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Exponential decay for the damped wave equation in unbounded domains

Romain JOLY*

Abstract

We discuss the exponential decay of the semigroup generated by the damped wave equation in an unbounded domain. A typical result is the following. Let $d \geq 1$ and let $\gamma \in L^\infty(\mathbb{R}^d)$ be a periodic non-negative damping such that there exists a length $L > 0$ such that any segment of \mathbb{R}^d of length L intersects the set $\{x \in \mathbb{R}^d, \gamma(x) > 0\}$. Then, there exist M and $\lambda > 0$ such that any solution of

$$\partial_{tt}^2 u + \gamma(x)\partial_t u = (\Delta - Id)u \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

satisfies

$$\|u(t)\|_{H^1(\mathbb{R}^d)} + \|\partial_t u(t)\|_{L^2(\mathbb{R}^d)} \leq M e^{-\lambda t} \left(\|u(0)\|_{H^1(\mathbb{R}^d)} + \|\partial_t u(0)\|_{L^2(\mathbb{R}^d)} \right) .$$

KEY WORDS: damped wave equation, exponential decay, uniform stabilisation, variable damping, unbounded domains.

AMS SUBJECT CLASSIFICATION: 35Q99, 93D15, 93B05, 35B41.

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1 Introduction

The damped wave equation:

In this article, we consider the linear damped wave equation in \mathbb{R}^d , with $d \geq 1$:

$$\begin{cases} \partial_{tt}^2 u(x, t) + \gamma(x) \partial_t u(x, t) = \operatorname{div}(K(x) \nabla u(x, t)) - u(x, t) & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \end{cases} \quad (1.1)$$

where $K \in \mathcal{C}^\infty(\mathbb{R}^d, \mathcal{M}_d(\mathbb{R}))$ is a smooth family of real symmetric matrices, which are uniformly positive in the sense that there exist two positive constants K_{inf} and K_{sup} such that

$$\forall \xi \in \mathbb{R}^d, \quad K_{sup} |\xi|^2 \geq \xi^\top \cdot K(x) \cdot \xi \geq K_{inf} |\xi|^2. \quad (1.2)$$

The damping coefficient $\gamma \in L^\infty(\mathbb{R}^d)$ is assumed to be a bounded and non-negative function. We set $X = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and

$$A = \begin{pmatrix} 0 & Id \\ (\operatorname{div}(K(x) \nabla) - Id) & -\gamma(x) \end{pmatrix} \quad D(A) = H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d). \quad (1.3)$$

We equipped $H^1(\mathbb{R}^d)$ with the scalar product

$$\langle u | v \rangle_{H^1} = \int_{\mathbb{R}^d} (\nabla u(x))^\top \cdot K(x) \cdot (\overline{\nabla v(x)}) + u(x) \overline{v(x)} dx. \quad (1.4)$$

Obviously, this scalar product is equivalent to the classical one and direct computations show that it satisfies

$$\langle (\operatorname{div}(K(x) \nabla) - Id) u | v \rangle_{L^2} = -\langle u | v \rangle_{H^1} \quad \text{and} \quad \operatorname{Re}(\langle AU | U \rangle_X) = - \int \gamma(x) |v(x)|^2 dx$$

for any $U = (u, v) \in D(A)$. Then, one easily checks that A is a dissipative operator and therefore generates a semigroup e^{At} on X .

Exponential decay and Hamiltonian flow:

The purpose of this paper is to investigate the exponential decay of the semigroup associated to (1.1): we wonder if there exist M and $\lambda > 0$ such that

$$\forall t \geq 0, \quad \| \| e^{At} \| \|_{\mathcal{L}(X)} \leq M e^{-\lambda t}. \quad (1.5)$$

For the damped wave equation in a bounded domain, it is well known that the exponential decay is almost equivalent to the fact that all the trajectories of the Hamiltonian flow intersect the support of the damping (see [16], [3] and [4]). More precisely, to the Laplacian operator with variable coefficients $\operatorname{div}(K(x) \nabla)$, we associate the symbol $g(x, \xi) = \xi^\top \cdot K(x) \cdot \xi$ and the Hamiltonian flow $\varphi_t(x_0, \xi_0) = (x(t), \xi(t))$ defined on \mathbb{R}^{2d} by

$$\varphi_0(x_0, \xi_0) = (x_0, \xi_0) \quad \text{and} \quad \partial_t \varphi_t(x, \xi) = (\partial_\xi g(x(t), \xi(t)), -\partial_x g(x(t), \xi(t))). \quad (1.6)$$

We introduce the mean value of the damping along a ray a length T :

$$\langle \gamma \rangle_T(x, \xi) = \frac{1}{T} \int_0^T \gamma(\varphi_t(x, \xi)) dt \quad (1.7)$$

where we use the obvious notation $\gamma(x, \xi) := \gamma(x)$. We also introduce the set Σ of rays of speed one, that is

$$\Sigma = \{(x, \xi) \in \mathbb{R}^{2d}, \xi^\top K(x) \xi = 1\}. \quad (1.8)$$

Some previous works:

If Ω is a bounded manifold, the uniform positivity of $\langle \gamma \rangle_T(x, \xi)$ in Σ for some $T > 0$ implies that the exponential decay (1.5) holds, as shown in the now famous articles [16], [3] and [4] of Bardos, Lebeau, Rauch and Taylor. The assumption that there exists $T > 0$ such that $\langle \gamma \rangle_T(x, \xi) > 0$ in Σ is called the *geometric control condition*. The article [12] underlines in addition the importance of the value of $\min_{(x, \xi) \in \Sigma} \langle \gamma \rangle_T(x, \xi)$ in order to control the decay of the high frequencies.

In the case of an unbounded manifold, two situations have been investigated. First, some authors have considered the free wave equation (1.1) in an exterior domain (with $\gamma \equiv 0$ or $\gamma > 0$ only on a compact subset the exterior domain). They have shown that the local energy decays to zero in the sense that, under suitable assumptions, the energy of any solution escapes away from any compact set, see [11], [15] and [2] and the references therein. Secondly, several works have studied the damped wave equation in an unbounded manifold and with a non-linearity, but assuming that the damping satisfies $\gamma(x) \geq \alpha > 0$ outside a compact set, see [19], [5] and [9].

Considering these previous works, it appears that one natural case has not been studied: the exponential decay of the semigroup e^{At} generated by the damped wave equation on a whole unbounded manifold, with the geometric control condition only, that is without assuming that $\gamma \geq \alpha > 0$ outside a compact set. To our knowledge, this case is surprisingly missing in the literature. The purpose of this article is to investigate this natural problem.

Main results:

We denote by $\mathcal{C}_b^k(\mathbb{R}^d)$ the set of functions in $\mathcal{C}^k(\mathbb{R}^d)$ which are bounded, as well as their k first derivatives. If $k = \infty$, the bound is not assumed to be uniform with respect to the derivatives. We recall that $\langle \tilde{\gamma} \rangle_T$ and Σ have been defined in (1.7) and (1.8). For $k \in \mathbb{R}^d$ and $n \in \mathbb{Z}^d$, we will use the notation $nk = (n_1 k_1, n_2 k_2, \dots)$. Our main result is as follows.

Theorem 1.1. *We assume that the non-negative damping $\gamma \in L^\infty(\mathbb{R}^d)$ and the metric $K(x) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathcal{M}_d(\mathbb{R}))$ satisfy:*

i) Geometric control condition for a regularized damping:

There exists a damping $\tilde{\gamma} \in \mathcal{C}_b^1(\mathbb{R}^d, \mathbb{R})$ satisfying $\gamma \geq \tilde{\gamma} \geq 0$ and such that there exist $T > 0$ and $\alpha > 0$ such that, for all $(x, \xi) \in \Sigma$, we have $\langle \tilde{\gamma} \rangle_T(x, \xi) \geq \alpha > 0$.

ii) A quasi-periodic framework:

The metric $K(x)$ belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d, \mathcal{M}_d(\mathbb{R}))$ and admits the splitting $K(x) = K_{per}(x) + K_{comp}(x)$ with K_{per} being a smooth family of real symmetric matrices satisfying (1.2) and K_{comp} satisfying $|K_{comp}(x)| + |D_x K_{comp}| \rightarrow 0$ when $|x| \rightarrow \infty$. Moreover, there exist $k \in \mathbb{R}^d$ with $k_i \neq 0$ for all $i = 1, \dots, d$ such that $K_{per}(x)$ is k -periodic and there exist a non-empty open set $\omega \subset \mathbb{R}^d$ and $\beta > 0$ such that $\gamma(x + nk) \geq \beta > 0$, for all $n \in \mathbb{Z}^d$ and $x \in \omega$.

Then, the semigroup generated by the damped wave equation (1.1) is exponentially decreasing that is that there exist M and $\lambda > 0$ such that

$$\forall t \geq 0, \quad \|e^{At}\|_{\mathcal{L}(X)} \leq M e^{-\lambda t}. \quad (1.9)$$

In the one-dimensional case, Hypothesis ii) of Theorem 1.1 is not necessary.

Theorem 1.2. *We consider the one-dimensional case $d = 1$. If Hypothesis i) of Theorem 1.1 holds, then the semigroup generated by the damped wave equation (1.1) is exponentially decreasing.*

Remarks:

- The simplest applications of Theorem 1.1 are the periodic frameworks satisfying the geometric control condition, see for example Figure 1.a). To our knowledge, the exponential decay of the semigroup was not known in this simple case. Notice that one cannot directly use the framework of the torus since the initial data (u_0, u_1) are not periodic.
- Of course, the ideas of the proof of Theorem 1.1 may apply to other situations. For example, if we consider an unbounded manifold without boundary as a cylinder instead of \mathbb{R}^d , then Theorem 1.1 will also hold with the obvious modifications of its statement.
- Theorem 1.2 is stronger than Theorem 1.1 applied to the one-dimensional setting, see Figure 1.c). The proof of Theorem 1.2 is given in Section 5.
- The exponential decay of the linear semigroup has important applications in the control theory and the study of dynamics for the wave equations. Some new results are obtained as corollaries of Theorems 1.1 and 1.2 as explained in Section 6.
- Using the finite speed of propagation of the energy in the damped wave equation, one could hope to obtain Theorem 1.1 by a basic argument. However, this seems not so

simple. Indeed, one could try to decompose the initial data into a sum of functions of bounded support, but the problem is that one cannot split a H^1 -function into a sum of compactly supported H^1 -functions without increasing the H^1 -energy. Another idea could be to restrict the initial data into a sufficiently large ball $B(0, R)$ and to observe it during a time T in the ball $B(0, R + T)$, which is a compact domain. However, the relevant time T , during which one has to wait to observe the exponential decay, may depend on the domain $B(0, R + T)$, itself depending on the time T . . . That is why our proof of Theorem 1.1 goes back to the fundamental arguments of the classical compact case: control the high frequencies by using semiclassical calculus and Hypothesis i) (see Section 3) and control the low frequencies by an analysis of the spectrum of a wave operator and by using Hypothesis ii) (see Section 4). The important of the controls of both high and low frequencies is made clear in [12].

- One of the novelties appearing in the statement of Theorems 1.1 and 1.2 is the requirement that γ can be regularized into a smaller C_b^1 -damping $\tilde{\gamma}$ satisfying the geometrical conditions. The fact that the derivative of $\tilde{\gamma}$ can be taken uniformly bounded is important in order to apply the pseudo-differential calculus or to use the multipliers techniques (see Sections 3 and 5). Of course, in the usual compact case, this assumption was automatically satisfied. In Figures 1.b) and 1.d), we show examples where all the Hypotheses of our main results apply, if one neglects the regularisation hypothesis. In these cases, it could natural to expect the exponential decay of the semigroup, but this is still an open problem. Notice that the simple requirement that γ belongs to L^∞ is not sufficient to define properly the mean value $\langle \gamma \rangle_T(x, \xi)$ everywhere. This could be a hint that the regularisation assumption is not just a technical one.

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2 Proof of Theorem 1.1

In this section, we outline the proof of our main theorem. The real technical parts of its proof will be detailed in Sections 3 and 4.

There exist several ways to obtain the exponential decay (1.9) of the semigroup e^{At} . The most classical one is to argue by contradiction to establish the observation inequality $E(v(0)) \leq C \int_0^T \gamma |\partial_t v|^2 dt$ for some $T > 0$ and any solution v of the free wave equation (see for example [7] for the relation between this observation estimate and the exponential decay of the damped semigroup). A less usual method consists in uniformly estimating the resolvent $(A - \lambda Id)^{-1}$ on the imaginary axis (see for example chapter 5 of [20]). We use here this last method as a direct corollary of the result of [8] stated below. With this method, the control of the low frequencies seems less tricky than the usual arguments since

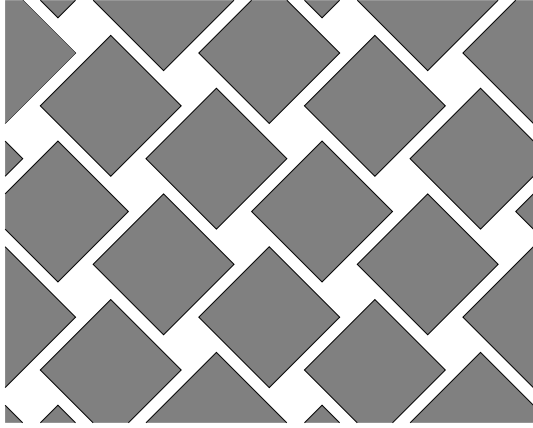


Figure 1.a): A periodic two-dimensional example for which Theorem 1.1 holds: the semigroup generated by the corresponding damped wave equation is decaying exponentially fast.

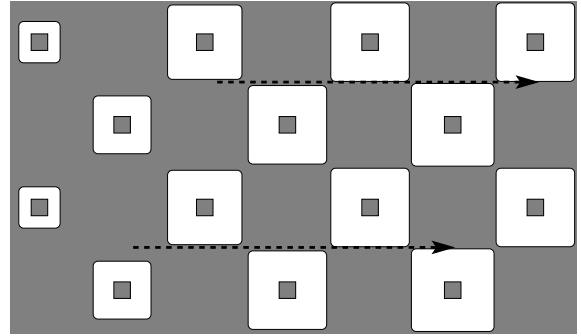


Figure 1.b): This two-dimensional quasi-periodic situation satisfies all the hypothesis of Theorem 1.1 if one does not care about the regularisation condition in Hypothesis i), since the minimum of the mean value $\langle \gamma \rangle_T$ in Σ is positive for T large enough. However, Theorem 1.1 fails to apply because for any \mathcal{C}_b^1 damping $\tilde{\gamma}$ satisfying $\tilde{\gamma} \leq \gamma$, the mean value $\langle \tilde{\gamma} \rangle$ must go to zero along the represented ray.

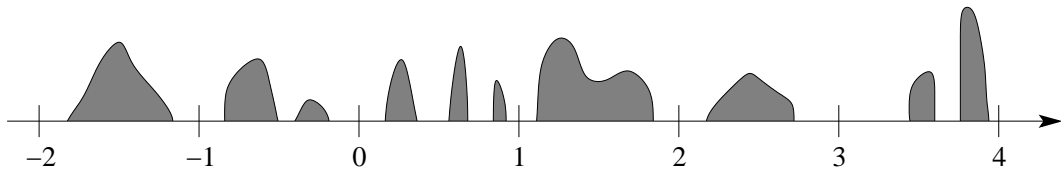


Figure 1.c): One can easily construct an example where Theorem 1.2 holds and where Hypothesis ii) of Theorem 1.1 fails. It is sufficient to place a uniformly bounded damping of integral larger than one in any interval $[n, n + 1]$ and to be careful to preclude any periodic pattern.

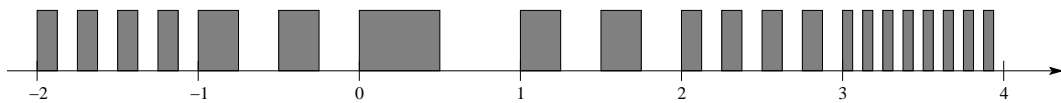


Figure 1.d): An one-dimensional example where the mean value of the damping $\langle \gamma \rangle_1(x, \xi)$ is uniformly positive in Σ , but where Theorem 1.2 does not apply since there is no regularisation $\tilde{\gamma} \in \mathcal{C}_b^1$ with mean value $\langle \tilde{\gamma} \rangle_T$ uniformly positive for some T and with a derivative uniformly bounded in \mathbb{R} .

Figure 1: *Discussion on some examples of damping. In the two-dimensional situations, the damping is equal to 1 on the grey regions and equal to 0 in the other regions. In the one-dimensional situations, the figure roughly represents the graph of the damping. In both cases, the metric is assumed to be flat, i.e. $K(x) = Id$.*

one simply has to estimate $(A - i\mu Id)^{-1}$ for a given $\mu \in \mathbb{R}$.

• *First step: a characterisation of exponential decay.*

We use here the characterisation given by Theorem 3 of [8].

Theorem 2.1. F.L. Huang (1985).

Let e^{At} be a C^0 - semigroup in a Hilbert space X and assume that there exists a positive constant $M > 0$ such that $\|e^{At}\| \leq M$ for all $t \geq 0$. Then e^{At} is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and

$$\sup_{\mu \in \mathbb{R}} \|(A - i\mu Id)^{-1}\|_{\mathcal{L}(X)} < +\infty. \quad (2.1)$$

Since the linear operator A associated to the damped wave equation is dissipative, we have $\|e^{At}\| \leq 1$ for all $t \geq 0$. To prove Theorem 1.1, it remains to show that (2.1) holds. We argue by contradiction and assume that there exist two sequences $(U_n) = (u_n, v_n) \subset D(A) = H^2(\mathbb{R}^d) \times H^1(\mathbb{R}^d)$ and $(\mu_n) \subset \mathbb{R}$ such that

$$\|U_n\|_X^2 = \|u_n\|_{H^1}^2 + \|v_n\|_{L^2}^2 = 1 \quad \text{and} \quad (A - i\mu_n)U_n \xrightarrow[n \rightarrow +\infty]{} 0 \quad \text{in } X. \quad (2.2)$$

Notice that, here, u_n and v_n are complex valued functions. Also notice that, without loss of generality, by a linear change of variables, one may also assume that $k = (1, 1, \dots, 1)$, that is that the periodic cell appearing in Hypothesis ii) of Theorem 1.1 is the hypercube $[0, 1]^d$.

• *Second step: replacing γ by a smooth damping.*

We recall that $H^1(\mathbb{R}^d)$ is equipped with the convenient scalar product (1.4). Let us denote the operator $\text{div}(K(x)\nabla)$ by Δ_K . We have

$$(A - i\mu_n Id)U_n = \begin{pmatrix} v_n - i\mu_n u_n \\ (\Delta_K - Id)u_n - \gamma(x)v_n - i\mu_n v_n \end{pmatrix}$$

and

$$\text{Re}(\langle (A - i\mu_n)U_n | U_n \rangle_X) = - \int \gamma(x) |v_n(x)|^2 dx.$$

Thus, (2.2) implies that $\int \gamma(x) |v_n(x)|^2 dx$ goes to zero. Therefore, we can replace γ by the C_b^1 damping $\tilde{\gamma}$ of Hypothesis i) of Theorem 1.1 without changing (2.2). We claim that we can even assume that $\gamma \in C_b^\infty(\mathbb{R}^d)$. Indeed, choose $\theta = \max(0, \tilde{\gamma} - \varepsilon)$. For a small enough $\varepsilon > 0$, the damping θ still satisfies that its mean value $\langle \theta \rangle_T(x, \xi)$ is uniformly bounded away from 0. Moreover, since the derivative of $\tilde{\gamma}$ is bounded, the support of θ stays at a uniform distance $\delta > 0$ of the set where $\tilde{\gamma}$ vanishes. Now, mollify θ into $\theta * \rho_\delta$ where ρ_δ is a C^∞ regularisation kernel with support in $B(0, \delta)$. We obtain a smooth damping γ with a support included in the one of $\tilde{\gamma}$. Thus, one can use this new damping without

changing (2.2). Moreover, this new damping γ belongs to $\mathcal{C}_b^\infty(\mathbb{R}^d)$, which ensures that the multiplication by γ is a pseudo-differential operator of order 0.

• *Third step: separation between high and low frequencies.*

We now work with $k = (1, 1, \dots, 1)$ and a smooth damping γ with bounded derivatives. To obtain a contradiction from (2.2), we consider two cases.

- **High frequencies:** assume that $|\mu_n|$ goes to $+\infty$. Since A is a real operator, by symmetry, we can assume that $\mu_n > 0$ and we set $h_n = 1/\mu_n$. We have to show that one cannot have $\|U_n\|_X = 1$ and $(A - i/h_n)U_n \rightarrow 0$. This will be shown in Section 3 by semiclassical pseudo-differential arguments, using hypothesis i) of Theorem 1.1.
- **Low frequencies:** assume that (μ_n) has a bounded subsequence. Then, up to extracting a subsequence, one can assume that (μ_n) converges to a real number μ . Then (2.2) is equivalent to have a sequence (U_n) with $\|U_n\|_X = 1$ and $(A - i\mu)U_n \rightarrow 0$. In Section 4, we will show that this is not possible due to Hypothesis ii) of Theorem 1.1, by using Floquet-Bloch theory.

Since Sections 3 and 4 provide a contradiction in both cases, Theorem 2.1 implies that our main result Theorem 1.1 is proved.

3 High frequencies

The purpose of this section is to obtain a contradiction from the existence of sequences (U_n) with $\|U_n\|_X = 1$ and (h_n) with $h_n \rightarrow 0$ satisfying $(A - i/h_n)U_n \rightarrow 0$. To simplify the notations, we may forget the index n for the remaining part of this section and set $U_n = U_h = (u_h, v_h)$. We have

$$\begin{cases} v_h - \frac{i}{h}u_h = o_{H^1}(1) \\ (\Delta_K - id)u_h - \gamma(x)v_h - \frac{i}{h}v_h = o_{L^2}(1) \end{cases}$$

and thus

$$\begin{cases} v_h - \frac{i}{h}u_h = o_{H^1}(1) \\ h^2(\Delta_K - Id)u_h - ih\gamma(x)u_h + u_h = o_{L^2}(h^2) + o_{H^1}(h) \end{cases} \quad (3.1)$$

To obtain a contradiction between (3.1) and Hypothesis i) of Theorem 1.1, we will use the semiclassical microlocal analysis and follow the ideas of the chapter 5 of [20]. Notice that the usual way to deal with high frequencies is to use semiclassical defect measures (see for example [20]). However, this is not possible in our case since we work in an unbounded domain and the semiclassical defect measure will only tell us what happens in compact subsets.

Lemma 3.1. *Assume that the operator $P_h = h^2(\Delta_K - Id) - ih\gamma(x) + Id$ has a resolvent in $L^2(\mathbb{R}^d)$ satisfying*

$$\|(P_h)^{-1}f\|_{L^2} \leq \frac{C}{h}\|f\|_{L^2} . \quad (3.2)$$

Then P_h has a resolvent in $H^1(\mathbb{R}^d)$ also satisfying

$$\|(P_h)^{-1}f\|_{H^1} \leq \frac{C}{h}\|f\|_{H^1} .$$

Proof: We argue by contradiction. Assume that there exists a sequence (u_h) with $\|u_h\|_{H^1} = 1$ and $P_h u_h = o_{H^1}(h)$. Multiplying by \bar{u}_h and integrating, we get that

$$-h^2\|u_h\|_{H^1}^2 - ih \int_{\mathbb{R}^d} \gamma(x)|u_h|^2 + \|u_h\|_{L^2}^2 = o(h)\|u_h\|_{L^2} .$$

Taking the real part and solving the equation in $\|u_h\|_{L^2}$, we get

$$\|u_h\|_{L^2} = \frac{1}{2} \left(o(h) + \sqrt{o(h^2) + 4h^2\|u_h\|_{H^1}^2} \right) \sim h .$$

We introduce the operator $\nabla_K = (-\Delta_K + Id)^{1/2}$. It is the particular case $h = 1$ of the semi-classical operator $(-h^2\Delta_K + Id)^{1/2}$, which has for principal symbol $\sqrt{\xi^\top K(x)\xi + 1}$ (see Section A.1 in Appendix for a brief recall about pseudo-differential semiclassical calculus). Obviously, it commutes with any polynomial of Δ_K . Moreover, applying i) of Corollary A.2 of the Appendix, with $h = 1$ fixed, we get that the commutator $[\nabla_K, \gamma(x)\cdot]$ is bounded in $L^2(\mathbb{R}^d)$. Thus, since $u_h = \mathcal{O}_{L^2}(h)$,

$$\nabla_K(P_h u_h) = P_h(\nabla_K u_h) - ih[\nabla_K, \gamma(x)]u_h = P_h(\nabla_K u_h) + \mathcal{O}_{L^2}(h^2) .$$

Since $P_h u_h = o_{H^1}(h)$, we obtain that $P_h(\nabla_K u_h) = o_{L^2}(h)$. Using the assumption on the resolvent of P_h , we obtain that $\nabla_K u_h$ goes to 0 in $L^2(\mathbb{R}^d)$ when h goes to 0. However, $\|\nabla_K u_h\|_{L^2}$ is equivalent to $\|u_h\|_{H^1}$ and we obtain a contradiction with the assumption $\|u_h\|_{H^1} = 1$. \square

Proposition 3.2. *If the operator $P_h = h^2(\Delta_K - Id) - ih\gamma(x) + Id$ has a resolvent in $L^2(\mathbb{R}^d)$ satisfying (3.2), then (3.1) cannot hold.*

Proof: We argue by contradiction again. Assume that P_h satisfies (3.2) and assume that there exists $U_h = (u_h, v_h)$ with $\|U_h\|_X = 1$ such that (3.1) holds. As in the beginning of the proof of Lemma 3.1, multiplying the second equation of (3.1) by \bar{u}_h , integrating, taking the real part and solving the equation of second degree in $\|u_h\|_{L^2}$, we get that

$$\|u_h\|_{L^2} = \frac{1}{2} \left(o(h) + \sqrt{o(h^2) + 4h^2\|u_h\|_{H^1}^2} \right) .$$

Due to the first equation of (3.1) and since $\|U_h\|_X = 1$, we must have $\|u_h\|_{H^1} \sim 1/\sqrt{2}$, $\|u_h\|_{L^2} \sim h/\sqrt{2}$ and $\|v_h\|_{L^2} \sim 1/\sqrt{2}$.

We introduce $w_h = P_h^{-1}(f_h)$ where f_h is the term $o_{H^1}(h)$ in the second equation of (3.1). By assumption and by Lemma 3.1, we have that $w_h = o_{H^1}(1)$. By the same straightforward computation than the one just above, we also have $w_h = o_{L^2}(h)$. Then, $u_h - w_h$ solves $P_h(u_h - w_h) = o_{L^2}(h^2)$ and by assumption, we get that $u_h - w_h = o_{L^2}(h)$ and thus that $u_h = o_{L^2}(h)$. This is a contradiction with the fact that $\|u_h\|_{L^2} \sim h/\sqrt{2}$, which was proved above. \square

Due to Proposition 3.2, to obtain a contradiction from (3.1), it remains to show the L^2 -resolvent estimate (3.2). Obtaining this estimate is the central argument for controlling the high frequencies. Here, we will use pseudo-differential calculus and we will see the importance of Hypothesis i) of Theorem 1.1. The remaining part of this section is thus devoted to the proof of the following result.

Proposition 3.3. *The operator*

$$P_h = h^2(\Delta_K - Id) - ih\gamma(x) + Id$$

has a resolvent in L^2 satisfying

$$\|(P_h)^{-1}f\|_{L^2} \leq \frac{C}{h}\|f\|_{L^2} .$$

Proof: As usual, we argue by contradiction and assume that there exists a sequence (h_n) going to zero and functions $(u_n) \subset H^2(\mathbb{R}^d)$ such that $\|u_n\|_{L^2} = 1$ and $P_{h_n}u_n = o_{L^2}(h_n)$. Once again, we may forget the indices and assume that $\|u_h\|_{L^2} = 1$ and

$$h^2(\Delta_K - Id)u_h - ih\gamma(x)u_h + u_h = o_{L^2}(h) . \quad (3.3)$$

In what follows, we will use the notations and the results of the pseudo-differential semiclassical calculus recalled in Section A.1. Our proof follows the lines of Chapter 5 of [20], omitting the notion of defect measures, which is not convenient in the case of unbounded domains.

• *First step: u_h is concentrating along the radial speeds $\xi^\top K(x)\xi = 1/h^2$.*

First notice that the main part of P_h is $h^2\Delta_K + Id$ in the sense that $(h^2\Delta_K + Id)u_h = o_{L^2}(1)$. As explained in Appendix, up to an error term $\mathcal{O}_{L^2}(h^2)$, this main part is a pseudo-differential semiclassical operator $\text{Op}_h(-\xi^\top K(x)\xi + 1)$. Let $\chi(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}_+)$ be a smooth cutting function which is equal to one in a neighbourhood of the sphere $\Sigma = \{(x, \xi), \xi^\top K(x)\xi = 1\}$ and equal to 0 outside the annulus $1/2K_{max} \leq |\xi| \leq 2/K_{min}$. Also assume that χ and its derivatives are bounded, which implies that $\chi(x, \xi)$ is a symbol

of order 0. We claim that u_h is concentrating on the microlocal set $\{(x, \xi), \xi^\top K(x)\xi = 1/h^2\}$ in the sense that $\langle \text{Op}_h(1 - \chi(x, \xi))u_h | u_h \rangle_{L^2}$ goes to 0 when h goes to 0.

To prove this claim, we introduce another smooth cutting function θ which is equal to 1 in a neighbourhood of the sphere $\Sigma = \{(x, \xi), \xi^\top K(x)\xi = 1\}$ and equal to 0 in the support of $1 - \chi$. The symbol $a(x, \xi) = -\xi^\top K(x)\xi + 1 + i\theta$ is of order 2 and uniformly bounded away from 0. By iii) of Corollary A.2 in Appendix, the symbol $b(x, \xi) = \frac{1}{a(x, \xi)}$ is of order -2 and satisfies

$$\text{Op}_h(a) \circ \text{Op}_h(b) = Id + \mathcal{O}_{L^2}(h) \quad \text{and} \quad \text{Op}_h(b) \circ \text{Op}_h(a) = Id + \mathcal{O}_{L^2}(h) .$$

Thus,

$$\langle \text{Op}_h(1 - \chi)u_h | u_h \rangle_{L^2} = \langle \text{Op}_h(1 - \chi) \circ \text{Op}_h(b) \circ \text{Op}_h(a)u_h | u_h \rangle_{L^2} + \mathcal{O}(h) .$$

On the other hand, $\text{Op}_h(a) = \text{Op}_h(-\xi^\top K(x)\xi + 1) + i\text{Op}_h(\theta)$ and thus $\text{Op}_h(a)u_h = o_{L^2}(1) + i\text{Op}_h(\theta)u_h$. Since $1 - \chi$ and b are of order 0 or less, their corresponding operators are bounded in $L^2(\mathbb{R}^d)$, uniformly with respect to h and

$$\langle \text{Op}_h(1 - \chi)u_h | u_h \rangle_{L^2} = i\langle \text{Op}_h(1 - \chi)\text{Op}_h(b)\text{Op}_h(\theta)u_h | u_h \rangle_{L^2} + o(1) .$$

Now, it remains to apply Proposition A.1 in Appendix to see that, since $1 - \chi$ and θ have disjoint supports,

$$\text{Op}_h(1 - \chi)\text{Op}_h(b)\text{Op}_h(\theta)u_h = \text{Op}_h((1 - \chi)b\theta) + o_{L^2}(1) = o_{L^2}(1) .$$

This shows that

$$\langle \text{Op}_h(1 - \chi)u_h | u_h \rangle_{L^2} \xrightarrow{h \rightarrow 0} 0 .$$

• *Second step: using the geometric control condition of Hypothesis i) of Theorem 1.1.*

First notice that

$$P_h = \text{Op}_h(-\xi^\top K(x)\xi + 1) - ih\text{Op}_h(\gamma(x)) + \mathcal{O}_{L^2 \rightarrow L^2}(h^2)$$

and that we may assume that γ is smooth and bounded and so that it is a symbol of order 0 (see Section 2). Let $a(x, \xi)$ be a symbol of order 0. By i) and ii) of Corollary A.2 in Appendix, the commutator of P_h and $\text{Op}_h(a)$ is

$$[\text{Op}_h(a), P_h] = -ih\text{Op}_h(\{\xi^\top K(x)\xi, a(x, \xi)\}) + \mathcal{O}_{L^2 \rightarrow L^2}(h^2) .$$

On the other hand, since $P_h u_h = o_{L^2}(h)$,

$$\begin{aligned} \langle [\text{Op}_h(a), P_h]u_h | u_h \rangle_{L^2} &= \langle \text{Op}_h(a)P_h u_h | u_h \rangle_{L^2} - \langle P_h \text{Op}_h(a)u_h | u_h \rangle_{L^2} \\ &= o(h) - \langle \text{Op}_h(a)u_h | P_h^* u_h \rangle_{L^2} \\ &= -\langle \text{Op}_h(a)u_h | (P_h + 2ih\gamma(x))u_h \rangle_{L^2} + o(h) \\ &= 2ih\langle \text{Op}_h(a)u_h | \gamma(x)u_h \rangle_{L^2} + o(h) \\ &= 2ih\langle \gamma(x)\text{Op}_h(a)u_h | u_h \rangle_{L^2} + o(h) \\ &= 2ih\langle \text{Op}_h(a\gamma)u_h | u_h \rangle_{L^2} + o(h) \end{aligned}$$

Thus, setting $g(x, \xi) = \xi^\top K(x)\xi$, we obtain that

$$\langle \text{Op}_h(2a\gamma + \{g, a\})u_h | u_h \rangle_{L^2} \xrightarrow{h \rightarrow 0} 0 . \quad (3.4)$$

Due to iii) of Corollary A.2 of Appendix, we will get a contradiction with $\|u_h\|_{L^2} = 1$ if we find a such that $2a\gamma + \{g, a\}$ is uniformly bounded away from zero. Assume that $a(x, \xi)$ is constant equal to 1 for large ξ , then $2a\gamma + \{g, a\}$ is a symbol of order 0. Moreover, the first step of this proof shows that modifying $2a\gamma + \{g, a\}$ away from the sphere $\Sigma = \{(x, \xi), \xi^\top K(x)\xi = 1\}$ has no influence on (3.4). Thus, it is sufficient to exhibit a symbol a such that $2a\gamma + \{g, a\}$ is uniformly bounded and stay uniformly away from zero on Σ .

Let us recall that φ_t is the Hamiltonian flow associated to g and that T is a time such that the mean value $\langle \gamma \rangle_T(x, \xi) = \frac{1}{T} \int_0^T \gamma(\varphi_t(x, \xi)) dt$ is uniformly bounded away from 0 away from Σ , according to Assumption i) of Theorem 1.1. We choose $a(x, \xi) = e^{c(x, \xi)}$ with

$$c(x, \xi) = \frac{2}{T} \int_0^T (T-t)\gamma(\varphi_t(x, \xi)) dt = \frac{2}{T} \int_0^T \int_0^t \gamma(\varphi_s(x, \xi)) ds dt .$$

By definition of the Hamiltonian flow, for any function $f \in \mathcal{C}^1(\mathbb{R}^{2d}, \mathbb{R})$, we have

$$\{g, f\}(x, \xi) = \partial_\tau f(\varphi_\tau(x, \xi))|_{\tau=0} .$$

Since

$$\begin{aligned} c(\varphi_\tau(x, \xi)) &= \frac{2}{T} \int_0^T (T-t)\gamma(\varphi_{t+\tau}(x, \xi)) dt \\ &= \frac{2}{T} \int_\tau^{T+\tau} (T-t+\tau)\gamma(\varphi_t(x, \xi)) dt \end{aligned}$$

we get that

$$\{g, c\}(x, \xi) = \frac{2}{T} \int_0^T \gamma(\varphi_t(x, \xi)) dt - 2\gamma(x, \xi) = 2\langle \gamma \rangle_T(x, \xi) - 2\gamma(x, \xi) .$$

Thus, we have

$$2a\gamma + \{g, a\} = 2e^{c(x, \xi)} \langle \gamma \rangle_T(x, \xi) .$$

By assumption i) of Theorem 1.1 and since $c \geq 0$, there exists $\alpha > 0$ such that, for all $(x, \xi) \in \Sigma$, $2a\gamma + \{g, a\} \geq \alpha > 0$. As explained above, we can neglect any (x, ξ) away from Σ and this yields that

$$\langle \text{Op}_h(2a\gamma + \{g, a\})u_h | u_h \rangle_{L^2} \sim \langle \text{Op}_h(2e^{c(x, \xi)} \langle \gamma \rangle_T(x, \xi)) u_h | u_h \rangle_{L^2} \geq 2\alpha \|u_h\|_{L^2}^2 ,$$

which contradicts (3.4) since $\|u_h\|_{L^2} = 1$. □

4 Low frequencies

In this section, we fix a real number μ and we assume that there is a sequence (U_n) with $\|U_n\|_X = 1$ and $(A - i\mu)U_n \rightarrow 0$, that is that $U_n = (u_n, v_n)$ satisfies $v_n = i\mu u_n + o_{H^1}(1)$ and

$$(\Delta_K - Id)u_n - i\mu\gamma(x)u_n + \mu^2 u_n = o_{L^2}(1). \quad (4.1)$$

As a first basic computation, we can multiply (4.1) by \bar{u}_n and integrating. We obtain

$$\|u_n\|_{H^1}^2 = \mu^2 \|u_n\|_{L^2}^2 + o(1) \quad \text{and} \quad \int_{\mathbb{R}^d} \gamma(x)|u_n(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0. \quad (4.2)$$

Also notice that $\|v_n\|_{L^2} \sim \mu \|u_n\|_{L^2}$ and thus $\|U_n\|_X$ is equivalent to $\|u_n\|_{H^1}$. Therefore, the purpose of this section is to obtain a contradiction from (4.1) and $\|u_n\|_{H^1} = 1$.

• *First step: u_n goes to zero on any compact set.*

Let R be a positive radius such that the ball $B(0, R)$ intersects the support of γ . We denote by \tilde{u}_n the restriction of u_n to the ball $B(0, R)$. Assume that $\|\tilde{u}_n\|_{L^2}$ does not go to zero. More precisely, up to extract a subsequence, assume that $\|\tilde{u}_n\|_{L^2} \geq \alpha > 0$. Since $\|u_n\|_{H^1}$ is bounded, up to extracting a subsequence, one can assume that (\tilde{u}_n) converges to a limit \tilde{u}_∞ , weakly in $H^1(B(0, R))$ and strongly in $L^2(B(0, R))$. By assumption, \tilde{u}_∞ is not equal to zero. Moreover, using the second part of (4.2), $\tilde{u}_\infty \equiv 0$ on the support of γ and

$$(\Delta_K - Id)\tilde{u}_\infty + \mu^2 \tilde{u}_\infty = 0 \quad \text{in } B(0, R)$$

in the sense of distributions. By the classical unique continuation property of the elliptic equations, one should have that $\tilde{u}_\infty \equiv 0$ everywhere in $B(0, R)$, which is an obvious contradiction. Thus (u_n) goes to zero in $L^2(B(0, R))$. Using (4.1), one gets that $(\Delta_K - Id)u_n$ also goes to zero in $L^2(B(0, R))$, which shows that (u_n) goes to zero in $H^2(B(0, R))$.

• *Second step: reduction to a periodic framework.*

Since (u_n) goes to zero in H^2 in any compact set, $\text{div}(K_{comp}(x)\nabla u_n)$ goes to zero in $L^2(\mathbb{R}^d)$, where K_{comp} is the non-periodic part of K introduced in Hypothesis ii) of Theorem 1.1. Due to (4.2), u_n goes to zero in the support of γ . Therefore, one can replace γ by a periodic damping γ_{per} satisfying $\gamma \geq \gamma_{per} \geq 0$ and Hypothesis ii) of Theorem 1.1 implies that γ_{per} may be chosen not vanishing everywhere. As a conclusion, one can replace K and γ by their periodic version K_{per} and γ_{per} and (4.1) will still hold.

• *Third step: bounding the resolvents in the periodic framework.*

From now on, we are in a periodic media described by periodic functions K and γ . We want to prove that (4.1) is absurd for $\|u_n\|_{H^1} = 1$. Due to (4.2), it is sufficient to show that $P = (\Delta_K - Id) - i\mu\gamma(x) + \mu^2 Id$ is invertible in $\mathcal{L}(L^2)$ and has a bounded resolvent from L^2 into L^2 . In order to work with periodic functions, we introduce the Floquet-Bloch

decomposition of a function $u \in L^2(\mathbb{R}^d)$ given in Theorem A.3 in Appendix:

$$u(x) = \frac{1}{(2\pi)^d} \int_{(0,1)^d} e^{i\sigma \cdot x} \hat{u}_\sigma(x) d\sigma ,$$

where the family of Bloch amplitudes $\sigma \mapsto \hat{u}_\sigma$ belongs to $L^2((0,1)^d, L^2(\mathbb{T}^d))$. We have

$$Pu(x) = \frac{1}{(2\pi)^d} \int_{(0,1)^d} e^{i\sigma \cdot x} P_\sigma \hat{u}_\sigma(x) d\sigma , \quad (4.3)$$

with

$$P_\sigma = (\nabla + i\sigma)^\top . K(x) . (\nabla + i\sigma) - i\mu\gamma(x) + (\mu^2 - 1)Id$$

where P_σ is well defined as an operator from $H^2(\mathbb{T}^d)$ into $L^2(\mathbb{T}^d)$, since K and γ are periodic. To bound the resolvent of P , it is sufficient to bound the resolvent of P_σ for all $\sigma \in [0, 1]^d$. Notice that $P_{\sigma'}$ is a relatively compact perturbation of P_σ when σ' is close to σ and thus the resolvent of P_σ depends continuously on σ . Since $[0, 1]^d$ is compact, we only have to show that, for one fixed $\sigma \in [0, 1]^d$, P_σ is invertible and has a bounded inverse in $L^2(\mathbb{T}^d)$.

Let us fix $\sigma \in [0, 1]^d$. Since our phase space \mathbb{T}^d is compact, P_σ has a compact resolvent. To show that it is invertible in $L^2(\mathbb{T}^d)$, it is sufficient to show that there is no non-trivial $u \in H^2(\mathbb{T}^d)$ such that $P_\sigma u = 0$. Assume that $P_\sigma u = 0$ with $\|u\|_{L^2} = 1$. Notice that

$$\begin{aligned} \int_{\mathbb{T}^d} \bar{u}(x) ((\nabla + i\sigma)^\top . K(x) . (\nabla + i\sigma)u(x)) dx &= \int_{\mathbb{T}^d} ((-\nabla + i\sigma)\bar{u}(x))^\top . K(x) . (\nabla + i\sigma)u(x) dx \\ &= - \int_{\mathbb{T}^d} \overline{(\nabla + i\sigma)u(x)}^\top . K(x) . (\nabla + i\sigma)u(x) dx \\ &\in (-\infty, 0] . \end{aligned}$$

Thus, multiplying $P_\sigma u = 0$ by \bar{u} , integrating and taking the imaginary part, we obtain that

$$\int_{\mathbb{T}^d} \gamma(x) |u(x)|^2 dx = 0$$

which shows that u must vanish in a non-empty open set of \mathbb{T}^d . To conclude, we have to use a unique continuation argument showing that $u \equiv 0$. The operator P_σ has complex coefficients, but the unique continuation property of classical real elliptic equations as $P\tilde{u} = 0$ still holds. Indeed, P_σ is locally conjugated to P by $P_\sigma = e^{-i\sigma \cdot x} P e^{i\sigma \cdot x}$ and so if u vanishes at infinite order at a point x_0 and satisfies $P_\sigma u = 0$, then u vanishes in a neighbourhood of x_0 . Notice that it is not correct to consider this conjugaison globally in \mathbb{T}^d since $e^{-i\sigma \cdot x}$ is not 1-periodic, but we only need to propagate the condition $u = 0$ locally to obtain that $u \equiv 0$ everywhere.

To conclude, we have shown that P_σ has a trivial kernel. Since it has compact resolvent, this shows that it is invertible with a bounded inverse in $L^2(\mathbb{T}^d)$. Using (4.3), this shows that P is invertible and of bounded inverse, which contradicts (4.1)

5 Proof of Theorem 1.2

First notice that, up to a change of spatial variable, one can assume that we are in the flat case $K(x) = 1$. The statement of Theorem 1.2 only differs from the one of Theorem 1.1 in the fact that Hypothesis ii) is not assumed. Thus, if we follow the proof of Theorem 1.1, one will be able to prove Theorem 1.2 if and only if one can deal with the low frequencies without using Hypothesis ii). In conclusion, using the first arguments of Section 4, we only have to consider a real number μ and a sequence (u_n) with $\|u_n\|_{H^1} = 1$ and

$$(\partial_{xx} - Id)u_n - i\mu\gamma(x)u_n + \mu^2u_n = o_{L^2}(1) , \quad (5.1)$$

and to obtain a contradiction from this, without assuming any periodic framework.

As usual, multiplying (5.1) by $\overline{u_n}$, we obtain that

$$\int_{\mathbb{R}} |u_n'(x)|^2 + (1 - \mu^2)|u_n(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \int_{\mathbb{R}} \gamma(x)|u_n(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0 . \quad (5.2)$$

We introduce a non-negative function $\varphi \in \mathcal{C}_b^1(\mathbb{R}, \mathbb{R}_+)$ such that $|\varphi| + |\varphi'| \leq C\gamma$ for some positive C and such that $\int_{kT}^{(k+1)T} \varphi(x)dx = T\alpha > 0$ for each $k \in \mathbb{Z}$. This is possible due to Hypothesis i), to the mollifying technique used in Section 2 and by slightly modifying φ on each interval $[kT, (k+1)T]$.

Then, we use multipliers techniques inspired by the ones of [13] and [18]. By multiplying (5.1) by $\varphi(x)\overline{u_n}$, integrating and taking the real part, we obtain that

$$\int_{\mathbb{R}} \varphi(x)(|u_n'(x)|^2 + (1 - \mu^2)|u_n(x)|^2) dx + \operatorname{Re} \left(\int_{\mathbb{R}} \varphi'(x)\overline{u_n(x)}u_n'(x) dx \right) \xrightarrow{n \rightarrow \infty} 0 .$$

First notice that $\int \varphi\mu^2|u_n|^2$ goes to zero due to (5.2) and the assumption $|\varphi| + |\varphi'| \leq C\gamma$. Moreover

$$\left| \int_{\mathbb{R}} \varphi'(x)\overline{u_n(x)}u_n'(x) dx \right| \leq \sqrt{\int_{\mathbb{R}} \varphi'|u_n|^2} \sqrt{\int_{\mathbb{R}} \varphi'|u_n'|^2}$$

which also goes to zero due to (5.2). Thus, we obtain that

$$\int_{\mathbb{R}} \varphi(x)|u_n'(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0 .$$

Now, we introduce $q(x) = \int_0^x (\alpha - \varphi(s)) ds$ and notice that q belongs to $L^\infty(\mathbb{R})$ because $\int_{kT}^{(k+1)T} \varphi(x)dx = T\alpha$ for each $k \in \mathbb{Z}$. We multiply (5.1) by $q(x)\overline{u_n}$ and we obtain that

$$\frac{1}{2} \int_{\mathbb{R}} q'(x)(|u_n'|^2 + (\mu^2 - 1)|u_n|^2) + i\mu \int_{\mathbb{R}} q(x)\gamma(x)u_n\overline{u_n'} \xrightarrow{n \rightarrow \infty} 0 .$$

The last term goes to zero since γu_n goes to zero in $L^2(\mathbb{R})$. Thus, we obtain that

$$\frac{\alpha}{2} \int_{\mathbb{R}} (|u'_n|^2 + (\mu^2 - 1)|u_n|^2) - \frac{1}{2} \int_{\mathbb{R}} \varphi(|u'_n|^2 + \mu^2|u_n|^2) \xrightarrow{n \rightarrow \infty} 0 .$$

As shown above, the second integral goes to zero and we obtain that $\int_{\mathbb{R}} (|u'_n|^2 + (\mu^2 - 1)|u_n|^2)$ goes to zero, which is in obvious contradiction with (5.2) and $\|u_n\|_{H^1} = 1$.

6 Applications to other problems

The exponential decay of the linear semigroup e^{At} is an essential assumption for obtaining several dynamical properties of the damped wave equations. In this section, we emphasise different results, which are corollaries of Theorems 1.1 and 1.2. Each result was already known with stronger assumptions implying the exponential decay of e^{At} . Since we have obtained this decay with weaker conditions, we can improve these results.

- *Linear control*

By HUM Method of Lions (see [13]), the exponential decay of the linear semigroup e^{At} is equivalent to the controllability of the linear wave equation. We denote by $\mathbb{1}_\omega$ the function $\mathbb{1}_\omega \equiv 1$ on ω and 0 elsewhere.

Corollary 6.1. *Let ω be an open subset of \mathbb{R}^d and assume that the hypotheses of Theorem 1.1, or the ones of Theorem 1.2, hold with $\gamma = \mathbb{1}_\omega$. Then, there exists $T > 0$ such that, for any $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and any $(\tilde{u}_0, \tilde{u}_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, there exists a control $v \in L^1((0, T), L^2(\omega))$ such that the solution u of*

$$\begin{cases} \partial_{tt}^2 u - \operatorname{div}(K(x)\nabla u) + u = \mathbb{1}_\omega v(x, t) & (t, x) \in (0, T) \times \mathbb{R}^d, \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \end{cases}$$

satisfies $(u, \partial u)(\cdot, T) = (\tilde{u}_0, \tilde{u}_1)$.

- *Stabilisation of the non-linear equation*

Another related problem is the stabilisation of the non-linear equation as in [19], [5] or [9]. One considers the non-linear equation

$$\begin{cases} \partial_{tt}^2 u + \gamma(x)\partial_t u = \operatorname{div}(K(x)\nabla u) - u - f(u) & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \end{cases} \quad (6.1)$$

with f satisfying

$$f(0) = 0, \quad sf(s) \geq 0, \quad |f(s)| \leq C(1 + |s|)^p \quad \text{and} \quad |f'(s)| \leq C(1 + |s|)^{p-1} \quad (6.2)$$

with $1 \leq p < (d+2)/(d-2)$ (or any $p \geq 1$ if $d < 3$). To each solution of (6.1), one can associate the energy

$$E(u) := E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^d} (|\partial_t u|^2 + |\nabla u^\top \cdot K(x) \cdot \nabla u| + |u|^2) + \int_{\mathbb{R}^d} V(u) ,$$

where $V(u) = \int_0^u f(s) ds$. One would like to show that the energy E of the solutions of (6.1) goes to zero, uniformly in any bounded set. Of course, assuming that the exponential decay of e^{At} is necessary, since it corresponds to the case $f \equiv 0$. In addition, one also needs a unique continuation property of the linear wave equation to prove that the only solutions having a constant energy E are the equilibrium points. In [19] and [5], the authors use a unique continuation property, which needs geometric assumptions stronger than the one required for the exponential decay of e^{At} . Thus, we cannot improve their results. However, we have shown in [9] that the geometric assumptions required for the exponential decay of e^{At} are sufficient if we assume that f is smooth and partially analytic. Thus, we can improve the result of [9] by using a weaker assumption than the one that $\gamma \geq \alpha > 0$ outside a compact set.

Corollary 6.2. *Assume that the hypotheses of Theorem 1.1 holds and that $f \in C^\infty(\mathbb{R})$ is real analytic and satisfies (6.2). Then the following exponential decay holds. For any $E_0 \geq 0$, there exist $K > 0$ and $\lambda > 0$ such that, for all solutions u of (6.1) with $E(u(0)) \leq E_0$, $E(u(t)) \leq Me^{-\lambda t} E(u(0))$ for all $t \geq 0$.*

Of course, in the one-dimensional case $d = 1$, the unique continuation property holds without any additional assumption. We obtain the following result, which is an improvement of the previous known result of [19] because one can omit the assumption $\gamma \geq \alpha > 0$ close to $\pm\infty$.

Corollary 6.3. *Assume that the hypotheses of Theorem 1.2 holds and that $f \in C^1(\mathbb{R})$ satisfies $f(0) = 0$ and $sf(s) \geq 0$. Then, for any $E_0 \geq 0$, there exist $K > 0$ and $\lambda > 0$ such that, for all solutions u of (6.1) with $E(u(0)) \leq E_0$, $E(u(t)) \leq Me^{-\lambda t} E(u(0))$ for all $t \geq 0$.*

This type of non-linear stabilisation results is also closely related to the problem of global control of the non-linear wave equation, see [5], [9] and [10]. For example, one get the following result in dimension $d = 1$.

Corollary 6.4. *Let ω be an open subset of \mathbb{R} . Assume that there exist $L > 0$ and $\varepsilon > 0$ such that ω contains an interval of length ε in any interval $[x, x + L]$, $x \in \mathbb{R}$. Let also $f \in C^1(\mathbb{R})$ satisfying $f(0) = 0$ and $sf(s) \geq 0$.*

Then, for all $E_0 \geq 0$, there exists $T > 0$ such that, for any (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with energy E less than E_0 , there exists a control $v \in L^1((0, T), L^2(\omega))$ such that the solution u of

$$\begin{cases} \partial_{tt}^2 u - \operatorname{div}(K(x) \nabla u) + u + f(u) = \mathbb{1}_\omega v(x, t) & (t, x) \in (0, T) \times \mathbb{R} , \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \end{cases}$$

satisfies $(u, \partial u)(\cdot, T) = (\tilde{u}_0, \tilde{u}_1)$.

• *Existence of a compact global attractor*

Instead of stabilising to zero the solutions of the non-linear equation (6.1), one may aim at stabilising the solutions to a compact set, not necessarily reduced to $\{0\}$. To this purpose, we consider a function $f(x, u) \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}, \mathbb{R})$ satisfying

$$|f(s)| \leq C(1 + |s|)^p \quad \text{and} \quad |f'(s)| \leq C(1 + |s|)^{p-1} \quad (6.3)$$

with $1 \leq p < (d+2)/(d-2)$ (or any $p \geq 1$ if $d < 3$) and we replace the first assumptions of (6.2) by

$$\exists R > 0, \quad \forall (x, s) \in \mathbb{R}^{d+1}, \quad |x| + |s| \geq R \Rightarrow f(x, s)s \geq 0. \quad (6.4)$$

Following the arguments of [9], one can prove that all the solutions of the non-linear equation

$$\begin{cases} \partial_{tt}^2 u + \gamma(x)\partial_t u = \operatorname{div}(K(x)\nabla u) - u - f(x, u) & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \partial_t u)(\cdot, 0) = (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \end{cases} \quad (6.5)$$

are attracted to a compact invariant set called the compact global attractor. This set is a central object of the theory of dynamical systems. It contains all the solutions $u(t)$, which exist for all $t \in \mathbb{R}$ and which are uniformly bounded in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ for all $t \in \mathbb{R}$ (as equilibrium points, periodic orbits, heteroclinic orbits, non-wandering points etc.). See for example [6] and [17] for a review on the concept of compact global attractors. In addition, the energy E associated to (6.5) is a Lyapounov function, that is that it is non-increasing and cannot be constant along a solution $u(t)$, except of course if $u(t)$ is an equilibrium point. Since (6.5) admits a Lyapounov function, one say that the corresponding dynamical system is gradient. In particular, it cannot admit periodic orbits, homoclinic orbits. . .

Thus, following the arguments of [9], one obtains the following generalisation of the result of [9], omitting the assumption that $\gamma \geq \alpha > 0$ outside a compact set.

Corollary 6.5. *Assume that $f \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ satisfies (6.3) and (6.4) and that:*

- either $d = 1$ and the hypothesis of Theorem 1.2 holds,
- or $d \geq 2$, the hypotheses of Theorem 1.1 hold and $f \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is analytic with respect to u .

Then the dynamical system generated by (6.5) in $H_0^1(\Omega) \times L^2(\Omega)$ is gradient and admits a compact global attractor \mathcal{A} .

A Appendix

A.1 Pseudo-differential semiclassical calculus

In section, we recall the main results and notations of pseudo-differential calculus, which are used in this paper. The details and the proofs could be found in many textbooks, as

[20], [14] or [1].

Let $h > 0$ be a small parameter, say that $h \in (0, 1]$. We say that $a(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is a *symbol* of order m if, for any multi-indices α and β , there exists $C_{\alpha, \beta}$ such that

$$\sup_{(x, \xi) \in \mathbb{R}^{2d}} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{m - |\beta|}$$

and m is the smallest number such that this bounds holds. To each symbol a , we associate the *pseudo-differential semiclassical operator* denoted by $\text{Op}_h(a)$ and defined by Weyl quantization

$$\text{Op}_h(a)u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2d}} e^{i(x-y) \cdot \xi} a\left(\frac{x+y}{2}, h\xi\right) u(y) dy d\xi .$$

If a is of order m , then, for any $s > 0$, $\text{Op}_h(a)$ is a bounded operator from $H^s(\mathbb{R}^d)$ into $H^{s-m}(\mathbb{R}^d)$, uniformly with respect to $h \in (0, 1]$.

Let $f \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ be a smooth bounded function such that all its derivative are also bounded functions of $L^\infty(\mathbb{R}^d)$. It is not so trivial but well known that the simple operator $u \mapsto f(x)u$ has for symbol $f(x)$, which is of order 0 (see for example Chapter 4 of [20]). More classically, we have that the operator $h\nabla$ has for symbol $i\xi$, which is of order 1. Using Proposition A.1 below, one can check that the operator $h^2 \Delta_K = h^2 \text{div}(K(x)\nabla \cdot)$ has for principal symbol $-\xi^\top \cdot K(x) \cdot \xi$ in the sense that

$$h^2 \Delta_K = \text{Op}_h(-\xi^\top \cdot K(x) \cdot \xi) + \mathcal{O}_{L^2 \rightarrow L^2}(h^2) .$$

Composing two pseudo-differential operators, one obtain a pseudo-differential operator, which symbol can be express by an asymptotic development. In this paper, we will simply use the following cases, see [20], [14], [1] or any other textbooks on pseudo-differential calculus for more precise developments and for proofs.

Proposition A.1. *Let a and b be two symbols of order m and n respectively. Assume that $m + n \leq 2$, then $\text{Op}_h(a) \circ \text{Op}_h(b)$ is a pseudo-differential operator of order $m + n$ and its symbol $(a \# b)$ satisfies*

$$a \# b = ab - \frac{ih}{2} \{a, b\} + \mathcal{O}_{L^2 \rightarrow L^2}(h^2)$$

where $\{a, b\} = \partial_\xi a \partial_x b - \partial_\xi b \partial_x a$ is the Poisson bracket of a and b and is of order $m + n - 1$. In particular, if $m + n \leq 1$, then

$$\text{Op}_h(a) \circ \text{Op}_h(b) = \text{Op}_h(ab) + \mathcal{O}_{L^2 \rightarrow L^2}(h) .$$

Corollary A.2.

- i) If a is of order 1 or less and if b is of order 0, then the commutator $[\text{Op}_h(a), \text{Op}_h(b)] = \text{Op}_h(a) \circ \text{Op}_h(b) - \text{Op}_h(b) \circ \text{Op}_h(a)$ is of order 0 or less and of estimate $\mathcal{O}_{L^2 \rightarrow L^2}(h)$.*

ii) If a is of order 2 and if b is of order 0, then their commutator is of order 1 and

$$[\text{Op}_h(a), \text{Op}_h(b)] = -ih\text{Op}_h(\{a, b\}) + \mathcal{O}_{L^2 \rightarrow L^2}(h^2) .$$

iii) Assume that a is a symbol of order $m \geq 0$ such that there exists $\alpha > 0$ such that $|a(x, \xi)| \geq \alpha$ for all $(x, \xi) \in \mathbb{R}^{2d}$. Then $b = 1/a$ is a symbol of order $-m$ and $\text{Op}_h(b)$ is a first order inverse of a in the sense that

$$\text{Op}_h(a) \circ \text{Op}_h(b) = Id + \mathcal{O}_{L^2 \rightarrow L^2}(h) \quad \text{and} \quad \text{Op}_h(b) \circ \text{Op}_h(a) = Id + \mathcal{O}_{L^2 \rightarrow L^2}(h)$$

In particular, $\text{Op}_h(a)$ is invertible for h sufficiently small.

A.2 Floquet-Bloch decomposition

In this section, we recall the Floquet-Bloch decomposition in the 1-periodic framework. For sake of completeness, we also recall a short proof of it. We denote by \mathbb{T}^d the torus $(\mathbb{R}/\mathbb{Z})^d$.

Theorem A.3. Let $u \in L^2(\mathbb{R}^d, \mathbb{C})$ and let $\hat{u}(y) = \int_{\mathbb{R}^d} e^{-iy \cdot x} u(x) dx$ be its Fourier transform. Then, we have the splitting

$$u(x) = \frac{1}{(2\pi)^d} \int_{(0,1)^d} e^{i\sigma \cdot x} \hat{u}_\sigma(x) d\sigma ,$$

where the Bloch amplitude \hat{u}_σ is defined by

$$\hat{u}_\sigma(x) = \sum_{n \in \mathbb{Z}^d} \hat{u}(\sigma + n) e^{in \cdot x} .$$

Moreover, each \hat{u}_σ is 1-periodic in the sense that, for any $n \in \mathbb{Z}^d$, $\hat{u}_\sigma(x + n) = \hat{u}_\sigma(x)$ and thus can be considered as a function defined in \mathbb{T}^d .

The family $\sigma \mapsto \hat{u}_\sigma$ is well defined for any function $u \in L^2(\mathbb{R}^d, \mathbb{C})$ as a family in $L^2((0, 1)^d, L^2(\mathbb{T}^d))$.

Proof: Let us first assume that u belongs to Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Floquet-Bloch decomposition is a simple rearrangement of the Fourier transform:

$$\begin{aligned} u(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iy \cdot x} \hat{u}(y) dy = \sum_{n \in \mathbb{Z}^d} \int_{(0,1)^d} e^{i(n+\sigma) \cdot x} \hat{u}(n + \sigma) d\sigma \\ &= \int_{(0,1)^d} e^{i\sigma \cdot x} \left(\sum_{n \in \mathbb{Z}^d} \hat{u}(n + \sigma) e^{in \cdot x} \right) d\sigma := \int_{(0,1)^d} e^{i\sigma \cdot x} \hat{u}_\sigma(x) d\sigma . \end{aligned}$$

The fact that \hat{u}_σ is 1-periodic is clear. To finish the proof, we have to extend the transformation from $L^2(\mathbb{R}^d)$ into $L^2((0, 1)^d, L^2(\mathbb{T}^d))$. First notice that, for each σ , $\|\hat{u}_\sigma\|_{L^2(\mathbb{T}^d)}^2 = \sum |\hat{u}(n + \sigma)|^2$. Thus,

$$\begin{aligned} \|\hat{u}_\sigma\|_{L^2((0,1)^d, L^2(\mathbb{T}^d))}^2 &= \int_{(0,1)^d} \left(\sum_{n \in \mathbb{Z}^d} |\hat{u}(n + \sigma)|^2 \right) d\sigma \\ &= \sum_{n \in \mathbb{Z}^d} \int_{(0,1)^d} |\hat{u}(n + \sigma)|^2 d\sigma \\ &= \|\hat{u}\|_{L^2}^2 \\ &= (2\pi)^d \|u\|_{L^2}^2 . \end{aligned}$$

This shows that we can extend the Floquet-Bloch decomposition in $L^2((0, 1)^d, L^2(\mathbb{T}^d))$ from $\mathcal{S}(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$. \square

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