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1 A note on Turing degree spectra of minimal
2 subshifts

3 Mike Hochman and Pascal Vanier

4 August 26, 2014

5 **Abstract**

6 Subshifts are shift invariant closed subsets of $\Sigma^{\mathbb{Z}^d}$, minimal subshifts
7 are subshifts in which all points contain the same patterns. It has been
8 proved by Jeandel and Vanier that the Turing degree spectra of non-
9 periodic minimal subshifts always contain the cone of Turing degrees
10 above any of its degree. It was however not known whether each minimal
11 subshift's spectrum was formed of exactly one cone or not. We construct
12 inductively a minimal subshift whose spectrum consists of an uncountable
13 number of cones with disjoint base.

14 A \mathbb{Z}^d -subshift is a closed shift invariant subset of $\Sigma^{\mathbb{Z}^d}$. Subshifts may be
15 seen as sets of colorings of \mathbb{Z}^d , with a finite number of colors, avoiding some
16 set of forbidden patterns. Minimal subshifts are subshifts containing no proper
17 subshift, or equivalently subshifts in which all configurations have the same
18 patterns. They are fundamental in the sense that all subshifts contain at least
19 a minimal subshift.

20 Degrees of unsolvability of subshifts have now been studied for a few years,
21 Cenzer, Dashti, and King [CDK08] and Cenzer, Dashti, Toska, and Wyman
22 [CDTW10; CDTW12] studied computability of one dimensional subshifts and
23 proved some results about their Turing degree spectra: the *Turing degree spec-*
24 *trum* of a subshift is the set of Turing degrees of its points, see Kent and Lewis
25 [KL10]. Simpson [Sim11], building on the work of Hanf [Han74] and Myers
26 [Mye74], noticed that the Medvedev and Muchnik degrees of subshifts of finite
27 type (SFTs) are the same as the Medvedev degrees of Π_1^0 classes: Π_1^0 classes
28 are the subsets of $\{0, 1\}^{\mathbb{N}}$ for which there exists a Turing machine halting only
29 on oracles not in the subset.

30 Subsequently Jeandel and Vanier [JV13] focused on Turing degree spectra
31 of different classes of multidimensional subshifts: SFTs, sofic and effective sub-
32 shifts. They proved in particular that the Turing degree spectra of SFTs are
33 almost the same as the spectra of Π_1^0 classes: adding a computable point to the
34 spectrum of any Π_1^0 class, one can construct an SFT with this spectrum. In
35 order to prove that one cannot get a stronger statement, they proved that the
36 spectrum of any non-periodic minimal subshift contains the cone above any of
37 its degrees:

38 **Theorem 1** (Jeandel and Vanier [JV13]). *Let X be a minimal non-finite sub-*
 39 *shift (i.e. non-periodic in at least one direction). For any point $x \in X$ and any*
 40 *degree $\mathbf{d} \geq_T \deg_T x$, there exists a point $y \in X$ such that $\mathbf{d} = \deg_T y$.*

41 Minimal subshifts are in particular interesting since any subshift contains
 42 a minimal subshift [Bir12]. Here we answer the followup question of whether
 43 a minimal subshift always corresponds to a single cone or if there exists one
 44 with at least two cones of disjoint base. It is quite easy to prove the following
 45 theorem:

46 **Theorem 2.** *For any Turing degree $\deg_T d$, there exists a minimal subshift X*
 47 *such that the set of Turing degrees of the points of X is a cone of base $\deg_T d$.*

48 For instance the spectrum of a Sturmian subshift [MH40] with an irrational
 49 angle is the cone whose base is the degree of the angle of the rotation. The
 50 theorem can also be seen as a corollary of Miller’s proof [Mil12][Proposition 3.1]
 51 of a result on Medvedev degrees.

52 In this paper, we prove the following result:

53 **Theorem 3.** *There exist a minimal subshift $X \subset \{0, 1\}^{\mathbb{Z}}$ and points $x_z \in X$*
 54 *with $z \in \{0, 1\}^{\mathbb{N}}$ such that for any $z \neq z' \in \{0, 1\}^{\mathbb{N}}$, $\deg_T x_z$ and $\deg_T x_{z'}$ are*
 55 *incomparable and such that there exists no point $y \in X$ with $\deg_T y \leq_T \deg_T x_z$*
 56 *and $\deg_T y \leq_T \deg_T x_{z'}$.*

57 The subshift constructed in this proof is not effective and cannot be “effectivized”,
 58 since minimal effective subshifts always contain a computable point and thus
 59 their spectra are the whole set of Turing degrees when they are non-periodic.

60 1 Preliminary definitions

61 We give here some standard definitions and facts about subshifts, one may
 62 consult the book of Lind and Marcus [LM95] for more details.

63 Let Σ be a finite alphabet, its elements are called *symbols*, the d -dimensional
 64 full shift on Σ is the set $\Sigma^{\mathbb{Z}^d}$ of all maps (colorings) from \mathbb{Z}^d to the Σ (the colors).
 65 For $v \in \mathbb{Z}^d$, the shift functions $\sigma_v : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$, are defined locally by $\sigma_v(c_x) =$
 66 c_{x+v} . The full shift equipped with the distance $d(x, y) = 2^{-\min\{\|v\| \mid v \in \mathbb{Z}^d, x_v \neq y_v\}}$
 67 is a compact metric space on which the shift functions act as homeomorphisms.
 68 An element of $\Sigma^{\mathbb{Z}^d}$ is called a *configuration*.

69 Every closed shift-invariant (invariant by application of any σ_v) subset X
 70 of $\Sigma^{\mathbb{Z}^d}$ is called a *subshift*. An element of a subshift is called a *point* of this
 71 subshift.

72 Alternatively, subshifts can be defined with the help of forbidden patterns.
 73 A *pattern* is a function $p : P \rightarrow \Sigma$, where P , the *support*, is a finite subset
 74 of \mathbb{Z}^d . We say that a configuration x contains a pattern $p : P \rightarrow \Sigma$, or
 75 equivalently that the pattern p appears in x , if there exists $z \in \mathbb{Z}^d$ such that
 76 $x|_{z+P} = p$.

77 Let \mathcal{F} be a collection of *forbidden* patterns, the subset $X_{\mathcal{F}}$ of $\Sigma^{\mathbb{Z}^d}$ containing
 78 the configurations having nowhere a pattern of \mathcal{F} . More formally, $X_{\mathcal{F}}$ is defined
 79 by

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} \mid \forall z \in \mathbb{Z}^d, \forall p \in \mathcal{F}, x_{|z+p} \neq p \right\}.$$

80 In particular, a subshift is said to be a *subshift of finite type* (SFT) when it
 81 can be defined by a collection of forbidden patterns that is finite. Similarly, an
 82 *effective subshift* is a subshift which can be defined by a recursively enumerable
 83 collection of forbidden patterns.

84 **Definition 1** (Minimal subshift). *A subshift X is called minimal if it verifies*
 85 *one of the following equivalent conditions:*

- 86 • *There is no subshift Y such that $Y \subsetneq X$.*
- 87 • *All the points of X contain the same patterns.*
- 88 • *It is the closure of the orbit of any of its points.*

89 We will use the two latter conditions.

90 For $x, y \in \{0, 1\}^{\mathbb{N}}$, we say that $x \leq_T y$ if there exists a Turing machine M
 91 such that M with oracle y computes x . Of course $x \equiv_T y$ when we have both
 92 $x \leq_T y$ and $y \leq_T x$. The Turing degree of x is the equivalence class of x with
 93 respect to \equiv_T . More details can be found in Rogers [Rog67].

94 For a possibly infinite word $w = w_0 \dots w_n$, we denote $w_{[i,j]} = w_i \dots w_j$.

95 2 Minimal subshifts with several cones

96 **Lemma 2.1.** *Let $\phi_1, \phi_2 : \{0, 1\}^{\mathbb{N}} \rightarrow \{0, 1\}^{\mathbb{N}}$ be two recursive operators and*
 97 *$L = \{w_1, w_2\} \subseteq \{0, 1\}^*$ two different words. There exists two words $w'_1, w'_2 \in L^*$*
 98 *starting respectively with w_1 and w_2 such that we have one of the following:*

- 99 (a) *either $\phi_1(w'_1 x) \neq \phi_2(w'_2 y)$ for any two sequences $x, y \in \{0, 1\}^{\mathbb{N}}$,*
- 100 (b) *or $\phi_1(w'_1 x)$ and $\phi_2(w'_2 y)$ are recursive for all sequences $x, y \in \{0, 1\}^{\mathbb{N}}$,*

101 *Proof.* Let M_1, M_2 be the Turing machines computing the functions $x \mapsto \phi_1(w_1 x), x \mapsto$
 102 $\phi_2(w_2 x)$ respectively. It is quite clear that:

- 103 • either there exists some sequences $x, y \in L^*$ such that $M_1(x)$'s output
 104 differs from $M_2(y)$'s output at some step,
- 105 • or the outputs of both machines M_1, M_2 do not depend on their inputs, in
 106 this latter case, we are in case **b**, since in particular recursive inputs have
 107 recursive images.

108 In the former case, there exist prefixes w'_1 and w'_2 of x and y such that the partial
 109 outputs of M_1 once it has read w'_1 already differs from the partial output of M_2
 110 once it has read w'_2 . \square

111 **Theorem 4.** *There exists a minimal subshift $X \subseteq \{0,1\}^{\mathbb{N}}$ whose set of Turing*
 112 *degrees contains 2^{\aleph_0} disjoint cones of Turing degrees.*

113 Note that the following proof is in no way effective. As a matter of fact, all
 114 effective minimal subshifts contain some recursive point [BJ10], and their set of
 115 Turing degrees is the cone of *all* degrees.

116 *Proof.* We construct a sequence of SFTs $(X_i)_{i \in \mathbb{N}}$ such that $X_{i+1} \subseteq X_i$ and such
 117 that the limit $X = \bigcap_{i \in \mathbb{N}} X_i$ is minimal. In the process of constructing the X_i ,
 118 which will be formed of concatenations of allowed words x_1^i, \dots, x_k^i , we ensure
 119 that no extensions of two distinct words may compute an identical sequence
 120 with any of the first i Turing machines. At the same time, we make sure that
 121 all allowed words of level $i + 1$ contain all words of level i , thus enforcing the
 122 minimality of the limit X . We also have to avoid that the limit X contains a
 123 computable point.

124 Let $(\mathcal{M}_i)_{i \in \mathbb{N}}$ be an enumeration of all minimal subshifts containing a recur-
 125 sive point. Such an enumeration exists since there are countably many compu-
 126 table sequences and minimal subshifts are the closure of any of their points.
 127 We do not know if such an enumeration is effective or not, but we do not need
 128 it to be for our purposes. We will also need an enumeration $(\phi_i)_{i \in \mathbb{N}}$ of the
 129 recursive operators from $\{0,1\}^{\mathbb{N}}$ to $\{0,1\}^{\mathbb{N}}$.

130 Now let us define the sequence of SFTs $(X_i)_{i \in \mathbb{N}}$. Each of these subshifts will
 131 be the shift invariant closure of the biinfinite words formed by concatenations
 132 of words of some language L_i which here will be finite languages. We define
 133 $X_0 = \{0,1\}^{\mathbb{N}}$ that is to say X_0 is generated by $L_0 = \{w_0 = 0, w_1 = 1\}$. Let
 134 us now give the induction step. At each step, L_{i+1} will contain 2^{i+1} words
 135 $w_{0\dots 0}, \dots, w_{1\dots 1}$, the indices being binary words of length $i + 1$, which will verify
 136 the following conditions:

- 137 1. The words w_{b_0}, w_{b_1} of L_{i+1} start with the word w_b of L_i .
- 138 2. The words w_b with $b \in \{0,1\}^{i+1}$ of L_{i+1} each contain all the words of $w_{b'}$
 139 with $b' \in \{0,1\}^i$ of L_i .
- 140 3. For any two words $w_b, w_{b'}$ of L_{i+1} and for all $j, j' \leq i$:
 - 141 • Either for all $x, y \in L_i^\omega$, $\phi_j(w_b x) \neq \phi_{j'}(w_{b'} y)$,
 - 142 • Or for all $x, y \in L_i^\omega$, $\phi_j(w_b x), \phi_{j'}(w_{b'} y)$ are computable.
- 143 4. The words w_{b_0}, w_{b_1} do not appear in any configuration of \mathcal{M}_j , for all $j \leq i$.

144 Conditions 1 and 2 are easy to ensure: just make w_{ba} start with $w' =$
 145 $w_b w_{0\dots 0} \dots w_{1\dots 1}$. We then use Lemma 2.1 to extend w' into a word w'' verifying
 146 condition 3, this is done several times, once for every quadruple w, w', j, j' . And
 147 finally, since X_i is not minimal, we can extend w'' so that it contains a pattern
 148 appearing in none of the \mathcal{M}_j 's for $j \geq i$, to obtain condition . Now we can
 149 extend w'' with two different words thus obtaining w_{b_0} and w_{b_1} .

150 Now let's check that this leads to the desired result:

- 151 • $X = \bigcap X_i$ is a countable intersection of compact shift-invariant spaces, it
152 is compact and shift-invariant, thus a subshift.
- 153 • Any pattern p appearing in some point of X is contained in a pattern
154 w_b , with $b \in \{0, 1\}^i$ for some i , by construction (condition 2), all $w_{b'}$
155 with $b \in \{0, 1\}^{i+1}$ contain w_b . Therefore, all points of X , since they are
156 contained in X_{i+1} , contain w_b and hence p . So X is minimal.
- 157 • For all $z \in \{0, 1\}^{\mathbb{N}}$, define the points $x_z = \lim_{i \rightarrow \infty} x_{z[0, i]}$, they are in X
158 because they belong to each X_i . Condition 3 ensures that if two of them
159 compute the same sequence $y \in \{0, 1\}^{\mathbb{N}}$, then this sequence is computable.
160 Therefore they can only belong to the cone with base $\mathbf{0}$. And condition 2
161 ensures that no point of X belongs to a subshift containing a computable
162 point.

163 □

164 It is quite straightforward to transform this proof in order to get a subshift
165 on $\{0, 1\}^{\mathbb{Z}}$ instead of $\{0, 1\}^{\mathbb{N}}$ and obtain the following corollary:

166 **Corollary 5.** *For any dimension d , there exists a minimal subshift $X \subseteq \{0, 1\}^{\mathbb{Z}^d}$*
167 *whose set of Turing degrees contains 2^{\aleph_0} disjoint cones of Turing degrees.*

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