3 of the physical space. The standard model uses only a left η_n wave for the neutrino. We always use the matrix representation (A.1) which allows to see the Clifford algebra $Cl_{1,3}$ as a sub-algebra of $M_4(\mathbb{C})$. Under the dilation R with ratio r induced by any M in $GL(2,\mathbb{C})$ we have (for more details, see [4]):

$$x' = MxM^\dagger \; ; \quad \det(M) = re^{i\theta} \; ; \quad x = x^\mu\sigma_\mu \; ; \quad x' = x'^\mu\sigma_\mu \tag{1.5}$$

$$\xi' = M\xi \; ; \quad \eta' = \widehat{M}\eta \; ; \quad \eta'_n = \widehat{M}\eta_n \; ; \quad \phi'_e = M\phi_e \; ; \quad \phi'_n = M\phi_n \tag{1.6}$$

$$\Psi'_l = \begin{pmatrix} \phi'_e & \phi'_n \\ \widehat{\phi}'_n & \widehat{\phi}'_e \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix}\begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_n & \widehat{\phi}_e \end{pmatrix} = N\Psi_l \tag{1.7}$$

The form (1.3) of the wave is compatible both with the form invariance of the Dirac theory and with the charge conjugation used in the standard model: the wave $\psi_{\overline{e}}$ of the positron satisfies

$$\psi_{\overline{e}} = i\gamma_2\psi^* \Leftrightarrow \widehat{\phi}_{\overline{e}} = \widehat{\phi}_e\sigma_1 \tag{1.8}$$

We can then think the Ψ_l wave as containing the electron wave ϕ_e, the neutrino wave ϕ_n and also the positron wave $\phi_{\overline{e}}$ and the antineutrino wave $\phi_{\overline{n}}$:

$$\Psi_l = \begin{pmatrix} \phi_e & \phi_n \\ \widehat{\phi}_{\overline{n}}\sigma_1 & \widehat{\phi}_{\overline{e}}\sigma_1 \end{pmatrix} \; ; \quad \phi_{\overline{e}} = \sqrt{2}\begin{pmatrix} \xi_{1\overline{e}} & -\eta^*_{2\overline{e}} \\ \xi_{2\overline{e}} & \eta^*_{1\overline{e}} \end{pmatrix} \; ; \quad \phi_{\overline{n}} = \sqrt{2}\begin{pmatrix} \xi_{1\overline{n}} & 0 \\ \xi_{2\overline{n}} & 0 \end{pmatrix} \tag{1.9}$$

And the antineutrino has only a right wave. The multivector $\Psi_l(x)$ is usually an invertible element of the space-time algebra because (See [10] (6.250)) with:

$$a_1 = \det(\phi_e) = \phi_e \overline{\phi}_e = 2(\xi_{1e}\eta^*_{1e} + \xi_{2e}\eta^*_{2e}) \tag{1.10}$$

$$a_2 = 2(\xi_{1\overline{e}}\eta^*_{1n} + \xi_{2\overline{e}}\eta^*_{2n}) = 2(\eta^*_{2e}\eta^*_{1n} - \eta^*_{1e}\eta^*_{2n}) \tag{1.11}$$

$$a_3 = 2(\xi_{1e}\eta^*_{1n} + \xi_{2e}\eta^*_{2n}) \tag{1.12}$$

we got

$$\det(\Psi_l) = a_1 a_1^* + a_2 a_2^*. \tag{1.13}$$

Most of the preceding presentation is easily extended to quarks. For each color $c = r, g, b$ the electro-weak theory needs only left waves:

$$\Psi_c = \begin{pmatrix} \phi_{dc} & \phi_{uc} \\ \widehat{\phi}_{uc} & \widehat{\phi}_{dc} \end{pmatrix};\ \widehat{\phi}_{dc} = \sqrt{2}\begin{pmatrix} \eta_{1dc} & 0 \\ \eta_{2dc} & 0 \end{pmatrix};\ \widehat{\phi}_{uc} = \sqrt{2}\begin{pmatrix} \eta_{1uc} & 0 \\ \eta_{2uc} & 0 \end{pmatrix} \tag{1.14}$$

The Ψ wave is now a function of space and time with value into $Cl_{1,5} = Cl_{5,1}$ which is a sub-algebra (on the real field) of $Cl_{5,2} = M_8(\mathbb{C})$:

$$\Psi = \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix};\ \widetilde{\Psi} = \begin{pmatrix} \widetilde{\Psi}_b & \widetilde{\Psi}_r \\ \widetilde{\Psi}_g & \widetilde{\Psi}_l \end{pmatrix} \tag{1.15}$$

The link between the reverse in $Cl_{1,5}$ and the reverse in $Cl_{1,3}$ is not trivial and is detailed in Appendix A. The wave equation for all objects of the first generation reads

$$0 = (\underline{\mathbf{D}}\Psi)L_{012} + \underline{\mathbf{M}} \tag{1.16}$$

The mass term reads

$$\underline{\mathbf{M}} = \begin{pmatrix} m_2\rho_2\chi_b & m_2\rho_2\chi_g \\ m_2\rho_2\chi_r & m_1\rho_1\chi_l \end{pmatrix} \tag{1.17}$$

where we use the scalar densities s_j and χ terms of Appendix B, with

$$\rho_1^2 = a_1 a_1^* + a_2 a_2^* + a_3 a_3^*\ ;\ \ \rho_2^2 = \sum_{j=1}^{j=15} s_j s_j^*. \tag{1.18}$$

The covariant derivative $\underline{\mathbf{D}}$ uses the matrix representation (A.1) and reads

$$\underline{\mathbf{D}} = \underline{\boldsymbol{\partial}} + \frac{g_1}{2}\underline{\mathbf{B}}\ \underline{P}_0 + \frac{g_2}{2}\underline{\mathbf{W}}^j\underline{P}_j + \frac{g_3}{2}\underline{\mathbf{G}}^k\underline{\mathbf{i}}\Gamma_k \tag{1.19}$$

$$\underline{\mathbf{D}} = \sum_{\mu=0}^{3} L^\mu D_\mu\ ;\ \ \underline{\boldsymbol{\partial}} = \sum_{\mu=0}^{3} L^\mu \partial_\mu\ ;\ \ \underline{\mathbf{B}} = \sum_{\mu=0}^{3} L^\mu B_\mu \tag{1.20}$$

$$\underline{\mathbf{W}}^j = \sum_{\mu=0}^{3} L^\mu W_\mu^j,\ j = 1, 2, 3 \tag{1.21}$$

$$\underline{\mathbf{G}}^k = \sum_{\mu=0}^{3} L^\mu G_\mu^k,\ k = 1, 2, \ldots, 8. \tag{1.22}$$

We use two projectors satisfying

$$\underline{P}_{\pm}(\Psi) = \frac{1}{2}(\Psi \pm \underline{\mathbf{i}}\Psi L_{21}) \; ; \quad \underline{\mathbf{i}} = L_{0123} \tag{1.23}$$

Three operators act on quarks like on leptons:

$$\underline{P}_1(\Psi) = \underline{P}_+(\Psi)L_{35} \tag{1.24}$$
$$\underline{P}_2(\Psi) = \underline{P}_+(\Psi)L_{5012} \tag{1.25}$$
$$\underline{P}_3(\Psi) = \underline{P}_+(\Psi)(-\underline{\mathbf{i}}). \tag{1.26}$$

The fourth operator acts differently on the leptonic and on the quark sector. Using projectors:

$$P^+ = \frac{1}{2}(I_8 + L_{012345}) = \begin{pmatrix} I_4 & 0 \\ 0 & 0 \end{pmatrix}; \; P^- = \frac{1}{2}(I_8 - L_{012345}) = \begin{pmatrix} 0 & 0 \\ 0 & I_4 \end{pmatrix} \tag{1.27}$$

we can separate the lepton part Ψ^l and the quark part Ψ^c of the wave:

$$\Psi^l = P^+\Psi P^+ = \begin{pmatrix} \Psi_l & 0 \\ 0 & 0 \end{pmatrix}; \; \Psi^c = \Psi - \Psi^l = \begin{pmatrix} 0 & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix}. \tag{1.28}$$

and we get (see [10] (B.4) with $a = b = 1$)

$$\underline{P}_0(\Psi^l) = \frac{1}{2}\underline{\mathbf{i}}\Psi^l L_{21} + \frac{3}{2}\Psi^l L_{21} \tag{1.29}$$
$$\underline{P}_0(\Psi^c) = -\frac{1}{3}\Psi^c L_{21}. \tag{1.30}$$

This last relation comes from the non-existence of the right part of the Ψ^c waves.

2 Chromodynamics

We start from generators λ_k of the $SU(3)$ gauge group of chromodynamics

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.1}$$

To simplify here notations we use now l, r, g, b instead Ψ_l, Ψ_r, Ψ_g, Ψ_b. So we have $\Psi = \begin{pmatrix} l & r \\ g & b \end{pmatrix}$. Then (C.1) gives

$$\begin{aligned} &\lambda_1 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} g \\ r \\ 0 \end{pmatrix},\ \lambda_2 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} -ig \\ ir \\ 0 \end{pmatrix},\ \lambda_3 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} r \\ -g \\ 0 \end{pmatrix} \\ &\lambda_4 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ r \end{pmatrix},\ \lambda_5 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ 0 \\ ir \end{pmatrix},\ \lambda_6 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ g \end{pmatrix} \\ &\lambda_7 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -ib \\ ig \end{pmatrix},\ \lambda_8 \begin{pmatrix} r \\ g \\ b \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} r \\ g \\ -2b \end{pmatrix}. \end{aligned} \tag{2.2}$$

We name Γ_k operators corresponding to λ_k acting on Ψ. We get with projectors P^+ and P^- in (1.27):

$$\Gamma_1(\Psi) = \frac{1}{2}(L_4\Psi L_4 + L_{01235}\Psi L_{01235}) = \begin{pmatrix} 0 & g \\ r & 0 \end{pmatrix} \tag{2.3}$$

$$\Gamma_2(\Psi) = \frac{1}{2}(L_5\Psi L_4 - L_{01234}\Psi L_{01235}) = \begin{pmatrix} 0 & -\mathbf{i}g \\ \mathbf{i}r & 0 \end{pmatrix} \tag{2.4}$$

$$\Gamma_3(\Psi) = P^+\Psi P^- - P^-\Psi P^+ = \begin{pmatrix} 0 & r \\ -g & 0 \end{pmatrix} \tag{2.5}$$

$$\Gamma_4(\Psi) = L_{01253}\Psi P^- = \begin{pmatrix} 0 & b \\ 0 & r \end{pmatrix}\ ;\ \Gamma_5(\Psi) = L_{01234}\Psi P^- = \begin{pmatrix} 0 & -\mathbf{i}b \\ 0 & \mathbf{i}r \end{pmatrix} \tag{2.6}$$

$$\Gamma_6(\Psi) = P^-\Psi L_{01253} = \begin{pmatrix} 0 & 0 \\ b & g \end{pmatrix}\ ;\ \Gamma_7(\Psi) = -\underline{\mathbf{i}}P^-\Psi L_4 = \begin{pmatrix} 0 & 0 \\ -\mathbf{i}b & \mathbf{i}g \end{pmatrix} \tag{2.7}$$

$$\Gamma_8(\Psi) = \frac{1}{\sqrt{3}}(P^-\Psi L_{012345} + L_{012345}\Psi P^-) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & r \\ g & -2b \end{pmatrix}. \tag{2.8}$$

Everywhere the left up term is 0, so all Γ_k project the wave Ψ on its quark sector.

We can extend the covariant derivative of electro-weak interactions in the electron-neutrino case:

$$D\Psi_l = \partial\Psi_l + \frac{g_1}{2}BP_0(\Psi_l) + \frac{g_2}{2}W^jP_j(\Psi_l) \tag{2.9}$$

to get the covariant derivative of the standard model

$$\underline{D}(\Psi) = \underline{\partial}(\Psi) + \frac{g_1}{2}\underline{B}\ \underline{P}_0(\Psi) + \frac{g_2}{2}\underline{W}^j\underline{P}_j(\Psi) + \frac{g_3}{2}\underline{G}^k\underline{\mathbf{i}}\Gamma_k(\Psi). \tag{2.10}$$

where g_3 is another constant and $\underline{G}^k$ are eight terms called "gluons". Since I_4 commute with any element of $Cl_{1,3}$ and since $P_j(\mathbf{i}\Psi_{ind}) = \mathbf{i}P_j(\Psi_{ind})$ for $j = 0, 1, 2, 3$ and $ind = l, r, g, b$ each operator $\underline{\mathbf{i}}\Gamma_k$ commutes with all operators $\underline{P}_j$.

Now we use 12 real numbers $a^0,\ a^j,\ j=1,2,3,\ b^k,\ k=1,2,...,8$, we let

$$S_0 = a^0 \underline{P}_0;\ S_1 = \sum_{j=1}^{j=3} a^j \underline{P}_j;\ S_2 = \sum_{k=1}^{k=8} b^k \underline{\mathbf{i}} \Gamma_k;\ S = S_0 + S_1 + S_2 \tag{2.11}$$

and we get, using exponentiation

$$\begin{aligned} \exp(S) &= \exp(S_0)\exp(S_1)\exp(S_2) = \exp(S_1)\exp(S_0)\exp(S_2) \\ &= \exp(S_0)\exp(S_2)\exp(S_1) = \dots \end{aligned} \tag{2.12}$$

in any order. The set of these operators $\exp(S)$ is a $U(1)\times SU(2)\times SU(3)$ Lie group. Only difference with the standard model: the structure of this group is not postulated but calculated. With

$$\Psi' = [\exp(S)](\Psi)\ ;\ \underline{\mathbf{D}} = L^\mu \underline{D}_\mu\ ;\ \underline{\mathbf{D}}' = L^\mu \underline{D}'_\mu \tag{2.13}$$

the gauge transformation reads

$$\underline{D}'_\mu \Psi' = [\exp(S)](\underline{D}_\mu \Psi) \tag{2.14}$$

$$B'_\mu = B_\mu - \frac{2}{g_1}\partial_\mu a^0 \tag{2.15}$$

$$W'^j_\mu \underline{P}_j = \Big[\exp(S_1) W^j_\mu \underline{P}_j - \frac{2}{g_2}\partial_\mu[\exp(S_1)]\Big]\exp(-S_1) \tag{2.16}$$

$$\underline{G}'^k_\mu \underline{\mathbf{i}} \Gamma_k = \Big[\exp(S_2)\underline{G}^k_\mu \underline{\mathbf{i}}\Gamma_k - \frac{2}{g_3}\partial_\mu[\exp(S_2)]\Big]\exp(-S_2). \tag{2.17}$$

The $SU(3)$ group generated by projectors Γ_k acts only on the quark sector of the wave:

$$P^+[\exp(b^k \underline{\mathbf{i}}\Gamma_k](\Psi)P^+ = P^+\Psi P^+ = \Psi^l \tag{2.18}$$

The physical translation is: Leptons do not act by strong interactions. This comes from the structure of the wave itself. It is fully satisfied in experiments. We get then a $U(1)\times SU(2)\times SU(3)$ gauge group for a wave including all fermions of the first generation. This group acts on the lepton sector only by its $U(1)\times SU(2)$ part. Consequently the wave equation is composed of a lepton wave equation and a quark wave equation:

$$0 = (\underline{\mathbf{D}}\Psi^l)L_{012} + m_1\rho_1 \begin{pmatrix} 0 & 0 \\ 0 & \chi_l \end{pmatrix} \tag{2.19}$$

$$0 = (\underline{\mathbf{D}}\Psi^c)L_{012} + m_2\rho_2\chi^c;\ \chi^c = \begin{pmatrix} \chi_b & \chi_g \\ \chi_r & 0 \end{pmatrix} \tag{2.20}$$

The wave equation (2.19) is equivalent to the wave equation

$$\mathbf{D}\Psi_l\gamma_{012} + m_1\rho_1\chi_l = 0\ ;\quad \gamma_{012} = \gamma_0\gamma_1\gamma_2 \tag{2.21}$$

studied in [9] [8], where

$$\chi_l = \frac{1}{\rho_1^2}\begin{pmatrix} a_1^*\phi_e + a_2^*\phi_n\sigma_1 + a_3^*\phi_n & -a_2^*\phi_{eL}\sigma_1 + a_3^*\phi_{eR} \\ a_2\widehat{\phi}_{eL}\sigma_1 + a_3\widehat{\phi}_{eR} & a_1\widehat{\phi}_e - a_2\widehat{\phi}_n\sigma_1 + a_3\widehat{\phi}_n \end{pmatrix} \tag{2.22}$$

$$\phi_{eR} = \phi_e\frac{1+\sigma_3}{2}\ ;\quad \phi_{eL} = \phi_e\frac{1-\sigma_3}{2} \tag{2.23}$$

This wave equation is equivalent to the invariant equation:

$$\widetilde{\Psi}_l(\mathbf{D}\Psi_l)\gamma_{012} + m\rho_1\widetilde{\Psi}_l\chi_l = 0\ ;\quad \widetilde{\Psi}_l = \begin{pmatrix} \overline{\phi}_e & \phi_n^\dagger \\ \overline{\phi}_n & \phi_e^\dagger \end{pmatrix}. \tag{2.24}$$

This wave equation is form invariant under the Lorentz dilation R induced by any invertible matrix M satisfying (1.5), (1.6), (1.7) [10]. It is gauge invariant under the $U(1)\times SU(2)$ group [9] generated by operators P_μ which are projections on the lepton sector of the operators defined in (1.23) to (1.29). Therefore we need to study only the quark sector and its wave equation (2.20).

We begin by the double link between wave equation and Lagrangian density that we have remarked firstly in the Dirac equation [6], next in the lepton case electron+neutrino [10].

3 Double link between wave equation and Lagrangian density

The existence of a Lagrangian mechanism in optics and mechanics is known since Fermat and Maupertuis. This principle of minimum is everywhere in quantum mechanics from its beginning, it is the main reason of the hypothesis of a wave linked to the move of any material particle made by L. de Broglie [11]. By the calculus of variations it is always possible to get the wave equation from the Lagrangian density. But another link exists : the Lagrangian density is the real scalar part of the invariant wave equation. This was obtained firstly for the electron alone [6], next for the pair electron-neutrino [9] where the Lagrangian density reads

$$\mathcal{L}_l = \mathcal{L}_0 + g_1\mathcal{L}_1 + g_2\mathcal{L}_2 + m_1\rho_1 \tag{3.1}$$

$$\mathcal{L}_0 = \Re[-i(\eta_e^\dagger\sigma^\mu\partial_\mu\eta_e + \xi_e^\dagger\widehat{\sigma}^\mu\partial_\mu\xi_e + \eta_n^\dagger\sigma^\mu\partial_\mu\eta_n)] \tag{3.2}$$

$$\mathcal{L}_1 = B_\mu(\frac{1}{2}\eta_e^\dagger\sigma^\mu\eta_e + \xi_e^\dagger\widehat{\sigma}^\mu\xi_e + \frac{1}{2}\eta_n^\dagger\sigma^\mu\eta_n) \tag{3.3}$$

$$\mathcal{L}_2 = -\Re[(W_\mu^1 + iW_\mu^2)\eta_e^\dagger\sigma^\mu\eta_n] + \frac{W_\mu^3}{2}(\eta_e^\dagger\sigma^\mu\eta_e - \eta_n^\dagger\sigma^\mu\eta_n). \tag{3.4}$$

We shall establish the double link now for the wave equation (1.16). It is sufficient to add the property for (2.20). This equation is equivalent to the

invariant equation:

$$0 = \widetilde{\Psi}^c(\underline{\mathbf{D}}\Psi^c)L_{012} + m_2\rho_2\widetilde{\Psi}^c\chi^c \tag{3.5}$$

$$\widetilde{\Psi}^c = \begin{pmatrix} \widetilde{\Psi}_b & \widetilde{\Psi}_r \\ \widetilde{\Psi}_g & 0 \end{pmatrix};\ \chi^c = \begin{pmatrix} \chi_b & \chi_g \\ \chi_r & 0 \end{pmatrix} \tag{3.6}$$

We get from the covariant derivative (1.19) with the operators $\underline{P}_j$ in (1.24), (1.25), (1.26) and (1.30) and Γ_k in (2.3) to (2.8) and with Ψ^c in (1.28)

$$\underline{\mathbf{D}}\Psi^c = \begin{pmatrix} A_g & A_b \\ 0 & A_r \end{pmatrix} \tag{3.7}$$

$$\begin{aligned} A_g = {} & \boldsymbol{\partial}\Psi_g - \frac{g_1}{6}\mathbf{B}\Psi_g\gamma_{21} + \frac{g_2}{2}(\mathbf{W}^1\Psi_g\gamma_3\mathbf{i} + \mathbf{W}^2\Psi_g\gamma_3 - \mathbf{W}^3\Psi_g\mathbf{i}) \\ & + \frac{g_3}{2}(\mathbf{G}^1\mathbf{i}\Psi_r - \mathbf{G}^2\Psi_r - \mathbf{G}^3\mathbf{i}\Psi_g + \mathbf{G}^6\mathbf{i}\Psi_b + \mathbf{G}^7\Psi_b + \frac{1}{\sqrt{3}}\mathbf{G}^8\mathbf{i}\Psi_g) \end{aligned} \tag{3.8}$$

$$\begin{aligned} A_b = {} & \boldsymbol{\partial}\Psi_b - \frac{g_1}{6}\mathbf{B}\Psi_b\gamma_{21} + \frac{g_2}{2}(\mathbf{W}^1\Psi_b\gamma_3\mathbf{i} + \mathbf{W}^2\Psi_b\gamma_3 - \mathbf{W}^3\Psi_b\mathbf{i}) \\ & + \frac{g_3}{2}(\mathbf{G}^4\mathbf{i}\Psi_r - \mathbf{G}^5\Psi_r + \mathbf{G}^6\mathbf{i}\Psi_g - \mathbf{G}^7\Psi_g - \frac{2}{\sqrt{3}}\mathbf{G}^8\mathbf{i}\Psi_b) \end{aligned} \tag{3.9}$$

$$\begin{aligned} A_r = {} & \boldsymbol{\partial}\Psi_r - \frac{g_1}{6}\mathbf{B}\Psi_r\gamma_{21} + \frac{g_2}{2}(\mathbf{W}^1\Psi_r\gamma_3\mathbf{i} + \mathbf{W}^2\Psi_r\gamma_3 - \mathbf{W}^3\Psi_r\mathbf{i}) \\ & + \frac{g_3}{2}(\mathbf{G}^1\mathbf{i}\Psi_g + \mathbf{G}^2\Psi_g + \mathbf{G}^3\mathbf{i}\Psi_r + \mathbf{G}^4\mathbf{i}\Psi_b + \mathbf{G}^5\Psi_b + \frac{1}{\sqrt{3}}\mathbf{G}^8\mathbf{i}\Psi_r) \end{aligned} \tag{3.10}$$

Next we get

$$\begin{aligned} & \widetilde{\Psi}^c(\underline{\mathbf{D}}\Psi^c)L_{012} + m_2\rho_2\widetilde{\Psi}^c\chi^c \\ & = \begin{pmatrix} \widetilde{\Psi}_b(A_b\gamma_{012} + m_2\rho_2\chi_b) + \widetilde{\Psi}_r(A_r\gamma_{012} + m_2\rho_2\chi_r) & \widetilde{\Psi}_b(A_g\gamma_{012} + m_2\rho_2\chi_g) \\ \widetilde{\Psi}_g(A_b\gamma_{012} + m_2\rho_2\chi_b) & \widetilde{\Psi}_g(A_g\gamma_{012} + m_2\rho_2\chi_g) \end{pmatrix} \end{aligned} \tag{3.11}$$

The calculation of the Lagrangian density in the general case is similar to the lepton case. We get

$$\mathcal{L} = \mathcal{L}_l + \mathcal{L}_c \tag{3.12}$$

$$\mathcal{L}_c = \sum_{c=r,g,b}\mathcal{L}_{0c} + g_1\sum_{c=r,g,b}\mathcal{L}_{1c} + g_2\sum_{c=r,g,b}\mathcal{L}_{2c} + g_3\mathcal{L}_3 + m_2\rho_2 \tag{3.13}$$

The calculation of $\mathcal{L}_{jc},\ j = 0, 1, 2$ replaces the pair e-n by the pair dc-uc and suppress the ξ terms, then (3.2) (3.3) (3.4) become

$$\mathcal{L}_{0c} = \Re[-i(\eta^\dagger_{dc}\sigma^\mu\partial_\mu\eta_{dc} + \eta^\dagger_{uc}\sigma^\mu\partial_\mu\eta_{uc})] \tag{3.14}$$

$$\mathcal{L}_{1c} = -\frac{B_\mu}{6}(\eta^\dagger_{dc}\sigma^\mu\eta_{dc} + \eta^\dagger_{uc}\sigma^\mu\eta_{uc}) \tag{3.15}$$

$$\mathcal{L}_{2c} = -\Re[(W^1_\mu + iW^2_\mu)\eta^\dagger_{dc}\sigma^\mu\eta_{uc}] + \frac{W^3_\mu}{2}(\eta^\dagger_{dc}\sigma^\mu\eta_{dc} - \eta^\dagger_{uc}\sigma^\mu\eta_{uc}) \tag{3.16}$$

Since three $SU(2)$ group are included in $SU(3)$ the calculation of $\mathcal{L}_3$ has similarities with the calculation of $\mathcal{L}_2$ and we get

$$\begin{aligned}\mathcal{L}_3 = &-\Re[(G^1_\mu + iG^2_\mu)(\eta^\dagger_{dr}\sigma^\mu\eta_{dg} + \eta^\dagger_{ur}\sigma^\mu\eta_{ug})] \\ &-\Re[(G^4_\mu + iG^5_\mu)(\eta^\dagger_{dr}\sigma^\mu\eta_{db} + \eta^\dagger_{ur}\sigma^\mu\eta_{ub})] \\ &-\Re[(G^6_\mu + iG^7_\mu)(\eta^\dagger_{dg}\sigma^\mu\eta_{db} + \eta^\dagger_{ug}\sigma^\mu\eta_{ub})] \\ &+\frac{G^3_\mu}{2}(-\eta^\dagger_{dr}\sigma^\mu\eta_{dr} - \eta^\dagger_{ur}\sigma^\mu\eta_{ur} + \eta^\dagger_{dg}\sigma^\mu\eta_{dg} + \eta^\dagger_{ug}\sigma^\mu\eta_{ug}) \\ &+\frac{G^8_\mu}{2\sqrt{3}}(-\eta^\dagger_{dr}\sigma^\mu\eta_{dr} - \eta^\dagger_{ur}\sigma^\mu\eta_{ur} + \eta^\dagger_{db}\sigma^\mu\eta_{db} + \eta^\dagger_{ub}\sigma^\mu\eta_{ub}) \\ &+\frac{G^8_\mu}{2\sqrt{3}}(-\eta^\dagger_{dg}\sigma^\mu\eta_{dg} - \eta^\dagger_{ug}\sigma^\mu\eta_{ug} + \eta^\dagger_{db}\sigma^\mu\eta_{db} + \eta^\dagger_{ub}\sigma^\mu\eta_{ub})\end{aligned} \tag{3.17}$$

This new link between the wave equation and the Lagrangian density is much stronger than the old one, because it comes from a simple separation of the different parts of a multivector in Clifford algebra. The old link, going from the Lagrangian density to the wave equation, supposes a condition of cancellation at infinity which is dubious in the case of a propagating wave. On the physical point of view, there are no difficulties in the case of a stationary wave. Difficulties begin when propagating waves are studied. Our wave equations, since they are compatible with an oriented time and an oriented space, appear as more general, more physical, than Lagrangians. These are only particular consequences of the wave equations.

On the mathematical point of view the old link is always available. It is from the Lagrangian density (3.12) and using Lagrange equations that we have obtained the wave equation (1.16).

4 Invariances

4.1 Form invariance of the wave equation

Under the Lorentz dilation R induced by an invertible M matrix satisfying

$$x' = MxM^\dagger \ ; \quad \det(M) = re^{i\theta} \ ; \quad x = x^\mu\sigma_\mu \ ; \quad x' = x'^\mu\sigma_\mu \tag{4.1}$$

$$\eta'_{uc} = \widehat{M}\eta_{uc} \ ; \quad \eta'_{dc} = \widehat{M}\eta_{dc} \ ; \quad \phi'_{dc} = M\phi_{dc} \ ; \quad \phi'_{uc} = M\phi_{uc} \tag{4.2}$$

$$\Psi'_c = \begin{pmatrix}\phi'_{dc} & \phi'_{uc}\\ \widehat{\phi}'_{uc} & \widehat{\phi}'_{dc}\end{pmatrix} = \begin{pmatrix}M & 0\\ 0 & \widehat{M}\end{pmatrix}\begin{pmatrix}\phi_{dc} & \phi_{uc}\\ \widehat{\phi}_{uc} & \widehat{\phi}_{dc}\end{pmatrix} = N\Psi_c \ ; \quad c = r, g, b. \tag{4.3}$$

We then let

$$\underline{N} = \begin{pmatrix}N & 0\\ 0 & N\end{pmatrix} ; \ \underline{\boldsymbol{\partial}} = L^\mu\partial_\mu = \begin{pmatrix}0 & \boldsymbol{\partial}\\ \boldsymbol{\partial} & 0\end{pmatrix} \tag{4.4}$$

which implies

$$\Psi'^c = \underline{N}\Psi^c; \ \widetilde{\Psi}'^c = \widetilde{\Psi}^c\underline{\widetilde{N}}; \ \underline{\widetilde{N}} = \begin{pmatrix}\widetilde{N} & 0\\ 0 & \widetilde{N}\end{pmatrix} ; \ \underline{\mathbf{D}} = \underline{\widetilde{N}}\ \underline{\mathbf{D}}'\underline{N}. \tag{4.5}$$

Then we get

$$\begin{aligned}\widetilde{\Psi}^c(\underline{\mathbf{D}}\Psi^c)L_{012} &= \widetilde{\Psi}^c\widetilde{\underline{N}}\ \underline{\mathbf{D}}'\underline{N}\Psi^c L_{012}\\ &= \widetilde{\Psi}'^c(\underline{\mathbf{D}}'\Psi'^c)L_{012}.\end{aligned} \tag{4.6}$$

and we shall now study the form invariance of the mass term. All s_j are determinants of a ϕ matrix, this implies

$$s'_j = \det(\phi') = \det(M\phi) = \det(M)\det(\phi) = re^{i\theta}s_j \tag{4.7}$$

$$s'^*_j = re^{-i\theta}s^*_j;\ \rho'_2 = r\rho_2. \tag{4.8}$$

This gives

$$\chi'^c = \begin{pmatrix}\chi'_b & \chi'_g\\ \chi'_r & 0\end{pmatrix} \tag{4.9}$$

$$r^2\rho_2^2\chi'^c = \rho'^2_2\chi'^c = \begin{pmatrix}re^{-i\theta}M & 0\\ 0 & re^{i\theta}\widehat{M}\end{pmatrix}\rho_2^2\chi^c \tag{4.10}$$

$$\chi'^c = \begin{pmatrix}r^{-1}e^{-i\theta}M & 0\\ 0 & r^{-1}e^{i\theta}\widehat{M}\end{pmatrix}\chi_c = \widetilde{N}^{-1}\chi^c \tag{4.11}$$

$$\widetilde{\Psi}'^c\chi'^c = \widetilde{\Psi}^c\widetilde{N}\widetilde{N}^{-1}\chi^c = \widetilde{\Psi}^c\chi^c \tag{4.12}$$

Then the form invariance of the wave equation is equivalent to the condition on the mass term

$$m'_2\rho'_2 = m_2\rho_2 \tag{4.13}$$

$$m'_2 r = m_2 \tag{4.14}$$

linked to the existence of the Planck factor [8].

4.2 Gauge invariance of the wave equation

Since we have previously proved the gauge invariance of the lepton part of the wave equation, it is reason enough to prove the gauge invariance of the quark part of the wave equation.

4.2.1 Gauge group generated by $\underline{P}_0$

We have here

$$\underline{P}_0(\Psi^c) = \Psi^c(-\frac{1}{3}L_{21}) \tag{4.15}$$

$$\Psi'^c = [\exp(\theta\underline{P}_0)](\Psi^c) = \Psi^c\exp(-\frac{\theta}{3}L_{21}) \tag{4.16}$$

$$B'_\mu = B_\mu - \frac{2}{g_1}B_\mu \tag{4.17}$$

To get the gauge invariance of the wave equation we must get

$$\chi'^{c} = \chi^{c} \exp(-\frac{\theta}{3} L_{21});\ \chi'_{c} = \chi_{c} \exp(-\frac{\theta}{3}\gamma_{21}),\ c = r, g, b. \tag{4.18}$$

This is satisfied because

$$\phi'_{dc} = \phi_{dc} e^{-i\frac{\theta}{3}\sigma_3};\ \phi'_{uc} = \phi_{uc} e^{-i\frac{\theta}{3}\sigma_3} \tag{4.19}$$

$$\eta'^{*}_{1dc} = e^{i\frac{\theta}{3}} \eta^{*}_{1dc};\ \eta'^{*}_{1uc} = e^{i\frac{\theta}{3}} \eta^{*}_{1uc}$$

$$\eta'^{*}_{2dc} = e^{i\frac{\theta}{3}} \eta^{*}_{2dc};\ \eta'^{*}_{2uc} = e^{i\frac{\theta}{3}} \eta^{*}_{2uc} \tag{4.20}$$

$$s'_{j} = e^{2i\frac{\theta}{3}} s_{j},\ j = 1, 2, \ldots, 15. \tag{4.21}$$

All up terms in the matrix χ_c contain $s^{*}_{j}\phi_{dc}\sigma_1$ and $s^{*}_{j}\phi_{uc}\sigma_1$ terms. We get

$$\phi'_{dc} = \phi_{dc} e^{-i\frac{\theta}{3}\sigma_3} = e^{i\frac{\theta}{3}} \phi_{dc} \tag{4.22}$$

$$s'^{*}_{j}\phi'_{dc}\sigma_1 = e^{-i\frac{\theta}{3}} \phi_{dc}\sigma_1 = \phi_{dc} e^{\frac{\theta}{3}\sigma_{12}}\sigma_1 = \phi_{dc}\sigma_1 e^{-\frac{\theta}{3}\sigma_{12}} \tag{4.23}$$

$$\chi'_{c} = \chi_{c} \exp(-\frac{\theta}{3}\gamma_{21}) \tag{4.24}$$

$$\chi'^{c} = \chi^{c} \exp(-\frac{\theta}{3} L_{21}). \tag{4.25}$$

And we finally get

$$(\underline{\mathbf{D}}'\Psi'^{c}) L_{012} + m_2 \rho'_2 \chi'^{c} = [(\underline{\mathbf{D}}\Psi^{c}) L_{012} + m_2 \rho_2 \chi^{c}] \exp(-\frac{\theta}{3} L_{21}) = 0 \tag{4.26}$$

The wave equation with mass term is gauge invariant under the group generated by $\underline{P}_0$.

4.2.2 Gauge group generated by $\underline{P}_1$

We have here

$$\underline{P}_1(\Psi^{c}) = \Psi^{c} L_{35} \tag{4.27}$$

$$\Psi'^{c} = [\exp(\theta \underline{P}_1)](\Psi^{c}) = \Psi^{c} \exp(\theta L_{35}) \tag{4.28}$$

$$W'^{1}_{\mu} = W^{1}_{\mu} - \frac{2}{g_2}\partial_{\mu}\theta \tag{4.29}$$

We put a more detailed calculation in C.1. We get

$$\begin{aligned}(\underline{\mathbf{D}}'\Psi'^{c}) L_{012} + m_2 \rho'_2 \chi'^{c} &= (\underline{\mathbf{D}}\Psi^{c}) \exp(\theta L_{35}) L_{012} + m_2 \rho'_2 \chi'^{c} \\ &= [(\underline{\mathbf{D}}\Psi^{c}) L_{012} + m_2 \rho_2 \chi^{c}] \exp(\theta L_{35}) = 0\end{aligned} \tag{4.30}$$

The wave equation with mass term is then gauge invariant under the group generated by $\underline{P}_1$.

4.2.3 Gauge group generated by $\underline{P}_2$

We have here

$$\underline{P}_2(\Psi^c) = \Psi^c L_{5012} \tag{4.31}$$

$$\Psi'^c = [\exp(\theta \underline{P}_2)](\Psi^c) = \Psi^c \exp(\theta L_{5012}) \tag{4.32}$$

$$W'^2_\mu = W^2_\mu - \frac{2}{g_2}\partial_\mu \theta \tag{4.33}$$

We have put a more detailed calculation in C.2. We get

$$\begin{aligned}(\underline{\mathbf{D}}'\Psi'^c)L_{012} + m_2\rho'_2\chi'^c &= (\underline{\mathbf{D}}\Psi^c)\exp(\theta L_{5012})L_{012} + m_2\rho'_2\chi'^c \\ &= [(\underline{\mathbf{D}}\Psi^c)L_{012} + m_2\rho_2\chi^c]\exp(-\theta L_{5012}) = 0\end{aligned} \tag{4.34}$$

The wave equation with mass term is then gauge invariant under the group generated by $\underline{P}_2$.

4.2.4 Gauge group generated by $\underline{P}_3$

We have here

$$\underline{P}_3(\Psi^c) = \Psi^c L_{3012} \tag{4.35}$$

$$\Psi'^c = [\exp(\theta \underline{P}_3)](\Psi^c) = \Psi^c \exp(\theta L_{3012}) \tag{4.36}$$

$$W'^3_\mu = W^3_\mu - \frac{2}{g_2}\partial_\mu \theta \tag{4.37}$$

We have put a more detailed calculation in C.3. We get

$$\begin{aligned}(\underline{\mathbf{D}}'\Psi'^c)L_{012} + m_2\rho'_2\chi'^c &= (\underline{\mathbf{D}}\Psi^c)\exp(\theta L_{3012})L_{012} + m_2\rho'_2\chi'^c \\ &= [(\underline{\mathbf{D}}\Psi^c)L_{012} + m_2\rho_2\chi^c]\exp(-\theta L_{3012}) = 0\end{aligned} \tag{4.38}$$

The wave equation with mass term is then gauge invariant under the group generated by $\underline{P}_3$.

4.2.5 Gauge group generated by Γ_1

We use now the gauge transformation

$$\Psi'_r = C\Psi_r + S\mathbf{i}\Psi_g;\ C = \cos(\theta);\ S = \sin(\theta) \tag{4.39}$$

$$\Psi'_g = C\Psi_g + S\mathbf{i}\Psi_r \tag{4.40}$$

$$\Psi'_b = \Psi_b \tag{4.41}$$

We can then forget here Ψ_b. The gauge invariance signifies that the system

$$\begin{aligned}\boldsymbol{\partial}\Psi_r &= -\frac{g_3}{2}\mathbf{G}^1\mathbf{i}\Psi_g + m_2\rho_2\chi_r\gamma_{012} \\ \boldsymbol{\partial}\Psi_g &= -\frac{g_3}{2}\mathbf{G}^1\mathbf{i}\Psi_r + m_2\rho_2\chi_g\gamma_{012}\end{aligned} \tag{4.42}$$

must be equivalent to the system

$$\begin{aligned}\boldsymbol{\partial}\Psi'_r &= -\frac{g_3}{2}\mathbf{G}'^1\mathbf{i}\Psi'_g + m_2\rho'_2\chi'_r\gamma_{012}\\ \boldsymbol{\partial}\Psi'_g &= -\frac{g_3}{2}\mathbf{G}'^1\mathbf{i}\Psi'_r + m_2\rho'_2\chi'_g\gamma_{012}\end{aligned} \tag{4.43}$$

Using relations (4.39) and (4.40) the system (4.43) is equivalent to (4.42) if and only if

$$\mathbf{G}'^1 = \mathbf{G}^1 - \frac{2}{g_3}\boldsymbol{\partial}\theta \tag{4.44}$$

because we get in C.4

$$\rho' = \rho \tag{4.45}$$
$$\chi'_r = C\chi_r - S\mathbf{i}\chi_g \tag{4.46}$$
$$\chi'_g = C\chi_g - S\mathbf{i}\chi_r \tag{4.47}$$

The change of sign of the phase between (4.39) and (4.46) comes from the anticommutation between $\mathbf{i}$ and $\boldsymbol{\partial}$.

4.2.6 Gauge groups generated by $\Gamma_k\ ,\ k>1$

We use with $k=2$ the gauge transformation

$$\Psi'_r = C\Psi_r + S\Psi_g;\ C=\cos(\theta);\ S=\sin(\theta) \tag{4.48}$$
$$\Psi'_g = C\Psi_g - S\Psi_r \tag{4.49}$$
$$\Psi'_b = \Psi_b \tag{4.50}$$

The gauge invariance signifies that the system

$$\begin{aligned}\boldsymbol{\partial}\Psi_r &= -\frac{g_3}{2}\mathbf{G}^2\Psi_g + m_2\rho_2\chi_r\gamma_{012}\\ \boldsymbol{\partial}\Psi_g &= \frac{g_3}{2}\mathbf{G}^2\Psi_r + m_2\rho_2\chi_g\gamma_{012}\end{aligned} \tag{4.51}$$

must be equivalent to the system

$$\begin{aligned}\boldsymbol{\partial}\Psi'_r &= -\frac{g_3}{2}\mathbf{G}'^2\Psi'_g + m_2\rho'_2\chi'_r\gamma_{012}\\ \boldsymbol{\partial}\Psi'_g &= \frac{g_3}{2}\mathbf{G}'^2\Psi'_r + m_2\rho'_2\chi'_g\gamma_{012}\end{aligned} \tag{4.52}$$

Using relations (4.48) and (4.49) the system (4.52) is equivalent to (4.51) if and only if

$$\mathbf{G}'^2 = \mathbf{G}^2 - \frac{2}{g_3}\boldsymbol{\partial}\theta \tag{4.53}$$

because we get

$$\rho' = \rho \tag{4.54}$$
$$\chi'_r = C\chi_r + S\chi_g \tag{4.55}$$
$$\chi'_g = C\chi_g - S\chi_r. \tag{4.56}$$

The case $k = 3$ is detailed in C.5 and the case $k = 8$ is detailed in C.6. Cases $k = 4$ and $k = 6$ are similar to $k = 1$ and cases $k = 5$ and $k = 7$ are similar to $k = 2$ by permutation of indexes of color.

5 Concluding remarks

From experimental results obtained in the accelerators physicists have built what is now known as the "standard model". This model is generally thought to be a part of quantum field theory, itself a part of axiomatic quantum mechanics. One of these axioms is that each state describing a physical situation follows a Schrödinger wave equation. Since this wave equation is not relativistic and does not account for the spin 1/2 which is necessary to any fermion, the standard model has evidently not followed the axiom and has used instead a Dirac equation to describe fermions. Our work also starts with the Dirac equation. This wave equation is the linear approximation of our nonlinear homogeneous equation of the electron.

The wave equation presented here is a wave equation for a classical wave, a function of space and time with value into a Clifford algebra. It is not a quantized wave with value into a Hilbertian space of operators. Nevertheless and consequently we get most of the aspects of the standard model, for instance the fact that leptons are insensitive to strong interactions. The standard model is much stronger than generally thought. For instance we firstly did not use the link between the wave of the particle and the wave of the antiparticle, but then we needed a greater Clifford algebra and we could not get the necessary link between reversions[1] that we use in our wave equation. We also needed the existence of the inverse to build the wave of a system of particles from the waves of its components. And we got two general identities which exist only if all parts of the general wave are left waves, only the electron having also a right wave.

The most important property of the general wave is its form invariance under a group including the covering group of the restricted Lorentz group. Our group does not explain why space and time are oriented, but it respects these orientations. The physical time is then compatible with thermodynamics, and the physical space is compatible with the violation of parity by weak interactions.

The wave accounts for all particles and anti-particles of the first generation. We have also given [6][7][8][9] the reason of the existence of three generations, it is simply the dimension of our physical space. Since the $SU(3)$ gauge group of chromodynamics acts independently from the index of generations, the physical quarks may be combinations of quarks of different generations. Quarks composing protons and neutrons are such combinations. Our wave equation allows only two masses at each generation, one for the lepton part of the wave, the other one for the two quarks. The mixing can give a different mass for the two quarks of each generation.

[1] The reversion is an anti-isomorphism changing the order of any product (see [10] 1.1). It is specific to each Clifford algebra. The Appendix A explains the link between the reversion in $Cl_{1,3}$ and the reversion in $Cl_{1,5}$

Since the wave equation with mass term is gauge invariant, there is no necessity to use the mechanism of spontaneous symmetry breaking. The scalar boson certainly exists, but it does not explain the masses.

A wave equation is only a beginning. It shall be necessary to study also the boson part of the standard model and the systems of fermions, from this wave equation. A construction of the wave of a system of identical particles is possible and compatible with the Pauli principle [5] [10].

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A Calculation of the reverse in $Cl_{1,5}$

Here indexes $\mu, \nu, \rho \ldots$ have value $0, 1, 2, 3$ and indexes a, b, c, d, e have value $0, 1, 2, 3, 4, 5$. We use[2] the following matrix representation of $Cl_{1,5}$:

$$L_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix};\ L_4 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix};\ L_5 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix};\ \mathbf{i} = \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}$$
$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix};\ \gamma_j = -\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix},\ j = 1, 2, 3 \tag{A.1}$$

where σ_j are Pauli matrices. This gives

$$L_{\mu\nu} = L_\mu L_\nu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \gamma_\nu \\ \gamma_\nu & 0 \end{pmatrix} = \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & \gamma_{\mu\nu} \end{pmatrix} \tag{A.2}$$

$$L_{\mu\nu\rho} = L_{\mu\nu} L_\rho = \begin{pmatrix} \gamma_{\mu\nu} & 0 \\ 0 & \gamma_{\mu\nu} \end{pmatrix} \begin{pmatrix} 0 & \gamma_\rho \\ \gamma_\rho & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_{\mu\nu\rho} \\ \gamma_{\mu\nu\rho} & 0 \end{pmatrix} \tag{A.3}$$

$$L_{0123} = L_{01} L_{23} = \begin{pmatrix} \gamma_{0123} & 0 \\ 0 & \gamma_{0123} \end{pmatrix} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \tag{A.4}$$

We get also

$$L_{45} = L_4 L_5 = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} = -L_{54} \tag{A.5}$$

$$L_{012345} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \tag{A.6}$$

$$L_{01235} = L_{0123} L_5 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I_4 \\ -I_4 & 0 \end{pmatrix}. \tag{A.7}$$

[2] I_2, I_4, I_8 are unit matrices. The identification process allowing to include $\mathbb{R}$ in each real Clifford algebra allows to read a instead of aI_n for any complex number a.

Similarly we get[3]

$$L_{\mu 4} = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} \; ; \quad L_{\mu 5} = \begin{pmatrix} \gamma_\mu \mathbf{i} & 0 \\ 0 & \gamma_\mu \mathbf{i} \end{pmatrix} \tag{A.8}$$

$$L_{\mu\nu 4} = \begin{pmatrix} 0 & -\gamma_{\mu\nu} \\ \gamma_{\mu\nu} & 0 \end{pmatrix} \; ; \quad L_{\mu\nu 5} = \begin{pmatrix} 0 & \gamma_{\mu\nu} \mathbf{i} \\ \gamma_{\mu\nu} \mathbf{i} & 0 \end{pmatrix} \tag{A.9}$$

$$L_{\mu\nu\rho 4} = \begin{pmatrix} \gamma_{\mu\nu\rho} & 0 \\ 0 & -\gamma_{\mu\nu\rho} \end{pmatrix} \; ; \quad L_{\mu\nu\rho 5} = \begin{pmatrix} \gamma_{\mu\nu\rho} \mathbf{i} & 0 \\ 0 & \gamma_{\mu\nu\rho} \mathbf{i} \end{pmatrix} \tag{A.10}$$

$$L_{\mu 45} = \begin{pmatrix} 0 & \gamma_\mu \mathbf{i} \\ -\gamma_\mu \mathbf{i} & 0 \end{pmatrix} \; ; \quad L_{\mu\nu 45} = \begin{pmatrix} -\gamma_{\mu\nu} \mathbf{i} & 0 \\ 0 & \gamma_{\mu\nu} \mathbf{i} \end{pmatrix} \tag{A.11}$$

$$L_{\mu\nu\rho 45} = \begin{pmatrix} 0 & \gamma_{\mu\nu\rho} \mathbf{i} \\ -\gamma_{\mu\nu\rho} \mathbf{i} & 0 \end{pmatrix} \; ; \quad L_{01234} = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} \tag{A.12}$$

Scalar and pseudo-scalar terms read

$$\alpha I_8 + \omega L_{012345} = \begin{pmatrix} (\alpha + \omega) I_4 & 0 \\ 0 & (\alpha - \omega) I_4 \end{pmatrix} \tag{A.13}$$

$$\alpha I_8 - \omega \Lambda_{012345} = \begin{pmatrix} (\alpha - \omega) I_4 & 0 \\ 0 & (\alpha + \omega) I_4 \end{pmatrix} \tag{A.14}$$

For the calculation of the 1-vector term

$$N^a L_a = N^4 L_4 + N^5 L_5 + N^\mu L_\mu$$

we let

$$\beta = N^4 \; ; \quad \delta = N^5 \; ; \quad \mathbf{a} = N^\mu \gamma_\mu. \tag{A.15}$$

This gives

$$N^a L_a = \begin{pmatrix} 0 & -\beta I_4 + \delta \mathbf{i} + \mathbf{a} \\ \beta I_4 + \delta \mathbf{i} + \mathbf{a} & 0 \end{pmatrix}. \tag{A.16}$$

For the calculation of the 2-vector term

$$N^{ab} L_{ab} = N^{45} L_{45} + N^{\mu 4} L_{\mu 4} + N^{\mu 5} L_{\mu 5} + N^{\mu\nu} L_{\mu\nu}$$

we let

$$\epsilon = N^{45} \; ; \quad \mathbf{b} = N^{\mu 4} \gamma_\mu \; ; \quad \mathbf{c} = N^{\mu 5} \gamma_\mu \; ; \quad \mathbf{A} = N^{\mu\nu} \gamma_{\mu\nu} \tag{A.17}$$

This gives

$$N^{ab} L_{ab} = \begin{pmatrix} -\epsilon \mathbf{i} + \mathbf{b} - \mathbf{i}\mathbf{c} + \mathbf{A} & 0 \\ 0 & \epsilon \mathbf{i} - \mathbf{b} - \mathbf{i}\mathbf{c} + \mathbf{A} \end{pmatrix}. \tag{A.18}$$

For the calculation of the 3-vector term

$$N^{abc} L_{abc} = N^{\mu 45} L_{\mu 45} + N^{\mu\nu 4} L_{\mu\nu 4} + N^{\mu\nu 5} L_{\mu\nu 5} + N^{\mu\nu\rho} L_{\mu\nu\rho}$$

[3] $\mathbf{i}$ anti-commutes with any odd element in space-time algebra and commutes with any even element.

we let

$$\mathbf{d} = N^{\mu 45}\gamma_\mu \ ; \quad \mathbf{B} = N^{\mu\nu 4}\gamma_{\mu\nu} \ ; \quad \mathbf{C} = N^{\mu\nu 5}\gamma_{\mu\nu} \ ; \quad \mathbf{ie} = N^{\mu\nu\rho}\gamma_{\mu\nu\rho} \tag{A.19}$$

This gives with (A.3) and (A.9)

$$N^{abc}L_{abc} = \begin{pmatrix} 0 & \mathbf{di} - \mathbf{B} + \mathbf{iC} + \mathbf{ie} \\ \mathbf{id} + \mathbf{B} + \mathbf{iC} + \mathbf{ie} & 0 \end{pmatrix}. \tag{A.20}$$

For the calculation of the 4-vector term

$$N^{abcd}L_{abcd} = N^{\mu\nu 45}L_{\mu\nu 45} + N^{\mu\nu\rho 4}L_{\mu\nu\rho 4} + N^{\mu\nu\rho 5}L_{\mu\nu\rho 5} + N^{0123}L_{0123}$$

we let

$$\mathbf{D} = N^{\mu\nu 45}\gamma_{\mu\nu} \ ; \quad \mathbf{if} = N^{\mu\nu\rho 4}\gamma_{\mu\nu\rho} \ ; \quad \mathbf{ig} = N^{\mu\nu\rho 5}\gamma_{\mu\nu\rho} \ ; \quad \zeta = N^{0123} \tag{A.21}$$

This gives with (A.4) and (A.10)

$$N^{abcd}L_{abcd} = \begin{pmatrix} -\mathbf{iD} + \mathbf{if} + \mathbf{g} + \zeta\mathbf{i} & 0 \\ 0 & \mathbf{iD} - \mathbf{if} + \mathbf{g} + \zeta\mathbf{i} \end{pmatrix}. \tag{A.22}$$

For the calculation of the pseudo-vector term

$$N^{abcde}L_{abcde} = N^{\mu\nu\rho 45}L_{\mu\nu\rho 45} + N^{01234}L_{01234} + N^{01235}L_{01235}$$

we let

$$\mathbf{ih} = N^{\mu\nu\rho 45}\gamma_{\mu\nu\rho} \ ; \quad \eta = N^{01234} \ ; \quad \theta = N^{01235} \tag{A.23}$$

This gives with (A.7) and (A.12)

$$N^{abcde}L_{abcde} = \begin{pmatrix} 0 & \mathbf{h} - \eta\mathbf{i} - \theta I_4 \\ -\mathbf{h} + \eta\mathbf{i} - \theta I_4 & \end{pmatrix}. \tag{A.24}$$

We then get

$$\Psi = \begin{pmatrix} \Psi_l & \Psi_r \\ \Psi_g & \Psi_b \end{pmatrix} \tag{A.25}$$

$$= \begin{pmatrix} (\alpha+\omega)I_4 + (\mathbf{b}+\mathbf{g}) + (\mathbf{A}-\mathbf{iD}) & -(\beta+\theta)I_4 + (\mathbf{a}+\mathbf{h}) + (-\mathbf{B}+\mathbf{iC}) \\ +\mathbf{i}(-\mathbf{c}+\mathbf{f}) + (\zeta-\epsilon)\mathbf{i} & +\mathbf{i}(-\mathbf{d}+\mathbf{e}) + (\delta-\eta)\mathbf{i} \\ (\beta-\theta)I_4 + (\mathbf{a}-\mathbf{h}) + (\mathbf{B}+\mathbf{iC}) & (\alpha-\omega)I_4 + (-\mathbf{b}+\mathbf{g}) + (\mathbf{A}+\mathbf{iD}) \\ +\mathbf{i}(\mathbf{d}+\mathbf{e}) + (\delta+\eta)\mathbf{i} & +\mathbf{i}(-\mathbf{c}-\mathbf{f}) + (\zeta+\epsilon)\mathbf{i} \end{pmatrix}$$

This implies

$$\Psi_l = (\alpha+\omega) + (\mathbf{b}+\mathbf{g}) + (\mathbf{A}-\mathbf{iD}) + \mathbf{i}(-\mathbf{c}+\mathbf{f}) + (\zeta-\epsilon)\mathbf{i} \tag{A.26}$$

$$\Psi_r = -(\beta+\theta) + (\mathbf{a}+\mathbf{h}) + (-\mathbf{B}+\mathbf{iC}) + \mathbf{i}(-\mathbf{d}+\mathbf{e}) + (\delta-\eta)\mathbf{i} \tag{A.27}$$

$$\Psi_g = (\beta-\theta) + (\mathbf{a}-\mathbf{h}) + (\mathbf{B}+\mathbf{iC}) + \mathbf{i}(\mathbf{d}+\mathbf{e}) + (\delta+\eta)\mathbf{i} \tag{A.28}$$

$$\Psi_b = (\alpha-\omega) + (-\mathbf{b}+\mathbf{g}) + (\mathbf{A}+\mathbf{iD}) + \mathbf{i}(-\mathbf{c}-\mathbf{f}) + (\zeta+\epsilon)\mathbf{i} \tag{A.29}$$

In $Cl_{1,3}$ the reverse of

$$A =< A >_0 + < A >_1 + < A >_2 + < A >_3 + < A >_4$$

is

$$\widetilde{A} =< A >_0 + < A >_1 - < A >_2 - < A >_3 + < A >_4$$

we must change the sign of bivectors **A**, **B**, **iC**, **iD**, and trivectors **ic**, **id**, **ie**, **if** and we then get

$$\widetilde{\Psi}_l = (\alpha + \omega) + (\mathbf{b} + \mathbf{g}) + (-\mathbf{A} + \mathbf{iD}) + \mathbf{i}(\mathbf{c} - \mathbf{f}) + (\zeta - \epsilon)\mathbf{i} \tag{A.30}$$

$$\widetilde{\Psi}_r = -(\beta + \theta) + (\mathbf{a} + \mathbf{h}) + (\mathbf{B} - \mathbf{iC}) + \mathbf{i}(\mathbf{d} - \mathbf{e}) + (\delta - \eta)\mathbf{i} \tag{A.31}$$

$$\widetilde{\Psi}_g = (\beta - \theta) + (\mathbf{a} - \mathbf{h}) - (\mathbf{B} + \mathbf{iC}) - \mathbf{i}(\mathbf{d} + \mathbf{e}) + (\delta + \eta)\mathbf{i} \tag{A.32}$$

$$\widetilde{\Psi}_b = (\alpha - \omega)I_4 + (-\mathbf{b} + \mathbf{g}) - (\mathbf{A} + \mathbf{iD}) + \mathbf{i}(\mathbf{c} + \mathbf{f}) + (\zeta + \epsilon)\mathbf{i} \tag{A.33}$$

The reverse, in $Cl_{1,5}$ now, of

$$A =< A >_0 + < A >_1 + < A >_2 + < A >_3 + < A >_4 + < A >_5 + < A >_6$$

is

$$\widetilde{A} =< A >_0 + < A >_1 - < A >_2 - < A >_3 + < A >_4 + < A >_5 - < A >_6$$

Only terms which change sign, with (A.13), (A.18) and (A.20), are scalars ϵ and ω, vectors **b**, **c**, **d**, **e** and bivectors **A**, **B**, **C**. These changes of sign are not the same in $Cl_{1,5}$ as in $Cl_{1,3}$. Differences are corrected by the fact that the reversion in $Cl_{1,5}$ also exchanges the place of Ψ_l and Ψ_b terms. We then get from (A.25)

$$\widetilde{\Psi} = \begin{pmatrix} (\alpha - \omega)I_4 + (-\mathbf{b} + \mathbf{g}) + (-\mathbf{A} - \mathbf{iD}) & -(\beta + \theta)I_4 + (\mathbf{a} + \mathbf{h}) + (\mathbf{B} - \mathbf{iC}) \\ +\mathbf{i}(\mathbf{c} + \mathbf{f}) + (\zeta + \epsilon)\mathbf{i} & +\mathbf{i}(\mathbf{d} - \mathbf{e}) + (\delta - \eta)\mathbf{i} \\ (\beta - \theta)I_4 + (\mathbf{a} - \mathbf{h}) - (\mathbf{B} + \mathbf{iC}) & (\alpha + \omega)I_4 + (\mathbf{b} + \mathbf{g}) + (-\mathbf{A} + \mathbf{iD}) \\ -\mathbf{i}(\mathbf{d} + \mathbf{e}) + (\delta + \eta)\mathbf{i} & +\mathbf{i}(\mathbf{c} - \mathbf{f}) + (\zeta - \epsilon)\mathbf{i} \end{pmatrix}$$

$$= \begin{pmatrix} \widetilde{\Psi}_b & \widetilde{\Psi}_r \\ \widetilde{\Psi}_g & \widetilde{\Psi}_l \end{pmatrix}. \tag{A.34}$$

This link between the reversion in $Cl_{1,3}$ and the reversion in $Cl_{1,5}$ is necessary to get an invariant wave equation. It is not general, for instance the reversion in Cl_3 is not linked to the reversion in $Cl_{2,3}$.

B Scalar densities and χ terms

There are $6 \times 5/2 = 15$ such complex scalar densities:

$$s_1 = 2(\xi_{1\overline{u}r}\eta^*_{1ug} + \xi_{2\overline{u}r}\eta^*_{2ug}) = 2(\eta^*_{2ur}\eta^*_{1ug} - \eta^*_{1ur}\eta^*_{2ug}) \tag{B.1}$$

$$s_2 = 2(\xi_{1\overline{u}g}\eta^*_{1ub} + \xi_{2\overline{u}g}\eta^*_{2ub}) = 2(\eta^*_{2ug}\eta^*_{1ub} - \eta^*_{1ug}\eta^*_{2ub}) \tag{B.2}$$

$$s_3 = -2(\xi_{1\overline{u}r}\eta^*_{1ub} + \xi_{2\overline{u}r}\eta^*_{2ub}) = 2(\eta^*_{2ub}\eta^*_{1ur} - \eta^*_{1ub}\eta^*_{2ur}) \tag{B.3}$$

$$s_4 = 2(\xi_{1\bar{d}r}\eta^*_{1dg} + \xi_{2\bar{d}r}\eta^*_{2dg}) = 2(\eta^*_{2dr}\eta^*_{1dg} - \eta^*_{1dr}\eta^*_{2dg}) \tag{B.4}$$
$$s_5 = 2(\xi_{1\bar{d}g}\eta^*_{1db} + \xi_{2\bar{d}g}\eta^*_{2db}) = 2(\eta^*_{2dg}\eta^*_{1db} - \eta^*_{1dg}\eta^*_{2db}) \tag{B.5}$$
$$s_6 = -2(\xi_{1\bar{d}r}\eta^*_{1db} + \xi_{2\bar{d}r}\eta^*_{2db}) = 2(\eta^*_{2db}\eta^*_{1dr} - \eta^*_{1db}\eta^*_{2dr}) \tag{B.6}$$

$$s_7 = 2(\xi_{1\bar{u}r}\eta^*_{1dr} + \xi_{2\bar{u}r}\eta^*_{2dr}) = 2(\eta^*_{2ur}\eta^*_{1dr} - \eta^*_{1ur}\eta^*_{2dr}) \tag{B.7}$$
$$s_8 = 2(\xi_{1\bar{u}g}\eta^*_{1dg} + \xi_{2\bar{u}g}\eta^*_{2dg}) = 2(\eta^*_{2ug}\eta^*_{1dg} - \eta^*_{1ug}\eta^*_{2dg}) \tag{B.8}$$
$$s_9 = 2(\xi_{1\bar{u}b}\eta^*_{1db} + \xi_{2\bar{u}b}\eta^*_{2db}) = 2(\eta^*_{2ub}\eta^*_{1db} - \eta^*_{1ub}\eta^*_{2db}) \tag{B.9}$$

$$s_{10} = 2(\xi_{1\bar{u}r}\eta^*_{1dg} + \xi_{2\bar{u}r}\eta^*_{2dg}) = 2(\eta^*_{2ur}\eta^*_{1dg} - \eta^*_{1ur}\eta^*_{2dg}) \tag{B.10}$$
$$s_{11} = 2(\xi_{1\bar{u}g}\eta^*_{1db} + \xi_{2\bar{u}g}\eta^*_{2db}) = 2(\eta^*_{2ug}\eta^*_{1db} - \eta^*_{1ug}\eta^*_{2db}) \tag{B.11}$$
$$s_{12} = -2(\xi_{1\bar{d}r}\eta^*_{1ub} + \xi_{2\bar{d}r}\eta^*_{2ub}) = 2(\eta^*_{2ub}\eta^*_{1dr} - \eta^*_{1ub}\eta^*_{2dr}) \tag{B.12}$$

$$s_{13} = 2(\xi_{1\bar{u}r}\eta^*_{1db} + \xi_{2\bar{u}r}\eta^*_{2db}) = 2(\eta^*_{2ur}\eta^*_{1db} - \eta^*_{1ur}\eta^*_{2db}) \tag{B.13}$$
$$s_{14} = -2(\xi_{1\bar{d}r}\eta^*_{1ug} + \xi_{2\bar{d}r}\eta^*_{2ug}) = 2(\eta^*_{2ug}\eta^*_{1dr} - \eta^*_{1ug}\eta^*_{2dr}) \tag{B.14}$$
$$s_{15} = -2(\xi_{1\bar{d}g}\eta^*_{1ub} + \xi_{2\bar{d}g}\eta^*_{2ub}) = 2(\eta^*_{2ub}\eta^*_{1dg} - \eta^*_{1ub}\eta^*_{2dg}). \tag{B.15}$$

We used in [9]

$$\chi_l = \frac{1}{\rho_1^2}\begin{pmatrix} a_1^*\phi_e + a_2^*\phi_n\sigma_1 + a_3^*\phi_n & -a_2^*\phi_{eL}\sigma_1 + a_3^*\phi_{eR} \\ a_2\widehat{\phi}_{eL}\sigma_1 + a_3\widehat{\phi}_{eR} & a_1\widehat{\phi}_e - a_2\widehat{\phi}_n\sigma_1 + a_3\widehat{\phi}_n \end{pmatrix} \tag{B.16}$$

with $\phi_{eR} = \phi_e(1+\sigma_3)/2$ and $\phi_{eL} = \phi_e(1-\sigma_3)/2$, and we need now

$$\rho_2^2\chi_r = \begin{pmatrix} \begin{pmatrix} s_4^*\phi_{dg} - s_6^*\phi_{db} - s_7^*\phi_{ur} \\ -s_{12}^*\phi_{ub} - s_{14}^*\phi_{ug} \end{pmatrix}\sigma_1 & \begin{pmatrix} s_1^*\phi_{ug} - s_3^*\phi_{ub} + s_7^*\phi_{dr} \\ +s_{10}^*\phi_{dg} + s_{13}^*\phi_{db} \end{pmatrix}\sigma_1 \\ \begin{pmatrix} -s_1\widehat{\phi}_{ug} + s_3\widehat{\phi}_{ub} - s_7\widehat{\phi}_{dr} \\ -s_{10}\widehat{\phi}_{dg} - s_{13}\widehat{\phi}_{db} \end{pmatrix}\sigma_1 & \begin{pmatrix} -s_4\widehat{\phi}_{dg} + s_6\widehat{\phi}_{db} + s_7\widehat{\phi}_{ur} \\ +s_{12}\widehat{\phi}_{ub} + s_{14}\widehat{\phi}_{ug} \end{pmatrix}\sigma_1 \end{pmatrix} \tag{B.17}$$

$$\rho_2^2\chi_g = \begin{pmatrix} \begin{pmatrix} s_5^*\phi_{db} - s_4^*\phi_{dr} - s_8^*\phi_{ug} \\ -s_{10}^*\phi_{ur} - s_{15}^*\phi_{ub} \end{pmatrix}\sigma_1 & \begin{pmatrix} s_2^*\phi_{ub} - s_1^*\phi_{ur} + s_8^*\phi_{dg} \\ +s_{11}^*\phi_{db} + s_{14}^*\phi_{dr} \end{pmatrix}\sigma_1 \\ \begin{pmatrix} -s_2\widehat{\phi}_{ub} + s_1\widehat{\phi}_{ur} - s_8\widehat{\phi}_{dg} \\ -s_{11}\widehat{\phi}_{db} - s_{14}\widehat{\phi}_{dr} \end{pmatrix}\sigma_1 & \begin{pmatrix} -s_5\widehat{\phi}_{db} + s_4\widehat{\phi}_{dr} + s_8\widehat{\phi}_{ug} \\ +s_{10}\widehat{\phi}_{ur} + s_{15}\widehat{\phi}_{ub} \end{pmatrix}\sigma_1 \end{pmatrix} \tag{B.18}$$

$$\rho_2^2\chi_b = \begin{pmatrix} \begin{pmatrix} s_6^*\phi_{dr} - s_5^*\phi_{dg} - s_9^*\phi_{ub} \\ -s_{11}^*\phi_{ug} - s_{13}^*\phi_{ur} \end{pmatrix}\sigma_1 & \begin{pmatrix} s_3^*\phi_{ur} - s_2^*\phi_{ug} + s_9^*\phi_{db} \\ +s_{12}^*\phi_{dr} + s_{15}^*\phi_{dg} \end{pmatrix}\sigma_1 \\ \begin{pmatrix} -s_3\widehat{\phi}_{ur} + s_2\widehat{\phi}_{ug} - s_9\widehat{\phi}_{db} \\ -s_{12}\widehat{\phi}_{dr} - s_{15}\widehat{\phi}_{dg} \end{pmatrix}\sigma_1 & \begin{pmatrix} -s_6\widehat{\phi}_{dr} + s_5\widehat{\phi}_{dg} + s_9\widehat{\phi}_{ub} \\ +s_{11}\widehat{\phi}_{ug} + s_{13}\widehat{\phi}_{ur} \end{pmatrix}\sigma_1 \end{pmatrix} \tag{B.19}$$

C Gauge invariance, details

C.1 Gauge group generated by $\underline{P}_1$

Since $\underline{P}_1(\Psi^c) = \Psi^c L_{35}$ we get

$$\Psi'^c = [\exp(\theta \underline{P}_1)](\Psi^c) = \Psi^c \exp(\theta L_{35}) \tag{C.1}$$

$$\Psi'_c = \Psi_c e^{\theta\gamma_3 \mathbf{i}},\ c = r, g, b. \tag{C.2}$$

We let

$$C = \cos(\theta)\ ;\quad S = \sin(\theta) \tag{C.3}$$

Then (C.2) is equivalent to the system

$$\widehat{\phi}'_{dc} = C\widehat{\phi}_{dc} - iS\widehat{\phi}_{uc}\sigma_3 \tag{C.4}$$

$$\widehat{\phi}'_{uc} = C\widehat{\phi}_{uc} - iS\widehat{\phi}_{dc}\sigma_3 \tag{C.5}$$

or to the system

$$\eta'_{1dc} = C\eta_{1dc} - iS\eta_{1uc};\ \eta'^*_{1dc} = C\eta^*_{1dc} + iS\eta^*_{1uc} \tag{C.6}$$

$$\eta'_{2dc} = C\eta_{2dc} - iS\eta_{2uc};\ \eta'^*_{2dc} = C\eta^*_{2dc} + iS\eta^*_{2uc} \tag{C.7}$$

$$\eta'_{1uc} = C\eta_{1uc} - iS\eta_{1dc};\ \eta'^*_{1uc} = C\eta^*_{1uc} + iS\eta^*_{1dc} \tag{C.8}$$

$$\eta'_{2uc} = C\eta_{2uc} - iS\eta_{2dc};\ \eta'^*_{2uc} = C\eta^*_{2uc} + iS\eta^*_{2dc} \tag{C.9}$$

We then get

$$s'_1 = C^2 s_1 - S^2 s_4 + iCS(s_{10} - s_{14}) \tag{C.10}$$

$$s'_4 = C^2 s_4 - S^2 s_1 + iCS(s_{10} - s_{14}) \tag{C.11}$$

$$s'_{10} = C^2 s_{10} + S^2 s_{14} + iCS(s_1 + s_4) \tag{C.12}$$

$$s'_{14} = C^2 s_{14} + S^2 s_{10} - iCS(s_1 + s_4). \tag{C.13}$$

This implies

$$s'_1 s'^*_1 + s'_4 s'^*_4 + s'_{10} s'^*_{10} + s'_{14} s'^*_{14} = s_1 s^*_1 + s_4 s^*_4 + s_{10} s^*_{10} + s_{14} s^*_{14}. \tag{C.14}$$

Similarly, permuting colors, we get

$$s'_2 = C^2 s_2 - S^2 s_5 + iCS(s_{11} - s_{15}) \tag{C.15}$$

$$s'_5 = C^2 s_5 - S^2 s_2 + iCS(s_{11} - s_{15}) \tag{C.16}$$

$$s'_{11} = C^2 s_{11} + S^2 s_{15} + iCS(s_2 + s_5) \tag{C.17}$$

$$s'_{15} = C^2 s_{15} + S^2 s_{11} - iCS(s_2 + s_5). \tag{C.18}$$

This implies

$$s'_2 s'^*_2 + s'_5 s'^*_5 + s'_{11} s'^*_{11} + s'_{15} s'^*_{15} = s_2 s^*_2 + s_5 s^*_5 + s_{11} s^*_{11} + s_{15} s^*_{15}. \tag{C.19}$$

and also

$$s'_3 = C^2 s_3 - S^2 s_6 + iCS(s_{12} - s_{13}) \tag{C.20}$$
$$s'_6 = C^2 s_6 - S^2 s_3 + iCS(s_{12} - s_{13}) \tag{C.21}$$
$$s'_{12} = C^2 s_{12} + S^2 s_{13} + iCS(s_3 + s_6) \tag{C.22}$$
$$s'_{13} = C^2 s_{13} + S^2 s_{12} - iCS(s_3 + s_6). \tag{C.23}$$

This implies

$$s'_3 s'^*_3 + s'_6 s'^*_6 + s'_{12} s'^*_{12} + s'_{13} s'^*_{13} = s_3 s^*_3 + s_6 s^*_6 + s_{12} s^*_{12} + s_{13} s^*_{13}. \tag{C.24}$$

Moreover we get

$$s'_7 = s_7;\ s'_8 = s_8;\ s'_9 = s_9. \tag{C.25}$$

We then get

$$\rho' = \rho \tag{C.26}$$

Next we have

$$\chi_r = \begin{pmatrix} A & B \\ \widehat{B} & \widehat{A} \end{pmatrix};\ \chi'_r = \begin{pmatrix} A' & B' \\ \widehat{B}' & \widehat{A}' \end{pmatrix} \tag{C.27}$$
$$\widehat{A} = (-s_4\widehat{\phi}_{dg} + s_6\widehat{\phi}_{db} + s_7\widehat{\phi}_{ur} + s_{12}\widehat{\phi}_{ub} + s_{14}\widehat{\phi}_{ug})\sigma_1 \tag{C.28}$$
$$\widehat{B} = (-s_1\widehat{\phi}_{ug} + s_3\widehat{\phi}_{ub} - s_7\widehat{\phi}_{dr} - s_{10}\widehat{\phi}_{dg} - s_{13}\widehat{\phi}_{db})\sigma_1. \tag{C.29}$$

and we get

$$\widehat{A}' = C\widehat{A} - iS\widehat{B}\sigma_3 \tag{C.30}$$
$$\widehat{B}' = C\widehat{B} - iS\widehat{A}\sigma_3 \tag{C.31}$$
$$\chi'_r = \chi_r \begin{pmatrix} C & -iS\sigma_3 \\ -iS\sigma_3 & C \end{pmatrix} = \chi_r e^{\theta\gamma_3 \mathbf{i}}. \tag{C.32}$$

Since we get the same relation for g and b colors we finally get

$$\chi'^c = \chi^c \exp(\theta L_{35}) \tag{C.33}$$

C.2 Gauge group generated by $\underline{P}_2$

Since $\underline{P}_2(\Psi^c) = \Psi^c L_{5012}$ we get

$$\Psi'^c = [\exp(\theta\underline{P}_2)](\Psi^c) = \Psi^c \exp(\theta L_{5012}) \tag{C.34}$$
$$\Psi'_c = \Psi_c e^{\theta\gamma_3},\ c = r, g, b. \tag{C.35}$$

We let

$$C = \cos(\theta)\ ;\quad S = \sin(\theta) \tag{C.36}$$

Then (C.35) is equivalent to the system

$$\widehat{\phi}'_{dc} = C\widehat{\phi}_{dc} + S\widehat{\phi}_{uc} \tag{C.37}$$
$$\widehat{\phi}'_{uc} = C\widehat{\phi}_{uc} - S\widehat{\phi}_{dc} \tag{C.38}$$

or to the system

$$\eta'_{1dc} = C\eta_{1dc} + S\eta_{1uc};\ \eta'^{*}_{1dc} = C\eta^{*}_{1dc} + S\eta^{*}_{1uc} \tag{C.39}$$
$$\eta'_{2dc} = C\eta_{2dc} + S\eta_{2uc};\ \eta'^{*}_{2dc} = C\eta^{*}_{2dc} + S\eta^{*}_{2uc} \tag{C.40}$$
$$\eta'_{1uc} = C\eta_{1uc} - S\eta_{1dc};\ \eta'^{*}_{1uc} = C\eta^{*}_{1uc} - S\eta^{*}_{1dc} \tag{C.41}$$
$$\eta'_{2uc} = C\eta_{2uc} - S\eta_{2dc};\ \eta'^{*}_{2uc} = C\eta^{*}_{2uc} - S\eta^{*}_{2dc} \tag{C.42}$$

We then get

$$s'_1 = C^2 s_1 + S^2 s_4 - CSs_{10} + CSs_{14} \tag{C.43}$$
$$s'_4 = C^2 s_4 + S^2 s_1 + CSs_{10} - CSs_{14} \tag{C.44}$$
$$s'_{10} = C^2 s_{10} + S^2 s_{14} + CSs_1 - CSs_4 \tag{C.45}$$
$$s'_{14} = C^2 s_{14} + S^2 s_{10} - CSs_1 + CSs_4. \tag{C.46}$$

This implies

$$s'_1 s'^{*}_1 + s'_4 s'^{*}_4 + s'_{10} s'^{*}_{10} + s'_{14} s'^{*}_{14} = s_1 s^{*}_1 + s_4 s^{*}_4 + s_{10} s^{*}_{10} + s_{14} s^{*}_{14}. \tag{C.47}$$

Similarly, permuting colors, we get

$$s'_2 = C^2 s_2 + S^2 s_5 - CSs_{11} + CSs_{15} \tag{C.48}$$
$$s'_5 = C^2 s_5 + S^2 s_2 + CSs_{11} - CSs_{15} \tag{C.49}$$
$$s'_{11} = C^2 s_{11} + S^2 s_{15} + CSs_2 - CSs_5 \tag{C.50}$$
$$s'_{15} = C^2 s_{15} + S^2 s_{11} - CSs_2 + CSs_5. \tag{C.51}$$

This implies

$$s'_2 s'^{*}_2 + s'_5 s'^{*}_5 + s'_{11} s'^{*}_{11} + s'_{15} s'^{*}_{15} = s_2 s^{*}_2 + s_5 s^{*}_5 + s_{11} s^{*}_{11} + s_{15} s^{*}_{15}. \tag{C.52}$$

and also

$$s'_3 = C^2 s_3 + S^2 s_6 - CSs_{12} + CSs_{13} \tag{C.53}$$
$$s'_6 = C^2 s_6 + S^2 s_3 + CSs_{12} - CSs_{13} \tag{C.54}$$
$$s'_{12} = C^2 s_{12} + S^2 s_{13} + CSs_3 - CSs_6 \tag{C.55}$$
$$s'_{13} = C^2 s_{13} + S^2 s_{12} - CSs_3 + CSs_6. \tag{C.56}$$

This implies

$$s'_3 s'^{*}_3 + s'_6 s'^{*}_6 + s'_{12} s'^{*}_{12} + s'_{13} s'^{*}_{13} = s_3 s^{*}_3 + s_6 s^{*}_6 + s_{12} s^{*}_{12} + s_{13} s^{*}_{13}. \tag{C.57}$$

Moreover we get

$$s'_7 = s_7;\ s'_8 = s_8;\ s'_9 = s_9. \tag{C.58}$$

We then get

$$\rho' = \rho \tag{C.59}$$

Next we get with (C.27)

$$\widehat{A}' = C\widehat{A} - S\widehat{B}\sigma_3 \tag{C.60}$$

$$\widehat{B}' = C\widehat{B} + S\widehat{A}\sigma_3 \tag{C.61}$$

$$\chi'_r = \chi_r \begin{pmatrix} C & -S\sigma_3 \\ S\sigma_3 & C \end{pmatrix} = \chi_r e^{-\theta\gamma_3}. \tag{C.62}$$

Since we get the same relation for g and b colors we finally get

$$\chi'^c = \chi^c \exp(-\theta L_{5012}) \tag{C.63}$$

C.3 Gauge group generated by $\underline{P}_3$

Since $\underline{P}_3(\Psi^c) = \Psi^c L_{3012}$ we get

$$\Psi'^c = [\exp(\theta\underline{P}_3)](\Psi^c) = \Psi^c \exp(\theta L_{3012}) \tag{C.64}$$

$$\Psi'_c = \Psi_c e^{\theta\gamma_{3012}},\ c = r, g, b. \tag{C.65}$$

Then (C.65) is equivalent to the system

$$\widehat{\phi}'_{dc} = e^{i\theta}\widehat{\phi}_{dc} \tag{C.66}$$

$$\widehat{\phi}'_{uc} = e^{-i\theta}\widehat{\phi}_{uc} \tag{C.67}$$

or to the system

$$\eta'_{1dc} = e^{i\theta}\eta_{1dc};\ \eta'^*_{1dc} = e^{-i\theta}\eta^*_{1dc} \tag{C.68}$$

$$\eta'_{2dc} = e^{i\theta}\eta_{2dc};\ \eta'^*_{2dc} = e^{-i\theta}\eta^*_{2dc} \tag{C.69}$$

$$\eta'_{1uc} = e^{-i\theta}\eta_{1uc};\ \eta'^*_{1uc} = e^{i\theta}\eta^*_{1uc} \tag{C.70}$$

$$\eta'_{2uc} = e^{-i\theta}\eta_{2uc};\ \eta'^*_{2uc} = e^{i\theta}\eta^*_{2uc} \tag{C.71}$$

We then get

$$s'_1 = e^{2i\theta}s_1;\ s'_2 = e^{2i\theta}s_2;\ s'_3 = e^{2i\theta}s_3 \tag{C.72}$$

$$s'_4 = e^{-2i\theta}s_4;\ s'_5 = e^{-2i\theta}s_5;\ s'_6 = e^{-2i\theta}s_6 \tag{C.73}$$

$$s'_7 = s_7;\ s'_8 = s_8;\ s'_9 = s_9 \tag{C.74}$$

$$s'_{10} = s_{10};\ s'_{11} = s_{11};\ s'_{12} = s_{12} \tag{C.75}$$

$$s'_{13} = s_{13};\ s'_{14} = s_{14};\ s'_{15} = s_{15}. \tag{C.76}$$

This implies

$$\rho' = \rho \tag{C.77}$$

Next we get with (C.27)

$$\widehat{A}' = e^{-i\theta}\widehat{A}\ ;\quad A' = e^{i\theta}A \tag{C.78}$$

$$\widehat{B}' = e^{i\theta}\widehat{B}\ ;\quad B' = e^{-i\theta}B \tag{C.79}$$

$$\chi'_r = \chi_r \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \chi_r e^{\theta \mathbf{i}}. \tag{C.80}$$

Since we get the same relation for g and b colors we finally get

$$\chi'^c = \chi^c \exp(-\theta L_{3012}) \tag{C.81}$$

C.4 Gauge group generated by $\underline{\mathbf{i}}\Gamma_1$

We name f_1 the gauge transformation

$$f_1 : \Psi^c \mapsto \underline{\mathbf{i}}\Gamma_1(\Psi^c) = \begin{pmatrix} 0 & \mathbf{i}\Psi_g \\ \mathbf{i}\Psi_r & 0 \end{pmatrix} \tag{C.82}$$

which implies with $C = \cos(\theta)$ and $S = \sin(\theta)$

$$[\exp(\theta f_1)](\Psi^c) = \begin{pmatrix} 0 & C\Psi_r + S\mathbf{i}\Psi_g \\ C\Psi_g + S\mathbf{i}\Psi_r & \Psi_b \end{pmatrix} = \begin{pmatrix} 0 & \Psi'_r \\ \Psi'_g & \Psi'_b \end{pmatrix} \tag{C.83}$$

$$\Psi'_r = C\Psi_r + S\mathbf{i}\Psi_g \tag{C.84}$$

$$\Psi'_g = C\Psi_g + S\mathbf{i}\Psi_r \tag{C.85}$$

$$\Psi'_b = \Psi_b \tag{C.86}$$

The equality (C.84) is equivalent to the system

$$\eta'^*_{1dr} = C\eta^*_{1dr} + iS\eta^*_{1dg};\ \eta'^*_{1ur} = C\eta^*_{1ur} + iS\eta^*_{1ug} \tag{C.87}$$

$$\eta'^*_{2dr} = C\eta^*_{2dr} + iS\eta^*_{2dg};\ \eta'^*_{2ur} = C\eta^*_{2ur} + iS\eta^*_{2ug} \tag{C.88}$$

The equality (C.85) is equivalent to the system

$$\eta'^*_{1dg} = C\eta^*_{1dg} + iS\eta^*_{1dr};\ \eta'^*_{1ug} = C\eta^*_{1ug} + iS\eta^*_{1ur} \tag{C.89}$$

$$\eta'^*_{2dg} = C\eta^*_{2dg} + iS\eta^*_{2dr};\ \eta'^*_{2ug} = C\eta^*_{2ug} + iS\eta^*_{2ur} \tag{C.90}$$

This gives for the invariant scalars s_j

$$s'_1 = s_1;\ s'_4 = s_4;\ s'_9 = s_9 \tag{C.91}$$

$$s'_2 = Cs_2 - iSs_3;\ s'_3 = Cs_3 - iSs_2 \tag{C.92}$$

$$s'_5 = Cs_5 - iSs_6;\ s'_6 = Cs_6 - iSs_5 \tag{C.93}$$

$$s'_{11} = Cs_{11} + iSs_{13};\ s'_{13} = Cs_{13} + iSs_{11} \tag{C.94}$$

$$s'_{12} = Cs_{12} + iSs_{15};\ s'_{15} = Cs_{15} + iSs_{12} \tag{C.95}$$

$$s'_7 = C^2 s_7 - S^2 s_8 + iCSs_{10} + iCSs_{14} \tag{C.96}$$
$$s'_8 = C^2 s_8 - S^2 s_7 + iCSs_{14} + iCSs_{10} \tag{C.97}$$
$$s'_{10} = C^2 s_{10} - S^2 s_{14} + iCSs_7 + iCSs_8 \tag{C.98}$$
$$s'_{14} = C^2 s_{14} - S^2 s_{10} + iCSs_8 + iCSs_7 \tag{C.99}$$

We then get

$$s'_2 s'^*_2 + s'_3 s'^*_3 = s_2 s^*_2 + s_3 s^*_3 \tag{C.100}$$
$$s'_5 s'^*_5 + s'_6 s'^*_6 = s_5 s^*_5 + s_6 s^*_6 \tag{C.101}$$
$$s'_{11} s'^*_{11} + s'_{13} s'^*_{13} = s_{11} s^*_{11} + s_{13} s^*_{13} \tag{C.102}$$
$$s'_{12} s'^*_{12} + s'_{15} s'^*_{15} = s_{12} s^*_{12} + s_{15} s^*_{15} \tag{C.103}$$
$$s'_7 s'^*_7 + s'_8 s'^*_8 + s'_{10} s'^*_{10} + s'_{14} s'^*_{14} = s_7 s^*_7 + s_8 s^*_8 + s_{10} s^*_{10} + s_{14} s^*_{14} \tag{C.104}$$
$$\rho' = \rho. \tag{C.105}$$

Next we let

$$\chi_r = \begin{pmatrix} A_r & B_r \\ \widehat{B}_r & \widehat{A}_r \end{pmatrix};\ \chi'_r = \begin{pmatrix} A'_r & B'_r \\ \widehat{B}'_r & \widehat{A}'_r \end{pmatrix} \tag{C.106}$$
$$\chi_g = \begin{pmatrix} A_g & B_g \\ \widehat{B}_g & \widehat{A}_g \end{pmatrix};\ \chi'_g = \begin{pmatrix} A'_g & B'_g \\ \widehat{B}'_g & \widehat{A}'_g \end{pmatrix} \tag{C.107}$$

and we get with (B.17) and (B.18)

$$A'_r = CA_r - iSA_g;\ B'_r = CB_r - iSB_g \tag{C.108}$$
$$A'_g = CA_g - iSA_r;\ B'_g = CB_g - iSB_r. \tag{C.109}$$

This gives the awaited result

$$\chi'_r = C\chi_r - \mathbf{i}S\chi_g;\ \chi'_g = C\chi_g - \mathbf{i}S\chi_r. \tag{C.110}$$

C.5 Gauge group generated by $\underline{\mathbf{i}}\Gamma_3$

We name f_3 the gauge transformation

$$f_3 : \Psi^c \mapsto \underline{\mathbf{i}}\Gamma_3(\Psi^c) = \begin{pmatrix} 0 & \mathbf{i}\Psi_r \\ -\mathbf{i}\Psi_g & 0 \end{pmatrix} \tag{C.111}$$

which implies

$$[\exp(\theta f_1)](\Psi^c) = \begin{pmatrix} 0 & e^{\theta\mathbf{i}}\Psi_r \\ e^{-\theta\mathbf{i}}\Psi_g & \Psi_b \end{pmatrix} = \begin{pmatrix} 0 & \Psi'_r \\ \Psi'_g & \Psi'_b \end{pmatrix} \tag{C.112}$$
$$\Psi'_r = e^{\theta\mathbf{i}}\Psi_r \tag{C.113}$$
$$\Psi'_g = e^{-\theta\mathbf{i}}\Psi_g \tag{C.114}$$
$$\Psi'_b = \Psi_b \tag{C.115}$$

The equality (C.113) is equivalent to

$$\begin{pmatrix} \phi'_{dr} & \phi'_{ur} \\ \widehat{\phi}'_{ur} & \widehat{\phi}'_{dr} \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} \phi_{dr} & \phi_{ur} \\ \widehat{\phi}_{ur} & \widehat{\phi}_{dr} \end{pmatrix} \tag{C.116}$$

The equality (C.114) is equivalent to

$$\begin{pmatrix} \phi'_{dg} & \phi'_{ug} \\ \widehat{\phi}'_{ug} & \widehat{\phi}'_{dg} \end{pmatrix} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} \phi_{dg} & \phi_{ug} \\ \widehat{\phi}_{ug} & \widehat{\phi}_{dg} \end{pmatrix} \tag{C.117}$$

We get

$$\eta'^{*}_{1dr} = e^{-i\theta}\eta^{*}_{1dr};\ \eta'^{*}_{1ur} = e^{-i\theta}\eta^{*}_{1ur} \tag{C.118}$$
$$\eta'^{*}_{2dr} = e^{-i\theta}\eta^{*}_{2dr};\ \eta'^{*}_{2ur} = e^{-i\theta}\eta^{*}_{2ur} \tag{C.119}$$
$$\eta'^{*}_{1dg} = e^{i\theta}\eta^{*}_{1dg};\ \eta'^{*}_{1ug} = e^{i\theta}\eta^{*}_{1ug} \tag{C.120}$$
$$\eta'^{*}_{2dg} = e^{i\theta}\eta^{*}_{2dg};\ \eta'^{*}_{2ug} = e^{i\theta}\eta^{*}_{2ug} \tag{C.121}$$

This gives

$$s'_1 = s_1\ ;\quad s'_2 = e^{-i\theta}s_2\ ;\quad s'_3 = e^{i\theta}s_3 \tag{C.122}$$
$$s'_4 = s_4\ ;\quad s'_5 = e^{-i\theta}s_5\ ;\quad s'_6 = e^{i\theta}s_6 \tag{C.123}$$
$$s'_9 = s_9\ ;\quad s'_8 = e^{-2i\theta}s_8\ ;\quad s'_7 = e^{2i\theta}s_7 \tag{C.124}$$
$$s'_{10} = s_{10}\ ;\quad s'_{11} = e^{-i\theta}s_{11}\ ;\quad s'_{12} = e^{i\theta}s_{12} \tag{C.125}$$
$$s'_{14} = s_{14}\ ;\quad s'_{15} = e^{-i\theta}s_{15}\ ;\quad s'_{13} = e^{i\theta}s_{13} \tag{C.126}$$

from which we get

$$s'_j s'^{*}_j = s_j s^{*}_j,\ j = 1, 2, \ldots, 15 \tag{C.127}$$
$$\rho' = \rho \tag{C.128}$$
$$\chi'_r = e^{-\mathbf{i}\theta}\chi_r \tag{C.129}$$
$$\chi'_g = e^{\mathbf{i}\theta}\chi_g \tag{C.130}$$

These relations are the awaited ones because

$$\begin{aligned} \boldsymbol{\partial}\Psi'_r &= \boldsymbol{\partial}(e^{\mathbf{i}\theta}\Psi_r) \\ &= e^{-\mathbf{i}\theta}(-\mathbf{i}\boldsymbol{\partial}\theta\Psi_r + \boldsymbol{\partial}\Psi_r) \end{aligned} \tag{C.131}$$
$$\begin{aligned} \boldsymbol{\partial}\Psi'_g &= \boldsymbol{\partial}(e^{-\mathbf{i}\theta}\Psi_g) \\ &= e^{\mathbf{i}\theta}(\mathbf{i}\boldsymbol{\partial}\theta\Psi_g + \boldsymbol{\partial}\Psi_g) \end{aligned} \tag{C.132}$$
$$\mathbf{G}'^{3} = \mathbf{G}^{3} - \frac{2}{g_3}\boldsymbol{\partial}\theta. \tag{C.133}$$

C.6 Gauge group generated by $\underline{\mathbf{i}}\Gamma_8$

We name f_8 the gauge transformation

$$f_8 : \Psi^c \mapsto \underline{\mathbf{i}}\Gamma_8(\Psi^c) = \begin{pmatrix} 0 & \frac{\mathbf{i}}{\sqrt{3}}\Psi_r \\ \frac{\mathbf{i}}{\sqrt{3}}\Psi_g & -\frac{2\mathbf{i}}{\sqrt{3}}\Psi_b \end{pmatrix} \tag{C.134}$$

which implies

$$[\exp(\theta f_1)](\Psi^c) = \begin{pmatrix} 0 & e^{\frac{\theta\mathbf{i}}{\sqrt{3}}}\Psi_r \\ e^{\frac{\theta\mathbf{i}}{\sqrt{3}}}\Psi_g & e^{-\frac{2\theta\mathbf{i}}{\sqrt{3}}}\Psi_b \end{pmatrix} = \begin{pmatrix} 0 & \Psi'_r \\ \Psi'_g & \Psi'_b \end{pmatrix} \tag{C.135}$$

$$\Psi'_r = \exp(\frac{\theta\mathbf{i}}{\sqrt{3}})\Psi_r \tag{C.136}$$

$$\Psi'_g = \exp(\frac{\theta\mathbf{i}}{\sqrt{3}})\Psi_g \tag{C.137}$$

$$\Psi'_b = \exp(-\frac{2\theta\mathbf{i}}{\sqrt{3}})\Psi_b \tag{C.138}$$

This gives

$$\phi'_{dr} = \exp(\frac{i\theta}{\sqrt{3}})\phi_{dr};\ \phi'_{ur} = \exp(\frac{i\theta}{\sqrt{3}})\phi_{ur} \tag{C.139}$$

$$\phi'_{dg} = \exp(\frac{i\theta}{\sqrt{3}})\phi_{dg};\ \phi'_{ug} = \exp(\frac{i\theta}{\sqrt{3}})\phi_{ug} \tag{C.140}$$

$$\phi'_{db} = \exp(-\frac{2i\theta}{\sqrt{3}})\phi_{db};\ \phi'_{ub} = \exp(-\frac{2i\theta}{\sqrt{3}})\phi_{ub} \tag{C.141}$$

We then get

$$\eta'^*_{1dr} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{1dr};\ \eta'^*_{1dg} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{1dg};\ \eta'^*_{1db} = \exp(-\frac{2i\theta}{\sqrt{3}})\eta^*_{1dg} \tag{C.142}$$

$$\eta'^*_{2dr} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{2dr};\ \eta'^*_{2dg} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{2dg};\ \eta'^*_{2db} = \exp(-\frac{2i\theta}{\sqrt{3}})\eta^*_{2dg} \tag{C.143}$$

$$\eta'^*_{1ur} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{1ur};\ \eta'^*_{1ug} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{1ug};\ \eta'^*_{1ub} = \exp(-\frac{2i\theta}{\sqrt{3}})\eta^*_{1ug} \tag{C.144}$$

$$\eta'^*_{2ur} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{2ur};\ \eta'^*_{2ug} = \exp(\frac{i\theta}{\sqrt{3}})\eta^*_{2ug};\ \eta'^*_{2ub} = \exp(-\frac{2i\theta}{\sqrt{3}})\eta^*_{2ug} \tag{C.145}$$

This implies

$$s'_1 = \exp(\frac{2i\theta}{\sqrt{3}})s_1;\ s'_2 = \exp(-\frac{i\theta}{\sqrt{3}})s_2;\ s'_3 = \exp(-\frac{i\theta}{\sqrt{3}})s_3 \tag{C.146}$$

$$s'_4 = \exp(\frac{2i\theta}{\sqrt{3}})s_4;\ s'_5 = \exp(-\frac{i\theta}{\sqrt{3}})s_5;\ s'_6 = \exp(-\frac{i\theta}{\sqrt{3}})s_6 \tag{C.147}$$

$$s'_7 = \exp(\frac{2i\theta}{\sqrt{3}})s_7;\ s'_8 = \exp(\frac{2i\theta}{\sqrt{3}})s_8;\ s'_9 = \exp(-\frac{4i\theta}{\sqrt{3}})s_9 \tag{C.148}$$

$$s'_{10} = \exp(\frac{2i\theta}{\sqrt{3}})s_{10};\ s'_{11} = \exp(-\frac{i\theta}{\sqrt{3}})s_{11};\ s'_{12} = \exp(-\frac{i\theta}{\sqrt{3}})s_{12} \tag{C.149}$$

$$s'_{13} = \exp(-\frac{i\theta}{\sqrt{3}})s_{13};\ s'_{14} = \exp(\frac{2i\theta}{\sqrt{3}})s_{14};\ s'_{15} = \exp(-\frac{i\theta}{\sqrt{3}})s_{15}. \tag{C.150}$$

We then get the awaited results

$$s'_j s'^*_j = s_j s^*_j,\ j = 1, 2, \ldots, 15 \tag{C.151}$$

$$\rho' = \rho \tag{C.152}$$

$$\chi'_r = \exp(-\frac{\mathbf{i}\theta}{\sqrt{3}})\chi_r \tag{C.153}$$

$$\chi'_g = \exp(-\frac{\mathbf{i}\theta}{\sqrt{3}})\chi_g \tag{C.154}$$

$$\chi'_b = \exp(\frac{2\mathbf{i}\theta}{\sqrt{3}})\chi_b. \tag{C.155}$$