



**HAL**  
open science

## Long time asymptotics for optimal investment

Huyen Pham

► **To cite this version:**

| Huyen Pham. Long time asymptotics for optimal investment. 2014. hal-01058657

**HAL Id: hal-01058657**

**<https://hal.science/hal-01058657>**

Preprint submitted on 27 Aug 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Long time asymptotics for optimal investment\*

Huyên PHAM

Laboratoire de Probabilités et

Modèles Aléatoires

CNRS, UMR 7599

Université Paris Diderot

pham at math.univ-paris-diderot.fr

and CREST-ENSAE

August 27, 2014

## Abstract

This survey reviews portfolio selection problem for long-term horizon. We consider two objectives: (i) maximize the probability for outperforming a target growth rate of wealth process (ii) minimize the probability of falling below a target growth rate. We study the asymptotic behavior of these criteria formulated as large deviations control problems, that we solve by duality method leading to ergodic risk-sensitive portfolio optimization problems. Special emphasis is placed on linear factor models where explicit solutions are obtained.

**MSC Classification (2000):** 60F10, 91G10, 93E20.

**Keywords:** Long-term investment, large deviations, risk-sensitive control, ergodic HJB equation.

---

\*Contribution to Springer Proceedings in Asymptotic Methods in Finance (Editors Friz-Gatheral-Gulisashvili-Jacquier-Teichmann), in memory of Peter Laurence.

# 1 Introduction

Dynamic portfolio selection looks for strategies maximizing some performance criterion. It is a main topic in mathematical finance, first solved in continuous time in the seminal paper [13], and extended in various directions by taking into account stochastic investment opportunities, market imperfections and/or transaction costs. We refer for instance to the textbooks [11], [10] or [19], and the recent survey paper [12] for developments on this subject.

Classical criterion for investment decision is the expected utility maximization from terminal wealth, which requires to specify on one hand the utility function representing the investor's preference, and subjective by nature, and on the other hand the finite horizon. We consider in this paper an alternative behavioral foundation, with an objective criterion over long term. More precisely, we are concerned with the performance of a portfolio relative to a given target, and are interested in maximizing (resp. minimizing) the probability to outperform (resp. to fall below) a target growth rate when time horizon goes to infinity. Such criterion, formulated as a large deviations portfolio optimization problem, has been proposed by [22] in a static framework, studied in a continuous-time framework for the maximization of upside chance probability by [17], and then by [9], see also [21] in discrete-time models. The asymptotics of minimizing the downside risk probability is studied in [8] and [15].

Large deviations portfolio optimization is a nonstandard stochastic control problem, and is tackled by duality approach. The dual control problem is an ergodic risk-sensitive portfolio optimization problem studied in [6] by dynamic programming PDE methods in a Markovian setting, see also [7], and leads to particularly tractable results with time-homogenous policies. A nice feature of the duality approach is also to relate the target level in the objective probability of upside chance maximization or downside risk minimization to the subjective degree of risk aversion, hence to make endogenous the utility function of the investor.

The rest of this paper is organized as follows. Section 2 formulates the large deviations criterion. In Section 3, we state the general duality relation for the large deviations optimization problem, both for the upside chance probability maximization and downside risk minimization. We illustrate in section 4 our results in the Black-Scholes toy model with constant proportion portfolio. Finally, we consider in Section 5 a factor model for assets price, and characterize the optimal strategy of the large deviations optimization problem via the resolution of an ergodic Hamilton-Jacobi-Bellman equation from the risk-sensitive dual control. Explicit solutions are provided in the linear Gaussian factor model.

## 2 Large deviations criterion

We study a portfolio choice criterion, which is preferences-free, i.e. objective, and horizon-free, i.e. over long term investment. This is formulated as a large deviations criterion that we now describe in an abstract set-up. On a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  supporting all the random quantities appearing in the sequel, we consider a frictionless

financial market with  $d$  assets of positive price process  $S = (S^1, \dots, S^d)$ . There is an agent investing at any time  $t$  a fraction  $\pi_t$  of her wealth in the assets based on the available information  $\mathcal{F}_t$ . We denote by  $\mathcal{A}$  the set of admissible control strategies  $\pi = (\pi_t)_{t \geq 0}$ , and  $X^\pi$  the associated positive wealth process of dynamics:

$$dX_t^\pi = X_t^\pi \pi_t' \text{diag}(S_t)^{-1} dS_t, \quad t \geq 0, \quad (2.1)$$

where  $\text{diag}(S_t)^{-1}$  denotes the diagonal  $d \times d$  matrix of  $i$ -th diagonal term  $1/S_t^i$ . We then define the so-called *growth rate portfolio*, i.e. the logarithm of the wealth process  $X^\pi$ :

$$L_t^\pi := \ln X_t^\pi, \quad t \geq 0.$$

We set by  $\bar{L}^\pi$  the average growth rate portfolio over time:

$$\bar{L}_t^\pi := \frac{L_t^\pi}{t}, \quad t > 0.$$

We shall then consider two problems on the long time asymptotics for the average growth rate:

- (i) **Upside chance probability:** given a target growth rate  $\ell$ , the agent wants to maximize over portfolio strategies  $\pi \in \mathcal{A}$

$$\mathbb{P}[\bar{L}_T^\pi \geq \ell] \quad \text{when } T \rightarrow \infty.$$

- (ii) **Downside risk probability:** given a target growth rate  $\ell$ , the agent wants to minimize over portfolio strategies  $\pi \in \mathcal{A}$

$$\mathbb{P}[\bar{L}_T^\pi \leq \ell] \quad \text{when } T \rightarrow \infty.$$

Actually, when horizon time  $T$  goes to infinity, the probabilities of upside chance or downside risk have typically an exponential decay in time, and we are led to the following mathematical formulations of large deviations criterion:

$$v_+(\ell) := \sup_{\pi \in \mathcal{A}} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln P[\bar{L}_T^\pi \geq \ell], \quad (2.2)$$

$$v_-(\ell) := \inf_{\pi \in \mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln P[\bar{L}_T^\pi \leq \ell]. \quad (2.3)$$

This criterion depends on the objective probability  $\mathbb{P}$ , and the target growth rate  $\ell$ , but there is no exogenous utility function, and finite horizon. Large deviations control problem (2.2) and (2.3) are nonstandard in the literature on stochastic control, and we shall study these problems by a duality approach.

### 3 Duality

We derive in this section the dual formulation of the large deviations criterion introduced in (2.2)-(2.3). Given  $\pi \in \mathcal{A}$ , if the average growth rate portfolio  $\bar{L}_T^\pi$  satisfies a large deviations

principle, then large deviations theory states that its rate function  $I(\cdot, \pi)$  should be related to its limiting log-Laplace transform  $\Gamma(\cdot, \pi)$  by duality via the Gärtner-Ellis theorem:

$$I(\ell, \pi) = \sup_{\theta} [\theta \ell - \Gamma(\theta, \pi)], \quad (3.1)$$

where  $I(\cdot, \pi)$  is the rate function associated to the LDP of  $\bar{L}_T^\pi$ :

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \ln P[\bar{L}_T^\pi \geq \ell] = -\inf_{\ell' \geq \ell} I(\ell', \pi) = I(\ell, \pi), \quad \ell \geq \lim_{T \rightarrow \infty} \bar{L}_T^\pi, \quad (3.2)$$

and  $\Gamma(\cdot, \pi)$  is the limiting log-Laplace transform of  $\bar{L}_T^\pi$ :

$$\Gamma(\theta, \pi) := \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{E}[e^{\theta T \bar{L}_T^\pi}], \quad \theta \in \mathbb{R},$$

The issue is now to extend this duality relation (3.1) when optimizing over control  $\pi$ . To fix the ideas, let us formally derive from (3.1)-(3.2) the maximization of upside chance probability.

$$\begin{aligned} \sup_{\pi} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln P[\bar{L}_T^\pi \geq \ell] &= \sup_{\pi} [-I(\ell, \pi)] \\ &= \sup_{\pi} \left[ -\sup_{\theta} [\theta \ell - \Gamma(\theta, \pi)] \right] \\ &= \sup_{\pi} \inf_{\theta} [\Gamma(\theta, \pi) - \theta \ell] \\ \text{(if we can invert sup and inf)} &= \inf_{\theta} \left[ \sup_{\pi} \Gamma(\theta, \pi) - \theta \ell \right]. \end{aligned}$$

We thus expect that

$$v_+(\ell) = \inf_{\theta} [\Lambda_+(\theta) - \theta \ell], \quad (3.3)$$

where  $\Lambda_+$  is defined by

$$\Lambda_+(\theta) = \sup_{\pi} \Gamma(\theta, \pi).$$

In other words, we should have a duality relation between the value function  $v_+$  of the large deviations control problem, and the value function  $\Lambda_+$ , which is known in the mathematical finance literature, as an ergodic risk-sensitive portfolio optimization problem.

Let us now state rigorously the duality relation in an abstract (model-free) setting. We first consider the upside chance large deviations probability, and define the corresponding dual control problem:

$$\Lambda_+(\theta) := \sup_{\pi \in \mathcal{A}} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{E}[e^{\theta T \bar{L}_T^\pi}], \quad \theta \geq 0. \quad (3.4)$$

We easily see from Hölder inequality that  $\Lambda_+$  is convex on  $\mathbb{R}_+$ . The following result is due to [17].

**Theorem 3.1** Suppose that  $\Lambda_+$  is finite and differentiable on  $(0, \bar{\theta})$  for some  $\bar{\theta} \in (0, \infty]$ , and there exists  $\hat{\pi}(\theta) \in \mathcal{A}$  solution to  $\Lambda_+(\theta)$  for any  $\theta \in (0, \bar{\theta})$ . Then, for all  $\ell < \Lambda'_+(\bar{\theta})$ , we have:

$$v_+(\ell) = \inf_{\theta \in (0, \bar{\theta})} [\Lambda_+(\theta) - \theta\ell].$$

Moreover, an optimal control for  $v_+(\ell)$ , when  $\ell \in (\Lambda'_+(0), \Lambda'_+(\bar{\theta}))$ , is

$$\pi^{+, \ell} = \hat{\pi}(\theta(\ell)), \quad \text{with } \Lambda'_+(\theta(\ell)) = \ell,$$

while a nearly-optimal control for  $v_+(\ell) = 0$ , when  $\ell \leq \Lambda'_+(0)$ , is:

$$\pi^{+(n)} = \hat{\pi}(\theta_n), \quad \text{with } \theta_n = \theta(\Lambda'_+(0) + \frac{1}{n}) \xrightarrow{n \rightarrow \infty} 0,$$

in the sense that

$$\lim_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^{\pi^{+(n)}} \geq \ell] = v_+(\ell).$$

**Proof.** *Step 1.* Let us consider the Fenchel-Legendre transform of the convex function  $\Lambda_+$  on  $[0, \bar{\theta})$ :

$$\Lambda_+^*(\ell) = \sup_{\theta \in [0, \bar{\theta})} [\theta\ell - \Lambda_+(\theta)], \quad \ell \in \mathbb{R}. \quad (3.5)$$

Since  $\Lambda_+$  is  $C^1$  on  $(0, \bar{\theta})$ , it is well-known (see e.g. Lemma 2.3.9 in [4]) that the function  $\Lambda_+^*$  is convex, nondecreasing and satisfies:

$$\Lambda_+^*(\ell) = \begin{cases} \theta(\ell)\ell - \Lambda_+(\theta(\ell)), & \text{if } \Lambda'_+(0) < \ell < \Lambda'_+(\bar{\theta}) \\ 0, & \text{if } \ell \leq \Lambda'_+(0), \end{cases} \quad (3.6)$$

$$\theta(\ell)\ell - \Lambda_+^*(\ell) > \theta(\ell)\ell' - \Lambda_+^*(\ell'), \quad \forall \Lambda'_+(0) < \ell < \Lambda'_+(\bar{\theta}), \quad \forall \ell' \neq \ell, \quad (3.7)$$

where  $\theta(\ell) \in (0, \bar{\theta})$  is s.t.  $\Lambda'_+(\theta(\ell)) = \ell \in (\Lambda'_+(0), \Lambda'_+(\bar{\theta}))$ . Moreover,  $\Lambda_+^*$  is continuous on  $(-\infty, \Lambda'_+(\bar{\theta}))$ .

*Step 2: Upper bound.* For all  $\ell \in \mathbb{R}$ ,  $\pi \in \mathcal{A}$ , an application of Chebycheff's inequality yields:

$$\mathbb{P}[\bar{L}_T^\pi \geq \ell] \leq \exp(-\theta\ell T) \mathbb{E}[\exp(\theta T \bar{L}_T^\pi)], \quad \forall \theta \in [0, \bar{\theta}),$$

and so

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^\pi \geq \ell] \leq -\theta\ell + \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{E}[\exp(\theta T \bar{L}_T^\pi)], \quad \forall \theta \in [0, \bar{\theta}).$$

By definitions of  $\Lambda_+$  and  $\Lambda_+^*$ , we deduce:

$$\sup_{\pi \in \mathcal{A}} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^\pi \geq \ell] \leq -\Lambda_+^*(\ell). \quad (3.8)$$

*Step 3: Lower bound.* Consider first the case  $\ell \in (\Lambda'_+(0), \Lambda'_+(\bar{\theta}))$ , and let us define the probability measure  $\mathbb{Q}_T$  on  $(\Omega, \mathcal{F}_T)$  via:

$$\frac{d\mathbb{Q}_T}{d\mathbb{P}} = \exp \left[ \theta(\ell) L_T^{\pi^+, \ell} - \Gamma_T(\theta(\ell), \pi^+, \ell) \right], \quad (3.9)$$

where

$$\Gamma_T(\theta, \pi) = \ln \mathbb{E}[\exp(\theta T \bar{L}_T^\pi)], \quad \theta \in [0, \bar{\theta}], \quad \pi \in \mathcal{A}.$$

For any  $\varepsilon > 0$ , we have:

$$\begin{aligned} \frac{1}{T} \ln \mathbb{P}[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon] &= \frac{1}{T} \ln \left( \int \frac{d\mathbb{P}}{d\mathbb{Q}_T} \mathbf{1}_{\{\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon\}} d\mathbb{Q}_T \right) \\ &\geq -\theta(\ell)(\ell + \varepsilon) + \frac{1}{T} \Gamma_T(\theta(\ell), \pi^+, \ell) \\ &\quad + \frac{1}{T} \ln \mathbb{Q}_T[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon], \end{aligned}$$

where we use (3.9) in the last inequality. By definition of the dual problem, this yields:

$$\begin{aligned} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon] &\geq -\theta(\ell)(\ell + \varepsilon) + \Lambda_+(\theta(\ell)) \\ &\quad + \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon] \\ &\geq -\Lambda_+^*(\ell) - \theta(\ell)\varepsilon \\ &\quad + \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon] \end{aligned} \quad (3.10)$$

where the second inequality follows by the definition of  $\Lambda_+^*$  (and actually holds with equality due to (3.6)). We now show that:

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon] = 0. \quad (3.11)$$

Denote by  $\tilde{\Gamma}_T$  the c.g.f. under  $\mathbb{Q}_T$  of  $L_T^{\pi^+, \ell}$ . For all  $\zeta \in \mathbb{R}$ , we have by (3.9):

$$\begin{aligned} \tilde{\Gamma}_T(\zeta) &:= \ln \mathbb{E}^{\mathbb{Q}_T}[\exp(\zeta L_T^{\pi^+, \ell})] \\ &= \Gamma_T(\theta(\ell) + \zeta, \pi^+, \ell) - \Gamma_T(\theta(\ell), \pi^+, \ell). \end{aligned}$$

Therefore, by definition of the dual control problem (3.4), we have for all  $\zeta \in [-\theta(\ell), \bar{\theta} - \theta(\ell)]$ :

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \tilde{\Gamma}_T(\zeta) \leq \Lambda_+(\theta(\ell) + \zeta) - \Lambda_+(\theta(\ell)). \quad (3.12)$$

As in part 1) of this proof, by Chebycheff's inequality, we have for all  $\zeta \in [0, \bar{\theta} - \theta(\ell)]$ :

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T[\bar{L}_T^{\pi^+, \ell} \geq \ell + \varepsilon] &\leq -\zeta(\ell + \varepsilon) + \limsup_{T \rightarrow \infty} \frac{1}{T} \tilde{\Gamma}_T(\zeta) \\ &\leq -\zeta(\ell + \varepsilon) + \Lambda_+(\zeta + \theta(\ell)) - \Lambda_+(\theta(\ell)), \end{aligned}$$

where the second inequality follows from (3.12). We deduce

$$\begin{aligned}
\limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T \left[ \bar{L}_T^{\pi^+, \ell} \geq \ell + \varepsilon \right] &\leq -\sup\{\zeta(\ell + \varepsilon) - \Lambda_+(\zeta) : \zeta \in [\theta(\ell), \bar{\theta}]\} \\
&\quad -\Lambda_+(\theta(\ell)) + \theta(\ell)(\ell + \varepsilon) \\
&\leq -\Lambda_+^*(\ell + \varepsilon) - \Lambda_+(\theta(\ell)) + \theta(\ell)(\ell + \varepsilon), \\
&= -\Lambda_+^*(\ell + \varepsilon) + \Lambda_+^*(\ell) + \varepsilon\theta(\ell), \tag{3.13}
\end{aligned}$$

where the second inequality and the last equality follow from (3.6). Similarly, we have for all  $\zeta \in [-\theta(\ell), 0]$ :

$$\begin{aligned}
\limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T \left[ \bar{L}_T^{\pi^+, \ell} \leq \ell - \varepsilon \right] &\leq -\zeta(\ell - \varepsilon) + \limsup_{T \rightarrow \infty} \frac{1}{T} \tilde{\Gamma}_T(\zeta) \\
&\leq -\zeta(\ell - \varepsilon) + \Lambda_+(\theta(\ell) + \zeta) - \Lambda_+(\theta(\ell)),
\end{aligned}$$

and so:

$$\begin{aligned}
\limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T \left[ \bar{L}_T^{\pi^+, \ell} \leq \ell - \varepsilon \right] &\leq -\sup\{\zeta(\ell - \varepsilon) - \Lambda_+(\zeta) : \zeta \in [0, \theta(\ell)]\} \\
&\quad -\Lambda_+(\theta(\ell)) + \theta(\ell)(\ell - \varepsilon) \\
&\leq -\Lambda_+^*(\ell - \varepsilon) + \Lambda_+^*(\theta(\ell)) - \varepsilon\theta(\ell). \tag{3.14}
\end{aligned}$$

By (3.13)-(3.14), we then get:

$$\begin{aligned}
&\limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T \left[ \left\{ \bar{L}_T^{\pi^+, \ell} \leq \ell - \varepsilon \right\} \cup \left\{ \bar{L}_T^{\pi^+, \ell} \geq \ell + \varepsilon \right\} \right] \\
&\leq \max \left\{ \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T \left[ \bar{L}_T^{\pi^+, \ell} \geq \ell + \varepsilon \right]; \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{Q}_T \left[ \bar{L}_T^{\pi^+, \ell} \leq \ell - \varepsilon \right] \right\} \\
&\leq \max \left\{ -\Lambda_+^*(\ell + \varepsilon) + \Lambda_+^*(\ell) + \varepsilon\theta(\ell); -\Lambda_+^*(\ell - \varepsilon) + \Lambda_+^*(\theta(\ell)) - \varepsilon\theta(\ell) \right\} \\
&< 0,
\end{aligned}$$

where the strict inequality follows from (3.7). This implies that  $\mathbb{Q}_T[\{\bar{L}_T^{\pi^+, \ell} \leq \ell - \varepsilon\} \cup \{\bar{L}_T^{\pi^+, \ell} \geq \ell + \varepsilon\}] \rightarrow 0$  and hence  $\mathbb{Q}_T[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon] \rightarrow 1$  as  $T$  goes to infinity. In particular (3.11) is satisfied, and by sending  $\varepsilon$  to zero in (3.10), we get for any  $\ell' < \ell < \Lambda'_+(\bar{\theta})$ :

$$\begin{aligned}
\liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^{\pi^+, \ell} > \ell'] &\geq \lim_{\varepsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\ell - \varepsilon < \bar{L}_T^{\pi^+, \ell} < \ell + \varepsilon] \\
&\geq -\Lambda_+^*(\ell).
\end{aligned}$$

By continuity of  $\Lambda_+^*$  on  $(-\infty, \Lambda'_+(\bar{\theta}))$ , we obtain

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^{\pi^+, \ell} \geq \ell] \geq -\Lambda_+^*(\ell).$$

This last inequality combined with (3.8) proves the assertion for  $v_+(\ell)$  when  $\ell \in (\Lambda'_+(0), \Lambda'_+(\bar{\theta}))$ .

Now, consider the case  $\ell \leq \Lambda'_+(0)$ , and define  $\ell_n = \Lambda'_+(0) + \frac{1}{n}$ ,  $\pi^{+(n)} = \hat{\pi}(\theta(\ell_n))$ . Then, by the same arguments as in (3.10) with  $\ell_n \in (\Lambda'_+(0), \Lambda'_+(\bar{\theta}))$ , we have

$$\begin{aligned}
\liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^{\pi^{+(n)}} \geq \ell] &\geq \lim_{\varepsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\ell_n - \varepsilon < \bar{L}_T^{\pi^{+(n)}} < \ell_n + \varepsilon] \\
&\geq -\Lambda_+^*(\ell_n).
\end{aligned}$$



By sending  $n$  to infinity, together with the continuity of  $\Lambda_+^*$ , we get

$$\liminf_{n \rightarrow \infty} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^{\pi^{+(n)}} \geq \ell] \geq -\Lambda_+^*(\Lambda_+'(0)) = 0,$$

which combined with (3.8), ends the proof.  $\square$

**Remark 3.1** Theorem 3.1 shows that the upside chance large deviations control problem can be solved via the resolution of the dual control problem. When the target growth rate level  $\ell$  is smaller than  $\Lambda_+'(0)$ , then one can achieve almost surely over long term an average growth term above  $\ell$ , in the sense that  $v_+(\ell) = 0$ , with a nearly optimal portfolio strategy which does not depend on this level. When the target level  $\ell$  lies between  $\Lambda_+'(0)$  and  $\Lambda_+'(\bar{\theta})$ , the optimal strategy depends on this level and is obtained from the optimal strategy for the dual control problem  $\Lambda_+(\theta)$  at point  $\theta = \theta(\ell)$ . When  $\Lambda_+'(\bar{\theta}) = \infty$ , i.e.  $\Lambda_+$  is steep, we have a complete resolution of the large deviations control problem for all values of  $\ell$ . Otherwise, the problem remains open for  $\ell > \Lambda_+'(\bar{\theta})$ .  $\square$

Let us next consider the downside risk probability, and define the corresponding dual control problem:

$$\Lambda_-(\theta) := \inf_{\pi \in \mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{E}[e^{\theta T \bar{L}_T^\pi}], \quad \theta \leq 0. \quad (3.15)$$

Convexity of  $\Lambda_-$  is not so straightforward as for  $\Lambda_+$ , and requires the additional condition that the set of admissible controls  $\mathcal{A}$  is convex. Indeed, under this condition, we observe from the dynamics (2.1) that a convex combination of wealth process is a wealth process. Thus, for any  $\theta_1, \theta_2 \in (-\infty, 0)$ ,  $\lambda \in (0, 1)$ ,  $\pi^1, \pi^2 \in \mathcal{A}$ , there exists  $\pi \in \mathcal{A}$  such that:

$$\frac{\lambda \theta_1}{\lambda \theta_1 + (1 - \lambda) \theta_2} X_T^{\pi^1} + \frac{(1 - \lambda) \theta_2}{\lambda \theta_1 + (1 - \lambda) \theta_2} X_T^{\pi^2} = X_T^\pi.$$

By concavity of the logarithm function, we then obtain

$$\ln X_T^\pi \geq \frac{\lambda \theta_1}{(\lambda \theta_1 + (1 - \lambda) \theta_2)} \ln X_T^{\pi^1} + \frac{(1 - \lambda) \theta_2}{(\lambda \theta_1 + (1 - \lambda) \theta_2)} \ln X_T^{\pi^2},$$

and so, by setting  $\theta = \lambda \theta_1 + (1 - \lambda) \theta_2 < 0$ :

$$\theta T \bar{L}_T^\pi \leq \lambda \theta_1 T \bar{L}_T^{\pi^1} + (1 - \lambda) \theta_2 T \bar{L}_T^{\pi^2}.$$

Taking exponential and expectation on both sides of this relation, and using Hölder inequality, we get:

$$\mathbb{E}[e^{\theta T \bar{L}_T^\pi}] \leq \left( \mathbb{E}[e^{\theta_1 T \bar{L}_T^{\pi^1}}] \right)^\lambda \left( \mathbb{E}[e^{\theta_2 T \bar{L}_T^{\pi^2}}] \right)^{1 - \lambda}.$$

Taking logarithm, dividing by  $T$ , sending  $T$  to infinity, and since  $\pi^1, \pi^2$  are arbitrary in  $\mathcal{A}$ , we obtain by definition of  $\Lambda_-$ :

$$\Lambda_-(\theta) \leq \lambda \Lambda_-(\theta_1) + (1 - \lambda) \Lambda_-(\theta_2),$$

i.e. the convexity of  $\Lambda_-$  on  $\mathbb{R}_-$ . Since  $\Lambda_-(0) = 0$ , the convex function  $\Lambda_-$  is either infinite on  $(-\infty, 0)$  or finite on  $\mathbb{R}_-$ . We now state the duality relation for downside risk large deviations probability, whose proof can be found in [15].

**Theorem 3.2** Suppose that  $\Lambda_-$  is differentiable on  $(-\infty, 0)$ , and there exists  $\hat{\pi}(\theta) \in \mathcal{A}$  solution to  $\Lambda_-(\theta)$  for any  $\theta < 0$ . Then, for all  $\ell < \Lambda'_-(0)$ , we have:

$$v_-(\ell) = \inf_{\theta \leq 0} [\Lambda_-(\theta) - \theta\ell],$$

and an optimal control for  $v_-(\ell)$ , when  $\ell \in (\Lambda'_-(-\infty), \Lambda'_-(0))$  is:

$$\pi^{-, \ell} = \hat{\pi}(\theta(\ell)), \quad \text{with } \Lambda'_-(\theta(\ell)) = \ell,$$

while  $v_-(\ell) = -\infty$  when  $\ell < \Lambda'_-(-\infty)$ .

**Remark 3.2** Theorem 3.2 shows that the downside risk large deviations control problem can be solved via the resolution of the dual control problem. When the target growth rate level  $\ell$  is smaller than  $\Lambda'_-(-\infty)$ , then one can find a portfolio strategy so that the average growth term almost never fall below  $\ell$  over the long term, in the sense that  $v_-(\ell) = -\infty$ . When the target level  $\ell$  lies between  $\Lambda'_-(-\infty)$  and  $\Lambda'_-(0)$ , the optimal strategy depends on this level and is obtained from the optimal strategy for the dual control problem  $\Lambda_-(\theta)$  at point  $\theta = \theta(\ell)$ .  $\square$

### Interpretation of the dual problem

For  $\theta \neq 0$ , the dual problem can be written as

$$\frac{1}{\theta} \Lambda_{\pm}(\theta) = \sup_{\pi \in \mathcal{A}} \limsup_{T \rightarrow \infty} J_T(\theta, \pi),$$

with

$$J_T(\theta, \pi) := \frac{1}{\theta T} \ln \mathbb{E}[e^{\theta T \bar{L}_T^\pi}],$$

and is known in the literature as a risk-sensitive control problem. A Taylor expansion around  $\theta = 0$  puts in evidence the role played by the risk sensitivity parameter  $\theta$ :

$$J_T(\theta, \pi) \simeq \mathbb{E}[\bar{L}_T^\pi] + \theta T \text{Var}(\bar{L}_T^\pi) + O(\theta^2).$$

This relation shows that risk-sensitive control amounts to making dynamic the Markowitz problem: one maximizes the expected average growth rate subject to a constraint on its variance. Risk-sensitive portfolio criterion on finite horizon  $T$  has been studied in [2] and [3], and in the ergodic case  $T \rightarrow \infty$ , by [6] and [16].

### Endogenous utility function

Recalling that growth rate is the logarithm of wealth process, the duality relation for the upside large deviations probability means formally that for large horizon  $T$ :

$$\begin{aligned} \mathbb{P}[\bar{L}_T^{\pi^+, \ell} \geq \ell] &\simeq \exp(v_+(\ell)T) \\ &= \exp(\Lambda_+(\theta(\ell))T - \theta(\ell)\ell T) \\ &\simeq \mathbb{E}\left[(X_T^{\pi^+, \ell})^{\theta(\ell)}\right] e^{-\theta(\ell)T}, \quad \text{with } \theta(\ell) > 0. \end{aligned}$$

Similarly, we have for the downside risk probability:

$$\mathbb{P}[\bar{L}_T^{\pi^-, \ell} \leq \ell] \simeq \mathbb{E}\left[(X_T^{\pi^-, \ell})^{\theta(\ell)}\right] e^{-\theta(\ell)T}, \quad \text{with } \theta(\ell) < 0.$$

In other words, the target growth rate level  $\ell$  determines endogenously the risk aversion parameter  $1 - \theta(\ell)$  of an agent with Constant Relative Risk Aversion (CRRA) utility function and large investment horizon. Moreover, the optimal strategy  $\pi^{\pm, \ell}$  for  $v_{\pm}(\ell)$  is expected to provide a good approximation for the solution to the CRRA utility maximization problem

$$\sup_{\pi \in \mathcal{A}} \mathbb{E}[(X_T^{\pi})^{\theta(\ell)}],$$

with a large but finite time horizon.

## 4 A toy model: the Black Scholes case

We illustrate the results of the previous section in a toy example, namely the Black-Scholes model, with one stock of price process

$$dS_t = S_t(bdt + \sigma dW_t), \quad t \geq 0.$$

We also consider an agent with constant proportion portfolio strategies. In other words, the set of admissible controls  $\mathcal{A}$  is equal to  $\mathbb{R}$ . Given a constant proportion  $\pi \in \mathbb{R}$  invested in the stock, and starting w.l.o.g. with unit capital, the average growth rate portfolio of the agent is equal to

$$\bar{L}_T^{\pi} = \frac{L_T^{\pi}}{T} = \left(b\pi - \frac{\sigma^2 \pi^2}{2}\right) + \sigma \pi \frac{W_T}{T}.$$

It follows that  $\bar{L}_T^{\pi}$  is distributed according to a Gaussian law:

$$\bar{L}_T^{\pi} \rightsquigarrow \mathcal{N}\left(b\pi - \frac{\sigma^2 \pi^2}{2}, \frac{\sigma^2 \pi^2}{T}\right),$$

and its (limiting) Log-Laplace function is equal to

$$\Gamma(\theta, \pi) := \left(\lim_{T \rightarrow \infty}\right) \frac{1}{T} \ln \mathbb{E}\left[e^{\theta T \bar{L}_T^{\pi}}\right] = \theta \left[b\pi - (1 - \theta) \frac{\sigma^2 \pi^2}{2}\right]$$

- **Upside chance probability.**

The dual control problem in the upside case is then given by

$$\Lambda_+(\theta) = \sup_{\pi \in \mathbb{R}} \Gamma(\theta, \pi) = \begin{cases} \infty, & \text{if } \theta \geq 1, \\ \Gamma(\theta, \hat{\pi}(\theta)) = \frac{b^2}{2\sigma^2} \frac{\theta}{1-\theta}, & \text{if } 0 \leq \theta < 1, \end{cases}$$

with

$$\hat{\pi}(\theta) = \frac{b}{\sigma^2(1-\theta)}.$$

Hence,  $\Lambda_+$  differentiable on  $[0, 1)$  with:  $\Lambda'_+(0) = \frac{b^2}{2\sigma^2}$ , and  $\Lambda'_+(1) = \infty$ , i.e.  $\Lambda_+$  is steep. From Theorem 3.1, the value function of the upside large deviations probability is explicitly computed as:

$$\begin{aligned} v_+(\ell) &:= \sup_{\pi \in \mathbb{R}} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^\pi \geq \ell] \\ &= \inf_{0 \leq \theta < 1} [\Lambda_+(\theta) - \theta \ell] \\ &= \begin{cases} 0, & \text{if } \ell \leq \Lambda'_+(0) = \frac{b^2}{2\sigma^2} \\ -(\sqrt{\Lambda'_+(0)} - \sqrt{\ell})^2, & \text{if } \ell > \Lambda'_+(0) \end{cases} \end{aligned}$$

with an optimal strategy:

$$\pi^{+, \ell} = \begin{cases} \frac{b}{\sigma^2}, & \text{if } \ell \leq \Lambda'_+(0) \\ \sqrt{\frac{2\ell}{\sigma^2}}, & \text{if } \ell > \Lambda'_+(0). \end{cases}$$

Notice that, when  $\ell \leq \Lambda'_+(0)$ , we have not only a nearly optimal control as stated in Theorem 3.1, but an optimal control given by  $\pi^+ = b/\sigma^2$ , which is precisely the optimal portfolio for the classical Merton problem with logarithm utility function. Indeed, in this model, we have by the law of large numbers:  $\bar{L}_T^{\pi^+} \rightarrow \frac{b^2}{2\sigma^2} = \Lambda'_+(0)$ , as  $T$  goes to infinity, and so  $\lim_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^{\pi^+} \geq \ell] = 0 = v_+(\ell)$ . Otherwise, when  $\ell > \Lambda'_+(0)$ , the optimal strategy depends on  $\ell$ , and the larger the target growth rate level, the more one has to invest in the stock.

• **Downside risk probability.**

The dual control problem in the downside case is then given by

$$\Lambda_-(\theta) = \inf_{\pi \in \mathbb{R}} \Gamma(\theta, \pi) = \Gamma(\theta, \hat{\pi}(\theta)) = \frac{b^2}{2\sigma^2} \frac{\theta}{1 - \theta}, \quad \theta \leq 0,$$

with

$$\hat{\pi}(\theta) = \frac{b}{\sigma^2(1 - \theta)}.$$

Hence,  $\Lambda_-$  is differentiable on  $\mathbb{R}_-$  with:  $\Lambda'_(-\infty) = 0$ , and  $\Lambda'_-(0) = \frac{b^2}{2\sigma^2}$ . From Theorem 3.1, the value function of the downside large deviations probability is explicitly computed as:

$$\begin{aligned} v_-(\ell) &:= \inf_{\pi \in \mathbb{R}} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{P}[\bar{L}_T^\pi \leq \ell] \\ &= \inf_{\theta \leq 0} [\Lambda_-(\theta) - \theta \ell] \\ &= \begin{cases} -\infty, & \text{if } \ell < 0 \\ -(\sqrt{\Lambda'_-(0)} - \sqrt{\ell})^2, & \text{if } 0 \leq \ell \leq \Lambda'_-(0) = \frac{b^2}{2\sigma^2} \end{cases} \end{aligned}$$

with an optimal strategy:

$$\pi^{-, \ell} = \sqrt{\frac{2\ell}{\sigma^2}}, \quad \text{if } 0 \leq \ell \leq \Lambda'_-(0).$$

Moreover, when  $\ell < 0$ , and by choosing  $\pi^- = 0$ , we have  $\bar{L}_T^{\pi^-} = 0$ , so that  $\mathbb{P}[\bar{L}_T^{\pi^-} \leq \ell] = 0$ , and thus  $v_-(\ell) = -\infty$ . In other words, when the target growth rate  $\ell < 0$ , by doing nothing, we have an optimal strategy for  $v_-(\ell)$ .

**Remark 4.1** The above direct calculations rely on the fact that we restrict portfolio  $\pi$  to be constant in proportion. Actually, the explicit forms of the value function and optimal strategy remain the same if we allow a priori portfolio strategies  $\pi \in \mathcal{A}$  to change over time based on the available information, i.e. to be  $\mathbb{F}$ -predictable. This requires more advanced tools from stochastic control and PDEs to be presented in the sequel in a more general framework.  $\square$

## 5 Factor model

We consider a market model with one riskless asset price  $S^0 = 1$ , and  $d$  stocks of price process  $S$  governed by

$$\begin{aligned} dS_t &= \text{diag}(S_t)(b(Y_t)dt + \sigma(Y_t)dW_t) \\ dY_t &= \eta(Y_t)dt + \gamma(Y_t)dW_t, \end{aligned}$$

where  $Y$  is a factor process valued in  $\mathbb{R}^m$ , and  $W$  is a  $d + m$  dimensional standard Brownian motion. The coefficients  $b, \sigma, \eta, \gamma$  are assumed to satisfy regular conditions ensuring existence of a unique strong solution to the above stochastic differential equation, and  $\sigma$  is also of full rank, i.e. the  $d \times d$ -matrix  $\sigma\sigma'$  is invertible.

A portfolio strategy  $\pi$  is an  $\mathbb{R}^d$ -valued adapted process, representing the fraction of wealth invested in the  $d$  stocks. The admissibility condition for  $\pi$  in  $\mathcal{A}$  will be precised later, but for the moment  $\pi$  is required to satisfy the integrability conditions:

$$\int_0^T |\pi_t' b(Y_t)| dt + \int_0^T |\pi_t' \sigma(Y_t)|^2 dt < \infty, \text{ a.s. for all } T > 0.$$

The growth rate portfolio is then given by:

$$L_T^\pi = \int_0^T \left( \pi_t' b(Y_t) - \frac{\pi_t' \sigma \sigma' (Y_t) \pi_t}{2} \right) dt + \int_0^T \pi_t' \sigma(Y_t) dW_t.$$

For any  $\theta \in \mathbb{R}$ , and  $\pi$ , we compute the Log-Laplace function of the growth rate portfolio:

$$\begin{aligned} \Gamma_T(\theta, \pi) &:= \ln \mathbb{E}[e^{\theta L_T^\pi}] \\ &= \ln \mathbb{E} \left[ \mathcal{E} \left( \int_0^T \theta \pi_t' \sigma(Y_t) dW_t \right) e^{\theta \int_0^T f(\theta, Y_t, \pi_t) dt} \right], \end{aligned}$$

where  $\mathcal{E}(\cdot)$  denotes the Doléans-Dade exponential, and  $f$  is the function:

$$f(\theta, y, \pi) = \pi' b(y) - \frac{1 - \theta}{2} \pi' \sigma \sigma' (y) \pi.$$

We now impose the admissibility condition that  $\pi$  lies in  $\mathcal{A}$  if the Doléans-Dade local martingale  $\mathcal{E} \left( \int_0^\cdot \theta \pi_t' \sigma(Y_t) dW_t \right)_{0 \leq t \leq T}$  is a true martingale for any  $T > 0$ , which is ensured,

for instance, by the Novikov condition. In this case, this Doléans-Dade exponential defines a probability measure  $\mathbb{Q}_\pi$  equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{F}_T)$ , and we have:

$$\Gamma_T(\theta, \pi) = \ln \mathbb{E}^{\mathbb{Q}_\pi} \left[ \exp \left( \theta \int_0^T f(\theta, Y_t, \pi_t) dt \right) \right],$$

where  $Y$  is governed under  $\mathbb{Q}_\pi$  by

$$dY_t = (\eta(Y_t) + \theta \gamma(Y_t) \sigma'(Y_t) \pi_t) dt + \gamma(Y_t) dW_t^\pi.$$

with  $W^\pi$  a  $\mathbb{Q}_\pi$ -Brownian motion from Girsanov's theorem.

We then consider the dual control problems:

- **Upside chance:** for  $\theta \geq 0$ ,

$$\Lambda_+(\theta) = \sup_{\pi \in \mathcal{A}} \limsup_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{E}^{\mathbb{Q}_\pi} \left[ \exp \left( \theta \int_0^T f(\theta, Y_t, \pi_t) dt \right) \right].$$

- **Downside risk:** for  $\theta \leq 0$ ,

$$\Lambda_-(\theta) = \inf_{\pi \in \mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{T} \ln \mathbb{E}^{\mathbb{Q}_\pi} \left[ \exp \left( \theta \int_0^T f(\theta, Y_t, \pi_t) dt \right) \right].$$

These problems are known in the literature as ergodic risk-sensitive control problems, and studied by dynamic programming methods in [1], [5] and [14]. Let us now formally derive the ergodic equations associated to these risk-sensitive control problems. We consider the finite horizon risk-sensitive stochastic control problems:

$$\begin{aligned} u_+(T, y; \theta) &= \sup_{\pi \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}_\pi} \left[ \exp \left( \theta \int_0^T f(\theta, Y_t, \pi_t) dt \right) \mid Y_0 = y \right], \quad \theta \geq 0 \\ u_-(T, y; \theta) &= \inf_{\pi \in \mathcal{A}} \mathbb{E}^{\mathbb{Q}_\pi} \left[ \exp \left( \theta \int_0^T f(\theta, Y_t, \pi_t) dt \right) \mid Y_0 = y \right], \quad \theta \leq 0, \end{aligned}$$

and by using the formal substitution:

$$\ln u_\pm(T, y; \theta) \simeq \Lambda_\pm(\theta)T + \varphi_\pm(y; \theta), \quad \text{for large } T,$$

in the corresponding Hamilton-Jacobi-Bellman (HJB) equations for  $u_\pm$ :

$$\frac{\partial u_\pm}{\partial T} = \sup_{\pi \in \mathbb{R}^d} \left[ \theta f(\theta, y, \pi) u_\pm + (\eta(y) + \theta \gamma(y) \sigma'(y) \pi)' D_y u_\pm + \frac{1}{2} \text{tr}(\gamma \gamma'(y) D_y^2 u_\pm) \right],$$

we obtain the ergodic HJB equation for the pair  $(\Lambda_\pm(\theta), \varphi_\pm(\cdot, \theta))$  as:

$$\begin{aligned} \Gamma(\theta) &= \eta(y)' D_y \varphi + \frac{1}{2} \text{tr}(\gamma \gamma'(y) D_y^2 \varphi) + \frac{1}{2} |\gamma'(y) D_y \varphi|^2 \\ &\quad + \theta \sup_{\pi \in \mathbb{R}^d} \left[ \pi' (b(y) + \sigma(y) \gamma'(y) D_y \varphi) - \frac{1-\theta}{2} \pi' \sigma \sigma'(y) \pi \right], \end{aligned}$$

which is well-defined for  $\theta < 1$ . In the above equation  $\Gamma(\theta)$  is a candidate for  $\Lambda_{\pm}(\theta)$  while  $\varphi$  is a candidate solution for  $\varphi_{\pm}$ . This can be rewritten as a semi-linear ergodic PDE with quadratic growth in the gradient:

$$\begin{aligned}\Gamma(\theta) &= \left(\eta(y) + \frac{\theta}{1-\theta}\gamma\sigma'(\sigma\sigma')^{-1}b(y)\right) \cdot D_y\varphi + \frac{1}{2}\text{tr}(\gamma\gamma'(y)D_y^2\varphi) \\ &\quad + \frac{1}{2}D_y\varphi'\gamma(y)\left[I_{d+m} + \frac{\theta}{1-\theta}\sigma'(\sigma\sigma')^{-1}\sigma(y)\right]\gamma'(y)D_y\varphi \\ &\quad + \frac{\theta}{2(1-\theta)}b'(\sigma\sigma')^{-1}b(y),\end{aligned}\tag{5.1}$$

and a candidate for optimal feedback control of the dual problem:

$$\hat{\pi}(y; \theta) = \frac{1}{1-\theta}(\sigma\sigma')^{-1}(y)[b(y) + \sigma\gamma'(y)D_y\varphi(y; \theta)].\tag{5.2}$$

We now face the questions:

- Existence of a pair solution  $(\Gamma(\theta), \varphi(\cdot, \theta))$  to the ergodic PDE (5.1)?
- Do we have  $\Gamma(\theta) = \Lambda_{\pm}(\theta)$ , and what is the domain of  $\Gamma$ ?

We give some assumptions, which allows us to answer the above issues.

**(H1)**  $b, \sigma, \eta$  and  $\gamma$  are smooth  $C^2$  and globally Lipschitz.

**(H2)**  $\sigma\sigma'(y)$  and  $\gamma\gamma'(y)$  are uniformly elliptic: there exist  $\delta_1, \delta_2 > 0$  s.t.

$$\begin{aligned}\delta_1|\xi|^2 &\leq \xi'\sigma\sigma'(y)\xi \leq \delta_2|\xi|^2, \quad \forall \xi, y \in \mathbb{R}^m, \\ \delta_1|\xi|^2 &\leq \xi'\gamma\gamma'(y)\xi \leq \delta_2|\xi|^2, \quad \forall \xi, y \in \mathbb{R}^m.\end{aligned}$$

**(H3)** There exist  $c_1 > 0$  and  $c_2 \geq 0$  s.t.

$$b(\sigma\sigma')^{-1}b(y) \geq c_1|y|^2 - c_2, \quad \forall y \in \mathbb{R}^m.$$

**(H4)** Stability condition: there exist  $c_3 > 0$  and  $c_4 \geq 0$  s.t.

$$(\eta(y) - \gamma\sigma'(\sigma\sigma')^{-1}b(y)) \cdot y \leq -c_3|y|^2 + c_4$$

According to [1] (see also [15] and [20]), the next result states the existence of a smooth solution to the ergodic equation.

**Proposition 5.1** *Under (H1)-(H4), there exists for any  $\theta < 1$ , a solution  $(\Gamma(\theta), \varphi(\cdot; \theta))$  with  $\varphi(\cdot; \theta) \in C^2$ , to the ergodic HJB equation s.t:*

- For  $\theta < 0$ ,  $\varphi(\cdot; \theta)$  is upper-bounded

$$\varphi(y; \theta) \longrightarrow -\infty, \quad \text{as } |y| \rightarrow \infty,$$

- For  $\theta \in (0, 1)$ ,  $\varphi(\cdot; \theta)$  is lower-bounded

$$\varphi(y; \theta) \longrightarrow \infty, \quad \text{as } |y| \rightarrow \infty,$$

and

$$|D_y\varphi(y; \theta)| \leq C_\theta(1 + |y|).$$

We now relate a solution to the ergodic equation to the dual risk-sensitive control problem. In other words, this means the convergence of the finite horizon risk-sensitive stochastic control to the component  $\Gamma$  of the ergodic equation. We distinguish the downside and upside cases.

• **Downside risk:** In this case, it is shown in [15] that for all  $\theta < 0$ , the solution  $(\Gamma(\theta), \varphi(\cdot; \theta))$  to (5.1), with  $\varphi(\cdot, \theta)$   $C^2$  and upper bounded, is unique (up to an additive constant for  $\varphi(\cdot; \theta)$ ), and we have:

$$\Gamma(\theta) = \Lambda_-(\theta), \quad \theta < 0.$$

Moreover, there is an admissible optimal feedback control  $\hat{\pi}(\cdot, \theta)$  for  $\Lambda_-(\theta)$  given by (5.2), and for which the factor process  $Y$  is ergodic under  $\mathbb{Q}_{\hat{\pi}}$ . It is also proved in [15] that  $\Gamma = \Lambda_-$  is differentiable on  $(-\infty, 0)$ . Therefore, from Theorem 3.2, the solution to the downside risk large deviations probability is given by:

$$v_-(\ell) = \inf_{\theta < 0} [\Gamma(\theta) - \theta\ell], \quad \ell < \Gamma'(0),$$

with an optimal control:

$$\pi_t^{-, \ell} = \hat{\pi}(Y_t; \theta(\ell)), \quad \Gamma'(\theta(\ell)) = \ell, \quad \forall \ell \in (\Gamma'(-\infty), \Gamma'(0)),$$

while  $v_-(\ell) = -\infty$  for  $\ell < \Gamma'(-\infty)$ .

• **Upside chance:** In this case,  $0 < \theta < 1$ , there is no unique solution  $(\Gamma(\theta), \varphi(\cdot; \theta))$  to the ergodic equation, with  $\varphi(\cdot; \theta)$   $C^2$  lower-bounded, even up to an additive constant, as pointed out in [6]. In general, we only have a verification type result, which states that if the process  $Y$  is ergodic under  $\mathbb{Q}_{\hat{\pi}}$ , then

$$\Gamma(\theta) = \Lambda_+(\theta),$$

and  $\hat{\pi}(\cdot, \theta)$  is an optimal feedback control for  $\Lambda_+(\theta)$ .

In the next paragraph, we consider a linear factor model for which explicit calculations can be derived.

## 5.1 Linear Gaussian factor model

We consider the linear factor model:

$$\begin{aligned} dS_t &= \text{diag}(S_t)((B_1 Y_t + B_0)dt + \sigma dW_t) \quad \text{in } \mathbb{R}^d, \\ dY_t &= KY_t dt + \gamma dW_t, \quad \text{in } \mathbb{R}^m, \end{aligned}$$

with  $K$  a stable matrix in  $\mathbb{R}^m$ ,  $B_1$  a constant  $d \times m$  matrix,  $B_0$  a non-zero vector in  $\mathbb{R}^d$ ,  $\sigma$  a  $d \times (d + m)$ -matrix of rank  $d$ , and  $\gamma$  a nonzero  $m \times (d + m)$  matrix. We are searching for a candidate solution to the ergodic equation (5.1) in the quadratic form:

$$\varphi(y; \theta) = \frac{1}{2}C(\theta)y \cdot y + D(\theta)y, \quad y \in \mathbb{R}^m,$$



for some  $m \times m$  matrices  $C(\theta)$  and  $D(\theta)$ . Plugging this form of  $\varphi$  into (5.1), we find that  $C(\theta)$  must solve the algebraic Riccati equation:

$$\begin{aligned} & \frac{1}{2}C(\theta)'\gamma(I_{d+m} + \frac{\theta}{1-\theta}\sigma'(\sigma\sigma')^{-1}\sigma)\gamma'C(\theta) \\ & + (K + \frac{\theta}{1-\theta}\gamma\sigma'(\sigma\sigma')^{-1}B_1)'C(\theta) + \frac{1}{2}\frac{\theta}{1-\theta}B_1'(\sigma\sigma')^{-1}B_1 = 0, \end{aligned} \quad (5.3)$$

while  $B(\theta)$  is determined by

$$\begin{aligned} & \left(K + \frac{\theta}{1-\theta}\gamma\sigma'(\sigma\sigma')^{-1}B_1 + \gamma(I_{d+m} + \frac{\theta}{1-\theta}\sigma'(\sigma\sigma')^{-1}\sigma)\gamma'C(\theta)\right)'D(\theta) \\ & + \frac{\theta}{1-\theta}(\sigma\gamma'C(\theta) + B_1)'(\sigma\sigma')^{-1}B_0 = 0. \end{aligned}$$

Then,  $\Gamma(\theta)$  is given by:

$$\begin{aligned} \Gamma(\theta) &= \frac{1}{2}\text{tr}(\gamma\gamma'C(\theta)) + \frac{1}{2}D(\theta)'\gamma(I_{d+m} + \frac{\theta}{1-\theta}\sigma'(\sigma\sigma')^{-1}\sigma)\gamma'D(\theta) \\ & + \frac{\theta}{1-\theta}B_0'(\sigma\sigma')^{-1}\sigma\gamma'D(\theta) + \frac{1}{2}\frac{\theta}{1-\theta}B_0'(\sigma\sigma')^{-1}B_0, \end{aligned}$$

and a candidate for the optimal feedback control is:

$$\hat{\pi}(y; \theta) = \frac{1}{1-\theta}(\sigma\sigma')^{-1}[(B_1 + \sigma\gamma'C(\theta))y + B_0 + \sigma\gamma'D(\theta)].$$

In [6], it is shown that there exists some positive  $\bar{\theta}$  small enough, s.t. for  $\theta < \bar{\theta}$ , there exists a solution  $C(\theta)$  to the Riccati equation (5.3) s.t.  $Y$  is ergodic under  $\mathbb{Q}_{\hat{\pi}}$ , and so by verification theorem,  $\Gamma(\theta) = \Lambda_{\pm}(\theta)$ . In the one-dimensional asset and factor model, as studied in [17], we obtain more precise results. Indeed, in this case:  $d = m = 1$ , the Riccati equation is a second-order polynomial equation in  $C(\theta)$ , which admits two explicit roots given by:

$$C_{\pm}(\theta) = -\frac{K}{|\gamma|^2} \left[ \frac{1 - \theta(1 - \rho \frac{|\gamma|B_1}{K|\sigma|}) \pm \sqrt{(1-\theta)(1-\theta\beta)}}{1 - \theta(1 - \rho^2)} \right],$$

for all  $\theta \leq \bar{\theta}$ , with

$$\bar{\theta} = \frac{1}{\beta} \wedge 1, \quad \beta = 1 - \rho^2 + \left(\rho - \frac{|\gamma|B_1}{K|\sigma|}\right)^2 > 0,$$

where  $|\gamma|$  (resp.  $|\sigma|$ ) is the Euclidian norm of  $\gamma$  (resp.  $\sigma$ ), and  $\rho \in [-1, 1]$  is the correlation between  $S$  and  $Y$ , i.e.  $\rho = \frac{\gamma\sigma'}{|\gamma||\sigma|}$ . Actually, only the solution  $C(\theta) = C_{-}(\theta)$  is relevant in the sense that for this root,  $Y$  is ergodic under  $\mathbb{Q}_{\hat{\pi}}$ , and thus by verification theorem:

$$\begin{aligned} \Lambda_{\pm}(\theta) = \Gamma(\theta) &= \frac{1}{2}|\gamma|^2 C_{-}(\theta) + \frac{1}{2}|\gamma|^2 D(\theta)^2 \left(1 + \frac{\theta}{1-\theta}\rho^2\right) \\ & + \frac{\theta}{1-\theta} \frac{B_0}{|\sigma|} \rho |\gamma| D(\theta) + \frac{1}{2} \frac{\theta}{1-\theta} \frac{B_0^2}{|\sigma|^2}, \quad \theta < \bar{\theta}, \end{aligned}$$

where

$$D(\theta) = -\frac{B_0}{K|\sigma|} \frac{\theta(\rho|\gamma|C_-(\theta) + \frac{B_1}{|\sigma|})}{\sqrt{(1-\theta)(1-\theta\beta)}},$$

and with optimal control for  $\Lambda_{\pm}(\theta)$  given by:

$$\hat{\pi}(y; \theta) = \frac{1}{(1-\theta)|\sigma|} \left[ \left( \frac{B_1}{|\sigma|} + \rho|\gamma|C_-(\theta) \right) y + \frac{B_0}{|\sigma|} + \rho|\gamma|D(\theta) \right].$$

Moreover, it is also proved in [17], that

$$\Gamma'(0) = \frac{B_0^2}{2|\sigma|^2} - \frac{B_1^2|\gamma|}{4|\sigma|^2K} > 0,$$

(recall that  $K < 0$ ) and the function  $\Gamma$  is steep, i.e.

$$\lim_{\theta \uparrow \bar{\theta}} \Gamma'(\theta) = \infty.$$

From Theorems 3.1 and 3.2, the solutions to the upside chance and downside risk large deviations probability are given by:

$$\begin{aligned} v_+(\ell) &= \inf_{0 \leq \theta < \bar{\theta}} [\Gamma(\theta) - \theta\ell], \quad \ell \in \mathbb{R}, \\ v_-(\ell) &= \inf_{\theta \leq 0} [\Gamma(\theta) - \theta\ell], \quad \ell < \Gamma'(0), \end{aligned}$$

with optimal control and nearly optimal control for  $v_+(\ell)$ :

$$\begin{aligned} \pi_t^{+, \ell} &= \hat{\pi}(Y_t; \theta(\ell)), \quad \Gamma'(\theta(\ell)) = \ell, \quad \text{when } \ell > \Gamma'(0), \\ \pi_t^{+(n)} &= \hat{\pi}(Y_t; \theta_n), \quad \text{with } \theta_n = \theta(\Gamma'(0) + \frac{1}{n}) \xrightarrow{n \rightarrow \infty} 0, \quad \text{when } \ell \leq \Gamma'(0), \end{aligned}$$

and optimal control for  $v_-(\ell)$ :

$$\pi_t^{-, \ell} = \hat{\pi}(Y_t; \theta(\ell)), \quad \Gamma'(\theta(\ell)) = \ell, \quad \forall \ell \in (\Gamma'(-\infty), \Gamma'(0)).$$

## 5.2 Examples

• **Black-Scholes model.** This corresponds to the case where  $B_1 = 0$ . Then,  $\beta = \bar{\theta} = 1$ ,  $C_-(\theta) = D(\theta) = 0$ , and so

$$\Lambda_{\pm}(\theta) = \Gamma(\theta) = \frac{1}{2} \frac{\theta}{1-\theta} \frac{B_0^2}{|\sigma|^2}, \quad \forall \theta < 1.$$

We thus obtain the same optimal strategy as described in Section 4.

• **Platen-Rebolledo model.** In this model, the logarithm of the stock price  $S$  is governed by an Ornstein-Uhlenbeck process  $Y$ , and this corresponds to the case where  $B_1 = K < 0$ ,  $B_0 = \frac{1}{2}|\gamma|^2 > 0$ ,  $\gamma = \sigma$ , and thus  $\rho = 1$ . Then,  $\beta = 0$ ,  $\bar{\theta} = 1$ ,

$$C_-(\theta) = \frac{|K|}{|\sigma|^2} [1 - \sqrt{1-\theta}], \quad D(\theta) = -\frac{1}{2}\theta,$$

and so

$$\begin{aligned}\Gamma(\theta) &= \frac{|K|}{2} [1 - \sqrt{1 - \theta}] + \theta \frac{|\sigma|^2}{8}, \quad \theta < 1, \\ \Gamma'(0) &= \bar{\ell} := \frac{|K|}{4} + \frac{|\sigma|^2}{8}, \quad \Gamma'(-\infty) = \underline{\ell} := \frac{|\sigma|^2}{8},\end{aligned}$$

$$\theta(\ell) = 1 - \left( \frac{\bar{\ell} - \underline{\ell}}{\ell - \underline{\ell}} \right)^2, \quad \forall \ell > \underline{\ell}.$$

The solution to the upside chance large deviations probability is then given by:

$$v_+(\ell) = \begin{cases} -\frac{(\ell - \bar{\ell})^2}{\ell - \bar{\ell} + \frac{|K|}{4}}, & \text{if } \ell > \bar{\ell} \\ 0, & \text{if } \ell \leq \bar{\ell}. \end{cases}$$

with optimal (resp. nearly optimal) portfolio strategy:

$$\begin{aligned}\pi_t^{+, \ell} &= \frac{K - 4(\ell - \bar{\ell})}{|\sigma|^2} Y_t + \frac{1}{2}, \quad \text{if } \ell > \bar{\ell} \\ \pi_t^{+(n)} &= \frac{K - 1/n}{|\sigma|^2} Y_t + \frac{1}{2}, \quad \text{if } \ell \leq \bar{\ell}.\end{aligned}$$

The solution to the downside risk large deviations probability is given by:

$$v_-(\ell) = \begin{cases} -\frac{(\ell - \bar{\ell})^2}{\ell - \underline{\ell}}, & \text{if } \underline{\ell} < \ell \leq \bar{\ell} \\ -\infty, & \text{if } \ell \leq \underline{\ell}, \end{cases}$$

with optimal portfolio strategy:

$$\pi_t^{-, \ell} = -\frac{4(\ell - \underline{\ell})}{|\sigma|^2} Y_t + \frac{1}{2}, \quad \text{if } \underline{\ell} < \ell \leq \bar{\ell}$$

## References

- [1] Bensoussan A. and J. Frehse (1992): “On Bellman equations of ergodic control in  $\mathbb{R}^N$ ”, *J. Reine Angew. Math.*, **429**, 125-160.
- [2] Bielecki T.R. and S.R. Pliska (1999): “Risk-sensitive dynamic asset management”, *Appl. Math. Optimizat.*, **39**, 337-360.
- [3] Davis M. and S. Lleo (2008): “Risk-sensitive benchmarked asset management”, *Quantitative Finance*, **8**, 415-426.
- [4] Dembo A. and O. Zeitouni (1998): Large deviations techniques and applications, 2nd edition, Springer Verlag.
- [5] Fleming W. and W. McEneaney (1995): “Risk sensitive control on an infinite horizon”, *SIAM J. Cont. and Optim.*, **33**, 1881-1915.

- [6] Fleming W. and S.J. Sheu (2000): “Risk sensitive control and an optimal investment model”, *Math. Finance*, **10**, 197-213.
- [7] Guasoni P. and S. Robertson (2012): “Portfolios and Risk Premia for the Long Run”, *Annals of Applied Probability*, **22**, 239-284.
- [8] Hata H., Nagai H. and S.J. Sheu (2010): “Asymptotics of the probability minimizing a down-side risk”, *Annals of Applied Probability*, **20**, 52-89.
- [9] Hata H. and J. Sekine (2005): “Solving long term investment problems with Cox-Ingersoll-Ross interest rates”, *Advances in Math. Econ.*, **8**, 231-255.
- [10] Karatzas I. and S. Shreve (1998): *Methods of Mathematical Finance*, Springer.
- [11] Korn R. (1997): *Optimal portfolios: stochastic models for optimal investment and risk management in continuous-time*, World scientific, Singapore.
- [12] Liu R. and J. Muhle-Karbe (2013): “Portfolio choice with stochastic investment opportunities: a user’s guide”, preprint.
- [13] Merton R. (1971): “Optimum consumption and portfolio rules in a continuous-time model”, *J. Economic Theory*, **3**, 373-413.
- [14] Nagai H. (1996): “Bellman equations of risk sensitive control”, *SIAM J. Cont. and Optim.*, **34**, 74-101.
- [15] Nagai H. (2012): “Downside risk minimization via a large deviation approach”, *Annals of Applied Probability*, **22**, 608-669.
- [16] Nagai H. and S. Peng (2002): “Risk-Sensitive Dynamic Portfolio Optimization with Partial Information on Infinite Time Horizon”, *Annals of Applied Probability*, **12**, 173-195.
- [17] Pham H. (2003): “A large deviations approach to optimal long term investment”, *Finance and Stochastics*, **7**, 169-195.
- [18] Pham H. (2007): “Some applications and methods of large deviations in finance and insurance”, Paris-Princeton lectures on Mathematical Finance, Lect. Notes in Mathematics, vol 1919.
- [19] Pham H. (2009): *Continuous time stochastic control and optimization with financial applications*, Springer, Series SMAP.
- [20] Robertson S. and H. Xing (2013): “Large time behavior of solutions to semi-linear equations with quadratic growth in the gradient”, arXiv:1307.5343
- [21] Stettner L. (2004): “Duality and risk-sensitive portfolio optimization”, *Mathematics of Finance*, eds G. Yin and Q. Zhang, Contemp. Maths. **351**, 333-347.
- [22] Stutzer M. (2003): “Portfolio choice with endogenous utility: A large deviations approach”, *J. Econometrics*, **116**, 365-386.