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► **To cite this version:**

Frédéric Klopp, Nikolaj Veniaminov. Interacting electrons in a random medium: a simple one-dimensional model. 2014. hal-01057798

HAL Id: hal-01057798

<https://hal.science/hal-01057798v1>

Preprint submitted on 25 Aug 2014

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INTERACTING ELECTRONS IN A RANDOM MEDIUM: A SIMPLE ONE-DIMENSIONAL MODEL

FRÉDÉRIC KLOPP AND NIKOLAJ A. VENIAMINOV

ABSTRACT. The present paper is devoted to the study of a simple model of interacting electrons in a random background. In a large interval Λ , we consider n one dimensional particles whose evolution is driven by the Luttinger-Sy model, i.e., the interval Λ is split into pieces delimited by the points of a Poisson process of intensity μ and, in each piece, the Hamiltonian is the Dirichlet Laplacian. The particles interact through a repulsive pair potential decaying polynomially fast at infinity. We assume that the particles have a positive density, i.e., $n/|\Lambda| \rightarrow \rho > 0$ as $|\Lambda| \rightarrow +\infty$. In the low density or large disorder regime, i.e., ρ/μ small, we obtain a two term asymptotic for the thermodynamic limit of the ground state energy per particle of the interacting system; the first order correction term to the non interacting ground state energy per particle is controlled by pairs of particles living in the same piece. The ground state is described in terms of its one and two-particles reduced density matrix. Comparing the interacting and the non interacting ground states, one sees that the effect of the repulsive interactions is to move a certain number of particles living together with another particle in a single piece to a new piece that was free of particles in the non interacting ground state.

RÉSUMÉ. Dans ce travail, nous considérons un modèle simple de électrons en interaction dans un environnement aléatoire. Dans un grand intervalle Λ , nous considérons n particules uni-dimensionnelles dont l'évolution est régie par le modèle de Luttinger-Sy : l'intervalle Λ est subdivisé en pièces délimitées par les points d'un processus de Poisson d'intensité μ et, dans chaque pièce, le hamiltonien est le laplacien de Dirichlet. Les particules interagissent par paires au travers d'un potentiel répulsif décroissant polynomiallement à l'infini. On suppose que la densité de particules est positive c'est-à-dire que $n/|\Lambda| \rightarrow \rho > 0$ quand $|\Lambda| \rightarrow +\infty$. Lorsque la densité est petite ou lorsque le désordre est grand, c'est-à-dire lorsque ρ/μ est petit, nous obtenons une asymptotique à deux termes de la limite thermodynamique de l'énergie fondamentale par particule du système ; le premier terme de correction à l'énergie fondamentale par particule du système sans interaction est contrôlé par les paires de particules vivant dans la même pièce. L'état fondamental est décrit au moyen de sa matrice de densité réduite à une et à deux particules. En comparant l'état fondamental avec interaction à l'état fondamental sans interaction, on voit que l'effet des interactions est de séparer un certain nombre de particules qui vivent en paire avec une autre particule dans la même pièce vers des pièces inoccupées dans l'état fondamental sans interaction.

1. INTRODUCTION: THE MODEL AND THE MAIN RESULTS

On \mathbb{R} , consider a Poisson point process $d\mu(\omega)$ of intensity μ . Let $(x_k(\omega))_{k \in \mathbb{Z}}$ denote its support (i.e., $d\mu(\omega) = \sum_{k \in \mathbb{Z}} \delta_{x_k(\omega)}$), the points being ordered increasingly.

On $L^2(\mathbb{R})$, define the Luttinger-Sy or pieces model (see e.g. [LS73, LGP88]), that is, the

Date: August 25, 2014.

2010 Mathematics Subject Classification. Primary 81V70, 82B44; Secondary 82D30.

Key words and phrases. Interacting electrons, random Schrödinger operators, thermodynamic limit.

This work is partially supported by the grant ANR-08-BLAN-0261-01. The authors also acknowledge the support of the IMS (NU Singapore) where part of this work was done. F.K. thanks T. Duquesne for his explaining the Palm formula.

random operator

$$H_\omega = \bigoplus_{k \in \mathbb{Z}} -\Delta_{[x_k, x_{k+1}]}^D$$

where, for an interval I , $-\Delta_I^D$ denotes the Dirichlet Laplacian on I .

Pick $L > 0$ and let $\Lambda = \Lambda_L = [0, L]$. Restrict H_ω to Λ with Dirichlet boundary conditions: on $\mathfrak{H} := L^2(\Lambda)$, define

$$H_\omega(L) = H_\omega(\Lambda) = \bigoplus_{k_- - 1 \leq k \leq k_+} -\Delta_{\Delta_k(\omega)}^D \quad (1.1)$$

where we have defined $\Delta_k(\omega) := [x_k(\omega), x_{k+1}(\omega)]$ to be the k -th piece and we have set

$$\begin{aligned} k_- &= \min\{k; x_k > 0\}, & x_{k_- - 1} &= 0, \\ k_+ &= \max\{k; x_k < L\}, & x_{k_+ + 1} &= L. \end{aligned}$$

From now on, we let $m(\omega)$ be the number of pieces and renumber them from 1 to $m(\omega)$ (i.e., $k_- = 2$ and $k_+ = m(\omega)$). For L large, with probability $1 - O(L^{-\infty})$, one has $m(\omega) = \mu L + O(L^{2/3})$.

The pieces model admits an integrated density of states that can be computed explicitly (see section 2.2 or [LGP88, Ven12]), namely,

$$\begin{aligned} N_\mu(E) &:= \lim_{L \rightarrow +\infty} \frac{\{\text{eigenvalues of } H_\omega(L) \text{ in } (-\infty, E]\}}{L} \\ &= \frac{\mu \cdot \exp(-\mu \ell_E)}{1 - \exp(-\mu \ell_E)} \mathbf{1}_{E \geq 0} \quad \text{where} \quad \ell_E := \frac{\pi}{\sqrt{E}}. \end{aligned} \quad (1.2)$$

1.1. Interacting electrons. Consider first n free electrons restricted to the box Λ in the background Hamiltonian $H_\omega(\Lambda)$, that is, on the space

$$\mathfrak{H}^n(\Lambda) = \mathfrak{H}^n(\Lambda_L) = \bigwedge_{j=1}^n L^2(\Lambda) = L^2_-(\Lambda^n), \quad (1.3)$$

consider the operator

$$H_\omega^0(\Lambda, n) = \sum_{i=1}^n \underbrace{\mathbf{1}_{\mathfrak{H}} \otimes \dots \otimes \mathbf{1}_{\mathfrak{H}}}_{i-1 \text{ times}} \otimes H_\omega(\Lambda) \otimes \underbrace{\mathbf{1}_{\mathfrak{H}} \otimes \dots \otimes \mathbf{1}_{\mathfrak{H}}}_{n-i \text{ times}}. \quad (1.4)$$

This operator is self-adjoint and lower semi-bounded. Let $E_\omega^0(\Lambda, n)$ be its ground state energy and $\Psi_\omega^0(\Lambda, n)$ be its ground state.

To $H_\omega^0(\Lambda, n)$, we now add a repulsive pair finite range interaction potential. Therefore, pick $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfying

(HU): U is a repulsive (i.e., non negative), even pair interaction potential decaying sufficiently fast at infinity. More precisely, we assume

$$x^3 \int_x^{+\infty} U(t) dt \xrightarrow{x \rightarrow +\infty} 0. \quad (1.5)$$

To control the possible local singularities of the interactions, we require that $U \in L^p(\mathbb{R})$ for some $p \in (1, +\infty]$.

On $\mathfrak{H}^n(\Lambda)$, we define

$$H_\omega^U(\Lambda, n) = H_\omega^0(\Lambda, n) + W_n \quad (1.6)$$

where

$$W_n(x^1, \dots, x^n) := \sum_{i < j} U(x^i - x^j) \quad (1.7)$$

on the domain

$$\mathcal{D}^n(\Lambda) := \mathcal{C}_0^\infty \left(\left(\bigcup_{k=1}^{m(\omega)}]x_k, x_{k+1}[\right)^n \right) \cap \mathfrak{H}^n(\Lambda). \quad (1.8)$$

As U is non negative, $H_\omega^U(\Lambda, n)$ is non negative. From now on, we let $H_\omega^U(\Lambda, n)$ be the Friedrichs extension of this operator. As W_n is a sum of pair interactions, the fact that $U \in L^p(\mathbb{R})$ for some $p > 1$ (see assumption **(HU)**) guarantees that W_n is $H_\omega^0(\Lambda, n)$ -form bounded with relative form bound 0 (see, e.g., [CFKS87, section 1.2]). Thus, the form domain of the operator $H_\omega^U(\Lambda, n)$ is

$$\mathfrak{H}_\infty^n(\Lambda) := \left(H_0^1 \left(\bigcup_{k=1}^{m(\omega)}]x_k, x_{k+1}[\right) \right)^{\otimes n} \cap \mathfrak{H}^n(\Lambda). \quad (1.9)$$

Moreover, $H_\omega^U(\Lambda, n)$ admits $\mathcal{D}^n(\Lambda)$ as a form core (see, e.g., [CFKS87, section 1.3]) and it has a compact resolvent, thus, only discrete spectrum.

We define $E_\omega^U(\Lambda, n)$ to be its ground state energy, that is,

$$E_\omega^U(\Lambda, n) := \inf_{\substack{\Psi \in \mathcal{D}^n(\Lambda) \\ \|\Psi\|=1}} \langle H_\omega^U(\Lambda, n)\Psi, \Psi \rangle \quad (1.10)$$

and $\Psi_\omega^U(\Lambda, n)$ to be a ground state, i.e., to be an eigenfunction associated to the eigenvalue $E_\omega^U(\Lambda, n)$.

By construction, there is no unique continuation principle for the pieces model (as the union of disjoint non empty intervals is not connected); so, one should not expect uniqueness for the ground state. Nevertheless due to the properties of the Poisson process, for the non interacting system, one easily sees that the ground state $\Psi_\omega^0(\Lambda, n)$ is unique ω almost surely (see section 2.4). For the interacting system, it is not as clear. Nonetheless, one proves

Theorem 1.1 (Almost sure non-degeneracy of the ground state). *Suppose that U is real analytic. Then, ω -almost surely, for any L and n , the ground state of $H_\omega^U(L, n)$ is non-degenerate.*

For a general U , while we don't know whether the ground state is degenerate or not, our analysis will show where the degeneracy may come from: we shall actually write $\mathfrak{H}^n(\Lambda)$ as an orthogonal sum of subspaces invariant by $H_\omega^U(L, n)$ such that on each such subspace, the ground state of $H_\omega^U(L, n)$ is unique. This will enable us to show that all the ground states of $H_\omega^U(L, n)$ on $\mathfrak{H}^n(\Lambda)$ are very similar to each other, i.e., they differ only by a small number of particles.

The goal of the present paper is to understand the thermodynamic limits of $E_\omega^U(\Lambda, n)$ and $\Psi_\omega^U(\Lambda, n)$. As usual, we define the *thermodynamic limit* to be the limit $L \rightarrow \infty$ and $n/L \rightarrow \rho$ where ρ is a positive constant. The constant ρ is the *density of particles*.

We will describe the thermodynamic limits of $E_\omega^U(\Lambda, n)$, or rather $n^{-1}E_\omega^U(\Lambda, n)$, and $\Psi_\omega^U(\Lambda, n)$ when ρ is positive and small (but independent of L and n). We will be specially interested in the influence of the interaction U , i.e., we will compare the thermodynamic limits for the non-interacting and the interacting systems.

1.2. The ground state energy per particle. Our first result describes the thermodynamic limit of $n^{-1}E_\omega^U(\Lambda, n)$ when we assume the density of particles n/L to be ρ . For the sake of comparison, we also included the corresponding result on the ground state energy of the free particles, i.e., on $n^{-1}E_\omega^0(\Lambda, n)$.

We prove

Theorem 1.2. *Under the assumptions made above, the following limits exist ω -almost surely and in L^1_ω*

$$\mathcal{E}^0(\rho, \mu) := \lim_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{E_\omega^0(\Lambda, n)}{n} \quad \text{and} \quad \mathcal{E}^U(\rho, \mu) := \lim_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{E_\omega^U(\Lambda, n)}{n} \quad (1.11)$$

and they are independent of ω .

In [Ven13] (see also [Ven12]), the almost sure existence of the thermodynamic limit of the ground state energy per particle is established for quite general systems of interacting electrons in a random medium if one assumes that the interaction has compact support. For decaying interactions (as in (HU)), only the L^2_ω convergence is proved. The improvement needed on the results of [Ven13] to obtain the almost sure convergence is the purpose of Theorem 5.1.

In [BL12b], the authors study the existence of the above limits in the grand canonical ensemble for Coulomb interactions.

The energy $\mathcal{E}^0(\rho, \mu)$ can be computed explicitly for our model (see section 2.4.1). We shall obtain a two term asymptotic formula for $\mathcal{E}^U(\rho, \mu)$ in the case when the disorder is not too large and the Fermi length $\ell_{\rho, \mu}$ is sufficiently large.

Define

- the *effective density* is defined as the ratio of the density of particles to the density of impurities, i.e., $\rho_\mu = \frac{\rho}{\mu}$,
- the *Fermi energy* $E_{\rho, \mu}$ is the unique solution to $N_\mu(E_{\rho, \mu}) = \rho$,
- the *Fermi length* $\ell_{\rho, \mu} := \ell_{E_{\rho, \mu}}$ where ℓ_E is defined in (1.2); the explicit formula for N_μ yields

$$\ell_{\rho, \mu} = \frac{1}{\mu} \left| \log \frac{\rho_\mu}{1 + \rho_\mu} \right| = \frac{1}{\mu} \left| \log \frac{\rho}{\mu + \rho} \right|. \quad (1.12)$$

For the free ground state energy per particle, a direct computation using (1.2) yields

$$\mathcal{E}^0(\rho, \mu) = \frac{1}{\rho} \int_{-\infty}^{E_{\rho, \mu}} E dN_\mu(E) = E_{\rho, \mu} \left(1 + O\left(\sqrt{E_{\rho, \mu}}\right) \right) \quad (1.13)$$

We prove

Theorem 1.3. *Under the assumptions made above, for $\mu > 0$ fixed, one computes*

$$\mathcal{E}^U(\rho, \mu) = \mathcal{E}^0(\rho, \mu) + \pi^2 \gamma_*^\mu \mu^{-1} \rho_\mu \ell_{\rho, \mu}^{-3} (1 + o(1)) \quad \text{where} \quad o(1) \xrightarrow{\rho_\mu \rightarrow 0} 0. \quad (1.14)$$

The positive constant γ_*^μ depends solely on U and μ ; it is defined in (1.17) below.

At fixed disorder, in the small density regime, the Fermi length is large and the Fermi energy is small. Moreover, the shift of ground state energy (per particle) due to the interaction is exponentially small compared to the free ground state energy: indeed it is of order $\rho |\log \rho|^{-3}$ while the ground state energy is of order $|\log \rho|^{-2}$.

For fixed μ , a coarse version of (1.14) was established, in the PhD thesis of the second author [Ven12], namely, for ρ sufficiently small, one has

$$\frac{1}{C_\mu} \rho |\log \rho|^{-3} \leq \mathcal{E}^U(\rho, \mu) - \mathcal{E}^0(\rho, \mu) \leq C_\mu \rho |\log \rho|^{-3}.$$

Moreover, from [Ven13, Propositions 3.6 and 3.7]), we know that the function $\rho \mapsto \mathcal{E}^U(\rho, \mu)$ is a non decreasing continuous function and that the function $\rho^{-1} \mapsto \mathcal{E}^U(\rho, \mu)$ is convex.

Let us now define the constant γ_*^μ . Therefore, we prove

Proposition 1.4. *Consider two electrons in $[0, \ell]$ interacting via an even non negative pair potential $U \in L^p(\mathbb{R}^+)$ for some $p > 1$ and such that*

$$\int_{\mathbb{R}} x^2 U(x) dx < +\infty.$$

That is, on $\mathfrak{H}^2([0, \ell]) = L^2([0, \ell]) \wedge L^2([0, \ell])$, consider the Hamiltonian

$$(-\Delta_{x_1|_{[0, \ell]}}^D) \otimes \mathbf{1}_{\mathfrak{H}} + \mathbf{1}_{\mathfrak{H}} \otimes (-\Delta_{x_2|_{[0, \ell]}}^D) + U(x_1 - x_2), \quad (1.15)$$

i.e., the Friedrichs extension of the same differential expression defined on the domain $\mathcal{C}^2([0, \ell])$ (see (1.8)).

For large ℓ , $E^U([0, \ell], 2)$, the ground state energy of this Hamiltonian, admits the following expansion

$$E^U([0, \ell], 2) = \frac{5\pi^2}{\ell^2} + \frac{\gamma}{\ell^3} + o\left(\frac{1}{\ell^3}\right) \quad (1.16)$$

where $\gamma = \gamma(U) > 0$ when U does not vanish a.e.

Let us first notice that the expansion (1.16) immediately implies that $U \mapsto \gamma(U)$ is a non decreasing concave function of the (non negative) interaction potential U such that $\gamma(0) = 0$; for α small positive, one computes

$$\frac{\gamma(\alpha U)}{\alpha} = 10\pi^2 \int_{\mathbb{R}} x^2 U(x) dx (1 + O(\alpha)).$$

Concavity and monotony follow immediately from the definition of $E^U([0, \ell], 2)$ and the form of (1.16).

In terms of γ , we then define

$$\gamma_*^\mu := 1 - \exp\left(-\frac{\mu \gamma}{8\pi^2}\right). \quad (1.17)$$

1.3. The ground state: its one- and two-particle density matrices. We shall now describe our results on the ground state. We start with a description of the spectral data of the one particle Luttinger-Sy model. Then, we describe the non interacting ground state.

1.3.1. The spectrum of the one particle Luttinger-Sy model. Let $(E_{j,\omega}^\Lambda)_{j \geq 1}$ and $(\varphi_{j,\omega}^\Lambda)_{j \geq 1}$ respectively denote the eigenvalues (ordered increasingly) and the associated eigenfunctions of $H_\omega(\Lambda)$ (see (1.1)). Clearly, the eigenvalues and the eigenfunctions are explicitly computable from the points $(x_k)_{1 \leq k \leq m(\omega)+1}$. In particular, one sees that the eigenvalues are simple ω almost surely.

As n/L is close to ρ and L is large, the n first eigenvalues are essentially all the eigenvalues below the Fermi energy $E_{\rho,\mu}$. These eigenvalues are the eigenvalues of $-\Delta_{|\Delta_k(\omega)}^D$ below $E_{\rho,\mu}$ for all the pieces $(\Delta_k(\omega))_{k_- - 1 \leq k \leq k_+}$ of length at least $\ell_{\rho,\mu}$ (see (1.2) and (1.13)). ω -almost surely, the number of pieces $(\Delta_k(\omega))_{1 \leq k \leq m(\omega)}$ longer than $\ell_{\rho,\mu}$ is asymptotic to n (see section 2.3), the number of those longer than $2\ell_{\rho,\mu}$ to $\rho_\mu n$, the number of those longer than $3\ell_{\rho,\mu}$ to $\rho_\mu^2 n$, etc. We refer to section 2.2 for more details.

1.3.2. The non interacting ground state. The ground state of the non interacting Hamiltonian $H_\omega^0(\Lambda, n)$ is given by the (normalized) Slater determinant

$$\Psi_\omega^0(\Lambda, n) = \bigwedge_{j=1}^n \varphi_{j,\omega}^\Lambda = \frac{1}{\sqrt{n!}} \text{Det} \left((\varphi_{j,\omega}^\Lambda(x_k))_{1 \leq j, k \leq n} \right). \quad (1.18)$$

Here and in the sequel, the exterior product is normalized so that the L^2 -norm of the product be equal to the product of the L^2 -norms of the factors (see (C.2) in section C).

It will be convenient to describe the interacting ground state using its one-particle and two-particles reduced density matrices. Let us define these now (see section 4 for more details). Let $\Psi \in \mathfrak{H}^n(\Lambda)$ be a normalized n -particle wave function. The corresponding *one-particle density matrix* is an operator on $\mathfrak{H}^1(\Lambda) = L^2(\Lambda)$ with the kernel

$$\gamma_\Psi(x, y) = \gamma_\Psi^{(1)}(x, y) = n \int_{\Lambda^{n-1}} \Psi(x, \tilde{x}) \Psi^*(y, \tilde{x}) d\tilde{x} \quad (1.19)$$

where $\tilde{x} = (x^2, \dots, x^n)$ and $d\tilde{x} = dx^2 \cdots dx^n$.

The *two-particles density matrix* of Ψ is an operator acting on $\mathfrak{H}^2(\Lambda) = \bigwedge_{j=1}^2 L^2(\Lambda)$ and its kernel is given by

$$\gamma_\Psi^{(2)}(x^1, x^2, y^1, y^2) = \frac{n(n-1)}{2} \int_{\Lambda^{n-2}} \Psi(x^1, x^2, \tilde{x}) \Psi^*(y^1, y^2, \tilde{x}) d\tilde{x} \quad (1.20)$$

where $\tilde{x} = (x^3, \dots, x^n)$ and $d\tilde{x} = dx^3 \cdots dx^n$.

Both γ_Ψ and $\gamma_\Psi^{(2)}$ are positive trace class operators satisfying

$$\text{Tr } \gamma_\Psi = n, \quad \text{and} \quad \text{Tr } \gamma_\Psi^{(2)} = \frac{n(n-1)}{2}. \quad (1.21)$$

So, for the non interacting ground state, using the description of the eigenvalues and eigenvectors of $H_\omega(\Lambda)$ given in section 1.3.1, as a consequence of Proposition 4.8, we obtain that

$$\gamma_{\Psi_\omega^0(\Lambda, n)} = \sum_{j=1}^n \gamma_{\varphi_{j, \omega}^\Lambda} = \sum_{\ell_{\rho, \mu} \leq |\Delta_k(\omega)| < 3\ell_{\rho, \mu}} \gamma_{\varphi_{\Delta_k(\omega)}^1} + \sum_{2\ell_{\rho, \mu} \leq |\Delta_k(\omega)| < 3\ell_{\rho, \mu}} \gamma_{\varphi_{\Delta_k(\omega)}^2} + R^{(1)} \quad (1.22)$$

where

- $|\Delta_k(\omega)|$ denotes the length of the piece $\Delta_k(\omega)$;
- $\varphi_{\Delta_k(\omega)}^j$ denotes the j -th normalized eigenvector of $-\Delta_{|\Delta_k(\omega)}^D$;
- the operator $R^{(1)}$ is trace class and $\|R^{(1)}\|_{\text{tr}} \leq 2n\rho_\mu^2$.

Here, $\|\cdot\|_{\text{tr}}$ denotes the trace norm in the ambient space, i.e., in $L^2(\Lambda)$ for the one particle density matrix, and in $L^2(\Lambda) \wedge L^2(\Lambda)$ for the two particles density matrix.

For the two-particles density matrix, again as a consequence of Proposition 4.8, we obtain

$$\gamma_{\Psi_\omega^0(\Lambda, n)}^{(2)} = \frac{1}{2}(\text{Id} - \text{Ex}) [\gamma_{\Psi_\omega^0(\Lambda, n)} \otimes \gamma_{\Psi_\omega^0(\Lambda, n)}] + R^{(2)} \quad (1.23)$$

where

- Id is the identity operator, Ex is the exchange operator on a two-particles space:

$$\text{Ex}[f \otimes g] = g \otimes f, \quad f, g \in \mathfrak{H},$$

- the operator $R^{(2)}$ is trace class and $\|R^{(2)}\|_{\text{tr}} \leq C_{\rho, \mu} n$.

One can represent graphically the ground state of the non interacting system by representing the distribution of its particles within the pieces: in abscissa, one puts the length of the pieces, in ordinate, the number of particles the ground state puts in a piece of that length. Figure 1 shows the picture thus obtained.

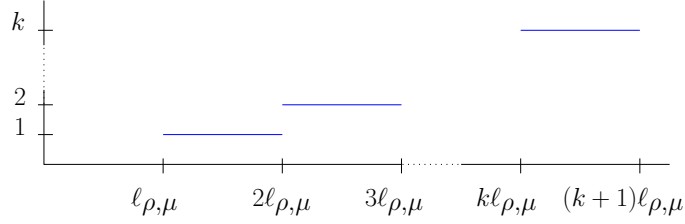


FIGURE 1. The distribution of particles in the non interacting ground state.

1.3.3. *The interacting ground state.* To describe the ground state of the interacting system, we shall describe its one-particle and two-particles reduced density matrices. Therefore, it will be useful to introduce the following approximate one-particle reduced density matrices. For a piece $\Delta_k(\omega)$, let $\zeta_{\Delta_k(\omega)}^j$ be the j -th normalized eigenvector of $-\Delta_{|\Delta_k(\omega) \times \Delta_k(\omega)}^D + U$ acting on $L^2(\Delta_k(\omega)) \wedge L^2(\Delta_k(\omega))$. We note that, for $U = 0$, the two-particles ground state can be rewritten as $\zeta_{\Delta_k(\omega)}^{1,U=0} = \varphi_{\Delta_k(\omega)}^1 \wedge \varphi_{\Delta_k(\omega)}^2$.

Define the following one-particle density matrix

$$\gamma_{\Psi_{\Lambda,n}^{\text{opt}}} = \sum_{\ell_{\rho,\mu} - \rho_\mu \gamma_*^\mu \leq |\Delta_k(\omega)| \leq 2\ell_{\rho,\mu} - \log(1 - \gamma_*^\mu)} \gamma_{\varphi_{\Delta_k(\omega)}^1} + \sum_{2\ell_{\rho,\mu} - \log(1 - \gamma_*^\mu) \leq |\Delta_k(\omega)|} \gamma_{\zeta_{\Delta_k(\omega)}^1}. \quad (1.24)$$

Because of the possible long range of the interaction U (see the remarks following Theorem 1.5 below), to describe our results precisely, it will be useful to introduce trace norms reduced to certain pieces. For $\ell \geq 0$, we define the projection onto the pieces shorter than ℓ

$$\mathbf{1}_{<\ell}^1 = \sum_{|\Delta_k(\omega)| < \ell} \mathbf{1}_{\Delta_k(\omega)}. \quad (1.25)$$

We shall use the following function to control remainder terms: define

$$Z(x) = \sup_{x \leq v} \left(v^3 \int_v^{+\infty} U(t) dt \right). \quad (1.26)$$

Under assumption (HU), the function Z is continuous and monotonously decreasing on $[0, +\infty)$ and tends to 0 at infinity.

We prove

Theorem 1.5. *Fix $\mu > 0$. Assume (HU) holds. Then, there exist $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0)$, ω -a.s., one has*

$$\limsup_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{1}{n} \left\| \left(\gamma_{\Psi_{\omega}^U(\Lambda,n)} - \gamma_{\Psi_{\Lambda,n}^{\text{opt}}} \right) \mathbf{1}_{<\ell_{\rho,\mu}+C}^1 \right\|_{\text{tr}} \leq \frac{1}{\rho_0} \max \left(\frac{\rho_\mu}{\ell_{\rho,\mu}}, \sqrt{\rho_\mu Z(\ell_{\rho,\mu})} \right),$$

$$\limsup_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{1}{n} \left\| \left(\gamma_{\Psi_{\omega}^U(\Lambda,n)} - \gamma_{\Psi_{\Lambda,n}^{\text{opt}}} \right) \left(\mathbf{1} - \mathbf{1}_{<\ell_{\rho,\mu}+C}^1 \right) \right\|_{\text{tr}} \leq \frac{1}{\rho_0} \max \left(\frac{\rho_\mu}{\ell_{\rho,\mu}}, \rho_\mu \sqrt{Z(\ell_{\rho,\mu})} \right).$$

Here, $\|\cdot\|_{\text{tr}}$ denotes the trace norm in $L^2(\Lambda)$.

This result calls for some comments. Let us first note that, if Z , that is, U , decays sufficiently fast at infinity, typically exponentially fast with a large rate, then the two estimates in Theorem 1.5 can be united into

$$\limsup_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho_\mu}} \frac{1}{n} \left\| \gamma_{\Psi_{\omega}^U(\Lambda,n)} - \gamma_{\Psi_{\Lambda,n}^{\text{opt}}} \right\|_{\text{tr}} \leq C \frac{\rho_\mu}{\ell_{\rho,\mu}}.$$

In this case, Theorem 1.5 can be summarized graphically. In Figure 2, using the same representation as in Figure 1, we compare the non interacting and the interacting ground state. The non interacting ground state distribution of particles is represented in blue, the interacting one in green. We assume that U has compact support and restrict ourselves to pieces shorter than $3\ell_{\rho,\mu}$.

Indeed, in this case, comparing (1.22) and (1.24), we see

$$\begin{aligned} \gamma_{\Psi_{\omega}^0(\Lambda,n)} - \gamma_{\Psi_{\Lambda,n}^{\text{opt}}} = & \sum_{2\ell_{\rho,\mu} - \log(1-\gamma_*^\mu) \leq |\Delta_k(\omega)|} \left(\gamma_{\varphi_{\Delta_k(\omega)}^1} + \gamma_{\varphi_{\Delta_k(\omega)}^2} - \gamma_{\zeta_{\Delta_k(\omega)}^1} \right) \\ & - \sum_{\ell_{\rho,\mu} - \rho\mu\gamma_*^\mu \leq |\Delta_k(\omega)| \leq \ell_{\rho,\mu}} \gamma_{\varphi_{\Delta_k(\omega)}^1} \\ & + \sum_{2\ell_{\rho,\mu} \leq |\Delta_k(\omega)| \leq 2\ell_{\rho,\mu} - \log(1-\gamma_*^\mu)} \gamma_{\varphi_{\Delta_k(\omega)}^2} + \tilde{R}^{(1)} \end{aligned} \quad (1.27)$$

where $\tilde{R}^{(1)}$ satisfies the same properties as $R^{(1)}$ in (1.22).

Thus, to obtain $\gamma_{\Psi_{\omega}^{\text{opt}}}$ from $\gamma_{\Psi_{\omega}^0(\Lambda,n)}$, we have displaced (roughly) $\gamma_*^\mu \rho_\mu n$ particles living in pieces of length within $[2\ell_{\rho,\mu}, 2\ell_{\rho,\mu} - \log(1-\gamma_*^\mu)]$ (i.e., pieces containing exactly two states below energy $E_{\rho,\mu}$ and the energy of the top state stays above $E_{\rho,\mu} \left(1 + \frac{\log(1-\gamma_*^\mu)}{\ell_{\rho,\mu}}\right)$ up to smaller order terms in $\ell_{\rho,\mu}^{-1}$) to pieces having lengths within $[\ell_{\rho,\mu} - \rho\gamma_*^\mu, \ell_{\rho,\mu}]$ (i.e., having ground state energy within the interval $[E_{\rho,\mu}, E_{\rho,\mu} \left(1 + \frac{2\rho\gamma_*^\mu}{\ell_{\rho,\mu}}\right)]$ up to smaller order terms in $\ell_{\rho,\mu}^{-1}$). In the remaining of (roughly) $(1-\gamma_*^\mu)\rho n$ pieces containing exactly two states below energy $E_{\rho,\mu}$ (that is, pieces of length within $[2\ell_{\rho,\mu} - \log(1-\gamma_*^\mu), 3\ell_{\rho,\mu}]$ or alternatively those with the top state below $E_{\rho,\mu} \left(1 + \frac{\log(1-\gamma_*^\mu)}{\ell_{\rho,\mu}}\right)$ (up to smaller order terms in $\ell_{\rho,\mu}^{-1}$), we have substituted the free two-particles ground state (given by the anti-symmetric tensor product of the first two Dirichlet levels in this piece) by the ground state of the interacting system (1.15). In particular, we compute (remark that the first sum in (1.27) contributes only to the error term according to Corollary 6.12)

$$\lim_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{1}{n} \left\| \gamma_{\Psi_{\omega}^0(\Lambda,n)} - \gamma_{\Psi_{\Lambda,n}^{\text{opt}}} \right\|_{\text{tr}} = 2\gamma_*^\mu \rho_\mu + O\left(\frac{\rho_\mu}{\ell_{\rho,\mu}}\right),$$

and, recalling (1.23), we then compute

$$\lim_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{1}{n^2} \left\| \gamma_{\Psi_{\omega}^0(\Lambda,n)}^{(2)} - \frac{1}{2}(\text{Id} - \text{Ex}) \left[\gamma_{\Psi_{\Lambda,n}^{\text{opt}}} \otimes \gamma_{\Psi_{\Lambda,n}^{\text{opt}}} \right] \right\|_{\text{tr}} = 2\gamma_*^\mu \rho_\mu + O\left(\frac{\rho_\mu}{\ell_{\rho,\mu}}\right). \quad (1.28)$$

So the main effect of the interaction is to shift a macroscopic (though small when ρ_μ is small) fraction of the particles to different pieces.

Let us now discuss what happens when the interaction does not decay so fast, typically, if it decays only polynomially. In this case, Theorem 1.5 tells us that one has to distinguish

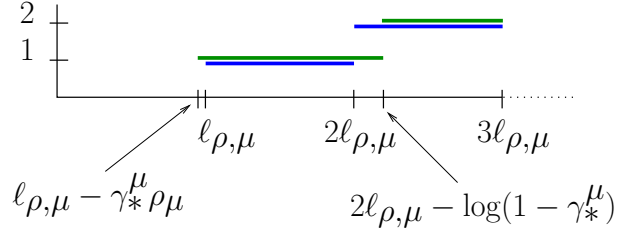


FIGURE 2. The distribution of particles in the interacting ground state.

between short and long pieces. In the long pieces, the description of the ground state is still quite good as the error estimate is still of order $o(\rho_\mu)$. Of course, this result only tells us something for the pieces of length at most $3\ell_{\rho,\mu}$: the larger ones are very few, thus, can only carry so few particles (see Lemma 3.27) that these can be integrated into the remainder term. For short intervals, the situation is quite different. Here, the remainder term becomes much larger, only of order $O\left(\sqrt{\rho_\mu}\ell_{\rho,\mu}^{-k/2}\right)$ if $Z(x) \asymp x^{-k}$ at infinity. This loss is explained in the following way. The short pieces carry the majority of the particles. When U is of longer range, particles in rather distant pieces start to interact in a way that is not negligible with respect to the second term of the expansion (1.14) (which gives an average surplus of energy per particle for the interacting ground state compared to the free one); thus, it may become energetically profitable to relocate some of these particles to new pieces so as to minimize the interaction energy. When the range of the interaction increases, the ground state will relocate more and more particles. Nevertheless, the shift in energy will still be smaller than the correction term obtained by relocating some of the particles living in pairs in not too long intervals; this is going to be the case as long as U satisfies the decay assumption (HU). When U decays slower than that, the main correction to the interacting ground state energy per particle can be expected to be given by the relocation of many particles living alone in their piece to new pieces so as to diminish the interaction energy.

We also obtain an analogue of Theorem 1.5 for the 2-particles density matrix of the ground state Ψ^U . We prove

Theorem 1.6. *Fix $\mu > 0$. Assume (HU) holds. Then, there exist $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0)$, ω -a.s., one has*

$$\begin{aligned} \limsup_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{1}{n^2} \left\| \left(\gamma_{\Psi_\omega^U(\Lambda, n)}^{(2)} - \frac{1}{2}(\text{Id} - \text{Ex}) \left[\gamma_{\Psi_{\Lambda, n}^{\text{opt}}} \otimes \gamma_{\Psi_{\Lambda, n}^{\text{opt}}} \right] \right) \mathbf{1}_{<\ell_{\rho, \mu} + C}^2 \right\|_{tr} \\ \leq \frac{1}{\rho_0} \max \left(\frac{\rho_\mu}{\ell_{\rho, \mu}}, \sqrt{\rho_\mu Z(\ell_{\rho, \mu})} \right) \end{aligned}$$

and

$$\begin{aligned} \limsup_{\substack{L \rightarrow +\infty \\ n/L \rightarrow \rho}} \frac{1}{n^2} \left\| \left(\gamma_{\Psi_\omega^U(\Lambda, n)}^{(2)} - \frac{1}{2}(\text{Id} - \text{Ex}) \left[\gamma_{\Psi_{\Lambda, n}^{\text{opt}}} \otimes \gamma_{\Psi_{\Lambda, n}^{\text{opt}}} \right] \right) \left(\mathbf{1} - \mathbf{1}_{<\ell_{\rho, \mu} + C}^2 \right) \right\|_{tr} \\ \leq \frac{1}{\rho_0} \max \left(\frac{\rho_\mu}{\ell_{\rho, \mu}}, \rho_\mu \sqrt{Z(\ell_{\rho, \mu})} \right) \end{aligned}$$

where, for $\ell \geq 0$, we recall that $\|\cdot\|_{tr}$ denotes the trace norm in $L^2(\Lambda) \wedge L^2(\Lambda)$, recall (1.25) and define

$$\mathbf{1}_{<\ell}^2 = \mathbf{1}_{<\ell}^1 \otimes \mathbf{1}_{<\ell}^1. \quad (1.29)$$

1.4. Discussion and perspectives. While a very large body of mathematical works has been devoted to one particle random Schrödinger operators (see e.g. [Kir08a, PF92]), there are only few works dealing with many interacting particles in a random medium (for the case of finitely many particles, see, for example, [AW09] or [CS09]).

The general Hamiltonian describing n electrons in a random background potential V_ω interacting via a pair potential U can be described as follows. In a d -dimensional domain Λ , consider the operator

$$H_\omega(\Lambda, n) = -\Delta_{nd}|_{\Lambda^n} + \sum_{i=1}^n V_\omega(x^i) + \sum_{i < j} U(x^i - x^j),$$

where, for $j \in \{1, \dots, n\}$, x^j denotes the coordinates of the j -th particle. The operator $H_\omega(\Lambda, n)$ acts on a space of totally anti-symmetric functions $\bigwedge_{i=1}^n L^2(\Lambda)$ which reflects the electronic nature of particles.

The general problem is to understand the behavior of $H_\omega(\Lambda, n)$ in the thermodynamic limit $\Lambda \rightarrow \infty$ while $n/|\Lambda| \rightarrow \rho > 0$; ρ is the particle density. One of the questions of interest is that of the behavior of the ground state energy, say, $E_\omega(\Lambda, n)$ and of the ground state $\Psi_\omega(\Lambda, n)$.

While the thermodynamic limit is known to exist for various quantities and in various settings (see [Ven13] for the micro-canonical ensemble that we study in the present paper and [BL12b] for the grand canonical ensemble), we don't know of examples, except for the model studied in the present paper, where the limiting quantities have been studied. In particular, it is of interest to study the dependence of these limiting quantities in the different physical parameters like the density of particles, the strength of the disorder or the interaction potential. As we shall argue now, for these questions to be tractable, one needs a good description of the spectral data of the underlying one particle random model.

1.4.1. *Why the pieces model?* In order to tackle the question of the behavior of n -electron ground state, let us first consider the system without interactions. This is not equivalent to a one-particle system as Fermi-Dirac statistics play a crucial role.

Let us assume our one particle model is ergodic and admits an integrated density of states (see (1.2) and e.g. [Kir08b, PF92]). As described above for the pieces model, the ground state of the n non interacting electrons is given by (1.18) and its energy per particle is given by

$$\frac{E_\omega^0(\Lambda, n)}{n} = \frac{1}{n} \sum_{j=1}^n E_{j,\omega}^\Lambda = |\Lambda| \int_{-\infty}^{E_{n,\omega}^\Lambda} E d \left[\frac{\#\{\text{eigenvalues of } H_\omega(\Lambda) \text{ below } E\}}{|\Lambda|} \right] \quad (1.30)$$

where $E_{n,\omega}^\Lambda$ is the n -th eigenvalue of the one particle random Hamiltonian $H_\omega(\Lambda)$, i.e., the smallest energy E such that

$$\frac{\#\{\text{eigenvalues of } H_\omega(\Lambda) \text{ below } E\}}{|\Lambda|} = \frac{n}{|\Lambda|}. \quad (1.31)$$

Here, we have kept the notations of the beginning of section 1.3.

The existence of the density of states, say $N(E)$, (see (1.2)), then, ensures the convergence of $E(\Lambda, n)$ to a solution to the equation $N(E) = \rho$, say E_ρ . Thus, to control the non interacting ground state, one needs to control all (or at least most of) the energies of the random operator $H_\omega(\Lambda)$ up to some macroscopic energy E_ρ . In particular, one needs to control simultaneously a number of energies of $H_\omega(\Lambda)$ that is of size the volume of Λ .

To our knowledge, up to now, there are no available mathematical results that give the simultaneous control over that many eigenvalues for general random systems. The results dealing with the spectral statistics of (one particle) random models deal with much smaller intervals: in [Min96], eigenvalues are controlled in intervals of size $K/|\Lambda|$ for arbitrary large K if Λ is sufficiently large; in [GK10, GK13], the interval is of size $|\Lambda|^{1-\beta}$ for some not too large positive β .

The second problem is that all these results only give a very rough picture of the eigenfunctions, a picture so rough that it actually is of no use to control the effect of the interaction on such states: the only information is that the eigenstates live in regions of linear size at most $\log |\Lambda|$ and decay exponentially outside such regions (see, e.g., [GK10] and references therein).

The pieces model that we deal with in the present paper exhibits the typical behavior of a random system in the localized regime: for $H_\omega(\Lambda)$,

- the eigenfunctions are localized (on a scale $\log |\Lambda|$)
- the localization centers and the eigenvalues satisfy Poisson statistics.

The advantage of the pieces model is that the eigenfunctions and eigenvalues are known explicitly and easily controlled. This is a consequence of the fact that a crucial quantum phenomenon is missing in the pieces model, namely, tunneling. Of course, once the particles do interact with each other, tunneling is again re-enabled.

All of this could lead one to think that the pieces model is very particular. Actually, at low energies, general one-dimensional random models exhibit the same characteristics as the pieces model up to some exponentially small errors which are essentially due to tunneling (see [Klo14]).

It seems reasonable to guess that the behavior will be comparable for general random operators in higher dimensions and, thus, that the results of the present paper on interacting electrons in a random potentials should find their analogues for these models.

1.4.2. *Outline of the paper.* In section 2, after rescaling the parameters of the problem so as to send μ to 1 and ρ to ρ/μ , we first discuss the validity of our results in a more general asymptotic regime in μ and ρ . We, then, gather some basic but crucial statistical properties of the distribution of the pieces. We first describe the free electrons. For the pieces model, a statistical analysis of the distribution of pieces gives exact expressions for the one-particle integrated density of states and the Fermi energy in Proposition 2.6. We also study the non interacting model and introduce notations for later use.

In section 3, we first introduce the occupation numbers (i.e., the number of particles a given state puts in each piece); the existence of the occupation numbers is tantamount to the existence of a particular orthogonal sum decomposition of the Hamiltonian $H_\omega^U(\Lambda, n)$. We prove that the ground state of $H_\omega^U(\Lambda, n)$ restricted to a fixed occupation space is non degenerate and, from this result, derive Theorem 1.1, the almost sure non degeneracy of the ground state for real analytic interaction.

Next, still in section 3, we prove the asymptotic formula for the interacting ground state energy per particle. The proof relies essentially on the minimizing properties of the ground state. This minimizing property yields a good description for the occupation numbers associated to a ground state. To get this description, we first study the ground state of the Hamiltonian $H_\omega^{U^p}(\Lambda, n)$ where the interactions have been cut-off at infinity (i.e., U^p is compactly supported). We construct an approximate ground state Ψ^{opt} which can essentially be thought of as the ground state for the Hamiltonian $H_\omega^{U^p}(\Lambda, n)$ restricted to the pieces shorter than $3\ell_{\rho, \mu}$. Then, letting $W^r(\Lambda, n) := H_\omega^U(\Lambda, n) - H_\omega^{U^p}(\Lambda, n)$ be the long range behavior of the interactions, one has

$$E_\omega^{U^p}(\Lambda, n) \leq E_\omega^U(\Lambda, n) \leq \langle H_\omega^{U^p}(\Lambda, n) \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle + \langle W^r(\Lambda, n) \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle$$

The minimizing property of Ψ^{opt} yields

$$E_\omega^{U^p}(\Lambda, n) \geq \langle H_\omega^{U^p}(\Lambda, n) \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle + n o(\rho_\mu \mu^{-1} \ell_{\rho, \mu}^{-3})$$

(see Theorem 3.28).

On the other hand, the decay assumption (HU) on U and the explicit construction of Ψ^{opt} yield

$$\langle W^r(\Lambda, n) \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle = n o(\rho_\mu \mu^{-1} \ell_{\rho, \mu}^{-3})$$

(see Proposition 2.7).

This yields the proof of Theorem 1.3.

In the course of these proofs, we also prove a certain number of estimates on the distance between the occupation numbers of the interacting ground state(s) to the state Ψ^{opt} .

Section 4 is devoted to the proofs of Theorems 1.5 and 1.6. Therefore, we transform the bounds of the distance between occupation numbers into bounds on the trace class norms of the difference between the one (and the two) particle densities of the interacting ground state(s) and the state Ψ^{opt} .

In Theorems 4.2 (resp. Theorem 4.4), we derive general formulas for the one particle (resp. two particles) density of a state expressed in a certain well chosen basis of $\mathfrak{H}^n(\Lambda)$. One of the main steps on the path going from occupation number bounds to the trace class norm bounds is to prove that, in most pieces, once the particle number is known, the state must be in the ground state for the given particle number. This is the purpose of Lemma 4.12; it relies on the minimizing properties of the ground state; actually, it is proved for a larger set of states, states satisfying a certain energy bound.

We then use Theorems 4.2 (resp. Theorem 4.4) to derive Theorems 1.5 (resp. Theorem 1.6).

Section 5 is devoted to the proof of the almost sure convergence of the ground state energy per particle. The proof is essentially identical to that found in [Ven13] except for the sub-additive estimate crucial to the proof. This estimate is provided by Theorem 5.1.

In section 6, we prove Proposition 1.4 as well as a number of estimates on the ground states and ground state energies for a finite number of electrons living in a fixed number of pieces and interacting.

In three appendices, we gather a number of results used in the main body of paper. In appendix A, we prove the results on the statistics of the pieces stated in section 2. Appendix B is devoted to a simple technical lemma used intensively in the derivation of Theorems 1.5 and 1.6 in section 4. Appendix C is devoted to anti-symmetric tensor products.

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2. PRELIMINARY RESULTS

In this section, we state a number results on the Luttinger-Sy model defined in section 1 on which our analysis is based. We first recall some results on the thermodynamic limit specialized to the pieces model. Then, we describe the statistics of the eigenvalues and eigenfunctions of the pieces model defined in (1.1); in the case of the pieces model, it suffices therefore to describe the statistics of the pieces (see section 2.2).

In section 2.4, we describe the non interacting system of n electrons.

2.1. Rescaling the operator. Consider the scaling $\tilde{x} = \mu x$, that is, define

$$S_\mu : \bigwedge_{j=1}^n L^2([0, L]) \rightarrow \bigwedge_{j=1}^n L^2([0, \tilde{L}]) \tag{2.1}$$

$$u \mapsto S_\mu u \quad \text{where } (S_\mu u)(x) = \mu^{n/2} u(\mu x) \quad \text{and} \quad \tilde{L} = \mu L.$$

One then computes

$$S_\mu^* H_\omega(L, n) S_\mu = \mu^2 \tilde{H}_\omega(\tilde{L}, n)$$

where $\tilde{H}_\omega(\tilde{L}, n)$ is the interacting pieces model on the interval $[0, \tilde{L}]$ defined by a Poisson process of intensity 1 and with pair interaction potential

$$U^\mu(\cdot) = \mu^{-2} U(\mu^{-1}\cdot). \tag{2.2}$$

For $\tilde{H}_\omega(\tilde{L}, n)$, the thermodynamic limit becomes

$$\frac{n}{\tilde{L}} = \frac{n}{\mu L} \rightarrow \frac{\rho}{\mu} = \rho_\mu.$$

We shall prove Theorems 1.3, 1.5 and 1.6 under the additional assumption $\mu = 1$. Let us now explain how Theorems 1.3, 1.5 and 1.6 get modified when one goes from $\mu = 1$ to arbitrary μ .

If one denotes by γ^μ the constant defined by Proposition 1.4 applied to the interaction potential U^μ instead of U , a direct computation yields $\gamma^\mu = \mu\gamma$.

In the same way, a direct computation yields that Z^μ , the analogue of Z in assumption (HU) for U^μ , is given by $Z^\mu(\cdot) = \mu^2 Z(\mu^{-1}\cdot)$. Thus, for the function f_{Z^μ} (see (1.26), (3.28) and (3.29)) defined for U^μ , see (2.2), one obtains $f_{Z^\mu}(\cdot) = \mu^2 f_Z(\mu^{-1}\cdot)$. This suffices to obtain Theorems 1.5 and 1.6 for μ arbitrary fixed from the case $\mu = 1$.

From now on, as we fix $\mu = 1$, we shall write drop the sub- or superscript μ and write, e.g., ℓ_ρ for $\ell_{\rho,\mu}$, E_ρ for $E_{\rho,\mu}$, etc. Similarly, the dependence on the random parameter ω will be frequently dropped so as to simplify notations.

2.1.1. Other asymptotic regimes. In the introduction, for the sake of simplicity we chose to state our results at fixed μ and sufficiently small ρ (depending on μ). Actually, the results that we obtained stay correct under less restrictive conditions on μ and ρ . The conditions that are required are the following. Fix $\mu_0 > 0$; then, Theorems 1.3, 1.5 and 1.6 stay correct as long as $\mu \in (0, \mu_0)$, ρ_μ be sufficiently small and $\ell_{\rho,\mu}$ sufficiently large depending only on μ_0 . Let us now explain this.

Therefore, we analyze the remainder terms of (3.80) (thus, of (3.82)). The second term in the last equality in (3.80) multiplied by μ^2 (to rescale energy properly, see above) becomes

$$\pi^2 \mu^2 \gamma_*^\mu \frac{\rho_\mu}{|\log \rho_\mu|^3} = \pi^2 \gamma_*^\mu \mu^{-1} \rho_\mu \ell_{\rho,\mu}^{-3} + o(\rho_\mu \ell_{\rho,\mu}^{-3})$$

by (1.12). Note that, by (1.17), $\gamma_*^\mu \mu^{-1}$ stays bound from above and below as $\mu \rightarrow 0^+$.

The remainder term in the last equality in (3.80) multiplied by μ^2 (to rescale energy properly, see above) becomes

$$\mu^2 \frac{\rho_\mu}{|\log \rho_\mu|^3} O(f_{Z^\mu}(|\log \rho_\mu|)) = \frac{\rho_\mu \mu^4}{\ell_{\rho,\mu}^3} O(f_Z[\ell_{\rho,\mu}(1 + o(1))]) = o\left(\frac{\rho_\mu \mu^{-1}}{\ell_{\rho,\mu}^3}\right)$$

when $\rho_\mu \rightarrow 0$ and $\ell_{\rho,\mu} \rightarrow +\infty$ while μ stays bounded.

This then yields Theorem 1.3 for (μ, ρ) arbitrary in the regime described above from the case $\mu = 1$ and ρ small.

To obtain Theorems 1.5 and 1.6 for μ arbitrary, we just use $Z^\mu(\cdot) = \mu^2 Z(\mu^{-1}\cdot)$ and the fact that Z is decaying; indeed, this implies that

$$Z^\mu(2|\log \rho_\mu|) = \mu^2 Z(2\ell_{\rho,\mu}(1 + o(1))) \leq \mu^2 Z(\ell_{\rho,\mu})$$

when $\rho_\mu \rightarrow 0$ and $\ell_{\rho,\mu} \rightarrow +\infty$ while μ stays bounded.

This suffices to obtain Theorems 1.5 and 1.6 for (μ, ρ) arbitrary in the regime described above from the case $\mu = 1$ and ρ small.

From now on, we fix $\mu = 1$ and assume ρ be small. Thus, we shall drop the sub- or superscript μ and write, e.g., ℓ_ρ for $\ell_{\rho,\mu}$, E_ρ for $E_{\rho,\mu}$, etc. Similarly, the dependence on the random parameter ω will be frequently dropped so as to simplify notations.

2.2. The analysis of the one-particle pieces model. Most of the proofs of the results stated in the present section can be found in Appendix A.

Recall that we partition $[0, L]$ using a Poisson process of intensity 1 and write

$$[0, L] = \bigcup_{j=1}^{m(\omega)} \Delta_j(\omega). \quad (2.3)$$

Note that, by a standard large deviation principle, for $\beta \in (0, 1/2)$, with probability at least $1 - e^{-L^\beta}$, one has $m = L + O(L^\beta)$.

Moreover, with probability one,

- $\min_{1 \leq j \leq m(\omega)} |\Delta_j(\omega)| > 0$,
- if $j \neq j'$ then $\frac{|\Delta_j(\omega)|^2}{|\Delta_{j'}(\omega)|^2} \notin \mathbb{Q}$.

Thus, distinct pieces generate distinct Dirichlet Laplacian energy levels. In particular, with probability one, all the eigenfunctions of the one-particle Hamiltonian $H_\omega(L) = H_\omega(L, 1)$ are supported on a single piece $\Delta_j(\omega)$ and the corresponding eigenvalues are simple.

Hence, we will enumerate the eigenvalues and the eigenfunctions of $H_\omega(L)$ using a two-component index (Δ_j, k) where

- Δ_j is the piece of the partition (2.3) on which the eigenfunction is supported,
- k is the index of the eigenvalue within the ordered list of eigenvalues of this piece,

i.e.,

$$\psi_{(\Delta_j, k)}(x) = \sqrt{\frac{2}{|\Delta_j|}} \sin\left(\frac{\pi k(x - \inf \Delta_j)}{|\Delta_j|}\right) \mathbf{1}_{\Delta_j}(x)$$

and the corresponding energy

$$E_{(\Delta_j, k)} = \left(\frac{\pi k}{|\Delta_j|}\right)^2. \quad (2.4)$$

Let $\mathcal{P} = \mathcal{P}(\omega)$ denote the set of all available indices enumerating single-particle states, i.e., $\mathcal{P} = \{\Delta_j\}_{j=1}^{m(\omega)} \times \mathbb{N}$.

In parallel to this two-component enumeration system, we will use a direct indexing procedure: $\{(E_j, \psi_j)\}_{j \in \mathbb{N}}$ are the eigenvalues and associated eigenfunctions of the one particle Hamiltonian $H_\omega(L)$ counted with multiplicity ordered with increasing energy.

2.3. The statistics of the pieces. We first study the statistical distribution of the pieces generated by the Poisson process. We will primarily be interested in the joint distributions of their lengths. These statistics immediately provide the statistics of the eigenvalues and eigenfunctions of the pieces model. These results are presumably well known; as we don't know a convenient reference, we provide their proofs in Appendix A for the sake of completeness.

In the sequel, the probability of the events will typically be $1 - O(L^{-\infty})$: we recall that $A_k = O(k^{-\infty})$ if $\forall N \geq 0, \lim_{k \rightarrow +\infty} k^N A_k = 0$. Actually, the proofs show that the probabilities

lie at an exponentially small distance from 1, i.e., $O(L^{-\infty}) = e^{-L^\beta}$ for some $\beta > 0$.

We prove

Proposition 2.1. *With probability $1 - O(L^{-\infty})$, the largest piece has length bounded by $\log L \cdot \log \log L$, i.e.,*

$$\max_{1 \leq k \leq m(\omega)} |\Delta_k(\omega)| \leq \log L \cdot \log \log L.$$

On the distribution of the length of the pieces, one proves

Proposition 2.2. *Fix $\beta \in (2/3, 1)$. Then, for L large, for any $(a_L, b_L) \in [0, \log L \cdot \log \log L]^2$, with probability $1 - O(L^{-\infty})$, the number of pieces of length contained in $[a_L, a_L + b_L]$ is equal to*

$$e^{-a_L}(1 - e^{-b_L}) \cdot L + R_L \cdot L^\beta \quad \text{where} \quad |R_L| \leq \kappa$$

and the positive constant κ is independent of a_L, b_L .

The proof of Proposition 2.2 is given in Appendix A.

We will also use the joint distributions of pairs and triplets of pieces that are close to each other. We prove

Proposition 2.3. *Fix $\beta \in (2/3, 1)$. Then, for any a, b positive and b, d, g, f all non negative, with probability $1 - O(L^{-\infty})$, the number of pairs of pieces such that*

- *the length of the left most piece is contained in $[a, a + b]$,*
- *the length of the right most piece is contained in $[c, c + d]$,*

- the distance between the two pieces belongs to $[g, g + f]$

is equal to

$$f e^{-a-c}(1 - e^{-b})(1 - e^{-d}) \cdot L + R_L \cdot L^\beta \quad \text{where} \quad |R_L| \leq \kappa \quad (2.5)$$

and the positive constant κ may depend on (a, b, c, d, f, g) .

For pairs of pieces, we shall also use

Proposition 2.4. *For $\ell, \ell', d > 0$, with probability $1 - O(L^{-\infty})$, one has*

$$\# \left\{ \begin{array}{l} \text{pairs of pieces at most at a dis-} \\ \text{tance } d \text{ from each other such that} \\ \text{the left most piece longer than } \ell, \\ \text{the right most piece longer than } \ell'. \end{array} \right\} \leq (2 + d)e^{-\ell - \ell'} L.$$

Finally, for triplets of pieces, we shall use

Proposition 2.5. *For $\ell, \ell', \ell'', d > 0$, with probability $1 - O(L^{-\infty})$, one has*

$$\# \left\{ (\Delta, \Delta', \Delta'') \text{ s.t. } \begin{array}{l} \Delta' \text{ between } \Delta \text{ and } \Delta'' \\ \text{dist}(\Delta, \Delta') \leq d, \text{ dist}(\Delta', \Delta'') \leq d \\ |\Delta| \leq \ell, |\Delta'| \leq \ell', |\Delta''| \leq \ell''. \end{array} \right\} \leq (2 + d^2)e^{-\ell - \ell' - \ell''} L.$$

As a straightforward consequence of Proposition 2.2, exploiting the formula (2.4) for the Dirichlet eigenvalues of the Laplacian on an interval, one obtains the explicit formula (1.2) for the one-particle integrated density of states for the pieces model defined in (1.2) (here, $\mu = 1$) That is, one proves

Proposition 2.6 (The one particle IDS). *The one-particle integrated density of states for the pieces model is given by*

$$N(E) = \frac{\exp(-\ell_E)}{1 - \exp(-\ell_E)} \mathbf{1}_{E>0} \quad (2.6)$$

where ℓ_E is defined in (1.2).

Formula (2.6) was already obtained in [LS73]; in Appendix A.1, we give a short proof for the readers convenience.

Recalling the scaling defined in section 2.1 immediately yields (1.2) for general μ .

2.4. Free electrons. Understanding the system without interactions will be key to answering the main questions raised in the present work. For free electrons, i.e., when the interactions are absent, $U \equiv 0$, the energy per particle $\mathcal{E}^0(\rho)$ can be expressed in terms of one-particle density of states measure.

2.4.1. The ground state energy per particle. Recall that (see Theorem 1.3), for a density of particles ρ , the Fermi energy E_ρ is a solution of the equation $N(E_\rho) = \rho$. In the present case, as N is continuous and strictly increasing from 0 to $+\infty$, the solution to this equation is unique for any $\rho > 0$. The length of the interval whose Dirichlet Laplacian has the Fermi energy E_ρ as ground state energy is the Fermi length ℓ_ρ given by

$$\ell_\rho := \pi / \sqrt{E_\rho} \quad (2.7)$$

As a direct corollary to (1.2) (recall that $\mu = 1$) or equivalently Proposition 2.6, we see that the Fermi energy is given by

$$E_\rho = \pi^2 (\log(\rho^{-1} + 1))^{-2} \sim \pi^2 |\log \rho|^{-2} \quad \text{when} \quad \rho \rightarrow 0 \quad (2.8)$$

and the Fermi length by:

$$\ell_\rho = \log(\rho^{-1} + 1) \sim |\log \rho| \quad \text{when } \rho \rightarrow 0. \quad (2.9)$$

We recall

Proposition 2.7 ([Ven13, Theorem 5.13 and Lemma 5.14]). *Let $E_{n,\omega}^\Lambda$ denote the n -th energy level of $H_\omega(L)$ (counting multiplicity). Then, ω -a.s., one has*

$$E_{n,\omega}^\Lambda \xrightarrow[n/L \rightarrow \rho]{L \rightarrow \infty} E_\rho \quad \text{and} \quad \mathcal{E}^0(\rho) = \frac{1}{\rho} \int_{-\infty}^{E_\rho} E \, dN(E). \quad (2.10)$$

Proposition 2.7 follows easily from Lemma 3.13, (1.30), (1.31) and (A.17).

We see that

- the highest energy level occupied by a system of non interacting electrons tends to the Fermi energy in the thermodynamic limit;
- the n -electron ground state energy per particle is the energy averaged with respect to the density of states measure of the one-particle system conditioned on energies less than the Fermi energy.

Combining formulas (2.8) and (2.10), one can expand $\mathcal{E}^0(\rho)$ into inverse powers of $\log \rho$ up to an arbitrary order. Taking the scaling defined in section 2.1 into account, (2.10) immediately implies (1.13).

2.4.2. *The eigenfunctions.* Let us now describe the eigenfunctions of $H_\omega^0(L, n)$. Let us recall that $(E_p)_{p \in \mathcal{P}}$ are the eigenvalues of the one-particle operator $H_\omega(L)$ and $(\psi_p)_{p \in \mathcal{P}}$ are the corresponding normalized eigenfunctions; here p in \mathcal{P} is a (piece - energy level) index. The n -electron eigenstates without interactions are given by the following procedure. Pick a set $\alpha := \{\alpha_1, \dots, \alpha_n\} \subset \mathcal{P}$ of n indices, $\text{card } \alpha = |\alpha| = n$. The normalized eigenstate associated to α is given by the Slater determinant

$$\Psi_\alpha(x^1, x^2, \dots, x^n) := \psi_{\alpha_1} \wedge \dots \wedge \psi_{\alpha_n} := \frac{1}{\sqrt{n!}} \det(\psi_p(x^j))_{\substack{p \in \alpha \\ 1 \leq j \leq n}}. \quad (2.11)$$

One easily checks that $(H_\omega^0(\Lambda, n) - E_\alpha)\Psi_\alpha = 0$ for the energy E_α defined by

$$E_\alpha = \sum_{p \in \alpha} E_p. \quad (2.12)$$

The subset α indicates which one-particle energy levels are occupied in the multi-particle state Ψ_α . For instance, in the ground state of n electrons, one chooses the states with lowest possible energy.

Notation 2.8. For a Slater determinant Ψ_α (see 2.11)) and $p \in \alpha$, we will refer to the one-particle functions ψ_p as *particles* that constitute the n -electron state indexed by α . Moreover, with a slight abuse of terminology, we will refer to an multi-index α as a (n -electron) state and to p in α as a particle.

3. THE ASYMPTOTICS FOR THE GROUND STATE ENERGY PER PARTICLE

In this section, we prove Theorem 1.3 on the asymptotic expansion of the ground state energy per particle in terms of small particle density. We assume that the pair interaction potential U satisfies condition (HU).

3.1. Decomposition by occupation numbers. We give a definition of the number of particles occupying a given piece. Therefore, we shall use the special structure of the Hamiltonian $H_\omega^0(\Lambda, n)$, that is, that of $H_\omega(L)$ (see (1.4) and (1.1)).

Fix ω . Recall that $(\Delta_j(\omega))_{1 \leq j \leq m}$ are the pieces defined in (2.3) ($m = m(\omega)$). The one particle space is then decomposed into

$$L^2(\Lambda) = L^2([0, L]) = \bigoplus_{1 \leq j \leq m}^\perp L^2(\Delta_j(\omega)). \quad (3.1)$$

Thus, for the n -particle space \mathfrak{H}^n (see (1.3)), we obtain the decomposition

$$\mathfrak{H}^n = \mathfrak{H}^n(\Lambda) = \bigwedge_{j=1}^n L^2(\Lambda) = \bigoplus_{\substack{Q=(Q_1, \dots, Q_m) \in \mathbb{N}^m \\ Q_1 + \dots + Q_m = n}} \mathfrak{H}_Q \quad (3.2)$$

where we have defined

Definition 3.1. For $Q = (Q_1, \dots, Q_m) \in \mathbb{N}^m$ s.t. $Q_1 + \dots + Q_m = n$, the space of states of fixed occupation Q denoted by \mathfrak{H}_Q is given by

$$\mathfrak{H}_Q = \bigwedge_{j=1}^m \left(\bigwedge_{k=1}^{Q_j} L^2(\Delta_j(\omega)) \right). \quad (3.3)$$

Here, as usual, we set $\bigwedge_{k=1}^0 L^2(\Delta_j(\omega)) = \mathbb{C}$.

An occupation Q is a multi-index of length m and of “modulus” n . Note that, as $\Delta_j(\omega) \cap \Delta_{j'}(\omega) = \emptyset$ for $j \neq j'$, we can identify

$$\mathfrak{H}_Q = \bigotimes_{j=1}^m \left(\bigwedge_{k=1}^{Q_j} L^2(\Delta_j(\omega)) \right).$$

Remark 3.2. The spaces of fixed occupation could also be defined starting from the eigenstates of $H_\omega^0(L, n)$ as in [Ven12]. Indeed, each of the eigenstates of $H_\omega^0(L, n)$, the non interacting Hamiltonian, belongs to a state of fixed occupation. More precisely, if $\Psi_\alpha \in \mathfrak{H}^n$ is the eigenstate of $H_\omega^0(L, n)$ given by (2.11) where $\alpha \subset \mathcal{P}$, $\text{card } \alpha = n$, then, defining the occupation $Q(\alpha) = (Q_1(\alpha), \dots, Q_m(\alpha))$ where, for $1 \leq j \leq m$, $Q_j(\alpha) := \#\{p \in \alpha \mid \text{supp } \psi_p = \Delta_j\}$, we see that $\Psi_\alpha \in \mathfrak{H}_Q$.

The following lemma is crucial in our analysis as it gives global information on the structure of the ground state of the Hamiltonian $H_\omega^U(L, n) = H_\omega^0(L, n) + W_n$. We prove

Lemma 3.3. *Let ω be fixed and let α and β be two n -electron indices corresponding each to an eigenstate of $H_\omega^0(L, n)$.*

If their occupations are different, then the corresponding n -particle states do not interact:

$$Q(\alpha) \neq Q(\beta) \Rightarrow \langle \Psi_\alpha, W_n \Psi_\beta \rangle = 0.$$

Proof. If α and β have different occupation numbers, the supports of Ψ_α and Ψ_β in Λ^n intersect at a set of measure zero: indeed, these supports are obtained by symmetrizing different collections of products of pieces (with repetitions for the pieces that are occupied more than once):

$$Q(\alpha) \neq Q(\beta) \Rightarrow \text{meas}(\text{supp } \Psi_\alpha \cap \text{supp } \Psi_\beta) = 0.$$

The latter means that $\Psi_\alpha \cdot \Psi_\beta \equiv 0$ as a function in $L^2(\Lambda^n)$. Then, clearly, by definition, for the matrix elements, one obtains

$$\langle \Psi_\alpha, W_n \Psi_\beta \rangle = \int_{\Lambda^n} W_n(\mathbf{x}) \Psi_\alpha(\mathbf{x}) \Psi_\beta^*(\mathbf{x}) d\mathbf{x} = 0.$$

Lemma 3.3 is proved. \square

As an immediate corollary to Lemma 3.3, we obtain

Corollary 3.4 (Decomposition by occupation). *Fix ω . For any $Q \in \mathbb{N}^m$ (here and in the sequel, $\mathbb{N} = \{0, 1, \dots\}$), $m = m(\omega)$, the subspace \mathfrak{H}_Q are invariant under the action of the n -particle Hamiltonian $H_\omega^U(L, n) = H_\omega^0(L, n) + W_n$, i.e.,*

$$(H_\omega^U(L, n) + i)^{-1} \mathfrak{H}_Q \subset \mathfrak{H}_Q. \quad (3.4)$$

Thus, the total Hamiltonian $H_\omega^U(L, n)$ is decomposed according to (3.2) in direct sum of its parts H_Q on subspaces of fixed occupation, i.e.,

$$H_\omega^U(L, n) = \bigoplus_{\substack{Q \in \mathbb{N}^m \\ Q_1 + \dots + Q_m = n}} H_Q, \quad (3.5)$$

where $H_Q = H_\omega^U(L, n)|_{\mathfrak{H}_Q}$.

Remark 3.5. All terms of this decomposition as well as the number of pieces m depend on the randomness ω , i.e., the configuration of pieces.

Proof of Corollary 3.4. Fix ω . The space

$$\mathcal{D}_\omega^n := \mathcal{C}_0^\infty \left(\left(\bigcup_{1 \leq j \leq m} \overset{\circ}{\Delta}_j(\omega) \right)^n \right) \cap \mathfrak{H}^n$$

is a core for $H_\omega^U(L, n)$. Here, $\overset{\circ}{\Delta}_j(\omega)$ denotes the interior of $\Delta_j(\omega)$.

It, thus, suffices to check that, for $H_\omega^U(L, n) (\mathfrak{H}_Q \cap \mathcal{D}_\omega^n) \subset \mathfrak{H}_Q$; this follows immediately from Lemma 3.3. This ensures the existence of the decomposition (3.5) and completes the proof of Corollary 3.4. \square

Corollary 3.4 states that the interaction operator W_n is partially diagonalized in the basis of eigenfunctions of $H_\omega^0(L, n)$, i.e., its matrix representation has a block structure corresponding to the subspaces of constant occupation.

3.2. Almost sure non-degeneracy of the interacting ground state. We first restrict ourselves to spaces with fixed occupation to prove

Lemma 3.6. *Fix an occupation Q . The ground state of $(H_\omega^U(L, n))|_{\mathfrak{H}_Q}$ is non-degenerate.*

Proof. To simplify notations, let us write $H = H_\omega^U(L, n)$ and $H^0 = H_\omega^0(L, n)$. Let $(\Delta_{j_p})_{1 \leq p \leq n}$ be the pieces such that $Q_{j_p} \geq 1$; in the list $(\Delta_{j_p})_{1 \leq p \leq n}$, each piece Δ_{j_p} is repeated Q_{j_p} times. We enumerate the pieces so that their left endpoints are non decreasing (i.e., from the leftmost piece to the rightmost piece). So, $p \mapsto j_p$ is non decreasing. Then, the operator H_Q^0 is the Dirichlet Laplacian on a space of anti-symmetric functions defined on the symmetrized domain

$$\Delta_Q = \text{Sym} \left(\bigtimes_{p=1}^n \Delta_{j_p} \right) := \bigcup_{\sigma \in \mathfrak{S}_n} \bigtimes_{p=1}^n \Delta_{\sigma(j_p)}. \quad (3.6)$$

Anti-symmetric functions on the domain (3.6) that vanish on the boundary $\partial(\Delta_Q)$ are in one-to-one correspondence with functions defined on the domain

$$\delta_Q = \{(x^1, \dots, x^n) \text{ s.t. } x^p \in \Delta_{j_p} \text{ and } x^p \leq x^q \text{ for } p < q\} \quad (3.7)$$

that vanish on $\partial(\delta_Q)$, the boundary of δ_Q . Actually,

$$\Delta_Q = \bigcup_{\sigma \in \mathfrak{S}_n} \sigma(\delta_Q) \quad \text{and, for } (\sigma, \sigma') \in \mathfrak{S}_n^2, \quad \sigma(\delta_Q) \cap \sigma'(\delta_Q) = \emptyset \text{ if } \sigma \neq \sigma'.$$

Here, for $\sigma \in \mathfrak{S}_n$, we have set $\sigma : (x^1, \dots, x^n) \mapsto (x^{\sigma(1)}, \dots, x^{\sigma(n)})$.

Thus, finding the ground state of $H_Q = H^0 + W$ is equivalent to finding the ground state of the Schrödinger operator $-\Delta + W$ with Dirichlet boundary conditions on the domain δ_Q . As the domain δ_Q is connected and has a piecewise linear boundary, the ground state of $-\Delta + W$ is non-degenerate (see [Dav90, Theorems 1.4.3, 1.8.2 and 3.3.5] and [RS78, Section XIII.12]). This completes the proof of Lemma 3.6. \square

3.3. The proof of Theorem 1.1. Considering the decomposition (3.5), Lemma 3.6 implies that the only possible source of degeneracy of the ground state is that different occupations, i.e., distributions of particles in the pieces, provide the same ground state energy. Let us show that, almost surely, this does not happen.

Let Π be the support of $d\mu(\omega)$, the Poisson process of intensity 1 on \mathbb{R}_+ . Let $\#(\Pi \cap [0, L])$ be the number of points the Poisson process puts into $(0, L)$. Suppose now that the probability that the ground state of $H_\omega^U(L, n)$ is degenerate is positive.

Thus, for some $m \geq 0$, conditioned on the fact that the Poisson process puts m points into $(0, L)$ (i.e., $\#(\Pi \cap [0, L]) = m$), the probability that the ground state of $H_\omega^U(L, n)$ be degenerate is positive. Let $(\ell_j)_j$ be the lengths of the pieces $(\Delta_j(\omega))_j$, i.e., the $(\Delta_j)_j$ are connected and $\cup_j \Delta_j(\omega) = (0, L) \setminus (\Pi \cap [0, L])$. Conditioned $\#(\Pi \cap [0, L]) = m$, the joint distribution of the vector $(\ell_j)_j$ is known.

Proposition 3.7 ([GMS83]). *Under the condition $\#(\Pi \cap [0, L]) = m$, the vector $(\ell_1, \dots, \ell_{m+1})$ has the same distribution as the random vector*

$$\left(\frac{L \cdot \eta_1}{\eta_1 + \dots + \eta_{m+1}}, \frac{L \cdot \eta_2}{\eta_1 + \dots + \eta_{m+1}}, \dots, \frac{L \cdot \eta_{m+1}}{\eta_1 + \dots + \eta_{m+1}} \right), \quad (3.8)$$

where $(\eta_i)_{1 \leq i \leq m+1}$ are i.i.d. exponential random variables of parameter 1.

As the lengths $(\ell_j)_j$ are continuous functions of the parameters $(\eta_j)_j$, we know that there exists an open set in $(\mathbb{R}^+)^{m+1}$, say O , such that, for each $(\ell_j)_{1 \leq j \leq m+1} \in O$, there are at least two occupations $Q_1((\ell_j)_{1 \leq j \leq m+1})$ and $Q_2((\ell_j)_{1 \leq j \leq m+1})$ that have the same ground state energy (which is at the same time the smallest possible among the ground state energies for all the occupations). Let us denote these branches of energy by $(\ell_j)_{1 \leq j \leq m+1} \mapsto E_1((\ell_j)_{1 \leq j \leq m+1})$ and $(\ell_j)_{1 \leq j \leq m+1} \mapsto E_2((\ell_j)_{1 \leq j \leq m+1})$ respectively.

For a fixed number of pieces, there are finitely many occupations and a change in the number of pieces occurs only when a wall, i.e., an endpoint of a piece, crosses 0 or L . Thus, there exists a subset $O_1 \subset O$ of positive measure, such that $Q_1((\ell_j)_{1 \leq j \leq m+1})$ and $Q_2((\ell_j)_{1 \leq j \leq m+1})$ are constant on O_1 .

Now, let us fix an initial set of lengths $(\ell_j^0)_{1 \leq j \leq m+1}$ in O_1 and move it continuously inside this exceptional set O_1 . This actually corresponds to moving continuously walls inside the interval $(0, L)$. As Q_1 and Q_2 are two different occupations, there exists a piece $[a, b] \subset [0, L]$, such that Q_1 and Q_2 put different number of particles in this piece, i.e., $Q_1([a, b]) \neq Q_2([a, b])$. Now, we move a continuously towards b ; if $a = 0$, we will move b towards a . Let a^0 be the value of a in the configuration $(\ell_j^0)_{1 \leq j \leq m+1}$. Let $E_1(a)$ and $E_2(a)$ be the ground state energies

corresponding to the two different occupations Q_1 and Q_2 . In a small neighborhood of a_0 , by the definition of O_1 , one has

$$E_1(a) = E_2(a)$$

As U is real analytic and as the ground state of H_Q is simple for any occupation Q , the functions $E_1(a)$ and $E_2(a)$ are analytic in the open interval (c, b) where c is the end of the piece $[c, a]$ to the left of the piece $[a, b]$. Indeed, E_1 (and E_2) is analytic around a_0 . Assume that $E_1(a)$ stops being analytic somewhere inside (c, b) . This would mean that the eigenvalue $E_1(a)$ of H_{Q_1} becomes degenerate, thus, that the ground state of H_{Q_1} becomes degenerate. This was already ruled out.

This immediately implies that $E_1(a) = E_2(a)$ for all $a \in (c, b)$.

But this cannot be. Indeed, if Q_1 puts k_1 particles in the piece $[a, b]$, and Q_2 puts k_2 particles in the piece $[a, b]$ with $k_1 \neq k_2$, the functions E_1 and E_2 have different asymptotics as a approaches b , indeed,

$$E_i(a) \sim k_i^3 / (b - a)^2 \quad \text{as } a \rightarrow b.$$

This contradicts the fact that the two functions agree on the whole interval. This completes the proof of Theorem 1.1. \square

Finally, we use the results from sections 3.1 together with Theorem 1.1 to obtain the following

Corollary 3.8. *Assume U is real analytic. Then, ω -almost surely, for any L and n , the ground state of $H_\omega^U(L, n)$ belongs to the a unique occupation subspace \mathfrak{H}_Q .*

Proof. Consider the orthogonal decomposition (3.5). As any projection of $\Psi_\omega(L, n)$ on \mathfrak{H}_Q is either a ground state or zero and as the ground state is ω -a.s. simple, only one of the projections of the ground state on a space of fixed occupation is different from zero. Thus, $\Psi_\omega(L, n)$ belongs to one of the subspaces \mathfrak{H}_Q . This completes the proof of Corollary 3.8. \square

3.4. The approximate ground state Ψ^{opt} . The basic idea of the construction of Ψ^{opt} is to find the optimal configuration with respect to different occupations. All the n -electron states are considered as deformations of the unperturbed ground state Ψ^0 which, we recall (2.11), is given by the Slater determinant:

$$\Psi^0 = \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_n.$$

When the interactions are turned on, the particles in the state Ψ^0 start to interact. For some particles, these interactions may be quite large. In particular, it may become energetically favorable to “decouple” some particles by moving them apart from each other to unoccupied pieces; obviously, it is better to move the more excited particles. One, thus, reduces the interaction energy but this will necessarily result in an increase of the “non interaction” energy of the state, i.e., of $\langle H_\omega^0(L, n)\Psi, \Psi \rangle$: indeed, in the non interacting ground state, the n particles occupy the n lowest levels of the system. Nevertheless the decrease of the interaction energy, i.e., $\langle W_n \Psi, \Psi \rangle$ may compensate the increase in “non interacting” energy. The “optimal” configuration then arises through the optimization on the occupation governed by the interplay between the loss of interaction energy and the gain of “non interacting” energy: it is achieved when loss and gain balance.

Let us note that a ground state Ψ is obviously the ground state of the Hamiltonian restricted to the appropriate fixed occupation subspace, i.e., Ψ is the ground state of $H_{Q(\Psi)}$ (see (3.5)). This corresponds to writing the minimization problem in the form

$$\inf_{\substack{\Phi \in \mathfrak{H}^n \\ \|\Phi\|=1}} \langle H_\omega(L, n)\Phi, \Phi \rangle = \inf_{\substack{Q \in \mathbb{N}^m \\ |Q|=n}} \inf_{\substack{\Phi \in \mathfrak{H}_Q \\ \|\Phi\|=1}} \langle H_Q \Phi, \Phi \rangle. \quad (3.9)$$

This reduces the problem to finding the optimal occupations rather than the optimal n -electron state itself.

Recalling that the constant γ is defined in Proposition 1.4, we set

$$A_* := \frac{\gamma}{8\pi^2}, \quad x_* := 1 - e^{-\frac{\gamma}{8\pi^2}}. \quad (3.10)$$

Note that

$$A_* = -\log(1 - x_*).$$

Let us now define Ψ^{opt} . Therefore, recall that the pieces in the model are denoted by $(\Delta_k(\omega))_{1 \leq k \leq m(\omega)}$ (see section 1) and that for $\Delta_k(\omega)$, a piece, we define (see sections 1.3.2 and 1.3.3)

- $\varphi_{\Delta_k(\omega)}^j$ to be the j -th normalized eigenvector of $-\Delta_{|\Delta_k(\omega)}^D$,
- $\zeta_{\Delta_k(\omega)}^j$ to be the j -th normalized eigenvector of $-\Delta_{|\Delta_k(\omega)}^D + U$ acting on $\bigwedge_{j=1}^2 L^2(\Delta_k(\omega))$.

We will define the state Ψ^{opt} in two steps. We first define Ψ_m^{opt} : it will contain less than n particles and will be the main part of Ψ^{opt} . We, then, add the missing particles to get the n -particle state Ψ^{opt} .

Definition 3.9. Consider all the pieces in $[0, L]$. For each piece, depending on its length, do one of the following:

- keep the pieces of length in $[0, \ell_\rho - \rho x_*] \cup [3\ell_\rho, \infty)$ empty;
- put one particle in its ground state in each piece of length in $[\ell_\rho - \rho x_*, 2\ell_\rho + A_*]$;
- in pieces of length in $[2\ell_\rho + A_*, 3\ell_\rho)$, put the ground state of a two-particles system with interactions (see Proposition 1.4 and section 6.1);

We define the state $\Psi_m^{\text{opt}} = \Psi_m^{\text{opt}}(L, n)$ to be the anti-symmetric tensor product of the thus constructed one- and two-particles sub-states, that is,

$$\Psi_m^{\text{opt}}(L, n) = \bigwedge_{|\Delta_j(\omega)| \in [\ell_\rho - \rho x_*, 2\ell_\rho + A_*]} \varphi_{\Delta_j(\omega)}^1 \wedge \bigwedge_{|\Delta_j(\omega)| \in [2\ell_\rho + A_*, 3\ell_\rho)} \zeta_{\Delta_j(\omega)}^1. \quad (3.11)$$

Note that, as the $(\zeta_{\Delta_j(\omega)}^1)_j$ carry two particles, $\Psi_m^{\text{opt}}(L, n)$ is not given by a Slater determinant; an explicit formula for such an anti-symmetric tensor product is given in (C.2) in Appendix C.

Remark 3.10. Note that, in step (c) of Definition 3.9, we put two interacting particles within these pieces. Because of the interactions, this is different from putting separately two particles on the two lowest one-particle energy levels (see appendix 6).

Let us now compute the total number of particles contained in Ψ_m^{opt} . We prove

Lemma 3.11. *With probability $1 - O(L^{-\infty})$, for L sufficiently large, in the thermodynamic limit, the total number of particles in Ψ_m^{opt} constructed in Definition 3.9 is given by*

$$\mathcal{N}(\Psi_m^{\text{opt}}) = n \left[1 - \rho^2 \left(3 - x_* - \frac{x_*^2}{2} \right) + O(\rho^3) \right].$$

Proof. It suffices to count the number of pieces of each type and multiply by the corresponding number of particles. We recall that, by (3.10), one has $\exp(-\ell_\rho) = \frac{\rho}{1 + \rho}$ and $\exp(-A_*) = 1 - x_*$. Thus, for $\beta \in (0, 1/2)$, using Proposition 2.2 and the second equation

in (3.10), with probability $1 - O(L^{-\infty})$, one computes

$$\begin{aligned}
\mathcal{N}(\Psi_m^{\text{opt}}) &= \#\{l \in [\ell_\rho - \rho x_*, 2\ell_\rho + A_*]\} + 2 \cdot \#\{l \in [2\ell_\rho + A_*, 3\ell_\rho]\} \\
&= L \left[e^{-(\ell_\rho - \rho x_*)} - e^{-(2\ell_\rho + A_*)} + 2e^{-(2\ell_\rho + A_*)} - 2e^{-3\ell_\rho} \right] + O(L^{1/2+\beta}) \\
&= \frac{L\rho}{1+\rho} \left[e^{\rho x_*} + \rho e^{-A_*} - \rho^2 e^{-A_*} - 2\rho^2 + O(\rho^3) \right] \\
&= \frac{L\rho}{1+\rho} \left[1 + \rho - \rho^2 \left(e^{-A_*} + 2 - \frac{x_*^2}{2} \right) + O(\rho^3) \right] \\
&= n \left[1 - \rho^2 \left(3 - x_* - \frac{x_*^2}{2} \right) + O(\rho^3) \right].
\end{aligned}$$

This completes the proof of Lemma 3.11. \square

Lemma 3.11 shows that, for ρ small, Ψ_m^{opt} contains less than n particles. Let us now add particles to Ψ_m^{opt} to complete it into Ψ^{opt} . Therefore, we prove

Lemma 3.12. *Let $(\tilde{\varphi}_k)_{1 \leq k \leq k_\rho(\omega)}$ be the particles that Ψ^0 , the non interacting ground state, puts in the pieces longer than $3\ell_\rho$ ordered by increasing energy.*

With probability $1 - O(L^{-\infty})$, for L sufficiently large, one has $k_\rho(\omega) \geq n\rho^2(3 - 18\rho)$.

Proof. By Proposition 2.2, with probability $1 - O(L^{-\infty})$, the number of pieces of length in $\ell_\rho[3 + \rho, 4)$ is equal to

$$n \frac{\rho^2}{(1+\rho)^3} \left(e^{-\rho} - \frac{\rho}{1+\rho} \right) + o(L) \geq n\rho^2(1 - 6\rho)$$

for L large.

To complete the proof of Lemma 3.12, let us now establish some auxiliary results. By (2.10) in Proposition 2.7, we know that $E_{n,\omega}^\Lambda$ converges to E_ρ in the thermodynamic limit. We will first investigate the rate of convergence in (2.10).

Lemma 3.13. *Denote by $\ell_{n,L}$ the length of an interval having a ground state energy equal to $E_{n,\omega}^\Lambda$, i.e.,*

$$\ell_{n,L} = \frac{\pi}{\sqrt{E_{n,\omega}^\Lambda}}.$$

Let $\rho > 0$ be fixed. For any $\delta > 0$, in the thermodynamic limit $L \rightarrow \infty$, $n/L \rightarrow \rho$, with probability $1 - O(L^{-\infty})$, one has

$$\begin{aligned}
\ell_{n,L} &= \ell_\rho + O(L^{-(1/2-\delta)}) + O\left(\left|\frac{n}{L} - \rho\right|\right), \\
E_{n,\omega}^\Lambda &= E_\rho + O(L^{-(1/2-\delta)}) + O\left(\left|\frac{n}{L} - \rho\right|\right).
\end{aligned}$$

In view of Lemma 3.13 and by the definition of Ψ^0 , for L sufficiently large, each piece of length in $\ell_\rho[3 + \rho, 4)$ contains at least 3 particles of Ψ^0 . This completes the proof of Lemma 3.12. \square

Proof of Lemma 3.13. By (A.17), with probability $1 - O(L^{-\infty})$, the normalized counting function for the Dirichlet eigenvalues of $H_\omega(L, 1)$ (see (2.4)) satisfies

$$\frac{n}{L} = N_L^D(E_{n,\omega}^\Lambda) = \frac{\exp(-\ell_{n,L})}{1 - \exp(-\ell_{n,L})} + O(L^{-(1/2-\delta)}).$$

Taking into account the fact that

$$\rho = N(E_\rho) = \frac{\exp(-\ell_\rho)}{1 - \exp(-\ell_\rho)},$$

we deduce that

$$\frac{\exp(-\ell_{n,L})}{1 - \exp(-\ell_{n,L})} = \frac{\exp(-\ell_\rho)}{1 - \exp(-\ell_\rho)} + O(L^{-(1/2-\delta)}) + O\left(\left|\frac{n}{L} - \rho\right|\right).$$

This immediately yields

$$\exp(-\ell_{n,L}) = \exp(-\ell_\rho) + O(L^{-(1/2-\delta)}) + O\left(\left|\frac{n}{L} - \rho\right|\right).$$

The proof of Lemma 3.13 is complete. \square

For ρ small, by Lemmas 3.11 and 3.12, one has $n - \mathcal{N}(\Psi_m^{\text{opt}}) < k_\rho(\omega)$. Thus, to construct Ψ^{opt} , we just add $n - \mathcal{N}(\Psi_m^{\text{opt}})$ particles of Ψ^0 living in pieces of length in $\ell_\rho[3 + \rho, 4)$ to Ψ_m^{opt} .

Definition 3.14. We define

$$\Psi^{\text{opt}} = \Psi^{\text{opt}}(L, n) := \Psi_m^{\text{opt}}(L, n) \wedge \bigwedge_{k=1}^{n - \mathcal{N}(\Psi_m^{\text{opt}})} \tilde{\varphi}_k. \quad (3.12)$$

Remark 3.15. Let us give an alternative approach to defining Ψ^{opt} which does not result in exactly the same Ψ^{opt} but which can serve exactly the same purpose in the subsequent arguments.

We start with the non interacting ground state Ψ^0 and describe how it is modified:

- for pairs of particles living in the same piece, the modification depends on the length of this piece:
 - for the pieces of length between $2\ell_\rho$ and $2\ell_\rho + A_*$, remove the more excited particle and put it into an unoccupied piece of length between $\ell_\rho - \rho x_*$ and ℓ_ρ ;
 - for the remaining pieces, i.e., the pieces of length between $2\ell_\rho + A_*$ and $3\ell_\rho$, the factorized two-particles state corresponding to Ψ^0 should be replaced by a true ground state of a two-particles system with interaction in this piece (see section 6.1 for a description of such a two-particle state);
- do not modify any of the particles in Ψ^0 that are either alone or live in groups of three or more pieces.

One can easily verify that, in the above procedure, up to a small relative error, the number of pieces to which the excited particles are displaced is equal to the number of pieces where we decouple the particles. Indeed, according to Proposition 2.2, with probability at least $1 - O(L^{-\infty})$, for the former, one has

$$\begin{aligned} \#\{l \in (2\ell_\rho, 2\ell_\rho - \log(1 - x_*))\} &= L \exp(-2\ell_\rho)x_* + O(L^{1/2+\beta}) \\ &= n\rho x_*(1 + O(\rho)), \end{aligned} \quad (3.13)$$

and, for the latter, one has

$$\begin{aligned} \#\{l \in (\ell_\rho - \rho x_*, \ell_\rho)\} &= L \exp(-\ell_\rho)(\exp(\rho x_*) - 1) + O(L^{1/2+\beta}) \\ &= n\rho x_*(1 + O(\rho)). \end{aligned} \quad (3.14)$$

Thus, both sets contain the same number of pieces (up to an error of order $n\rho^2$). This completes the construction of Ψ^{opt} .

3.5. Comparing Ψ^{opt} with the ground state of the interacting system. Our goal in the sections to come is to estimate how much Ψ^{opt} differs from a true ground state $\Psi^U = \Psi_\omega^U(L, n)$ (and to show that it doesn't differ much). This will be done through the comparison of their occupation numbers. We shall see that the ground states of the interacting Hamiltonian must live in subspaces with special occupation numbers (see Corollary 3.32). To compare occupation numbers, we introduce the distance dist_1 .

Definition 3.16. Let $m = m(\omega)$ be the number of pieces in $[0, L]$. For $j \in \{1, 2\}$, pick an occupation

$$Q^j = (Q_1^j, Q_2^j, \dots, Q_m^j) \in \mathbb{N}^m, \quad |Q^j| = n.$$

Define

$$\text{dist}_1(Q^1, Q^2) = \sum_{i=1}^m |Q_i^1 - Q_i^2|.$$

Remark 3.17. Recall that the non interacting ground state Ψ^0 has a single occupation $Q(\Psi^0)$: all the states with energy below $E_{n,\omega}^\Lambda$ (where we recall that $E_{n,\omega}^\Lambda$ denote the n -th (counting multiplicity) energy level of the one-particle Hamiltonian $H_\omega(L)$); moreover, only those states are occupied. In [Ven12], for U compactly supported, for Ψ^U an interacting ground state, it was proved that

$$C^{-1}n\rho \leq \text{dist}_0(Q(\Psi^U), Q(\Psi^0)) \leq Cn\rho. \quad (3.15)$$

where dist_0 is defined by $\text{dist}_0(Q^1, Q^2) = \sum_{i=1}^m \mathbf{1}_{Q_i^1 \neq Q_i^2}$. Clearly, one has $\text{dist}_0 \leq \text{dist}_1$.

In the sequel, we shall prove that Ψ^{opt} is a better approximation of a ground state of the interacting system than is the non-interacting ground state Ψ^0 (compare (3.83) with (3.15)).

For interaction potentials that decrease at infinity sufficiently fast (see **(HU)**), we will prove that the main modification to the ground state energy comes from U restricted to some (sufficiently large) compact set.

Fix a constant $B > 2$. We decompose the interaction potential in the sum of the “principal” and “residual” parts that is, write $U = U^p + U^r$ where

$$U^p := \mathbf{1}_{[-B\ell_\rho, B\ell_\rho]}U \quad \text{and} \quad U^r := \mathbf{1}_{(-\infty, -B\ell_\rho) \cup (B\ell_\rho, +\infty)}U. \quad (3.16)$$

As the sum of pair interactions W_n is linear in U , this yields the following decomposition for the full Hamiltonian:

$$H^U = H^0 + W_n = H^0 + W_n^{U^p} + W_n^{U^r} = H^{U^p} + W_n^r. \quad (3.17)$$

Our analysis is done in the following steps:

- (a) first, we prove that Ψ^{opt} approximates well the ground state for the system with compactified interactions Ψ^{U^p} ;
- (b) second, we show that the quadratic form of the residual interactions W^r on Ψ^{opt} contributes only to the error term; this will imply (1.16);
- (c) finally, we will conclude that the same Ψ^{opt} gives also a good approximation for the full Hamiltonian H^U ground state Ψ^U in terms of the distance dist_1 for the respective occupations.

Remark 3.18. Let us clarify a point of terminology: we will minimize the quadratic form $\langle H_Q \Psi, \Psi \rangle = \langle H_Q^0 \Psi, \Psi \rangle + \langle W_n \Psi, \Psi \rangle$; the term $\langle H_Q^0 \Psi, \Psi \rangle$ is referred to as the “non interacting energy” term and $\langle W_n \Psi, \Psi \rangle$ the “interaction energy” term; we use the same decomposition and terminology for smaller groups of particles or at the single particle level.

3.6. The analysis of H^{U^p} . We start with the analysis of H^{U^p} , in particular, of its ground state energy and ground state(s). Later, we show that the addition of W_n^r will not change much in the ground state energy and ground state(s).

First, we compute the energy of Ψ^{opt} . We prove

Theorem 3.19. *There exists $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0)$, in the thermodynamic limit, with probability 1, one has*

$$\begin{aligned} \lim_{\substack{L \rightarrow \infty \\ n/L \rightarrow \rho}} \frac{1}{n} \langle H_\omega^{Up}(L, n) \Psi^{\text{opt}}(L, n), \Psi^{\text{opt}}(L, n) \rangle \\ = \mathcal{E}^0(\rho) + \pi^2 \gamma_* \rho |\log \rho|^{-3} (1 + O(f_Z(|\log \rho|))) \end{aligned} \quad (3.18)$$

where γ_* is defined in (1.17) and f_Z is a continuous function satisfying $f_Z(x) \rightarrow 0$ as $x \rightarrow +\infty$ no faster than $1/x$ (for more details, see (3.29)).

Proof. To shorten the notations, we will frequently drop the arguments L, n and the subscript ω in this proof. We will show that, up to error terms, the only terms that contribute to $\langle H^{Up} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle - \langle H^0 \Psi^0, \Psi^0 \rangle$ are those due to

- (a) the interactions between two particles in the same piece,
- (b) the decoupling of a fraction of these particles following the construction of Ψ^{opt} .

In (3.18), the interactions between neighboring distinct pieces will be shown to contribute only to the error term where we have defined

Definition 3.20. A pair of *neighboring* or *interacting* pieces is a pair of distinct pieces at distance at most $B\ell_\rho$ from one another, particular, particles in two such pieces can still interact via the potential U^p .

Let us now outline the main idea of the proof of Theorem 3.19. The pieces longer than $2\ell_\rho + A_*$ contain two particles both in Ψ^0 and Ψ^{opt} . Hence, for each piece of this type, the energy difference is given by the second term in the asymptotics (1.16) in Proposition 1.4. On the contrary, in pieces of length between $2\ell_\rho$ and $2\ell_\rho + A_*$, in Ψ^0 , the two particles were decoupled in order to construct Ψ^{opt} , keeping one intact and displacing another to a piece of length between $\ell_\rho - \rho x_*$ and ℓ_ρ . In this case, the energy difference is given by the increase of non interacting energy of the second (displaced) particle. The single particles in Ψ^0 remain untouched in Ψ^{opt} and groups of three and more particles contribute only to the error term (as they carry only a small number of particles).

To put the above arguments into a rigorous form, we will use the following partition of the set of available pieces according to their length. Choose K large but independent of L . For $k \in \{1, \dots, K\}$, consider the sets of pieces

$$\begin{aligned} \mathcal{L}_k^1 &= \{ \text{pieces of length in } [\ell_\rho - \frac{k}{K}\rho, \ell_\rho - \frac{k-1}{K}\rho] \}, \\ \mathcal{L}_k^2 &= \{ \text{pieces of length in } [2\ell_\rho - \log(1 - \frac{k-1}{K}), 2\ell_\rho - \log(1 - \frac{k}{K})] \}. \end{aligned}$$

As K is independent of L , with probability $1 - O(L^{-\infty})$, the number of pieces in the classes $(\mathcal{L}_k^j)_{\substack{j \in \{1,2\} \\ k \in \{1, \dots, K\}}}$ is given by Proposition 2.2. We will, henceforth, use these estimates without reference to probabilities.

As in (3.13) and (3.14), one shows that these two sets map one-to-one onto one another up to an error estimated as follows

$$\text{card } \mathcal{L}_k^1 = \text{card } \mathcal{L}_k^2 + O(n\rho^2 K^{-1}) = n\rho K^{-1} (1 + O(\rho)).$$

Recall that x_* is defined in (3.10). For $k \leq Kx_*$, according to our scheme, the pairs of particles in pieces belonging to \mathcal{L}_k^2 get decoupled, one of the particles being sent to occupy a piece belonging to \mathcal{L}_k^1 . For $k > Kx_*$, the pairs of particles in the pieces of \mathcal{L}_k^2 are kept untouched. The latter pieces are those of size at least $2\ell_\rho + A_*$. It is easily seen that the number of such pieces is given by

$$\#\{j : |\Delta_j(\omega)| \geq 2\ell_\rho + A_*\} = n\rho e^{-A_*} (1 + O(\rho)) = n\rho(1 - x_*) + O(n\rho^2).$$

The majority of these pieces is smaller than $2\ell_\rho + A_* + \log \ell_\rho$; indeed,

$$\#\{j : |\Delta_j(\omega)| \in 2\ell_\rho + A_* + [0, \log \ell_\rho]\} = n\rho(1 - x_*) + O(n\rho|\log \rho|^{-1}).$$

By Proposition 1.4, for a piece of length ℓ in $2\ell_\rho + A_* + [0, \log \ell_\rho]$, the interaction energy of the two-particles system is given by

$$\frac{\gamma}{\ell^3} + o(\ell^{-3}) = \frac{\gamma}{8\ell_\rho^3} + o(\ell_\rho^{-3}).$$

For the difference of energies, this yields

$$\begin{aligned} \langle H^{Up} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle - \langle H^0 \Psi^0, \Psi^0 \rangle &= \frac{n\rho}{K} \sum_{k=1}^{Kx_*} \left[\frac{\pi^2}{(\ell_\rho - \frac{k}{K}\rho)^2} - \frac{4\pi^2}{(2\ell_\rho - \log(1 - \frac{k}{K}))^2} \right] \\ &\quad + \frac{\gamma}{8\ell_\rho^3} n\rho(1 - x_*) + o(n\rho|\log \rho|^{-3}). \end{aligned} \quad (3.19)$$

Taking K large, we approximate the Riemann sum in the last expression by an integral

$$\begin{aligned} &\frac{1}{K} \sum_{k=1}^{Kx_*} \left[\frac{\pi^2}{(\ell_\rho - \frac{k}{K}\rho)^2} - \frac{4\pi^2}{(2\ell_\rho - \log(1 - \frac{k}{K}))^2} \right] \\ &= x_* \int_0^1 \left[\frac{\pi^2}{(\ell_\rho - tx_*)^2} - \frac{\pi^2}{(\ell_\rho - \frac{1}{2} \log(1 - tx_*))^2} \right] dt + O\left(\frac{1}{K}\right) \\ &= x_* \left(- \int_0^1 \frac{\pi^2}{\ell_\rho^3} \log(1 - tx_*) dt + o(\ell_\rho^{-3}) \right) + O\left(\frac{1}{K}\right) \\ &= \pi^2 \ell_\rho^{-3} (x_* - (1 - x_*)A_*) (1 + o(1)) + O\left(\frac{1}{K}\right). \end{aligned}$$

Picking $\delta \in (0, 1)$, letting $K = \rho^{-\delta}$ and recalling (3.10) for A_* and x_* , for δ small, we get

$$\begin{aligned} \langle H^{Up} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle - \langle H^0 \Psi^0, \Psi^0 \rangle &= n\rho \ell_\rho^{-3} \left(\pi^2 (x_* - (1 - x_*)A_*) + \frac{\gamma}{8} (1 - x_*) \right) \\ &\quad + o(n\rho \ell_\rho^{-3}) \\ &= n\rho \ell_\rho^{-3} \pi^2 \left(1 - e^{-\frac{\gamma}{8\pi^2}} \right) + o(n\rho \ell_\rho^{-3}). \end{aligned} \quad (3.20)$$

In order to finish the proof of (3.18) and, thus, of Theorem 3.19, it suffices to upper bound the interactions between distinct pieces. Recall that Ψ^{opt} is an anti-symmetric exterior product of one- and two-particles eigenstates (see (3.11) and (3.12)):

$$\Psi^{\text{opt}} = \bigwedge_{i=1}^{\hat{k}_1} \varphi_i \wedge \bigwedge_{j=1}^{k_2} \zeta_j \wedge \bigwedge_{i=1}^{\tilde{k}_1} \tilde{\phi}_i, \quad (3.21)$$

where the numbers of sub-states in each group are respectively

$$\begin{aligned} \hat{k}_1 &= n \left(1 - 2\rho(1 - x_*) + \rho^2 \left(3(1 - x_*) + \frac{x_*^2}{2} \right) + O(\rho^3) \right), \\ k_2 &= n\rho(1 - x_* - \rho(3 - 2x_*)) + O(\rho^2), \\ \tilde{k}_1 &= n - \mathcal{N}(\Psi_m^{\text{opt}}) = n\rho^2 \left(3 - x_* - \frac{x_*^2}{2} \right) (1 + O(\rho)). \end{aligned}$$

The functions φ_i and $\tilde{\phi}_i$ are one-particle ground states in certain and the functions ζ_j are two-particles ground states in certain pieces. Of course, $\hat{k}_1 + k_2 + \tilde{k}_1 = n$. As in what follows

we will only need to distinguish between one- and two-particles states, let us put the two groups of one-particle sub-states from (3.21) together, i.e. write

$$\Psi^{\text{opt}} = \bigwedge_{i=1}^{k_1} \phi_i \wedge \bigwedge_{j=1}^{k_2} \zeta_j, \quad (3.22)$$

where $k_1 = \hat{k}_1 + \tilde{k}_1$ and $\{\phi_i\}_{i=1}^{k_1} = \{\varphi_i\}_{i=1}^{\hat{k}_1} \cup \{\tilde{\phi}_i\}_{i=1}^{\tilde{k}_1}$. As W^p is a totally symmetric sum of pair interaction potentials, one computes

$$\begin{aligned} \langle W^p \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle &= \sum_{1 \leq i < j \leq n} \int_{[0,L]^n} U(x_i - x_j) |\Psi^{\text{opt}}(x)|^2 dx \\ &= \frac{n(n-1)}{2} \int_{[0,L]^n} U(x_1 - x_2) |\Psi^{\text{opt}}(x)|^2 dx = \text{Tr} \left(U^p \gamma_{\Psi^{\text{opt}}}^{(2)} \right). \end{aligned} \quad (3.23)$$

According to Proposition 4.8, for Ψ^{opt} having the structure (3.22), its two-particle density matrix is given by

$$\begin{aligned} \gamma_{\Psi^{\text{opt}}}^{(2)} &= \sum_{j=1}^{k_2} \gamma_{\zeta_j}^{(2)} + (\text{Id} - \text{Ex}) \sum_{\substack{i,j=1,\dots,k_1 \\ i < j}} \gamma_{\phi_i} \otimes^s \gamma_{\phi_j} + (\text{Id} - \text{Ex}) \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \gamma_{\phi_i} \otimes^s \gamma_{\zeta_j} \\ &\quad + (\text{Id} - \text{Ex}) \sum_{\substack{i,j=1,\dots,k_2 \\ i < j}} \gamma_{\zeta_i} \otimes^s \gamma_{\zeta_j}. \end{aligned} \quad (3.24)$$

As ζ_j is a two-particle state and ϕ_j is a one-particle state, one has

$$\gamma_{\zeta_j}^{(2)} = \langle \cdot, \zeta_j \rangle \zeta_j \quad \text{and} \quad \gamma_{\phi_j} = \langle \cdot, \phi_j \rangle \phi_j.$$

The decomposition (3.24) being plugged in the r.h.s. of (3.23) reads as follows:

- (a) the first term corresponds to the interaction of two particles living in the same piece; this term is the leading one in the difference (3.19) and has been already taken into account in the first part of the proof;
- (b) the second term is the interaction between two one-particle sub-states living in distinct pieces;
- (c) the third term is due to the interaction between a one-particle sub-state in one piece and a two-particle sub-state (represented by its one-particle reduced density matrix) in another piece;
- (d) finally, the last term describes the interaction between two distinct two-particle sub-states.

Thus, we are interested in upper bounds on $\text{Tr}(U^p \beta)$ where β is any of the last three terms in (3.24). Let γ_1 and γ_2 be two arbitrary one-particle density matrices encountered in the above expressions. Then, the kernel of $(\text{Id} - \text{Ex}) \gamma_1 \otimes^s \gamma_2$ is given by

$$\begin{aligned} (\text{Id} - \text{Ex})(\gamma_1 \otimes^s \gamma_2)(x, y, x', y') &= \frac{1}{2} (\gamma_1(x, x') \gamma_2(y, y') + \gamma_2(x, x') \gamma_1(y, y') \\ &\quad - \gamma_1(y, x') \gamma_2(x, y') - \gamma_2(y, x') \gamma_1(x, y')). \end{aligned} \quad (3.25)$$

Taking into account the fact that in our case γ_1 and γ_2 live on distinct pieces Δ_1 and Δ_2 respectively, (3.25) implies

$$\begin{aligned} \text{Tr} (U^p (\text{Id} - \text{Ex}) \gamma_1 \otimes^s \gamma_2) &= \int_{\mathbb{R}^2} U^p(x-y) (\text{Id} - \text{Ex})(\gamma_1 \otimes^s \gamma_2)(x, y, x, y) dx dy \\ &= \int_{\Delta_1} \int_{\Delta_2} U^p(x-y) \gamma_1(x, x) \gamma_2(y, y) dx dy. \end{aligned} \quad (3.26)$$

To upper bound the last expression, we use the estimates proved in section 6.2. We now study the different sums in (3.24).

For pairs of one-particle states, we estimate the number of pairs of pieces at a certain distance by Proposition 2.3 and we bound individual terms by Lemma 6.18. We compute that, for any $\eta > 0$ and $\varepsilon > 0$, for L sufficiently large, with probability $1 - O(L^{-\infty})$, one has

$$\begin{aligned}
\mathrm{Tr}\left(U^p(\mathrm{Id} - \mathrm{Ex}) \sum_{\substack{i,j=1,\dots,k_1 \\ i < j}} \gamma_{\phi_i} \otimes^s \gamma_{\phi_j}\right) &\leq \sum_{\substack{|\Delta_i| \geq \ell_\rho - \rho x_* \\ |\Delta_j| \geq \ell_\rho - \rho x_* \\ \mathrm{dist}(\Delta_i, \Delta_j) \leq B\ell_\rho}} \int_{\Delta_i \times \Delta_j} U(x-y) |\varphi_{\Delta_i}^1(x)|^2 |\varphi_{\Delta_j}^1(y)|^2 dx dy \\
&\leq \sum_{k=0}^{B\ell_\rho/\eta} \sum_{\substack{|\Delta_i| \geq \ell_\rho - \rho x_* \\ |\Delta_j| \geq \ell_\rho - \rho x_* \\ k\eta \leq \mathrm{dist}(\Delta_i, \Delta_j) < (k+1)\eta}} \int_{\Delta_i \times \Delta_j} U(x-y) |\varphi_{\Delta_i}^1(x)|^2 |\varphi_{\Delta_j}^1(y)|^2 dx dy \\
&\leq C \sum_{k=0}^{B\ell_\rho/\eta} \# \left\{ \begin{array}{l} |\Delta_i| \geq \ell_\rho - \rho x_*, \\ |\Delta_j| \geq \ell_\rho - \rho x_* \\ k\eta \leq \mathrm{dist}(\Delta_i, \Delta_j) < (k+1)\eta \end{array} \right\} \ell_\rho^{-4+\varepsilon} ((k+1)\eta)^{-\varepsilon} Z((k+1)\eta) \\
&\leq CL e^{-2\ell_\rho} \ell_\rho^{-4+\varepsilon} \sum_{k=0}^{B\ell_\rho/\eta} ((k+1)\eta)^{-\varepsilon} Z((k+1)\eta)\eta.
\end{aligned}$$

Here, to get line three from line two, we have used Lemma 6.18, and to get line four from line three, we have used Proposition 2.3 to bound the counting function with a probability $1 - O(L^{-\infty})$.

Thus, by the continuity and local integrability of $x \mapsto x^{-\varepsilon} Z(x)$, choosing η small and $\varepsilon \in [0, 1)$, we obtain that, for L sufficiently large, with probability $1 - O(L^{-\infty})$, one has

$$\mathrm{Tr}\left(U^p(\mathrm{Id} - \mathrm{Ex}) \sum_{\substack{i,j=1,\dots,k_1 \\ i < j}} \gamma_{\phi_i} \otimes^s \gamma_{\phi_j}\right) \leq C n \rho \ell_\rho^{-4+\varepsilon} \int_0^{B\ell_\rho} a^{-\varepsilon} Z(a) da. \quad (3.27)$$

Let us now estimate the last integral. For $\varepsilon \in [0, 1)$ and $0 \leq Y < X$, one computes

$$\begin{aligned}
\int_0^X a^{-\varepsilon} Z(a) da &\leq \left(\int_0^Y + \int_Y^X \right) a^{-\varepsilon} Z(a) da \\
&\leq (1-\varepsilon)^{-1} [Z(0)Y^{1-\varepsilon} + Z(Y)X^{1-\varepsilon} - Z(Y)Y^{1-\varepsilon}] \\
&= (1-\varepsilon)^{-1} X^{1-\varepsilon} [(Y/X)^{1-\varepsilon} (Z(0) - Z(Y)) + Z(Y)].
\end{aligned}$$

Let us now optimize the last expression with respect to $\alpha = Y/X \in [0, 1]$. Consider

$$f(X, \alpha) := \alpha^{1-\varepsilon} (Z(0) - Z(\alpha X)) + Z(\alpha X). \quad (3.28)$$

In general, the more rapidly Z goes to zero at infinity, the smaller the optimal α and, thus, the smaller is the minimal value. Let us define the following functional of Z (depending also on X):

$$f_Z(X) = \inf_{\alpha \in [0, 1]} f(X, \alpha). \quad (3.29)$$

Obviously, as soon as $Z(X) = o(1)$ for $X \rightarrow +\infty$, one finds that $f_Z(X) = o(1)$ for $X \rightarrow +\infty$. Then, plugging this into the estimate (3.27), we obtain

$$\mathrm{Tr}\left(U^p(\mathrm{Id} - \mathrm{Ex}) \sum_{\substack{i,j=1,\dots,k_1 \\ i < j}} \gamma_{\phi_i} \otimes^s \gamma_{\phi_j}\right) \leq C_1 n \rho \ell_\rho^{-3} \cdot f_Z(B\ell_\rho). \quad (3.30)$$

In particular, the last expression is $o(n\rho\ell_\rho^{-3})$. Note also that, it can never be made better than $O(n\rho\ell_\rho^{-4})$ as there is no control of the size of Z near the origin.

To estimate the interactions between a one-particle state and a one-particle density matrix of a two-particle state, we use the bound derived in Lemma 6.20. We estimate the number of pairs of pieces of this type at a certain distance by Proposition 2.4 (in this case, there is no need in for the more precise Proposition 2.3 as in the derivation of (3.30) above). This yields

$$\begin{aligned} & \text{Tr}\left(U^p(\text{Id} - \text{Ex}) \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \gamma_{\phi_i} \otimes^s \gamma_{\zeta_j}\right) \\ & \leq \sum_{\substack{|\Delta_i| \geq \ell_\rho - \rho x_* \\ |\Delta_j| \in [2\ell_\rho + A_*, 3\ell_\rho] \\ \text{dist}(\Delta_i, \Delta_j) \leq B\ell_\rho}} \int_{\Delta_i \times \Delta_j} U(x-y) |\varphi_{\Delta_i}^1(x)|^2 \gamma_{\zeta_{\Delta_j}^1}(y, y) dx dy \\ & \leq C n \rho^2 \ell_\rho \ell_\rho^{-7/2+\varepsilon} \int_0^{B\ell_\rho} a^{-\varepsilon} Z(a) da. \end{aligned} \quad (3.31)$$

Finally, for interactions between two reduced density matrices of two-particles sub-states, we proceed as before; using Lemma 6.21 for each term, we compute

$$\begin{aligned} & \text{Tr}\left(U^p(\text{Id} - \text{Ex}) \sum_{\substack{i,j=1,\dots,k_2 \\ i < j}} \gamma_{\zeta_i} \otimes^s \gamma_{\zeta_j}\right) \\ & = \sum_{\substack{|\Delta_i|, |\Delta_j| \in [2\ell_\rho + A_*, 3\ell_\rho] \\ i < j \\ \text{dist}(\Delta_i, \Delta_j) \leq B\ell_\rho}} \int_{\Delta_i \times \Delta_j} U(x-y) \gamma_{\zeta_{\Delta_i}^1}(x, x) \gamma_{\zeta_{\Delta_j}^1}(y, y) dx dy \\ & \leq C n \rho^3 \int_0^{B\ell_\rho} \min(1, a^{-2} Z(a)) da. \end{aligned} \quad (3.32)$$

Summing (3.30), (3.31), (3.32), we obtain

$$\langle W^p \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle \leq C n \rho \ell_\rho^{-3} \cdot f_Z(B\ell_\rho). \quad (3.33)$$

Taking (3.20) into account, this completes the proof of Theorem 3.19. \square

To formulate our next result, we will first need to define the notion of occupation restricted to a subset of the total set of pieces.

Definition 3.21. Let $\mathcal{P}_\omega = \{\Delta_k(\omega)\}_{k=1}^{m(\omega)}$ be the total set of pieces and let $Q \in \mathbb{N}^m$ be an occupation. For $P \subset \mathcal{P}_\omega$ a subset of pieces, define the corresponding sub-occupation (or a restriction of occupation) as an occupation vector containing only those components that are singled out by P :

$$Q|_P = (Q_k)_{k: \Delta_k \in P}.$$

When the subset P is defined by a condition on the length of the pieces, we will use a shorthand notation involving only this condition, e.g., $Q|_{>\ell_\rho}$ stands for the occupation Q restricted to the pieces of length greater than the Fermi length ℓ_ρ .

Recall that Ψ^{opt} is constructed in Definition 3.14.

Theorem 3.22. *For any non negative function $r : [0, \rho_0] \rightarrow \mathbb{R}^+$ such that $r(\rho) = o(1)$ when $\rho \rightarrow 0^+$, there exist $C > 0$ and $\rho_r > 0$ such that, for $\rho \in (0, \rho_r)$, in the thermodynamic limit,*

with probability $1 - O(L^{-\infty})$, if Ψ is a normalized n -particles state in $\mathfrak{H}_{Q(\Psi)} \cap \mathfrak{H}_{\infty}^n([0, L])$ (see (3.3)) satisfying

$$\frac{1}{n} \langle H_{\omega}^{Up}(L, n) \Psi, \Psi \rangle \leq \frac{1}{n} \langle H_{\omega}^{Up}(L, n) \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle + \rho |\log \rho|^{-3} (r(\rho))^2, \quad (3.34)$$

then

$$\begin{aligned} \text{dist}_1(Q|_{\geq \ell_{\rho} + C}(\Psi), Q|_{\geq \ell_{\rho} + C}(\Psi^{\text{opt}})) &\leq Cn\rho \cdot \max(r(\rho), |\log \rho|^{-1}), \\ \text{dist}_1(Q|_{< \ell_{\rho} + C}(\Psi), Q|_{< \ell_{\rho} + C}(\Psi^{\text{opt}})) &\leq Cn \max(\sqrt{\rho} \cdot r(\rho), \rho |\log \rho|^{-1}). \end{aligned} \quad (3.35)$$

Proof of Theorem 3.22. First of all, taking into account the form of the first inequality in (3.35), while dealing with its proof we may suppose without loss of generality that $|\log \rho|^{-1}$ is asymptotically bounded by $r(\rho)$, i.e., for ρ small,

$$|\log \rho|^{-1} \lesssim r(\rho). \quad (3.36)$$

For the proof of the second inequality in (3.35), we will no longer assume (3.36).

Consider now the pieces $(\Delta_k(\omega))_{1 \leq k \leq m(\omega)}$ (see section 1). Fix $\varepsilon > 0$. We say that a piece $\Delta_k(\omega)$ is of ε -type

- (a) if $|\Delta_k(\omega)| \geq 3\ell_{\rho}(1 - \varepsilon)$, that is, it has length at least $3\ell_{\rho}(1 - \varepsilon)$;
- (b) if $|\Delta_k(\omega)| \geq 2\ell_{\rho}(1 - \varepsilon)$ and $\Delta_k(\omega)$ has at least one neighbor (in the sense of interactions U^p from (3.16)) of length at least $\ell_{\rho}(1 - \varepsilon)$;
- (c) if $|\Delta_k(\omega)| \geq \ell_{\rho}(1 - \varepsilon)$ and $\Delta_k(\omega)$ has at least two neighbors, each of length at least $\ell_{\rho}(1 - \varepsilon)$.

Note that, by (3.16), as U^p is of compact support of radius at most $B\ell_{\rho}$, there exists $\rho_0 > 0$ such that for $\rho \in (0, \rho_0)$ and $\varepsilon \in (0, 1/2)$, a given piece can have at most $2B$ neighbors of length at least $\ell_{\rho}(1 - \varepsilon)$.

We first prove that ‘‘exceptional’’ pieces contribute only to the error term.

Lemma 3.23. *Fix $\eta \in (0, 1/3)$. There exists $\varepsilon \in (0, 1/2)$ and $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0)$, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, if $\Psi \in \mathfrak{H}_{Q(\Psi)} \cap \mathfrak{H}_{\infty}^n([0, L])$ satisfies*

$$\langle H_{\omega}^{Up}(L, n) \Psi, \Psi \rangle \leq 2\mathcal{E}^0(\rho)n\|\Psi\|^2, \quad (3.37)$$

then

$$\sum_{\bullet \in \{a, b, c\}} \sum_{\Delta_k(\omega) \text{ of } \varepsilon\text{-type } (\bullet)} Q_k(\Psi) \leq n\rho^{1+\eta}/2. \quad (3.38)$$

and

$$\sum_{\Delta_k(\omega) \text{ of } \varepsilon\text{-type } (a)} [Q_k(\Psi)]^2 \lesssim \mathcal{E}^0(\rho)n \cdot \log n \cdot \log \log n. \quad (3.39)$$

Let us postpone the proof of this result for a while and continue with the proof of Theorem 3.22. The following lemma estimates the total contribution of ‘‘normal’’ pieces (i.e., that are not of ε -type) that carry too many particles.

Lemma 3.24. *Recall that $\{\Delta_k\}_{k=1}^{m(\omega)}$ denote the pieces.*

There exists $C > 0$ such that, for L sufficiently large, with probability $1 - O(L^{-\infty})$, for a normalized n -state Ψ in $\mathfrak{H}_{Q(\Psi)} \cap \mathfrak{H}_{\infty}^n([0, L])$ satisfying (3.34) and $Q(\Psi) = (Q_k)_{1 \leq k \leq m(\omega)}$, the occupation number of the state Ψ , one has

$$\sum_{\substack{|\Delta_k| \leq \ell_{\rho}(1-\rho^2) \\ Q_k \geq 2}} Q_k + \sum_{\substack{|\Delta_k| \in [\ell_{\rho}(1-\rho^2), 2\ell_{\rho}(1-\rho^2)) \\ Q_k \geq 3}} Q_k + \sum_{\substack{|\Delta_k| \in [2\ell_{\rho}(1-\rho^2), 3\ell_{\rho}(1-\rho^2)) \\ Q_k \geq 4}} Q_k \leq Cn\rho\ell_{\rho}^{-1} \quad (3.40)$$

and

$$\sum_{|\Delta_k| \leq 3\ell_\rho(1-\rho^2)} Q_k^2 \leq Cn\rho\ell_\rho^{-1} \quad (3.41)$$

and, for $\varepsilon \in (\rho^2, 1/4)$,

$$\sum_{\substack{|\Delta_k| \leq \ell_\rho(1-\varepsilon) \\ Q_k \geq 1}} Q_k + \sum_{\substack{|\Delta_k| \leq 2\ell_\rho(1-\varepsilon) \\ Q_k \geq 2}} Q_k + \sum_{\substack{|\Delta_k| \leq 3\ell_\rho(1-\varepsilon) \\ Q_k \geq 3}} Q_k \leq Cn \frac{\rho}{\varepsilon - \rho^2} \ell_\rho^{-1}. \quad (3.42)$$

Proof. First, note that by Theorem 3.19 and (3.34), there exists a constant \tilde{C} such that

$$\langle H_\omega^{Up} \Psi, \Psi \rangle \leq \langle H_\omega^{Up} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle + n\rho |\log \rho|^{-3} (r(\rho))^2 \leq \langle H_\omega^0 \Psi^0, \Psi^0 \rangle + \tilde{C}n\rho\ell_\rho^{-3}. \quad (3.43)$$

Moreover, if $-\Delta_{\Delta_k}^{Q_k}$ denotes the Laplacian with Dirichlet boundary conditions on $\bigwedge^{Q_k} L^2(\Delta_k)$, one has

$$(H_\omega^{Up})_{\mathfrak{H}_{Q(\Psi)}} \geq (H_\omega^0)_{\mathfrak{H}_{Q(\Psi)}} \geq \sum_{k=1}^{m(\omega)} \inf(\sigma(-\Delta_{\Delta_k}^{Q_k})) = \sum_{k=1}^{m(\omega)} \sum_{j=1}^{Q_k} \frac{\pi^2 j^2}{|\Delta_k|^2} = \sum_{k=1}^{m(\omega)} \frac{\pi^2 P(Q_k)}{|\Delta_k|^2} \quad (3.44)$$

where $P(X) := \frac{(2X+1)(X+1)X}{6}$.

On the other hand, by the description of Ψ^0 , for some $C > 0$, one has

$$\langle H_\omega^0 \Psi^0, \Psi^0 \rangle \leq \sum_{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2))} \frac{P(1)\pi^2}{|\Delta_k|^2} + \sum_{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2))} \frac{P(2)\pi^2}{|\Delta_k|^2} + Cn\rho^2$$

Plugging this and (3.44) into (3.43), we obtain

$$\begin{aligned} & \sum_{|\Delta_k| \leq \ell_\rho(1-\rho^2)} \frac{\pi^2}{|\Delta_k|^2} P(Q_k) + \sum_{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2))} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(1)) \\ & + \sum_{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2))} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(2)) \leq Cn\rho\ell_\rho^{-3}. \end{aligned} \quad (3.45)$$

By Lemma 3.23 and the explicit description of the non interacting ground state Ψ^0 (see the beginning of section 3.5), for some $C > 0$ and ρ sufficiently small, for L sufficiently large, with probability $1 - O(L^{-\infty})$, one has

$$\begin{aligned} & \sum_{|\Delta_k| \leq \ell_\rho(1-\rho^2)} Q_k + \sum_{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2))} Q_k + \sum_{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2))} Q_k \\ & \geq n(1 - C\rho^2) \\ & \geq \left[\sum_{|\Delta_k| \in [\ell_\rho(1+\rho^2), 2\ell_\rho(1-\rho^2))} 1 + \sum_{|\Delta_k| \in [2\ell_\rho(1+\rho^2), 3\ell_\rho(1-\rho^2))} 2 \right] - 2Cn\rho^2 \\ & \geq \left[\sum_{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2))} 1 + \sum_{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2))} 2 \right] - 3Cn\rho^2 \end{aligned} \quad (3.46)$$

as

$$\#\{k; |\Delta_k| \in [\ell_\rho(1-\rho^2), \ell_\rho(1+\rho^2)) \cup [2\ell_\rho(1-\rho^2), 2\ell_\rho(1+\rho^2))\} \leq Cn\rho^2.$$

Thus, (3.46) yields

$$\begin{aligned}
& \sum_{\substack{|\Delta_k| \leq \ell_\rho(1-\rho^2) \\ Q_k \geq 1}} Q_k + \sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k \geq 2}} (Q_k - 1) + \sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \geq 3}} (Q_k - 2) \\
& \geq \left[\sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k = 0}} 1 + \sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \leq 1}} 2 \right] - 3n\rho^{1+\eta}
\end{aligned} \tag{3.47}$$

Rewrite (3.45) as

$$\begin{aligned}
Cn\rho\ell_\rho^{-1} & \geq \sum_{\substack{|\Delta_k| \leq \ell_\rho(1-\rho^2) \\ Q_k \geq 1}} \frac{\pi^2}{|\Delta_k|^2} P(Q_k) + \sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k \geq 2}} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(1)) \\
& \quad + \sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \geq 3}} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(2)) \\
& \quad - \sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k = 0}} \frac{P(1)\pi^2}{|\Delta_k|^2} - \sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \leq 1}} \frac{(P(2) - P(Q_k))\pi^2}{|\Delta_k|^2} \\
& \geq \sum_{\substack{|\Delta_k| \leq \ell_\rho(1-\rho^2) \\ Q_k \geq 1}} \frac{\pi^2}{|\Delta_k|^2} P(Q_k) + \sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k \geq 2}} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(1)) \\
& \quad + \sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \geq 3}} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(2)) \\
& \quad - P(1) \left(\sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k = 0}} \frac{\pi^2}{|\Delta_k|^2} \right) - P(2) \left(\sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \leq 1}} \frac{\pi^2}{|\Delta_k|^2} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
Cn\rho\ell_\rho^{-1} & \geq \sum_{\substack{|\Delta_k| \leq \ell_\rho(1-\rho^2) \\ Q_k \geq 1}} \frac{\pi^2}{|\Delta_k|^2} P(Q_k) + \sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k \geq 2}} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(1)) \\
& \quad + \sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \geq 3}} \frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(2)) \\
& \quad - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} \left[\sum_{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2))} 1 + \sum_{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2))} 2 \right]
\end{aligned}$$

as $P(1) = 1$ and $P(2) = 5 \leq 8 = 2^3 P(1)$.

Using (3.47), we then obtain

$$\begin{aligned}
Cn\rho\ell_\rho^{-1} &\geq \sum_{\substack{|\Delta_k| \leq \ell_\rho(1-\rho^2) \\ Q_k \geq 1}} \left(\frac{\pi^2}{|\Delta_k|^2} P(Q_k) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} Q_k \right) \\
&+ \sum_{\substack{|\Delta_k| \in [\ell_\rho(1-\rho^2), 2\ell_\rho(1-\rho^2)) \\ Q_k \geq 2}} \left(\frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(1)) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} (Q_k - 1) \right) \\
&+ \sum_{\substack{|\Delta_k| \in [2\ell_\rho(1-\rho^2), 3\ell_\rho(1-\rho^2)) \\ Q_k \geq 3}} \left(\frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(2)) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} (Q_k - 2) \right).
\end{aligned} \tag{3.48}$$

Now, we note that, for $X \geq n+1$, X integer, one has

$$P(X) - P(n) = \sum_{k=n+1}^X k^2 \geq (n+1)^2(X-n). \tag{3.49}$$

This yields

- for $Q_k \geq 1$ and $|\Delta_k| \leq \ell_\rho(1-\rho^2)$, one has

$$\frac{\pi^2}{|\Delta_k|^2} P(Q_k) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} Q_k > \frac{\pi^2 Q_k (Q_k - 1) (2Q_k + 3)}{6|\ell_\rho(1-\rho^2)|^2} \geq 0; \tag{3.50}$$

if, moreover, $|\Delta_k| \leq \ell_\rho(1-\varepsilon)$ ($\rho^2 < \varepsilon < 1/2$), by (3.49), one has

$$\frac{\pi^2}{|\Delta_k|^2} P(Q_k) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} Q_k \geq \left(\frac{\pi^2}{|\Delta_k|^2} - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} \right) Q_k \geq \frac{(8\pi)^2(\varepsilon - \rho^2)}{|\ell_\rho|^2} Q_k; \tag{3.51}$$

- for $Q_k \geq 2$ and $|\Delta_k| \leq 2\ell_\rho(1-\rho^2)$, one has

$$\begin{aligned}
\frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(1)) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} (Q_k - 1) &> \frac{\pi^2(2Q_k + 9)(Q_k - 2)(Q_k - 1)}{24|\ell_\rho(1-\rho^2)|^2} \\
&\geq 0;
\end{aligned} \tag{3.52}$$

if, moreover, $|\Delta_k| \leq 2\ell_\rho(1-\varepsilon)$ ($\rho^2 < \varepsilon < 1/2$), by (3.49), one has

$$\begin{aligned}
\frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(1)) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} (Q_k - 1) &\geq \left(\frac{4\pi^2}{|\Delta_k|^2} - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} \right) (Q_k - 1) \\
&\geq \frac{(8\pi)^2(\varepsilon - \rho^2)}{|\ell_\rho|^2} (Q_k - 1);
\end{aligned} \tag{3.53}$$

- for $Q_k \geq 3$ and $|\Delta_k| \leq 3\ell_\rho(1-\rho^2)$, one has

$$\begin{aligned}
\frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(2)) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} (Q_k - 2) &> \frac{\pi^2(2Q_k + 13)(Q_k - 3)(Q_k - 2)}{|\ell_\rho(1-\rho^2)|^2} \\
&\geq 0;
\end{aligned} \tag{3.54}$$

if, moreover, $|\Delta_k| \leq 3\ell_\rho(1-\varepsilon)$ ($\rho^2 < \varepsilon < 1/2$), by (3.49), one has

$$\begin{aligned}
\frac{\pi^2}{|\Delta_k|^2} (P(Q_k) - P(2)) - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} (Q_k - 2) &\geq \left(\frac{9\pi^2}{|\Delta_k|^2} - \frac{\pi^2}{|\ell_\rho(1-\rho^2)|^2} \right) (Q_k - 2) \\
&\geq \frac{(8\pi)^2(\varepsilon - \rho^2)}{|\ell_\rho|^2} (Q_k - 2).
\end{aligned} \tag{3.55}$$

Plugging (3.50) - (3.55) into (3.48) immediately yields (3.40) and (3.42), thus, completes the proof of (3.40) and (3.42) in Lemma 3.24.

To derive (3.41), we proceed as follows. Clearly, for $Q_k \geq 4$, the right hand sides of (3.50), (3.52) and (3.54) is larger than $\delta \cdot Q_k^2$ (for some $\delta \in (0, 1)$). Thus, (3.48) implies

$$\sum_{\substack{|\Delta_k| \leq 3\ell_\rho(1-\rho^2) \\ Q_k \geq 4}} Q_k^2 \leq Cn\rho\ell_\rho^{-1}.$$

On the other hand, by (3.40), one clearly has

$$\sum_{\substack{|\Delta_k| \leq 3\ell_\rho(1-\rho^2) \\ Q_k \leq 3}} Q_k^2 \leq 3 \sum_{\substack{|\Delta_k| \leq 3\ell_\rho(1-\rho^2) \\ Q_k \leq 3}} Q_k \leq Cn\rho\ell_\rho^{-1}.$$

Thus, the proof of (3.41) is complete. This completes the proof of Lemma 3.24. \square

We also remark the following

Lemma 3.25. Consider Ψ_ω^{Up} , the ground state of $H_\omega^{Up}(L, n)$.

There exists $C > 0$ such that for L sufficiently large, with probability at least $1 - O(L^{-\infty})$, no piece of length smaller than

$$\ell_{min} = \ell_\rho - C\rho\ell_\rho \tag{3.56}$$

is occupied by particles of Ψ^{Up} .

Remark 3.26. The proof of Lemma 3.25 shows that it suffices to take $C > 4B + 4$ for ρ sufficiently small; here, B is the constant defining U^p (see (3.16)).

Proof. Suppose that the claim of the lemma is false. Then, a piece shorter than ℓ_{min} is occupied.

Let us show now that, as there are too many such pieces, pieces longer than ℓ_{min} cannot be all in interaction with n particles, no matter where these n particles are.

First of all, according to Proposition 2.2, the total number of pieces longer than ℓ_{min} is

$$\begin{aligned} \#\{j : |\Delta_j(\omega)| \geq \ell_{min}\} &= Le^{-\ell_{min}} + O(L^{1/2+0}) = L \frac{\rho}{1+\rho} (1 + C\rho\ell_\rho + O(\rho^2\ell_\rho^2)) \\ &= n(1 + C\rho\ell_\rho + O(\rho)). \end{aligned}$$

The number of pieces of length larger than $2\ell_\rho$ is $n\rho(1 + O(\rho))$. If a particle lies in one of these pieces, it can interact with at most $2B$ other pieces of length greater than ℓ_{min} .

For pieces smaller than $2\ell_\rho$ (but as always larger than ℓ_{min}), we remark that if two such pieces are at a distance greater than $(2B + 2)\ell_\rho$ from one another then they cannot interact with the same particle, except for the cases already taken into account above.

Moreover, according to Proposition 2.3, the number of pairs of such pieces at distance at most $(2B + 2)\ell_\rho$ is given by

$$\begin{aligned} &\#\{(\Delta_i, \Delta_j), |\Delta_i| > \ell_{min}, |\Delta_j| > \ell_{min}, \text{dist}(\Delta_i, \Delta_j) \leq (2B + 2)\ell_\rho\} \\ &= 2(2B + 2)\ell_\rho L (e^{-\ell_{min}})^2 + O(L^{3/4}) \\ &= (4B + 4)n\rho\ell_\rho(1 + O(\rho\ell_\rho)). \end{aligned}$$

Consequently, the rest of these pieces are at larger distances from each other. This leaves at least

$$\begin{aligned} n(1 + C\rho\ell_\rho + O(\rho)) - (2B + 1)n\rho(1 + O(\rho)) - (4B + 4)n\rho\ell_\rho(1 + O(\rho\ell_\rho)) \\ = n(1 + (C - 4B - 4)\rho\ell_\rho + O(\rho)) \end{aligned}$$

pieces such that no two of them can interact with the same particle. Remark that it suffices to take $C > 4B + 4$ to ensure that this number is larger than n for ρ small. This proves that there exists at least one piece longer than ℓ_{min} which neither occupied nor interacting with any particle in a ground state $\Psi_\omega^{Up}(L, n)$.

This leads to a contradiction with the fact that the ground state $\Psi_\omega^{Up}(L, n)$ puts at least one particle in a piece smaller than ℓ_{min} : indeed, moving this particle to the piece longer than ℓ_{min} which was singled out just above would result in a decrease of energy as no interaction energy would be added and non interacting energy would obviously decrease with the increase of the piece's length. This completes the proof of Lemma 3.25. \square

Let us now resume the proof of Theorem 3.22. In what follows, Ψ is a function satisfying condition (3.34). By Theorem 3.19, using $\Psi^{opt}(L, n)$ as a trial function, we see that both Ψ and $\Psi^{opt}(L, n)$ satisfy the assumptions of Lemma 3.23. Thus, picking $\eta \in (0, 1/3)$ and ε sufficiently small, by Lemma 3.23, for ρ sufficiently small and L sufficiently large, with probability $1 - O(L^{-\infty})$, we have

$$\sum_{\bullet \in \{a,b,c\}} \sum_{\Delta_k(\omega) \text{ of } \varepsilon\text{-type } (\bullet)} (Q_k(\Psi^{opt}(L, n)) + Q_k(\Psi)) \leq n\rho^{1+\eta}. \quad (3.57)$$

We will now reason on the particles in $\Psi_\omega^{Up}(L, n)$ that live in pieces that are not of ε -type (a), (b) or (c).

Recall that, by definition (see Definitions 3.9 and 3.14), $\Psi^{opt}(L, n)$ puts

- no particle in each piece of length in $(0, \ell_\rho - x_*\rho)$;
- one particle in each piece of length in $[\ell_\rho - x_*\rho, 2\ell_\rho + A_*]$;
- two particles (as a true two-particles state) in each piece of length in $[2\ell_\rho + A_*, 3\ell_\rho]$;

Let C be the constant from the claim of Theorem 3.22 that we will fix later on. Define

- n_0^+ to be the total number of pieces of length in $(0, \ell_\rho - x_*\rho)$ where Ψ puts exactly 1 particle;
- n_1^- to be the total number of pieces of length in $[\ell_\rho - x_*\rho, \ell_\rho + C)$ where Ψ puts no particle;
- n_1^+ to be the total number of pieces of length in $[\ell_\rho - x_*\rho, \ell_\rho + C)$ where Ψ puts exactly 2 particles;
- \tilde{n}_1^- to be the total number of pieces of length in $[\ell_\rho + C, 2\ell_\rho + A_*)$ where Ψ puts no particle;
- \tilde{n}_1^+ to be the total number of pieces of length in $[\ell_\rho + C, 2\ell_\rho + A_*)$ where Ψ puts exactly 2 particles;
- n_2^- to be the total number of pieces of length in $[2\ell_\rho + A_*, 3\ell_\rho(1 - \varepsilon))$ where Ψ puts exactly 1 particle;
- n_2^+ to be the total number of pieces of length in $[2\ell_\rho + A_*, 3\ell_\rho(1 - \varepsilon))$ where Ψ puts exactly 3 particles.

The general idea of the forthcoming proof is the following. On the one hand, Lemma 3.23 tells that pieces with too many neighbors are a sort of exception in a sense that they occur relatively rarely and carry relatively few particles. From the other hand, according to Lemma 3.24, pieces with too many particles are also relatively exceptional.

Finally, let us complement these two observations by noting that no particle in a piece of length in $[2\ell_\rho + A_*, 3\ell_\rho(1 - \varepsilon))$ can also occur for a small fraction of them. Therefore, we first note that it is sufficient to argue for pieces that are not of ε -type (as those of ε -type are already handled by Lemma 3.23). Let us now take a look at the distribution of particles in the state Ψ^{opt} in the pieces of length in $[2\ell_\rho + A_*, 3\ell_\rho(1 - \varepsilon))$ that have no particles and no neighbors (as they are not of ε -type) in Ψ . Obviously, moving a particle from a piece of

length greater than $2\ell_\rho + A_*$ to a smaller piece induces an increase of the non interacting energy of order ℓ_ρ^{-2} just because the pieces longer than $\ell_\rho - \rho x_*$ are already occupied by at least one particle (thus the non interacting energy of a second particle is a best $4\pi^2/(2\ell_\rho + A_*)^2$ and $\pi^2/(\ell_\rho - \rho x_*)^2$ if a particle is placed in a non occupied piece). Thus, the total number of pieces of length greater than $2\ell_\rho + A_*$ with no particles is bounded by $O(n\rho\ell_\rho^{-1})$.

The last three arguments together prove essentially that the distances dist_0 and dist_1 coincide for the matter of the current proof up to an admissible error i.e. of size $O(n\rho\ell_\rho^{-1})$. Namely, by the definition of the distance dist_1 , one has

$$\begin{aligned} \text{dist}_1(Q|_{<\ell_\rho+C}(\Psi), Q|_{<\ell_\rho+C}(\Psi^{\text{opt}})) &= n_0^+ + n_1^+ + n_1^- + r, \\ \text{dist}_1(Q|_{\geq\ell_\rho+C}(\Psi), Q|_{\geq\ell_\rho+C}(\Psi^{\text{opt}})) &= \tilde{n}_1^+ + \tilde{n}_1^- + n_2^+ + n_2^- + r', \end{aligned} \quad (3.58)$$

and, by the fact that the total number of particles in both states is the same, one gets

$$n_0^+ + n_1^+ + \tilde{n}_1^+ + n_2^+ + r'' = n_1^- + \tilde{n}_1^- + n_2^- + r''' \quad (3.59)$$

where

$$\max(r, r', r'', r''') \leq Cn\rho\ell_\rho^{-1}. \quad (3.60)$$

Recall that $r(\rho)$ is of order at most $|\log \rho|^{-1}$. Hence, if (3.35) does not hold, for any constant C_1 , if L is large enough, either one has

$$\tilde{n}_1^+ + \tilde{n}_1^- + n_2^+ + n_2^- \geq C_1 n\rho \cdot r(\rho) \quad (3.61)$$

or one has

$$n_0^+ + n_1^+ + n_1^- \geq C_1 n\sqrt{\rho} \cdot r(\rho). \quad (3.62)$$

First, we simplify (3.61). Suppose that, for some C_1 large, one has

$$n_2^+ \geq \frac{C_1}{4} n\rho \cdot r(\rho). \quad (3.63)$$

The number of pieces of length in $[\frac{5}{2}\ell_\rho, 3\ell_\rho(1-\varepsilon))$ is given by

$$\#\left\{j : |\Delta_j(\omega)| \in \left[\frac{5}{2}\ell_\rho, 3\ell_\rho(1-\varepsilon)\right]\right\} = O(n\rho^{3/2}).$$

Thus, at least $\frac{C_1}{5} n\rho \cdot r(\rho)$ of the pieces with three particles (as given by (3.63)) have their length in $[2\ell_\rho + A_*, \frac{5}{2}\ell_\rho)$. Hence, the non interacting energy excess (compared to the non interacting energy in the ground state) for each of these pieces is lower bounded by $O(\ell_\rho^{-2})$ which, in turn, being multiplied by their total number, contradicts (3.34). This simplifies (3.61) into

$$\tilde{n}_1^+ + \tilde{n}_1^- + n_2^- \geq C_1 n\rho \cdot r(\rho). \quad (3.64)$$

The conditions (3.59), (3.60) and either (3.62) or (3.64) lead us to a number of possibilities that we will now study one by one. More precisely, there are nine possible variants as at least one among n_1^- , \tilde{n}_1^- and n_2^- should be ‘‘large’’ and the same is true for either n_0^+ , n_1^+ , n_2^+ and \tilde{n}_1^+ . We now discuss these cases.

(a) Consider first the case when

$$\min(\tilde{n}_1^+, n_2^-) \geq C_2 n\rho \cdot r(\rho) \quad (3.65)$$

with $C_2 < C_1/3$.

This corresponds to taking the same configuration of particles as in Ψ^{opt} and move some of them from pieces of length in $[2\ell_\rho + A_*, 3\ell_\rho(1-\varepsilon))$ to pieces of length in $[\ell_\rho + C, 2\ell_\rho + A_*)$ that already contain one particle each. As we are now dealing only with pieces that are not of ε -type, this implies in particular that the pieces of length in $[2\ell_\rho + A_*, 3\ell_\rho(1-\varepsilon))$ from which we withdraw particles and that originally contain 2 particles, do not have any neighbors.

Taking the smallest available pieces for particle donors and the largest available for particle acceptors gives a lower bound on the total energy increase induced by this operation. Suppose that $C_2 n \rho r(\rho)$ smallest pieces have their length between $2\ell_\rho + A_*$ and $2\ell_\rho + A_* + \delta$. Then, choosing C_1 (thus, C_2) much larger than the constant in Lemma 3.24 for the case when $r(\rho) \asymp |\log \rho|^{-1}$, we obtain

$$L e^{-2\ell_\rho - A_*} (1 - e^{-\delta}) \geq \frac{C_2}{2} n \rho \cdot r(\rho),$$

which yields

$$\delta \geq \frac{C_2 e^{A_*}}{2} r(\rho). \quad (3.66)$$

Moreover, analogous calculations show that at least $\frac{C_2}{3} n \rho r(\rho)$ of these pieces have length in $(2\ell_\rho + A_* + \delta/2, 2\ell_\rho + A_* + \delta)$. For the particles in these pieces, the increase of energy is lower bounded by

$$\begin{aligned} \frac{4\pi^2}{(2\ell_\rho + A_* + \delta/2)^2} + \frac{\gamma}{(2\ell_\rho + A_* + \delta/2)^3} \\ - \frac{4\pi^2}{(2\ell_\rho + A_*)^2} - \frac{\gamma}{(2\ell_\rho + A_*)^3} + O(\ell_\rho^{-4}) \geq C_3 r(\rho) \ell_\rho^{-3}, \end{aligned} \quad (3.67)$$

where $C_3 > 0$. Multiplying the number of pieces by the lower bound (3.67) gives a total energy excess that contradicts (3.34) if we choose C_2 (hence, C_1) sufficiently large.

(b) The case

$$\min(n_1^+, n_2^-) \geq C_2 n \rho \cdot r(\rho)$$

is even simpler than the previous one. Indeed, in Ψ^{opt} , the occupations of the pieces of length in $[\ell_\rho - \rho x_*, \ell_\rho + C)$ and in $[\ell_\rho + C, 2\ell_\rho + A_*)$ are the same but the lengths considered in the previous case are smaller. Hence, the arguments developed in point (a) above enable one to conclude with the only difference that the increase of energy is even larger. Moreover, there is no need to remove the small interval of size δ .

(c) Next, the situation when

$$\min(n_0^+, n_2^-) \geq C_2 n \rho \cdot r(\rho) \quad (3.68)$$

corresponds to moving excited particles, i.e., particles occupying the second energy level, from pieces of length in $[2\ell_\rho + A_*, 3\ell_\rho(1 - \varepsilon))$ to empty pieces of length smaller than $\ell_\rho - \rho x_*$. Recall that actually the approximate equilibrium between the gain in interaction energy due to decoupling and the increase of non-interaction energy was part of the definition of values of x_* and A_* , i.e.,

$$\frac{4\pi^2}{(2\ell_\rho + A_*)^2} + \gamma \ell_\rho^{-3} = \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} + O(\ell_\rho^{-4}). \quad (3.69)$$

Obviously, the smaller the piece we choose to remove the second particle from, the more energy one gains. On the other hand, the larger the piece where one puts the particle, the smaller the non interacting energy increase, thus, the better.

According to these two observations, we choose to move particles from the $C_2 n \rho \cdot r(\rho)$ smallest pieces longer than $2\ell_\rho + A_*$. Suppose that the largest of these pieces has length $2\ell_\rho + A_* + B_2$. Then, by Proposition 2.2, B_2 satisfies

$$L e^{-2\ell_\rho - A_*} (1 - e^{-B_2}) + O(L^{1/2+0}) = C_2 n \rho \cdot r(\rho).$$

Hence, $B_2 = C_2 e^{A_*} r(\rho)(1 + O(r(\rho)))$. Moreover, the number of such pieces with length in $[2\ell_\rho + A_* + B_2/2, 2\ell_\rho + A_* + B_2)$ is

$$\begin{aligned} \#\{k; |\Delta_k(\omega)| - 2\ell_\rho - A_* \in [B_2/2, B_2)\} &= L e^{-2\ell_\rho - A_*} (e^{-B_2/2} - e^{-B_2}) + O(L^{1/2+0}) \\ &\geq \frac{C_2}{3} n\rho \cdot r(\rho). \end{aligned} \quad (3.70)$$

Clearly, for all these $\frac{C_2}{3} n\rho \ell_\rho^{-1}$ pieces, the non interacting energy excess is proportional to $C_2 \ell_\rho^{-3} r(\rho)$; thus, multiplied by their total number (3.70), for large C_2 , this energy excess does not fit within the margin allowed by (3.34).

(d) Yet another possibility for (3.64) is that

$$\min(\max(n_1^+, \tilde{n}_1^+), \max(n_1^-, \tilde{n}_1^-)) \geq C_2 n\rho \cdot r(\rho).$$

Obviously, the variant

$$\min(\tilde{n}_1^+, n_1^-) \geq C_2 n\rho \cdot r(\rho).$$

is more advantageous from the energetic point of view. The question here is whether it is worth moving a particle from a piece of length close to the lower bound of the corresponding group, i.e., $\ell_\rho - \rho x_*$, to another piece (but as the second particle because there is already another particle in that piece) of length close to the upper bound, i.e., $2\ell_\rho + A_*$. In a certain sense, this is the opposite to the case (c) as the latter tells that the threshold value A_* is not too small, while the current case will explain why A_* is not too big.

As above, one shows that, in order to choose the $C_2 n\rho \cdot r(\rho)$ largest pieces of length in $[\ell_\rho - \rho x_*, 2\ell_\rho + A_*)$, it is sufficient to solve

$$L e^{-2\ell_\rho - A_*} (e^{B_1} - 1) + O(L^{1/2+0}) = C_2 n\rho \cdot r(\rho),$$

which also implies $B_1 = C_2 e^{A_*} r(\rho)(1 + O(r(\rho)))$. Then, as above, the energy excess is proportional to $C_2 \ell_\rho^{-3} r(\rho)$ (where the constant C_2 can be chosen arbitrarily large) whereas the interaction terms are uniformly bounded by $O(\ell_\rho^{-4+0})$. Thus, the total energy gained by such an operation exceeds the limits imposed by (3.34).

(e) The next possible option is that

$$\min(n_0^+, \tilde{n}_1^-) \geq C_2 n\rho \cdot r(\rho). \quad (3.71)$$

This corresponds to moving particles in Ψ^{opt} from pieces of longer than $\ell_\rho + C$ to pieces shorter than $\ell_\rho - \rho x_*$. Remark first that the increase of non interacting energy is at least

$$\frac{\pi^2}{(\ell_\rho - \rho x_*)^2} - \frac{\pi^2}{(\ell_\rho + C)^2} \geq \frac{2\pi^2 C}{\ell_\rho^3}, \quad (3.72)$$

which always dominates the possible interaction with a particle in a neighboring piece: this interaction is $O(\ell_\rho^{-4+0})$ by Lemma 6.18. Multiplying the left hand sides of (3.71) and (3.72) gives a lower estimate on the energy excess that contradicts (3.34) because $r(\rho) = o(1)$.

(f) Finally, the only case left is when

$$\min(n_0^+, n_1^-) \geq C_2 n\sqrt{\rho} \cdot r(\rho). \quad (3.73)$$

Informally speaking, this is about the question if the threshold $\ell_\rho - \rho x_*$ between occupation zero and occupation one is placed correctly.

It is also remarkable that the allowed number of particle displacements for this case is much larger than in the other cases: one has to compare $o(n\sqrt{\rho})$ to $o(n\rho)$. This is due to the following mechanism. First, note that moving a particle that interacts with another particle in a neighboring piece may result to a decrease of the total energy.

Obviously, the contribution of the displacement of such particles is upper bounded by $O(n\rho\ell_\rho^{-4+0})$ because there are at most $O(n\rho)$ neighboring particles and the size of interaction is $O(\ell_\rho^{-4+0})$ by Lemma 6.18. Thus, these particles may be neglected for the precision of the current proof.

Then, reasoning as we did many times above, we observe that, at least $\frac{C_2}{2}n\sqrt{\rho}r(\rho)$ of particles that are removed from pieces of length in $[\ell_\rho - \rho x_*, \ell_\rho + C)$ have their length greater than $\ell_\rho + C_3\sqrt{\rho}r(\rho)$, where the constant C_3 grows together with C_2 . But, for each of these particles the non interacting energy increase is of order $C_3\ell_\rho^{-3}\sqrt{\rho}\cdot r(\rho)$. As above, multiplying the number of particles involved by the lower bound on the energy change, we get a contradiction with (3.34).

This completes the proof of Theorem 3.22. \square

We are now left with proving Lemmas 3.23.

The proof of Lemma 3.23. We first prove the estimate (3.38). It will be a consequence of the fact that the number of pieces in any of the three type is small and of the following

Lemma 3.27. *Pick k pieces of respective lengths $l_1 \leq l_2 \leq \dots \leq l_k$. Assume that, for $1 \leq i \leq k$, the state $\Psi \in \mathfrak{H}_{Q(\Psi)}^n \cap \mathfrak{H}_\infty^n([0, L])$ puts exactly ν_i particles in the piece i so that $\nu_1 + \dots + \nu_k = \nu$. Then, one has*

$$\frac{\pi^2\nu^3}{3l_k^2k^2} \leq \langle H^0(L, n)\Psi, \Psi \rangle \leq \langle H_\omega^{Up}(L, n)\Psi, \Psi \rangle \leq \langle H_\omega^U(L, n)\Psi, \Psi \rangle. \quad (3.74)$$

Let us postpone the proof of this result for a while and complete the proof of Lemma 3.23. We shall write out the proof for pieces of type (a). Those for pieces of type (b) and (c) is similar.

Pick $\eta \in (0, 1)$ and $\varepsilon > 0$ such that $\eta + 2\varepsilon < 1/6$. The proof of Propositions 2.2 and 2.1 show that there exists $\rho_\varepsilon > 0$ such that, for $\rho \in (0, \rho_\varepsilon)$, for L sufficiently large, with probability $1 - O(L^{-\infty})$, for one has

$$\#\{k; |\Delta_k(\omega)| \in [3\ell_\rho(1 - \varepsilon), 4\ell_\rho]\} \leq n\rho^{2-3\varepsilon} \quad (3.75)$$

and, for $4 \leq k \leq \log L \cdot \log \log L$,

$$\#\{k; |\Delta_k(\omega)| \in [k\ell_\rho, (k+1)\ell_\rho]\} \leq n\rho^{k-1-\varepsilon}. \quad (3.76)$$

Now, if Ψ places more than $n\rho^{1+\eta}$ particles in pieces of type a then

- either it places at least $2^{-1}n\rho^{1+\eta}$ particles in pieces of length in $[3\ell_\rho(1 - \varepsilon), 4\ell_\rho]$; in this case, by Lemma 3.27, as $3(\eta + 2\varepsilon) < 1$, we know that

$$\langle H^0(L, n)\Psi, \Psi \rangle \geq \frac{\pi^2(n\rho^{1+\eta})^3}{8(4\ell_\rho)^2(n\rho^{2-3\varepsilon})^2} \gtrsim n\ell_\rho^{-2}\rho^{-1+3(\eta+2\varepsilon)} \gg n\ell_\rho^{-2} \quad (3.77)$$

for ρ small;

- or, for some $4 \leq k \leq \log L$, it places at least $n\rho^{1+\eta}2^{-k+2}$ particles in pieces of length in $[k\ell_\rho, (k+1)\ell_\rho]$; in this case, by Lemma 3.27, we know that

$$\langle H^0(L, n)\Psi, \Psi \rangle \gtrsim \frac{n\rho^{3+3\eta-2k-2\varepsilon}}{((k+1)\ell_\rho)^2 2^{3k}} \gtrsim n\ell_\rho^{-2}\rho^{-1} \frac{(8\rho)^{-k}}{(k+1)^2} \geq n\ell_\rho^{-2}\rho^{-1} \quad (3.78)$$

for ρ sufficiently small.

Hence, for ρ sufficiently small, recalling (1.13) and (2.7) (and that here $\mu = 1$), one has $\langle H^0(L, n)\Psi, \Psi \rangle > 2\mathcal{E}^0(\rho)n$.

This completes the proof of (3.38) in Lemma 3.23 for particles of type (a).

To deal with the particles of type (b) (resp. (c)), we replace the upper bounds (3.75) and (3.76) obtained using Proposition 2.2 by analogous upper bounds on the numbers of

pieces of type (b) (resp. (c)) obtained through Proposition 2.4 (resp. Proposition 2.5).

This completes the proof of (3.38) in Lemma 3.23.

Let us now prove (3.39). By (3.44), one has

$$\sum_{k=1}^{m(\omega)} \frac{\pi^2 P(Q_k(\Psi))}{|\Delta_k|^2} \leq \langle H_\omega^{Up}(L, n)\Psi, \Psi \rangle \leq 2\mathcal{E}^0(\rho)n$$

where P is defined in (3.44).

Taking Proposition 2.1 into account immediately yields (3.39) and completes the proof of Lemma 3.23. \square

The proof of Lemma 3.27. The form of the Hamiltonians (1.4), (3.16) (the definition of U^p), (1.6) and the non negativity of the interactions guarantee that

$$\langle H_\omega^{Up}(L, n)\Psi, \Psi \rangle \geq \langle H^0(L, n)\Psi, \Psi \rangle \geq \sum_{i=1}^k \sum_{m=1}^{\nu_i} \left(\frac{\pi \alpha_m^i}{l_i} \right)^2$$

where $(\alpha_m^i)_{1 \leq m \leq \nu_i} \in (\mathbb{N}^*)^{\nu_i}$ and $\alpha_1^i < \alpha_2^i < \dots < \alpha_{\nu_i}^i$.

Thus

$$\langle H^0(L, n)\Psi, \Psi \rangle \geq \sum_{i=1}^k \sum_{m=1}^{\nu_i} \left(\frac{\pi m}{l_i} \right)^2 \geq \frac{\pi^2}{3l_k^2} \sum_{i=1}^k \nu_i^3 \geq \frac{\pi^2 \nu^3}{3l_k^2 k^2}$$

as $\nu_1 + \dots + \nu_k = \nu$.

This completes the proof of Lemma 3.27. \square

Theorem 3.28. *For ρ sufficiently small, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, for any function $\Psi \in \mathfrak{H}^n \cap \mathfrak{H}_\infty^n([0, L])$,*

$$\frac{1}{n} \langle H_\omega^{Up}(L, n)\Psi, \Psi \rangle \geq \frac{1}{n} \langle H_\omega^{Up}(L, n)\Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle - o(\rho |\log \rho|^{-3}). \quad (3.79)$$

Proof. This result can easily be traced throughout the proof of Theorem 3.22 by considering each of the cases. Before doing so, let us give some preliminary remarks that correspond exactly to the three remarks found in the beginning of the proof of Theorem 3.22.

First, the energy gain due to moving a single particle is always bounded by $O(\ell_\rho^{-2})$ just because each individual particle in Ψ^{opt} brings to the system at most this amount of energy. Next, the number of pieces of ε -type is $O(n\rho^{1+\eta})$ (see Lemma 3.23); thus, the energy gain due to them is at most $O(n\rho^{1+\eta}\ell_\rho^{-2})$.

The pieces with too many particles are also rare by Lemma 3.24. Moreover, the many particles in these pieces always bring an excess of energy and never an energy gain.

Finally, the analysis of n_2^+ large (see (3.63)) shows that moving an extra particle to the majority of these pieces results in an energy increase of order of $O(\ell_\rho^{-2})$, whereas for only $O(n\rho^{3/2})$ of them adding a particle may be energetically favorable.

We treat now the cases from (a) to (f) of the last part of the proof of Theorem 3.22. For the matter of the current proof we shall put $r(\rho) = 0$ (because we are interested only in those states that have the energy smaller than Ψ^{opt}), thus, reducing the claim of Theorem 3.22 to

$$\text{dist}_1(Q(\Psi), Q(\Psi^{\text{opt}})) = O(n\rho\ell_\rho^{-1}).$$

- For those displacements when the possible energy gain is due to removing interaction with neighbors (this includes the cases (d), (e) and (f)), it suffices to remark that, by Lemma 6.18, the size of the interacting energy is bounded by $O(\ell_\rho^{-4+0})$. Combined with the fact that, in total, there are $O(n\rho)$ pairs of neighboring particles, this yields a total energy gain of size $O(n\rho\ell_\rho^{-4+0})$.

- For those displacements when the possible energy gain is due to decoupling particles living in the same piece (cases (a), (b) and (c)), the individual interacting energy is of size $O(\ell_\rho^{-3})$ while their total number is $O(n\rho\ell_\rho^{-1})$. This yields a total energy gain of size $O(n\rho\ell_\rho^{-4})$.
- Finally, when the energy gain results from a non interacting energy decrease (like in the case (d)), it is at most $O(\ell_\rho^{-3})$ and the total number of displacements that result in energy decrease is $O(n\rho\ell_\rho^{-1})$. This again yields a total energy gain of size $O(n\rho\ell_\rho^{-4})$.

This concludes the proof of (3.79). \square

Corollary 3.29. *There exists $\rho_0 > 0$ such that for $\rho \in (0, \rho_0)$, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$,*

$$\begin{aligned} \frac{1}{n} \langle H_\omega^{Up}(L, n) \Psi^{Up}, \Psi^{Up} \rangle &= \frac{1}{n} \langle H_\omega^{Up}(L, n) \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle + O(\rho |\log \rho|^{-4}) \\ &= \mathcal{E}^0(\rho) + \pi^2 \gamma_* \frac{\rho}{|\log \rho|^3} + \frac{\rho}{|\log \rho|^3} O(f_Z(|\log \rho|)), \end{aligned} \quad (3.80)$$

where the constant γ_* is given in (1.17), Z describes the behavior of U at infinity and f_Z is defined in Theorem 3.19.

Proof. The upper bound is given by the fact that Ψ^{Up} is the ground state of H_ω^{Up} . The lower bound is a direct consequence of (3.79) and (3.18). This proves (3.80). \square

Remark 3.30. The ground state Ψ^{Up} satisfies the conditions of Theorem 3.22. Hence, the inequalities (3.35) hold for the distance between the occupations of Ψ^{Up} and Ψ^{opt} .

3.7. The proof of Theorem 1.3. Theorem 3.22 and Theorem 3.28 give a rather complete description of the ground state for the operator with compactified interactions $H_\omega^{Up}(L, n)$. The description is given in terms of comparison with Ψ^{opt} (see Definitions 3.9 and 3.14). In this section, we complement it with estimates on the residual part of interactions W^r (see (3.16)).

Proposition 3.31. *There exists ρ_0 such that, for $\rho \in (0, \rho_0)$, in the thermodynamic limit, for L sufficiently large, with probability $1 - O(L^{-\infty})$, one has*

$$\frac{1}{n} \langle W^r \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle = O(\rho |\log \rho|^{-3} Z(2|\log \rho|)). \quad (3.81)$$

Proof. We will mostly follow the lines of the second part of the proof of Theorem 3.19 (see formula (3.21) and what follows). First, as in (3.23), one computes

$$\langle W^r \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle = \text{Tr} \left(U^r \gamma_{\Psi^{\text{opt}}}^{(2)} \right)$$

where $\gamma_{\Psi^{\text{opt}}}^{(2)}$ is given by (3.24). Let us treat here only the contribution of the second sum (3.24). It corresponds to interactions between single particles in pieces of length in $[\ell_\rho - \rho x_*, 2\ell_\rho + A_*]$. The other three sums only contribute error terms as the number of 2-particles sub-states in Ψ^{opt} is by a factor ρ smaller than that of single-particle sub-states. For the

second sum in (3.24)., using Lemma 6.17, one obtains

$$\begin{aligned}
& \text{Tr} \left(U^r (\text{Id} - \text{Ex}) \sum_{\substack{i,j=1,\dots,k_1 \\ i < j}} \gamma_{\phi_i} \otimes^s \gamma_{\phi_j} \right) \\
& \leq \sum_{\substack{|\Delta_i|, |\Delta_j| \in [\ell_\rho - \rho x_*, 2\ell_\rho + A_*] \cup [3\ell_\rho, +\infty) \\ i < j \\ \text{dist}(\Delta_i, \Delta_j) > B\ell_\rho}} \int_{\Delta_i \times \Delta_j} U(x-y) |\varphi_{\Delta_i}^1(x)|^2 |\varphi_{\Delta_j}^1(y)|^2 dx dy \\
& \leq C_1 n \rho \int_{B\ell_\rho}^{+\infty} \ell_\rho^{-1} a^{-3} Z(a) da.
\end{aligned}$$

Recall that Z is defined in (1.26).

We compute next

$$\int_{B\ell_\rho}^{+\infty} a^{-3} Z(a) da = \int_{B\ell_\rho}^{+\infty} \int_a^{+\infty} U(x) dx da \leq \int_{B\ell_\rho}^{+\infty} x U(x) dx \leq C \ell_\rho^{-2} Z(B\ell_\rho),$$

where the last inequality is just (6.61) for $\varepsilon = 2$. This completes the proof of (3.81). \square

Proof of Theorem 1.3. Proposition 3.31 immediately entails the asymptotics of the interacting ground state energy $\mathcal{E}^U(\rho)$. Indeed, as $H^{U^p} \leq H^U$, one has $\mathcal{E}^{U^p}(\rho) \leq \mathcal{E}^U(\rho)$; thus, the announced lower bound is given by (3.80). On the other hand, by Theorem 3.19 and Proposition 3.31, one has

$$\begin{aligned}
\langle H^U \Psi^U, \Psi^U \rangle & \leq \langle H^U \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle = \langle H^{U^p} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle + \langle W^r \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle \\
& = \mathcal{E}^0(\rho) + \pi^2 \gamma_* \rho |\log \rho|^{-3} (1 + O(f_Z(|\log \rho|))),
\end{aligned} \tag{3.82}$$

which gives the announced upper bound.

This, the facts that $B > 2$ and that Z is decreasing complete the proof of Theorem 1.3. \square

Our analysis yields the following description for the possible occupations of the ground state of the full Hamiltonian.

Corollary 3.32. *There exists $C > 0$ such that, ω almost surely, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, for any Ψ^U , ground state of the full Hamiltonian of fixed occupation $Q(\Psi^U)$, one has*

$$Q(\Psi^U) \in \mathcal{Q}_\rho := \left\{ Q \text{ occ.}; \begin{array}{l} \text{dist}_1(Q|_{\geq \ell_\rho + C}, Q|_{\geq \ell_\rho + C}(\Psi^{\text{opt}})) \\ \leq C n \rho \max \left(\sqrt{Z(2|\log \rho|)}, |\log \rho|^{-1} \right), \\ \text{dist}_1(Q|_{< \ell_\rho + C}, Q|_{< \ell_\rho + C}(\Psi^{\text{opt}})) \\ \leq C n \max \left(\sqrt{\rho Z(2|\log \rho|)}, \rho |\log \rho|^{-1} \right). \end{array} \right\} \tag{3.83}$$

Proof. Note that

$$\langle H^{U^p} \Psi^U, \Psi^U \rangle \leq \langle H^U \Psi^U, \Psi^U \rangle \leq \langle H^{U^p} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle + \langle W^r \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle.$$

Thus, according to Proposition 3.31, Ψ^U satisfies the condition (3.34) with

$$r(\rho) = C \sqrt{Z(2|\log \rho|)}$$

for some $C > 0$ sufficiently large.

Then, Theorem 3.22 is applicable and yields (3.83). This completes the proof of Corollary 3.32. \square

4. FROM THE OCCUPATION AND ENERGY BOUNDS TO THE CONTROL OF THE DENSITY MATRICES

In this section, we will derive Theorem 1.5 from Theorem 1.3, Corollary 3.32 and a computation of the reduced one particle and two particles density matrix of a (non factorized) state. More precisely, from Theorem 1.3 and Corollary 3.32, we will infer a description of the ground state Ψ^U in most of the pieces: roughly, in most of the pieces, the only occupied state is the ground state (up to a controllable error). We then use this knowledge to compute the reduced one particle and two particles density matrix of Ψ^U (up to a controllable error).

4.1. From the occupation decomposition to the reduced density matrices. Fix a configuration of the Poisson points, say, ω , and a state $\Psi \in \mathfrak{H}^n(\Lambda)$. Recall that, in the configuration ω , the pieces are denoted by $(\Delta_j(\omega))_{1 \leq j \leq m} = (\Delta_j)_{1 \leq j \leq m}$ (where $m = m(\omega)$, see section 2.2). For $1 \leq j \leq m$ and $q \geq 1$, let $(E_{q,n}^j)_{1 \leq n}$ be the eigenvalues (ordered increasingly) and $(\varphi_{q,n}^j)_{1 \leq n}$ be the associated eigenvectors of q interacting electronic particles in the piece $\Delta_j(\omega)$ i.e. the eigenvalues and eigenvectors of the Hamiltonian

$$H_{\Delta_j(\omega)}^q = - \sum_{l=1}^q \frac{d^2}{dx_l^2} + \sum_{1 \leq l < l' \leq q} U^p(x_l - x_{l'}) \quad (4.1)$$

acting on $\bigwedge_{l=1}^q L^2(\Delta_j(\omega))$ with Dirichlet boundary conditions. Recall that U^p is defined in section 3.5 (see (3.16)).

The occupation number decomposition (see section 3.1) implies that one can write

$$\Psi = \sum_Q \Psi_Q \quad \text{and} \quad \Psi_Q = \sum_{\bar{n} \in \mathbb{N}^m} a_{\bar{n}}^Q \Phi_{\bar{n}}^Q = \sum_{\substack{(n_j)_{1 \leq j \leq m} \\ \forall j, n_j \geq 1}} a_{n_1, \dots, n_m}^Q(\Psi) \bigwedge_{j=1}^m \varphi_{Q_j, n_j}^j \quad (4.2)$$

where

- the first sum is taken over the occupation number $Q = (Q_j)_{1 \leq j \leq m}$; recall $\sum_{j=1}^m Q_j = n$;
- we have defined $\Phi_{\bar{n}}^Q := \bigwedge_{j=1}^m \varphi_{Q_j, n_j}^j$; we refer to (C.2) in section C for an explicit description of the anti-symmetric tensor product.

Remark 4.1. In (4.2), the convention in the exterior product is that, if $Q_j = 0$, then the corresponding basis vector drops out of the exterior product. Thus, the product is only at most n fold. Moreover, in this case, $a_{n_1, \dots, n_m}^Q = 0$ if $n_j \geq 2$.

For $\bar{n} = (n_1, \dots, n_m) \in \mathbb{N}^m$, we write $a_{\bar{n}}^Q = a_{n_1, \dots, n_m}^Q = a_{n_1, \dots, n_m}^Q(\Psi)$. These coefficients are uniquely determined by Ψ .

4.1.1. The one-particle density matrix. We shall first compute the 1 particle reduced density matrix in terms of the coefficients $(a_{\bar{n}}^Q)_{Q, \bar{n}}$ coming up in the occupation number decomposition (4.2). We prove

Theorem 4.2. *The 1-particle density $\gamma_{\Psi}^{(1)}$ (see (1.19)) is written as $\gamma_{\Psi}^{(1)} = \gamma_{\Psi}^{(1),d} + \gamma_{\Psi}^{(1),o}$ where*

$$\gamma_{\Psi}^{(1),d} = \sum_{j=1}^m \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 1}} \sum_{\substack{n_j \geq 1 \\ n'_j \geq 1}} \sum_{\tilde{n} \in \mathbb{N}^{m-1}} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^{Q'}} \gamma_{Q_j}^{(1)}(n_j, n'_j) \quad (4.3)$$

$$\gamma_{\Psi}^{(1),o} = \sum_{\substack{i,j=1 \\ i \neq j}}^m \sum_{\substack{Q, \text{ occ.} \\ Q_j \geq 1 \\ Q'_k = Q_k \text{ if } k \notin \{i,j\} \\ Q'_i = Q_i + 1 \\ Q'_j = Q_j - 1}} C_1(Q, i, j) \sum_{\tilde{n} \in \mathbb{N}^{m-2}} \sum_{\substack{n_i, n_j \geq 1 \\ n'_i, n'_j \geq 1}} a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}} \gamma_{Q_i, Q_j}^{(1)}(n_i, n_j, n'_i, n'_j) \quad (4.4)$$

and

- we have used the shorthands
 - \tilde{n}_j for the vector $(\tilde{n}_1, \dots, \tilde{n}_{j-1}, n_j, \tilde{n}_j, \dots, \tilde{n}_{m-1})$ when $\tilde{n} = (\tilde{n}_1, \dots, \tilde{n}_{m-1})$,
 - $\tilde{n}_{i,j}$ for $(\tilde{n}_1, \dots, \tilde{n}_{i-1}, n_i, \tilde{n}_i, \dots, \tilde{n}_{j-2}, n_j, \tilde{n}_{j-1}, \dots, \tilde{n}_{m-2})$ when $i < j$ and $\tilde{n} = (\tilde{n}_1, \dots, \tilde{n}_{m-2})$,
- the trace class operator $\gamma_{Q_j}^{(1)} : L^2(\Delta_j) \rightarrow L^2(\Delta_j)$ has the kernel

$$\gamma_{Q_j}^{(1)}(x, y) = Q_j \int_{\Delta_j} \varphi_{Q_j, n_j}^j(x, z) \overline{\varphi_{Q_j, n'_j}^j(y, z)} dz,$$

- $C_1(Q, i, j) = \frac{(n - Q_j - Q_i - 1)! Q_i! Q_j!}{(n - 1)!}$;
- the rank one operator $\gamma_{Q_i, Q_j}^{(1)} : L^2(\Delta_i) \rightarrow L^2(\Delta_j)$ has the kernel

$$\gamma_{Q_i, Q_j}^{(1)}(x, y) = \int_{\Delta_j} \varphi_{Q_j, n_j}^j(x, z) \overline{\varphi_{Q_j-1, n'_j}^j(z)} dz \int_{\Delta_i} \varphi_{Q_i, n_i}^i(z) \overline{\varphi_{Q_i+1, n'_i}^i(y, z)} dz. \quad (4.5)$$

Remark 4.3. In (4.5), in accordance with remark 4.1, we use the following convention

- if $Q_j = 1$ and $Q_i = 0$ then $n'_j = 1$ and $n_i = 1$ (i.e. for different indices, the coefficient $a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}}$ vanishes) and

$$\gamma_{Q_i, Q_j}^{(1)}(x, y) = \varphi_{1, n_j}^j(x) \cdot \overline{\varphi_{1, n'_i}^i(y)}, \quad (4.6)$$

- if $Q_j \geq 2$ and $Q_i = 0$ then $n_i = 1$ and

$$\gamma_{Q_i, Q_j}^{(1)}(x, y) = \overline{\varphi_{1, n'_i}^i(y)} \int_{\Delta_j} \overline{\varphi_{Q_j-1, n'_j}^j(z)} \varphi_{Q_j, n_j}^j(x, z) dz, \quad (4.7)$$

- if $Q_j = 1$ and $Q_i \geq 1$ then $n'_j = 1$ and

$$\gamma_{Q_i, Q_j}^{(1)}(x, y) = \varphi_{1, n_j}^j(x) \int_{\Delta_i} \varphi_{Q_i, n_i}^i(z) \overline{\varphi_{Q_i+1, n'_i}^i(y, z)} dz. \quad (4.8)$$

Proof. Theorem 4.2 follows from a direct computation that we now perform. First, by the bilinearity of formula (1.19), one has

$$\gamma_{\Psi}^{(1)} = n \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \sum_{\substack{Q' \text{ occ.} \\ \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q a_{\bar{n}'}^{Q'} \gamma_{Q, \bar{n}}^{(1)} \gamma_{Q', \bar{n}'}^{(1)} \quad (4.9)$$

where the trace class operator $\gamma_{Q, n, Q', n'}^{(1)}$ acts on $L^2([0, L])$ and has the kernel

$$\gamma_{Q, \bar{n}}^{(1)} \gamma_{Q', \bar{n}'}^{(1)}(x, y) := \int_{[0, L]^{n-1}} \left[\bigwedge_{j=1}^m \varphi_{Q_j, n_j}^j \right](x, z) \overline{\left[\bigwedge_{j=1}^m \varphi_{Q'_j, n'_j}^j \right]}(y, z) dz.$$

Recall (C.2), that is, in the present case

$$\begin{aligned} & \left[\bigwedge_{j=1}^m \varphi_{Q_j, n_j}^j \right](z_1, z_2, \dots, z_n) \\ &= c(Q) \cdot \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \varepsilon(A_1, \dots, A_m) \prod_{j=1}^m \varphi_{Q_j, n_j}^j((z_l)_{l \in A_j}) \quad (4.10) \end{aligned}$$

where

- $\varepsilon(A_1, \dots, A_m)$ is the signature of $\sigma(A_1, \dots, A_m)$, the unique permutation of $\{1, \dots, n\}$ such that, if $A_j = \{a_{ij}; 1 \leq i \leq Q_j\}$ for $1 \leq j \leq m$ then $\sigma(a_{ij}) = Q_1 + \dots + Q_{j-1} + i$,
- and $c(Q)$ is such that $\| \bigwedge_j \varphi_{Q_j, n_j}^j \| = 1$ i.e.

$$c(Q) = \sqrt{\frac{\prod_{j=1}^m Q_j!}{n!}}. \quad (4.11)$$

Thus, by (1.19), one has

$$\frac{\gamma_{Q, \bar{n}}^{(1)} \gamma_{Q', \bar{n}'}^{(1)}(x, y)}{c(Q)c(Q')} = \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{|A'_j|=Q'_j, \forall 1 \leq j \leq m \\ A'_1 \cup \dots \cup A'_m = \{1, \dots, n\} \\ A'_j \cap A'_{j'} = \emptyset \text{ if } j \neq j'}} (-1)^{\varepsilon((A_j)) + \varepsilon((A'_j))} I((A_j)_j, (A'_j)_j) \quad (4.12)$$

where

$$\begin{aligned} I(A, A') &:= I((A_j)_j, (A'_j)_j) \\ &= \int_{[0, L]^{n-1}} \left[\prod_{j=1}^m \varphi_{Q_j, n_j}^j((x_l)_{l \in A_j}) \overline{\varphi_{Q'_j, n'_j}^j((y_l)_{l \in A'_j})} \right]_{\substack{x_1=x \\ y_1=y \\ x_j=y_j \text{ if } j \geq 2}} dx_2 \cdots dx_n. \quad (4.13) \end{aligned}$$

To evaluate this last integral, we note that, for any pair of partitions $(A_j)_j$ and $(A'_j)_j$ (as in the indices of the sum in (4.12)), if there exists $j \neq j'$ such that $A_j \cap A'_{j'} \cap \{2, \dots, n\} \neq \emptyset$, then the integral $I(A, A')$ vanishes.

Now, note that, if $d_1(Q, Q') > 2$, then, for any pair of partitions $(A_j)_j$ and $(A'_j)_j$, there exists $j \neq j'$ such that $A_j \cap A'_{j'} \cap \{2, \dots, n\} \neq \emptyset$; thus, the integral $I(A, A')$ above always vanishes and, summing this, one has

$$\gamma_{Q, \bar{n}}^{(1)} \gamma_{Q', \bar{n}'}^{(1)} = 0 \quad \text{if} \quad d_1(Q, Q') > 2.$$

So we are left with the case $Q = Q'$ or $d_1(Q, Q') = 2$.

Assume first $Q = Q'$. Consider the sums in (4.12). If $1 \in A_{j_0}$ and $1 \notin A'_{j_0}$, then, as $\forall j$,

$|A'_j| = |A_j|$, there exists $\alpha \in A'_{j_0} = A'_{j_0} \cap \{2, \dots, n\}$ and $j \neq j_0$ such that $\alpha \in A_j$. That is, there exists $j \neq j'$ such that $A_j \cap A'_{j'} \cap \{2, \dots, n\} \neq \emptyset$, thus, the integral $I(A, A')$ vanishes. Thus, we rewrite

$$\begin{aligned} \frac{\gamma_{Q, \bar{n}, Q, \bar{n}}^{(1)}(x, y)}{c^2(Q)} &= \sum_{\substack{j_0=1 \\ Q_{j_0} \geq 1}}^m \sum_{\substack{1 \in A_{j_0} \\ |A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{1 \in A'_{j_0} \\ |A'_{j_0}|=Q_{j_0} \\ A'_j=A_j \text{ if } j \neq j_0}} (-1)^{\varepsilon((A_j)) + \varepsilon((A'_j))} I(A, A') \\ &= \sum_{\substack{j_0=1 \\ Q_{j_0} \geq 1}}^m \sum_{\substack{1 \in A_{j_0} \\ |A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} I(A) \end{aligned} \quad (4.14)$$

where, using the support and orthonormality properties of the functions $(\varphi_{Q, n}^j)_{1 \leq n}$, one computes

$$\begin{aligned} I(A) &:= \left(\int_{\Delta_{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, z) \overline{\varphi_{Q_{j_0}, n'_{j_0}}^{j_0}(y, z)} dz \right) \prod_{\substack{j=1 \\ j \neq j_0}}^m \int_{\Delta_{Q_j}} \varphi_{Q_j, n_j}^j(z) \overline{\varphi_{Q_j, n'_j}^j(z)} dz \\ &= \prod_{j \neq j_0} \delta_{n_j = n'_j} \cdot \left(\int_{\Delta_{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, z) \overline{\varphi_{Q_{j_0}, n'_{j_0}}^{j_0}(y, z)} dz \right). \end{aligned}$$

As

$$\#\{(A_j)_j; 1 \in A_{j_0}, \forall j, |A_j| = Q_j\} = \frac{(n-1)! Q_{j_0}}{\prod_{j=1}^m Q_j!}$$

by (4.11) and (4.14), one computes

$$\gamma_{Q, \bar{n}, Q, \bar{n}}^{(1)}(x, y) = \sum_{\substack{j=1 \\ Q_j \geq 1}}^m \frac{Q_j}{n} \int_{\Delta_{Q_j-1}} \varphi_{Q_j, n_j}^j(x, z) \overline{\varphi_{Q_j, n'_j}^j(y, z)} dz \prod_{j \neq j_0} \delta_{n_j = n'_j} = \frac{1}{n} \sum_{\substack{j=1 \\ n_j, n'_j \geq 1}}^m \gamma_{Q_j}^{(1)}(x, y).$$

We now assume that $d_1(Q, Q') = 2$. Thus, there exist $1 \leq i_0 \neq j_0 \leq m$ such that $Q_{j_0} \geq 1$, $Q'_{i_0} = Q_{i_0} + 1$, $Q_{j_0} = Q'_{j_0} + 1$ and $Q_k = Q'_k$ for $k \notin \{i_0, j_0\}$.

Consider the sums in (4.12). If $1 \notin A_{j_0}$ (or $1 \notin A'_{i_0}$), then as $|A'_{j_0}| = Q'_{j_0} = Q_{j_0} - 1$, there exists $\alpha \in A_{j_0} = A_{j_0} \cap \{2, \dots, n\}$ and $i \neq j_0$ such that $\alpha \in A'_i$. That is, there exists $j \neq j'$ such that $A_j \cap A'_{j'} \cap \{2, \dots, n\} \neq \emptyset$, thus, the integral $I(A, A')$ vanishes. The reasoning is the same if $1 \notin A'_{i_0}$. Moreover, if $1 \in A_{j_0}$ and $1 \in A'_{i_0}$, then, as in the derivation of (4.14), we see that $I(A, A') = 0$ except if $A_j = A'_j$ for all $j \notin \{i_0, j_0\}$. Therefore, if $d_1(Q, Q') = 2$, we rewrite

$$\frac{\gamma_{Q, \bar{n}, Q', \bar{n}'}^{(1)}(x, y)}{c^2(Q)} = \sum_{\substack{j_0, i_0=1 \\ i_0 \neq j_0 \\ Q_{j_0} \geq 1}}^m \sum_{\substack{1 \in A_{j_0} \\ |A_l|=Q_l, \forall 1 \leq l \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{i_0} = \{1\} \cup A_{i_0} \\ A'_{j_0} = A_{j_0} \setminus \{1\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} (-1)^{\varepsilon((A_j)) + \varepsilon((A'_j))} I(A, A'). \quad (4.15)$$

For such $(A_j)_j$ and $(A'_j)_j$, one has $(-1)^{\varepsilon((A_j)) + \varepsilon((A'_j))} = 1$ and we compute

$$I(A, A') = \int_{\Delta_{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, z) \overline{\varphi_{Q_{j_0}-1, n'_{j_0}}^{j_0}(z)} dz + \int_{\Delta_{Q_{i_0}}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(z) \overline{\varphi_{Q_{i_0}+1, n'_{i_0}}^{i_0}(y, z)} dz \prod_{j \notin \{i_0, j_0\}} \delta_{n_j = n'_j} \quad (4.16)$$

with the convention described in Remark 4.3.

The number of partitions coming up in (4.15) is given by

$$\sum_{\substack{1 \in A_{j_0} \\ |A_l| = Q_l, \forall 1 \leq l \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{i_0} = \{1\} \cup A_{i_0} \\ A'_{j_0} = A_{j_0} \setminus \{1\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} 1 = \frac{(n - Q_{j_0} - Q_{i_0} - 1)! Q_{i_0}! Q_{j_0}!}{Q_1! \dots Q_m!}.$$

Plugging this and (4.16) into (4.15), we obtain (4.5). This completes the proof of Theorem 4.2. \square

4.1.2. *The two-particle density matrix.* We shall now compute the 2 particles reduced density matrix in terms of the coefficients $(a_{\bar{n}}^Q)_{Q, \bar{n}}$ coming up in the occupation number decomposition (4.2). We prove

Theorem 4.4. *The 2-particle density $\gamma_{\Psi}^{(2)}$ (see (1.19)) is written as*

$$\gamma_{\Psi}^{(2)} = \gamma_{\Psi}^{(2), d, d} + \gamma_{\Psi}^{(2), d, o} + \gamma_{\Psi}^{(2), 2} + \gamma_{\Psi}^{(2), 4, 2} + \gamma_{\Psi}^{(2), 4, 3} + \gamma_{\Psi}^{(2), 4, 3'} + \gamma_{\Psi}^{(2), 4, 4} \quad (4.17)$$

where

$$\gamma_{\Psi}^{(2), d, d} = \sum_{j=1}^m \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2}} \sum_{\substack{n_j \geq 1 \\ n'_j \geq 1}} \sum_{\tilde{n} \in \mathbb{N}^{m-1}} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^Q} \gamma_{n_j, n'_j}^{(2), d, d} \quad (4.18)$$

$$\gamma_{\Psi}^{(2), d, o} = \sum_{\substack{1 \leq i < j \leq m \\ Q_i \geq 1 \\ Q_j \geq 1}} \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1}} \sum_{\substack{\tilde{n} \in \mathbb{N}^{m-2} \\ n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^Q} \gamma_{n_i, n'_i, n_j, n'_j}^{(2), d, o} \quad (4.19)$$

$$\gamma_{\Psi}^{(2), 2} = \sum_{\substack{i, j=1 \\ i \neq j}}^m \sum_{\substack{Q, \text{ occ.} \\ Q'_k = Q_k \text{ if } k \notin \{i, j\} \\ Q'_i = Q_i + 1 \\ Q'_j = Q_j - 1}} \sum_{\substack{Q_j \geq 1 \\ \tilde{n} \in \mathbb{N}^{m-1}}} C_2(Q, i, j) \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}} \gamma_{n_i, n'_i, n_j, n'_j}^{(2), 2} \quad (4.20)$$

$$\gamma_{\Psi}^{(2), 4, 2} = \sum_{i \neq j} \sum_{\tilde{n} \in \mathbb{N}^{m-2}} \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2 \\ Q'_k = Q_k \text{ if } k \notin \{i, j\} \\ Q'_i = Q_i + 2 \\ Q'_j = Q_j - 2}} C_2(Q, i, j) \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}} \gamma_{n_i, n'_i, n_j, n'_j}^{(2), 4, 2} \quad (4.21)$$

$$\gamma_{\Psi}^{(2),4,3} = \sum_{\substack{i,j,k \\ \text{distinct}}} \sum_{\tilde{n} \in \mathbb{N}^{m-3}} \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2 \\ Q': Q'_i = Q_i \text{ if } l \notin \{i,j,k\} \\ Q'_i = Q_i + 1 \\ Q'_j = Q_j - 2 \\ Q'_k = Q_k + 1}} C_3(Q, i, j, k) \sum_{\substack{n_i, n_j, n_k \geq 1 \\ n'_i, n'_j, n'_k \geq 1}} a_{\tilde{n}_{i,j,k}}^Q \overline{a_{\tilde{n}'_{i,j,k}}^{Q'}} \gamma_{Q_i, Q_j, Q_k}^{(2),4,3} \quad (4.22)$$

$$\gamma_{\Psi}^{(2),4,3'} = \sum_{\substack{i,j,k \\ \text{distinct}}} \sum_{\tilde{n} \in \mathbb{N}^{m-3}} \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1, Q_k \geq 1 \\ Q': Q'_i = Q_i \text{ if } l \notin \{i,j,k\} \\ Q'_i = Q_i - 1 \\ Q'_j = Q_j + 2 \\ Q'_k = Q_k - 1}} C_3(Q, i, j, k) \sum_{\substack{n_i, n_j, n_k \geq 1 \\ n'_i, n'_j, n'_k \geq 1}} a_{\tilde{n}_{i,j,k}}^Q \overline{a_{\tilde{n}'_{i,j,k}}^{Q'}} \gamma_{Q_i, Q_j, Q_k}^{(2),4,3'} \quad (4.23)$$

and

$$\gamma_{\Psi}^{(2),4,4} = \sum_{\substack{i,j,k,l \\ \text{distinct}}} \sum_{\tilde{n} \in \mathbb{N}^{m-4}} \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1, Q_j \geq 1 \\ Q': Q'_i = Q_i \text{ if } l \notin \{i,j,k,l\} \\ Q'_i = Q_i - 1, Q'_j = Q_j - 1 \\ Q'_k = Q_k + 1, Q'_l = Q_l + 1}} C_4(Q, i, j, k, l) \sum_{\substack{n_i, n_j, n_k, n_l \geq 1 \\ n'_i, n'_j, n'_k, n'_l \geq 1}} a_{\tilde{n}_{i,j,k,l}}^Q \overline{a_{\tilde{n}'_{i,j,k,l}}^{Q'}} \gamma_{Q_i, Q_j, Q_k, Q_l}^{(2),4,4} \quad (4.24)$$

where

- we have used the shorthands defined in Theorem 4.2 and defined
 - $\tilde{n}_{i,j,k}$ for $(\tilde{n}_1, \dots, \tilde{n}_{i-1}, n_i, \tilde{n}_i, \dots, \tilde{n}_{j-2}, n_j, \tilde{n}_{j-1}, \dots, \tilde{n}_{k-3}, n_k, \tilde{n}_{k-2}, \dots, \tilde{n}_{m-3})$ when $i < j < k$ and $\tilde{n} = (\tilde{n}_1, \dots, \tilde{n}_{m-3})$,
 - $\tilde{n}_{i,j,k,l}$ for $(\tilde{n}_1, \dots, \tilde{n}_{i-1}, n_i, \tilde{n}_i, \dots, \tilde{n}_{j-2}, n_j, \tilde{n}_{j-1}, \dots, \tilde{n}_{k-3}, n_k, \tilde{n}_{k-2}, \dots, \tilde{n}_{l-4}, n_l, \tilde{n}_{l-3}, \dots, \tilde{n}_{m-4})$ when $i < j < k < l$ and $\tilde{n} = (\tilde{n}_1, \dots, \tilde{n}_{m-4})$,
- the trace class operator $\gamma_{Q_j}^{(2),d,d} : L^2(\Delta_j) \wedge L^2(\Delta_j) \rightarrow L^2(\Delta_j) \wedge L^2(\Delta_j)$ has the kernel

$$\gamma_{Q_j}^{(2),d,d}(x, x', y, y') = \frac{Q_j(Q_j - 1)}{2} \int_{\Delta_j^{Q_j-2}} \varphi_{Q_j, n_j}^j(x, x', z) \overline{\varphi_{Q_j, n'_j}^j(y, y', z)} dz \quad (4.25)$$

- the trace class operator $\gamma_{Q_i, Q_j}^{(2),d,o} : L^2(\Delta_i) \otimes L^2(\Delta_j) \rightarrow L^2(\Delta_i) \otimes L^2(\Delta_j)$ has the kernel

$$\gamma_{Q_i, Q_j}^{(2),d,o}(x, x', y, y') = \frac{Q_i Q_j}{2} \times \int_{\Delta_i^{Q_i-1} \times \Delta_j^{Q_j-1}} dz dz' \quad (4.26)$$

$$\begin{vmatrix} \varphi_{Q_i, n_i}^i(x, z) & \varphi_{Q_i, n_i}^i(x, z') \\ \varphi_{Q_j, n_j}^j(x', z) & \varphi_{Q_j, n_j}^j(x', z') \end{vmatrix} \cdot \overline{\begin{vmatrix} \varphi_{Q_i, n'_i}^i(y, z) & \varphi_{Q_i, n'_i}^i(y', z) \\ \varphi_{Q_j, n'_j}^j(y, z') & \varphi_{Q_j, n'_j}^j(y', z') \end{vmatrix}}$$

- $C_2(Q, i, j) = \frac{(n - Q_j - Q_i - 2)! Q_i! Q_j!}{2(n-2)!}$;

- the trace-class operator $\gamma_{\substack{Q_i, Q_j \\ n_i, n_j \\ n'_i, n'_j}}^{(2),2} : L^2(\Delta_j) \wedge L^2(\Delta_j) \rightarrow L^2(\Delta_i) \wedge L^2(\Delta_i)$ has the kernel

$$\begin{aligned} \gamma_{\substack{Q_i, Q_j \\ n_i, n_j \\ n'_i, n'_j}}^{(2),2}(x, y) &= \mathbf{1}_{Q_j \geq 2} \int_{\Delta_j^{Q_j-2} \times \Delta_i^{Q_i}} \varphi_{Q_j, n_j}^j(x, x', z) \varphi_{Q_i, n_i}^i(z') \overline{\begin{vmatrix} \varphi_{Q_j-1, n'_j}^j(y', z) & \varphi_{Q_j-1, n'_j}^j(y, z) \\ \varphi_{Q_i+1, n'_i}^i(y', z') & \varphi_{Q_i+1, n'_i}^i(y, z') \end{vmatrix}} dz dz' \\ &+ \mathbf{1}_{Q_i \geq 1} \int_{\Delta_j^{Q_j-1} \times \Delta_i^{Q_i-1}} \left| \begin{vmatrix} \varphi_{Q_j, n_j}^j(x', z) & \varphi_{Q_j, n_j}^j(x, z) \\ \varphi_{Q_i, n_i}^i(x', z') & \varphi_{Q_i, n_i}^i(x, z') \end{vmatrix} \overline{\varphi_{Q_j-1, n'_j}^j(z) \varphi_{Q_i+1, n'_i}^i(y, y', z')} \right| dz dz', \end{aligned} \quad (4.27)$$

- the rank one operator $\gamma_{\substack{Q_i, Q_j \\ n_i, n_j \\ n'_i, n'_j}}^{(2),4,2} : L^2(\Delta_j) \wedge L^2(\Delta_j) \rightarrow L^2(\Delta_i) \wedge L^2(\Delta_i)$ has the kernel

$$\gamma_{\substack{Q_i, Q_j \\ n_i, n_j \\ n'_i, n'_j}}^{(2),4,2}(x, x', y, y') = \int_{\Delta_j^{Q_j-2}} \varphi_{Q_j, n_j}^j(x, x', z) \overline{\varphi_{Q_j-2, n'_j}^j(z)} dz \int_{\Delta_i^{Q_i}} \varphi_{Q_i, n_i}^i(z) \overline{\varphi_{Q_i+2, n'_i}^i(y, y', z)} dz. \quad (4.28)$$

- the rank 2 operator $\gamma_{\substack{Q_i, Q_j, Q_k \\ n_i, n_j, n_k \\ n'_i, n'_j, n'_k}}^{(2),4,3} : L^2(\Delta_i) \otimes L^2(\Delta_k) \rightarrow L^2(\Delta_j) \wedge L^2(\Delta_j)$ has the kernel

$$\begin{aligned} \gamma_{\substack{Q_i, Q_j, Q_k \\ n_i, n_j, n_k \\ n'_i, n'_j, n'_k}}^{(2),4,3}(x, x', y, y') &= \int_{\Delta_j^{Q_j-2}} \varphi_{Q_j, n_j}^j(x, x', z) \overline{\varphi_{Q_j-2, n'_j}^j(z)} dz \\ &\times \left| \begin{array}{cc} \int_{\Delta_i^{Q_i}} \varphi_{Q_i, n_i}^i(z) \overline{\varphi_{Q_i+1, n'_i}^i(y, z)} dz & \int_{\Delta_i^{Q_i}} \varphi_{Q_i, n_i}^i(z) \overline{\varphi_{Q_i+1, n'_i}^i(y', z)} dz \\ \int_{\Delta_k^{Q_k}} \varphi_{Q_k, n_k}^k(z) \overline{\varphi_{Q_k+1, n'_k}^k(y, z)} dz & \int_{\Delta_k^{Q_k}} \varphi_{Q_k, n_k}^k(z) \overline{\varphi_{Q_k+1, n'_k}^k(y', z)} dz \end{array} \right|, \end{aligned} \quad (4.29)$$

$$\bullet C_3(Q, i, j, k) = \frac{(n - Q_i - Q_j - Q_k - 2)! Q_i! Q_j! Q_k!}{2(n-2)!};$$

- the rank 2 operator $\gamma_{\substack{Q_i, Q_j, Q_k \\ n_i, n_j, n_k \\ n'_i, n'_j, n'_k}}^{(2),4,3'} : L^2(\Delta_j) \wedge L^2(\Delta_j) \rightarrow L^2(\Delta_i) \otimes L^2(\Delta_k)$ has the kernel

$$\begin{aligned} \gamma_{\substack{Q_i, Q_j, Q_k \\ n_i, n_j, n_k \\ n'_i, n'_j, n'_k}}^{(2),4,3'}(x, x', y, y') &= \left| \begin{array}{cc} \int_{\Delta_i^{Q_i-1}} \varphi_{Q_i, n_i}^i(x, z) \overline{\varphi_{Q_i-1, n'_i}^i(z)} dz & \int_{\Delta_i^{Q_i-1}} \varphi_{Q_i, n_i}^i(x', z) \overline{\varphi_{Q_i-1, n'_i}^i(z)} dz \\ \int_{\Delta_k^{Q_k-1}} \varphi_{Q_k, n_k}^k(x, z) \overline{\varphi_{Q_k-1, n'_k}^k(z)} dz & \int_{\Delta_k^{Q_k-1}} \varphi_{Q_k, n_k}^k(x', z) \overline{\varphi_{Q_k-1, n'_k}^k(z)} dz \end{array} \right| \\ &\times \int_{\Delta_j^{Q_j}} \varphi_{Q_j, n_j}^j(z) \overline{\varphi_{Q_j+2, n'_j}^j(y, y', z)} dz, \end{aligned} \quad (4.30)$$

- the rank 4 operator $\gamma_{\substack{Q_i, Q_j, Q_k, Q_l \\ n_i, n_j, n_k, n_l \\ n'_i, n'_j, n'_k, n'_l}}^{(2),4,4} : L^2(\Delta_k) \otimes L^2(\Delta_l) \rightarrow L^2(\Delta_i) \otimes L^2(\Delta_j)$ has the kernel

$$\begin{aligned} \gamma_{\substack{Q_i, Q_j, Q_k, Q_l \\ n_i, n_j, n_k, n_l \\ n'_i, n'_j, n'_k, n'_l}}^{(2),4,4}(x, x', y, y') &= \left| \begin{array}{cc} \int_{\Delta_i^{Q_i-1}} \varphi_{Q_i, n_i}^i(x, z) \overline{\varphi_{Q_i-1, n'_i}^i(z)} dz & \int_{\Delta_i^{Q_i-1}} \varphi_{Q_i, n_i}^i(x', z) \overline{\varphi_{Q_i-1, n'_i}^i(z)} dz \\ \int_{\Delta_j^{Q_j-1}} \varphi_{Q_j, n_j}^j(x, z) \overline{\varphi_{Q_j-1, n'_j}^j(z)} dz & \int_{\Delta_j^{Q_j-1}} \varphi_{Q_j, n_j}^j(x', z) \overline{\varphi_{Q_j-1, n'_j}^j(z)} dz \end{array} \right| \\ &\times \left| \begin{array}{cc} \int_{\Delta_k^{Q_k}} \varphi_{Q_k, n_k}^k(z) \overline{\varphi_{Q_k+1, n'_k}^k(y, z)} dz & \int_{\Delta_k^{Q_k}} \varphi_{Q_k, n_k}^k(z) \overline{\varphi_{Q_k+1, n'_k}^k(y', z)} dz \\ \int_{\Delta_l^{Q_l}} \varphi_{Q_l, n_l}^l(z) \overline{\varphi_{Q_l+1, n'_l}^l(y, z)} dz & \int_{\Delta_l^{Q_l}} \varphi_{Q_l, n_l}^l(z) \overline{\varphi_{Q_l+1, n'_l}^l(y', z)} dz \end{array} \right| \end{aligned} \quad (4.31)$$

$$\bullet C_4(Q, i, j, k, l) = \frac{(n - Q_i - Q_j - Q_k - Q_l - 2)! Q_i! Q_j! Q_k! Q_l!}{2(n-2)!};$$

Remark 4.5. In (4.25) - (4.31), in accordance with Remark 4.1, in the degenerate cases, we use the conventions derived from those in Remark 4.3 in a obvious way.

For example, in (4.26), if $Q_i = Q_j = 1$, one has

$$\gamma_{\substack{Q_i, Q_j \\ n_i, n_j \\ n'_i, n'_j}}^{(2), d, o}(x, x', y, y') = \frac{Q_i Q_j}{2} \begin{vmatrix} \varphi_{Q_i, n_i}^i(x) & \varphi_{Q_i, n_i}^i(x') \\ \varphi_{Q_j, n_j}^j(x) & \varphi_{Q_j, n_j}^j(x') \end{vmatrix} \cdot \overline{\begin{vmatrix} \varphi_{Q_i, n'_i}^i(y) & \varphi_{Q_i, n'_i}^i(y') \\ \varphi_{Q_j, n'_j}^j(y) & \varphi_{Q_j, n'_j}^j(y') \end{vmatrix}}. \quad (4.32)$$

Proof of Theorem 4.4. Theorem 4.4 follows from a direct computation that we now perform. First, by the bilinearity of formula (1.20), one has

$$\gamma_{\Psi}^{(2)} = \frac{n(n-1)}{2} \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \sum_{\substack{Q' \text{ occ.} \\ \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q \overline{a_{\bar{n}'}^{Q'}} \gamma_{\substack{Q, \bar{n} \\ Q', \bar{n}'}}^{(2)} \quad (4.33)$$

where the trace class operator $\gamma_{\substack{Q, \bar{n} \\ Q', \bar{n}'}}^{(2)}$ acts on $L^2([0, L]) \wedge L^2([0, L])$ and has the kernel

$$\gamma_{\substack{Q, \bar{n} \\ Q', \bar{n}'}}^{(2)}(x, x', y, y') := \int_{[0, L]^{n-2}} \left[\bigwedge_{j=1}^m \varphi_{Q_j, n_j}^j \right] (x, x', z_3, \dots, z_n) \overline{\left[\bigwedge_{j=1}^m \varphi_{Q'_j, n'_j}^j \right] (y, y', z_3, \dots, z_n)} dz_3 \cdots dz_n. \quad (4.34)$$

By (4.10), one has

$$\frac{\gamma_{\substack{Q, \bar{n} \\ Q', \bar{n}'}}^{(2)}(x, x', y, y')}{c(Q)c(Q')} = \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{|A'_j|=Q'_j, \forall 1 \leq j \leq m \\ A'_1 \cup \dots \cup A'_m = \{1, \dots, n\} \\ A'_j \cap A'_{j'} = \emptyset \text{ if } j \neq j'}} (-1)^{\varepsilon((A_j)) + \varepsilon((A'_j))} I(A, A') \quad (4.35)$$

where

$$I(A, A') := \int_{[0, L]^{n-2}} \left[\prod_{j=1}^m \varphi_{Q_j, n_j}^j((z_l)_{l \in A_j}) \overline{\varphi_{Q'_j, n'_j}^j((y_l)_{l \in A'_j})} \right]_{\substack{x_1=x, x_2=x' \\ y_1=y, y_2=y' \\ x_j=y_j \text{ if } j \geq 3}} dx_3 \cdots dx_n \quad (4.36)$$

To evaluate this last integral, we note that, for any pair of partitions $(A_j)_j$ and $(A'_j)_j$ (as in the indices of the above sum), if there exists $j \neq j'$ such that $A_j \cap A'_{j'} \cap \{3, \dots, n\} \neq \emptyset$, then the integral $I(A, A')$ vanishes.

Now, note that, if $d_1(Q, Q') > 4$, then, for any pair of partitions $(A_j)_j$ and $(A'_j)_j$, there exists $j \neq j'$ such that $A_j \cap A'_{j'} \cap \{3, \dots, n\} \neq \emptyset$; thus, the integral $I(A, A')$ above always vanishes and, summing this, one has

$$\gamma_{\substack{Q, \bar{n} \\ Q', \bar{n}'}}^{(2)} = 0 \quad \text{if} \quad d_1(Q, Q') > 4.$$

So we are left with the cases $Q = Q'$, $d_1(Q, Q') = 2$ or $d_1(Q, Q') = 4$.

Assume first $Q = Q'$. Consider the sums in (4.12). If $\{1, 2\} \subset A_{i_0} \cup A_{j_0}$ and $\{1, 2\} \not\subset A'_{i_0} \cup A'_{j_0}$ then, as $\forall j, |A'_j| = |A_j|$, there exists $\alpha \in (A'_{i_0} \cup A'_{j_0}) \cap \{3, \dots, n\}$ and $j \notin \{i_0, j_0\}$ such that $\alpha \in A_j$. That is, there exists $j \neq j'$ such that $A_j \cap A'_{j'} \cap \{3, \dots, n\} \neq \emptyset$, thus, the integral $I(A, A')$ vanishes. Moreover, if $\{1, 2\} \subset A_{j_0}$ and $\{1, 2\} \not\subset A'_{j_0}$ then, there exists

$\alpha \in A'_{j_0} \cap \{3, \dots, n\}$ and $j \neq j_0$ such that $\alpha \in A_j$, thus, the integral $I(A, A')$ vanishes. Thus, we rewrite

$$\begin{aligned} \frac{\gamma_{Q, \bar{n}, Q, \bar{n}}^{(2)}(x, x', y, y')}{c^2(Q)} &= \sum_{i_0, j_0=1}^m \sum_{\substack{\{1,2\} \subset A_{i_0} \cup A_{j_0} \\ |A_l|=Q_l, \forall 1 \leq l \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{\{1,2\} \subset A'_{i_0} \cup A'_{j_0} \\ |A'_{i_0}|=Q_{i_0} \\ |A'_{j_0}|=Q_{j_0} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} (-1)^{\varepsilon((A_j)) + \varepsilon((A'_j))} I(A, A') \\ &= \sum_{\substack{j_0=1 \\ Q_{j_0} \geq 2}}^m \sum_{\substack{\{1,2\} \subset A_{j_0} \\ |A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} I(A) + \sum_{\substack{i_0 \neq j_0 \\ Q_{i_0} \geq 1 \\ Q_{j_0} \geq 1}} \sum_{\substack{1 \in A_{i_0}, 2 \in A_{j_0} \\ |A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} J(A) \end{aligned} \quad (4.37)$$

where

$$I(A) := \prod_{j \neq j_0} \delta_{n_j = n'_j} \int_{\Delta_{Q_{j_0}-2}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, x', z) \overline{\varphi_{Q_{j_0}, n'_{j_0}}^{j_0}(y, y', z)} dz$$

and

$$\begin{aligned} J(A) := \prod_{j \notin \{i_0, j_0\}} \delta_{n_j = n'_j} &\left(\int_{\Delta_{Q_{i_0}-1}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(x, z) \overline{\varphi_{Q_{i_0}, n'_{i_0}}^{i_0}(y, z)} dz \right. \\ &\cdot \int_{\Delta_{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x', z') \overline{\varphi_{Q_{j_0}, n'_{j_0}}^{j_0}(y', z')} dz' \\ &- \int_{\Delta_{Q_{i_0}-1}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(x, z) \overline{\varphi_{Q_{i_0}, n'_{i_0}}^{i_0}(y', z)} dz \\ &\left. \cdot \int_{\Delta_{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x', z') \overline{\varphi_{Q_{j_0}, n'_{j_0}}^{j_0}(y, z')} dz' \right). \end{aligned}$$

As

$$\#\{(A_j)_j; \{1, 2\} \subset A_{j_0}, \forall j, |A_j| = Q_j\} = \frac{(n-2)! Q_{j_0} (Q_{j_0} - 1)}{\prod_{j=1}^m Q_j!}$$

and

$$\#\{(A_j)_j; 1 \in A_{i_0}, 2 \in A_{j_0}, \forall j, |A_j| = Q_j\} = \frac{(n-2)! Q_{i_0} Q_{j_0}}{\prod_{j=1}^m Q_j!} \quad \text{if } i_0 \neq j_0$$

by (4.11) and (4.37), one obtains

$$\sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m \\ \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q a_{\bar{n}'}^{\overline{Q}} \gamma_{Q, \bar{n}, Q, \bar{n}'}^{(2)} = \gamma_{\Psi}^{(2), d, d} + \gamma_{\Psi}^{(2), d, o} \quad (4.38)$$

where $\gamma_{\Psi}^{(2), d, d}$ and $\gamma_{\Psi}^{(2), d, o}$ are defined in Theorem 4.4.

Let us now assume $d_1(Q, Q') = 2$. Thus, there exists $1 \leq i_0 \neq j_0 \leq m$ such that $Q_{j_0} \geq 1$, $Q'_{i_0} = Q_{i_0} + 1$, $Q_{j_0} = Q'_{j_0} + 1$ and $Q_k = Q'_k$ for $k \notin \{i_0, j_0\}$.

Consider now the sums in (4.35). If $\{1, 2\} \cap A_{j_0} = \emptyset$, then as $|A'_{j_0}| = Q'_{j_0} = Q_{j_0} - 1$, there exists $\alpha \in A_{j_0} = A_{j_0} \cap \{3, \dots, n\}$ and $i \neq j_0$ such that $\alpha \in A'_i$. Thus, the integral $I(A, A')$ vanishes. If $A_{j_0} = \{1\} \cup B$ (resp. $A_{j_0} = \{2\} \cup B$) with $B \subset \{3, \dots, n\}$, either $A'_{j_0} = B$ (and $\{1, 2\} \subset A'_{i_0}$) or the integral $I(A, A')$ vanishes. Finally, if $A_{j_0} = \{1, 2\} \cup B$ with $B \subset \{3, \dots, n\}$, then, $A'_{j_0} = \{1\} \cup B$ or $A'_{j_0} = \{2\} \cup B$ or $I(A, A') = 0$. The same holds

true for A_{j_0} replaced with A'_{i_0} .

Therefore, using the definition of $\varepsilon((A_j))$, if $d_1(Q, Q') = 2$, we rewrite

$$\frac{\gamma_{Q, \bar{n}}^{(2)}(x, y)}{c^2(Q)} = \sum_{\substack{i_0 \neq j_0 \\ Q_{j_0} \geq 2}} \Sigma_1(i_0, j_0) - \Sigma_2(i_0, j_0) + \sum_{\substack{i_0 \neq j_0 \\ Q_{i_0} \geq 1}} \Sigma_3(i_0, j_0) - \Sigma_4(i_0, j_0) \quad (4.39)$$

where

$$\Sigma_1(i_0, j_0) := \sum_{\substack{\{1,2\} \subset A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{i_0} = \{1\} \cup A_{i_0} \\ A'_{j_0} = A_{j_0} \setminus \{1\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} I(A, A'), \quad (4.40)$$

$$\Sigma_2(i_0, j_0) := \sum_{\substack{\{1,2\} \subset A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{i_0} = \{2\} \cup A_{i_0} \\ A'_{j_0} = A_{j_0} \setminus \{2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} I(A, A'), \quad (4.41)$$

$$\Sigma_3(i_0, j_0) := \sum_{\substack{\{1,2\} \subset A'_{i_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A_{i_0} = A'_{i_0} \setminus \{1\} \\ A_{j_0} = A'_{j_0} \cup \{1\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} I(A, A'), \quad (4.42)$$

$$\Sigma_4(i_0, j_0) := \sum_{\substack{\{1,2\} \subset A'_{i_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A_{i_0} = A'_{i_0} \setminus \{2\} \\ A_{j_0} = A'_{j_0} \cup \{2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} I(A, A') \quad (4.43)$$

and

- for the summands in $\Sigma_1(i_0, j_0)$:

$$I(A, A') = \int_{\Delta_{j_0}^{Q_{j_0}-2}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, x', z) \overline{\varphi_{Q_{j_0}-1, n'_{j_0}}^{j_0}(y', z)} dz \\ \int_{\Delta_{i_0}^{Q_{i_0}}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(z') \overline{\varphi_{Q_{i_0}+1, n'_{i_0}}^{i_0}(y, z')} dz' \prod_{j \notin \{i_0, j_0\}} \delta_{n_j = n'_j}$$

- for the summands in $\Sigma_2(i_0, j_0)$:

$$I(A, A') = \int_{\Delta_{j_0}^{Q_{j_0}-2}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, x', z) \overline{\varphi_{Q_{j_0}-1, n'_{j_0}}^{j_0}(y, z)} dz \\ \int_{\Delta_{i_0}^{Q_{i_0}}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(z') \overline{\varphi_{Q_{i_0}+1, n'_{i_0}}^{i_0}(y', z')} dz' \prod_{j \notin \{i_0, j_0\}} \delta_{n_j = n'_j}$$

- for the summands in $\Sigma_3(i_0, j_0)$:

$$I(A, A') = \int_{\Delta_{j_0}^{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x', z) \overline{\varphi_{Q_{j_0}-1, n'_{j_0}}^{j_0}(z)} dz \\ \int_{\Delta_{i_0}^{Q_{i_0}-1}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(x, z') \overline{\varphi_{Q_{i_0}+1, n'_{i_0}}^{i_0}(y, y', z')} dz' \prod_{j \notin \{i_0, j_0\}} \delta_{n_j = n'_j}$$

- for the summands in $\Sigma_4(i_0, j_0)$:

$$I(A, A') = \int_{\Delta_{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, z) \overline{\varphi_{Q_{j_0}-1, n'_{j_0}}^{j_0}(z)} dz \\ \int_{\Delta_{Q_{i_0}-1}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(x', z') \overline{\varphi_{Q_{i_0}+1, n'_{i_0}}^{i_0}(y, y', z')} dz' \prod_{j \notin \{i_0, j_0\}} \delta_{n_j = n'_j}$$

with the convention described in Remark 4.1.

The number of partitions coming up in (4.40), (4.41), (4.42) and (4.43) are the same: indeed, it suffices to invert the roles of 1 and 2 and i_0 and j_0 . We compute

$$\sum_{\substack{\{1,2\} \subset A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{i_0} = \{1\} \cup A_{i_0} \\ A'_{j_0} = A_{j_0} \setminus \{1\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} 1 = \frac{(n - Q_{j_0} - Q_{i_0} - 2)! Q_{i_0}! Q_{j_0}!}{Q_1! \dots Q_m!}.$$

Hence, we get that

$$\frac{n(n-1)}{2} \sum_{\substack{Q, Q' \text{ occ.} \\ d_1(Q, Q') = 2 \\ \bar{n}, \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q \overline{a_{\bar{n}'}^{Q'}} \gamma_{Q, \bar{n}}^{(2)} = \sum_{i \neq j} \sum_{\bar{n} \in \mathbb{N}^{m-2}} \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 1 \\ Q': Q'_k = Q_k \text{ if } k \notin \{i, j\} \\ Q'_i = Q_i + 1 \\ Q'_j = Q_j - 1}} \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\bar{n}_{i,j}}^Q \overline{a_{\bar{n}'_{i,j}}^{Q'}} \gamma_{Q_i, Q_j}^{(2), 2} \quad (4.44)$$

where $\gamma_{Q_i, Q_j}^{(2), 2}$ is defined in (4.27).

Let us now assume $d_1(Q, Q') = 4$. Thus,

- (a) either there exist $1 \leq i_0 \neq j_0 \leq m$ such that $Q_{j_0} \geq 2$, $Q'_{i_0} = Q_{i_0} + 2$, $Q_{j_0} = Q'_{j_0} + 2$ and $Q_k = Q'_k$ for $k \notin \{i_0, j_0\}$.

In this case, either $A_{j_0} = \{1, 2\} \cup A'_{j_0}$ and $A'_{i_0} = \{1, 2\} \cup A_{i_0}$ with $A_{i_0}, A'_{j_0} \subset \{3, \dots, n\}$ or $I(A, A') = 0$ vanishes. Thus,

$$\frac{\gamma_{Q, \bar{n}}^{(2)}(x, y)}{c^2(Q)} = \sum_{\substack{\{1,2\} \subset A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{i_0} = \{1,2\} \cup A_{i_0} \\ A'_{j_0} = A_{j_0} \setminus \{1,2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0\}}} (-1)^{\varepsilon((A_j)) + \varepsilon((A'_{j_0}))} I(A, A'), \quad (4.45)$$

and

$$I(A, A') = \int_{\Delta_{Q_{j_0}-2}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, x', z) \overline{\varphi_{Q_{j_0}-2, n'_{j_0}}^{j_0}(z)} dz \\ \int_{\Delta_{Q_{i_0}}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(z') \overline{\varphi_{Q_{i_0}+2, n'_{i_0}}^{i_0}(y, y', z')} dz' \prod_{j \notin \{i_0, j_0\}} \delta_{n_j = n'_j}.$$

Hence, taking (4.28) into account, we get

$$\begin{aligned}
 & \frac{n(n-1)}{2} \sum_{\substack{Q, Q' \text{ occ.} \\ \exists i \neq j, Q_j \geq 2 \\ Q': Q'_k = Q_k \text{ if } k \notin \{i, j\} \\ Q'_i = Q_i + 2 \\ Q'_j = Q_j - 2}} \sum_{\substack{\bar{n} \in \mathbb{N}^m \\ \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q \overline{a_{\bar{n}'}}^{Q'} \gamma_{Q, \bar{n}}^{(2)} \\
 & = \sum_{i \neq j} \sum_{\bar{n} \in \mathbb{N}^{m-2}} \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2 \\ Q': Q'_k = Q_k \text{ if } k \notin \{i, j\} \\ Q'_i = Q_i + 2 \\ Q'_j = Q_j - 2}} C_2(Q, i, j) \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\bar{n}_{i,j}}^Q \overline{a_{\bar{n}'_{i,j}}}^{Q'} \gamma_{Q_i, Q_j}^{(2), 4, 2} \quad (4.46)
 \end{aligned}$$

as

$$\sum_{\substack{\{1,2\} \subset A_j \\ |A_l| = Q_l, \forall 1 \leq l \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_l \cap A_{l'} = \emptyset \text{ if } l \neq l'}} \sum_{\substack{A'_i = \{1,2\} \cup A_i \\ A'_j = A_j \setminus \{1,2\} \\ A'_l = A_l \text{ if } j \notin \{i,j\}}} 1 = \frac{(n - Q_j - Q_i - 2)! Q_i! Q_j!}{Q_1! \dots Q_m!} = \frac{2 C_2(Q, i, j)}{n(n-1) c(Q)^2}.$$

- (b) or there exist $1 \leq i_0, j_0, k_0 \leq m$ distinct such that $Q_{j_0} \geq 2$, $Q'_{j_0} = Q_{j_0} - 2$, $Q_{i_0} = Q'_{i_0} + 1$, $Q_{k_0} = Q'_{k_0} + 1$, and $Q_k = Q'_k$ for $k \notin \{i_0, j_0, k_0\}$. In this case, either $A_{j_0} = \{1, 2\} \cup A'_{j_0}$ and $((A'_{i_0} = \{1\} \cup A_{i_0}$ and $A'_{k_0} = \{2\} \cup A_{k_0})$ or $(A'_{i_0} = \{2\} \cup A_{i_0}$ and $A'_{k_0} = \{1\} \cup A_{k_0})$ with $A_{j_0}, A'_{i_0}, A'_{k_0} \subset \{3, \dots, n\}$ or $I(A, A') = 0$ vanishes. Thus,

$$\frac{\gamma_{Q, \bar{n}}^{(2)}(x, y)}{c^2(Q)} = \sum_{\substack{\{1,2\} \subset A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \left(\sum_{\substack{A'_{i_0} = \{1\} \cup A_{i_0} \\ A'_{k_0} = \{2\} \cup A_{k_0} \\ A'_{j_0} = A_{j_0} \setminus \{1,2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0\}}} I(A, A') - \sum_{\substack{A'_{i_0} = \{2\} \cup A_{i_0} \\ A'_{k_0} = \{1\} \cup A_{k_0} \\ A'_{j_0} = A_{j_0} \setminus \{1,2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0\}}} I(A, A') \right) \quad (4.47)$$

and, if $A'_{i_0} = \{1\} \cup A_{i_0}$ and $A'_{k_0} = \{2\} \cup A_{k_0}$, one has

$$\begin{aligned}
 I(A, A') &= \int_{\Delta_{j_0}^{Q_{j_0}-2}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, x', z) \overline{\varphi_{Q_{j_0}-2, n'_{j_0}}^{j_0}(z)} dz \\
 & \int_{\Delta_{i_0}^{Q_{i_0}}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(z') \overline{\varphi_{Q_{i_0}+1, n'_{i_0}}^{i_0}(y, z')} dz' \\
 & \int_{\Delta_{k_0}^{Q_{k_0}}} \varphi_{Q_{k_0}, n_{k_0}}^{k_0}(z'') \overline{\varphi_{Q_{k_0}+1, n'_{k_0}}^{k_0}(y', z'')} dz'' \prod_{j \notin \{i_0, j_0, k_0\}} \delta_{n_j = n'_j}
 \end{aligned}$$

and, if $A'_{i_0} = \{2\} \cup A_{i_0}$ and $A'_{k_0} = \{1\} \cup A_{k_0}$, one has

$$I(A, A') = \int_{\Delta_{j_0-2}} \varphi_{Q_{j_0}, n_{j_0}}^{j_0}(x, x', z) \overline{\varphi_{Q_{j_0-2}, n'_{j_0}}(z)} dz \\ \int_{\Delta_{i_0}} \varphi_{Q_{i_0}, n_{i_0}}^{i_0}(z') \overline{\varphi_{Q_{i_0+1}, n'_{i_0}}(y', z')} dz' \\ \int_{\Delta_{k_0}} \varphi_{Q_{k_0}, n_{k_0}}^{i_0}(z'') \overline{\varphi_{Q_{k_0+1}, n'_{k_0}}(y, z'')} dz'' \prod_{j \notin \{i_0, j_0, k_0\}} \delta_{n_j = n'_j}.$$

For i_0, j_0, k_0 distinct, one has

$$\sum_{\substack{\{1,2\} \subset A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{i_0} = \{1\} \cup A_{i_0} \\ A'_{k_0} = \{2\} \cup A_{k_0} \\ A'_{j_0} = A_{j_0} \setminus \{1, 2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0\}}} 1 = \frac{(n - Q_{j_0} - Q_{i_0} - Q_{k_0} - 2)! Q_{i_0}! Q_{j_0}! Q_{k_0}!}{Q_1! \dots Q_m!} \\ = \frac{2 C_3(Q, i_0, j_0, k_0)}{n(n-1) c(Q)^2}.$$

Inverting the roles of 1 and 2 we see that the number of partitions coming up in the second sum in (4.47) is the same. Thus, taking (4.28) into account, we get

$$\frac{n(n-1)}{2} \sum_{\substack{Q, Q' \text{ occ.} \\ \exists i, j, k \text{ distinct} \\ Q_j \geq 2 \\ Q': Q'_i = Q_i \text{ if } i \notin \{i, j, k\} \\ Q'_j = Q_j - 2 \\ Q'_i = Q_i + 1, Q'_k = Q_k + 1}} \sum_{\substack{\bar{n} \in \mathbb{N}^m \\ \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q \overline{a_{\bar{n}'}^{Q'}} \gamma_{Q, \bar{n}}^{(2)} \\ Q': Q'_i = Q_i \text{ if } i \notin \{i, j, k\} \\ Q'_j = Q_j - 2 \\ Q'_i = Q_i + 1, Q'_k = Q_k + 1} C_3(Q, i, j, k) \sum_{\substack{n_i, n_j, n_k \geq 1 \\ n'_i, n'_j, n'_k \geq 1}} a_{\bar{n}_{i,j,k}}^Q \overline{a_{\bar{n}'_{i,j,k}}^{Q'}} \gamma_{Q_i, Q_j, Q_k}^{(2), 4, 3}. \quad (4.48)$$

(c) or there exist $1 \leq i_0, j_0, k_0 \leq m$ distinct such that $Q_{i_0} \geq 1, Q_{k_0} \geq 1, Q'_{j_0} = Q_{j_0} + 2, Q_{i_0} = Q'_{i_0} - 1, Q_{k_0} = Q'_{k_0} - 1$, and $Q_k = Q'_k$ for $k \notin \{i_0, j_0, k_0\}$.

We see that we are back to case (b) if we invert the roles of Q and Q' . Thus, we get

$$\frac{n(n-1)}{2} \sum_{\substack{Q, Q' \text{ occ.} \\ \exists i, j, k \text{ distinct} \\ Q_i \geq 1, Q_k \geq 1 \\ Q': Q'_i = Q_i \text{ if } i \notin \{i, j, k\} \\ Q'_j = Q_j + 2 \\ Q'_i = Q_i - 1, Q'_k = Q_k - 1}} \sum_{\substack{\bar{n} \in \mathbb{N}^m \\ \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q \overline{a_{\bar{n}'}^{Q'}} \gamma_{Q, \bar{n}}^{(2)} \\ Q': Q'_i = Q_i \text{ if } i \notin \{i, j, k\} \\ Q'_j = Q_j + 2 \\ Q'_i = Q_i - 1, Q'_k = Q_k - 1} C_3(Q, i, j, k) \sum_{\substack{n_i, n_j, n_k \geq 1 \\ n'_i, n'_j, n'_k \geq 1}} a_{\bar{n}_{i,j,k}}^Q \overline{a_{\bar{n}'_{i,j,k}}^{Q'}} \gamma_{Q_i, Q_j, Q_k}^{(2), 4, 3'}. \quad (4.49)$$

(d) or there exist $1 \leq i_0, j_0, k_0, l_0 \leq m$ distinct such that $Q_{j_0} \geq 1$, $Q_{l_0} \geq 1$, $Q'_{i_0} = Q_{i_0} - 1$, $Q_{j_0} = Q'_{j_0} - 1$, $Q'_{k_0} = Q_{k_0} + 1$, $Q_{l_0} = Q'_{l_0} + 1$ and $Q_k = Q'_k$ for $k \notin \{i_0, j_0, k_0, l_0\}$.

Then, either $I(A, A') = 0$ or

(i) either $A_{i_0} = \{1\} \cup A'_{i_0}$ and $A_{j_0} = \{2\} \cup A'_{j_0}$ and $A'_{i_0}, A'_{j_0} \subset \{3, \dots, n\}$,

(ii) or $A_{i_0} = \{2\} \cup A'_{i_0}$ and $A_{j_0} = \{1\} \cup A'_{j_0}$ and $A'_{i_0}, A'_{j_0} \subset \{3, \dots, n\}$, in which case

Moreover, in each of the cases (i) and (ii), either $I(A, A') = 0$ or

(i) either $A'_{k_0} = \{1\} \cup A_{k_0}$ and $A'_{l_0} = \{2\} \cup A_{l_0}$ and $A_{k_0}, A_{l_0} \subset \{3, \dots, n\}$,

(ii) or $A'_{k_0} = \{2\} \cup A_{k_0}$ and $A'_{l_0} = \{1\} \cup A_{l_0}$ and $A_{k_0}, A_{l_0} \subset \{3, \dots, n\}$

In the 4 cases when $I(A, A')$ does not vanish, one computes

- $I(A, A') = \alpha(x, x', y, y')$ in case (i.i),
- $I(A, A') = \alpha(x', x, y, y')$ in case (ii.i),
- $I(A, A') = \alpha(x, x', y, y')$ in case (i.ii),
- $I(A, A') = \alpha(x', x, y', y)$ in case (ii.ii),

where

$$\begin{aligned} \alpha(x, x', y, y') &:= \int_{\Delta_{i_0}^{Q_{i_0}-1}} \varphi_{Q_{i_0}, n_i}^i(x, z) \overline{\varphi_{Q_{i_0}-1, n'_i}^i(z)} dz \int_{\Delta_{j_0}^{Q_{j_0}-1}} \varphi_{Q_{j_0}, n_j}^j(x', z) \overline{\varphi_{Q_{j_0}-1, n'_j}^j(z)} dz \\ &\quad \times \int_{\Delta_{k_0}^{Q_{k_0}}} \varphi_{Q_{k_0}, n_k}^k(z) \overline{\varphi_{Q_{k_0}+1, n'_k}^k(y, z)} dz \int_{\Delta_{l_0}^{Q_{l_0}}} \varphi_{Q_{l_0}, n_l}^l(z) \overline{\varphi_{Q_{l_0}+1, n'_l}^l(y', z)} dz. \end{aligned}$$

Hence, if $d_1(Q, Q') = 4$, we obtain

$$\begin{aligned} \frac{\gamma_{Q, \bar{n}}^{(2)}(x, y)}{c^2(Q)} &= \sum_{\substack{1 \in A_{i_0}, 2 \in A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \left(\sum_{\substack{A'_{k_0} = \{1\} \cup A_{k_0}, A'_{l_0} = \{2\} \cup A_{l_0} \\ A'_{i_0} = A_{i_0} \setminus \{1\}, A'_{j_0} = A_{j_0} \setminus \{2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0, l_0\}}} I(A, A') - \sum_{\substack{A'_{k_0} = \{2\} \cup A_{k_0}, A'_{l_0} = \{1\} \cup A_{l_0} \\ A'_{i_0} = A_{i_0} \setminus \{1\}, A'_{j_0} = A_{j_0} \setminus \{2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0, l_0\}}} I(A, A') \right) \\ &- \sum_{\substack{2 \in A_{i_0}, 1 \in A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \left(\sum_{\substack{A'_{k_0} = \{1\} \cup A_{k_0}, A'_{l_0} = \{2\} \cup A_{l_0} \\ A'_{i_0} = A_{i_0} \setminus \{2\}, A'_{j_0} = A_{j_0} \setminus \{1\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0, l_0\}}} I(A, A') - \sum_{\substack{A'_{k_0} = \{2\} \cup A_{k_0}, A'_{l_0} = \{1\} \cup A_{l_0} \\ A'_{i_0} = A_{i_0} \setminus \{2\}, A'_{j_0} = A_{j_0} \setminus \{1\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0, l_0\}}} I(A, A') \right). \end{aligned} \quad (4.50)$$

For i_0, j_0, k_0, l_0 distinct, the number of partitions coming up in the first sum in (4.50) is given by

$$\begin{aligned} \sum_{\substack{1 \in A_{i_0}, 2 \in A_{j_0} \\ |A_j| = Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{A'_{k_0} = \{1\} \cup A_{k_0}, A'_{l_0} = \{2\} \cup A_{l_0} \\ A'_{i_0} = A_{i_0} \setminus \{1\}, A'_{j_0} = A_{j_0} \setminus \{2\} \\ A'_j = A_j \text{ if } j \notin \{i_0, j_0, k_0, l_0\}}} 1 &= \frac{(n - Q_{j_0} - Q_{i_0} - Q_{k_0} - Q_{l_0} - 2)! Q_{i_0}! Q_{j_0}! Q_{k_0}! Q_{l_0}!}{Q_1! \dots Q_m!} \\ &= \frac{2 C_4(Q, i_0, j_0, k_0, l_0)}{n(n-1) c(Q)^2}. \end{aligned}$$

Inverting the roles of i_0, j_0, k_0, l_0 , we see that the number of partitions involved is the same in the three remaining sums of (4.50).

Thus, taking (4.28) into account, we get

$$\begin{aligned}
& \frac{n(n-1)}{2} \sum_{\substack{Q, Q' \text{ occ.} \\ \exists i, j, k, l \text{ distinct} \\ Q_i \geq 1, Q_j \geq 1 \\ Q'_p = Q_p \text{ if } p \notin \{i, j, k, l\} \\ Q'_i = Q_i - 1, Q'_j = Q_j - 1 \\ Q'_k = Q_k + 1, Q'_l = Q_l + 1}} \sum_{\substack{\bar{n} \in \mathbb{N}^m \\ \bar{n}' \in \mathbb{N}^m}} a_{\bar{n}}^Q a_{\bar{n}'}^{\overline{Q'}} \gamma_{Q, \bar{n}}^{(2)} \gamma_{Q', \bar{n}'}^{(2)} \\
&= \sum_{\substack{i, j, k, l \\ \text{distinct}}} \sum_{\bar{n} \in \mathbb{N}^{m-4}} \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1, Q_j \geq 1 \\ Q'_p = Q_p \text{ if } p \notin \{i, j, k, l\} \\ Q'_i = Q_i - 1, Q'_j = Q_j - 1 \\ Q'_k = Q_k + 1, Q'_l = Q_l + 1}} C_4(Q, i, j, k) \sum_{\substack{n_i, n_j, n_k, n_l \geq 1 \\ n'_i, n'_j, n'_k, n'_l \geq 1}} a_{\bar{n}_{i, j, k, l}}^Q a_{\bar{n}'_{i, j, k, l}}^{\overline{Q'}} \gamma_{Q_i, Q_j, Q_k, Q_l}^{(2), 4, 4} \gamma_{n'_i, n'_j, n'_k, n'_l}^{(2), 4, 4}. \quad (4.51)
\end{aligned}$$

Plugging this, (4.46) and (4.44) into (4.33), we obtain (4.17). This completes the proof of Theorem 4.4. \square

4.1.3. *A particular case.* Let us now explain how the structure of the one-particle and two-particles density matrices may be simplified in the particular case when the ground state is factorized. This in particular immediately yields the expansions (1.22) and (1.23) for the one-particle and two-particles density matrices of the non interacting ground state.

Definition 4.6. Let $\alpha \in \mathfrak{H}^i(L)$ and $\beta \in \mathfrak{H}^j(L)$ be two states describing i and j electrons respectively. We say α and β *do not interact* if for all $(x^2, \dots, x^i, y^2, \dots, y^j) \in [0, L]^{i+j-2}$,

$$\int_0^L \alpha(x^1, \dots, x^i) \beta^*(y^1, \dots, y^j) \Big|_{x^1=y^1} dx^1 = 0. \quad (4.52)$$

To denote this complete orthogonality, we will write $\alpha \perp\!\!\!\perp \beta$.

Remark 4.7. Because of the anti-symmetric nature of the states α and β in the above definition, it is sufficient to impose the orthogonality only on the first variables. Thus, an integral of the type (4.52) vanishes for any pair of coordinates $x^{i_1} = y^{j_1}$ for $i_1 \in \{1, \dots, i\}$, and $j_1 \in \{1, \dots, j\}$.

We prove

Proposition 4.8. *Suppose that a n -particle state $\Psi \in \mathfrak{H}^n(L)$ is decomposed in its non interacting parts:*

$$\Psi = \bigwedge_{j=1}^k \zeta_j,$$

where each $\zeta_j \in \mathfrak{H}^{k_j}(L)$ is a k_j -particle state describing a packet of particles that do not interact with other packets, i.e., for $i \neq j$, $\zeta_i \perp\!\!\!\perp \zeta_j$ in the sens of Definition 4.6. Then

$$\gamma_\Psi = \sum_{j=1}^k \gamma_{\zeta_j} \quad (4.53)$$

and

$$\gamma_\Psi^{(2)} = \sum_{j=1}^k \left[\gamma_{\zeta_j}^{(2)} - \frac{1}{2} (\text{Id} - \text{Ex}) \gamma_{\zeta_j} \otimes \gamma_{\zeta_j} \right] + \frac{1}{2} (\text{Id} - \text{Ex}) \gamma_\Psi \otimes \gamma_\Psi, \quad (4.54)$$

where Id is the identity, Ex is the exchange operator on the two-particles space defined as

$$\text{Ex } f \otimes g = g \otimes f, \quad f, g \in \mathfrak{H},$$

and with the obvious convention that $\gamma_{\zeta_j}^{(2)} = 0$ if ζ_j is a one-particle state.

While Proposition 4.8 could be obtained as a consequence of Theorems 4.2 and 4.4, we will derive it from the following auxiliary lemma.

Lemma 4.9. *Let $\alpha \in \mathfrak{H}^n(L)$ and $\beta \in \mathfrak{H}^m(L)$ be two vectors describing n and m electrons respectively. Suppose that α and β do not interact:*

$$\alpha \perp \beta.$$

Then,

$$\gamma_{\alpha \wedge \beta} = \gamma_\alpha + \gamma_\beta \quad (4.55)$$

and

$$\gamma_{\alpha \wedge \beta}^{(2)} = \gamma_\alpha^{(2)} + \gamma_\beta^{(2)} + (\text{Id} - \text{Ex})\gamma_\alpha \otimes^s \gamma_\beta \quad (4.56)$$

where \otimes^s denotes the symmetrized tensor product:

$$A \otimes^s B = \frac{1}{2}(A \otimes B + B \otimes A).$$

Proof. Define $\mathbb{N}_n := \{1, \dots, n\}$. Consider the two-particles density matrix. By (C.2), the anti-symmetrized product of two eigenfunctions in respectively n and m variables is given by

$$(\alpha \wedge \beta)(x^1, \dots, x^{n+m}) = \frac{1}{\sqrt{\binom{n+m}{n}}} \sum_{\substack{J \cup J' = \mathbb{N}_{n+m} \\ J \cap J' = \emptyset, |J|=n}} (-1)^{\text{sign } J} \alpha(x^J) \beta(x^{J'}).$$

Thus, the corresponding two-particles density matrix can be written as

$$\begin{aligned} & \gamma_{\alpha \wedge \beta}^{(2)}(x^1, x^2, y^1, y^2) \\ &= \frac{n(n-1)}{2} \int_{[0,L]^{n+m-2}} (\alpha \wedge \beta)(x^1, x^2, \bar{x}) (\alpha \wedge \beta)^*(y^1, y^2, \bar{x}) d\bar{x} \\ &= \frac{n(n-1)}{2 \binom{n+m}{n}} \sum_{\substack{I \cup I' = \mathbb{N}_{n+m} \\ I \cap I' = \emptyset, |I|=n \\ J \cup J' = \mathbb{N}_{n+m} \\ J \cap J' = \emptyset, |J|=n}} \int_{[0,L]^{n+m-2}} (-1)^{\text{sign } I + \text{sign } J} \alpha(x^I) \beta(x^{I'}) \alpha^*(y^J) \beta^*(y^{J'}) \Big|_{\substack{y^j = x^j \\ j \in \{3, \dots, n+m\}}} d\bar{x}. \end{aligned} \quad (4.57)$$

As α and β do not interact, the integrals in the sum in the last part of (4.57) vanish if I differs from J by more than two elements, i.e., $|I \setminus J| \geq 2$. Moreover, if such an integral does not vanish, one distinguishes the following cases:

- (a) if $\{1, 2\} \subset I$, then $I = J$; indeed, otherwise J would contain an index in I' and the integration of $\beta(x^{I'}) \alpha^*(y^J) \Big|_{\substack{y^j = x^j \\ j \in \{3, \dots, n+m\}}}$ over the corresponding variable would produce zero because $\alpha \perp \beta$.
- (b) if $\{1, 2\} \subset J$, then $I = J$.
- (c) if $(1, 2) \in (I \times I') \cup (I' \times I)$ then $(1, 2) \in (J \times J') \cup (J' \times J)$ by the same argument as above.

As the functions α and β are completely anti-symmetric under permutations of variables, the terms of the sums over I and J corresponding to different cases described above are all the same. If we denote $\hat{x}^k = x^3, \dots, x^k$ and $d\hat{x}^k = dx^3 \dots dx^k$ for $k \in \{n, m, n+m\}$, this finally yields

$$\gamma_{\alpha \wedge \beta}^{(2)}(x^1, x^2, y^1, y^2) = A + B + C$$

where

$$\begin{aligned} A &:= \frac{n(n-1)}{2} \frac{1}{\binom{n+m}{n}} \binom{n+m-2}{n-2} \int_{[0,L]^{n-2}} \alpha(x^1, x^2, \hat{x}^n) \alpha^*(y^1, y^2, \hat{x}^n) d\hat{x}^n \\ &= \gamma_\alpha^{(2)}(x^1, x^2, y^1, y^2), \end{aligned}$$

$$\begin{aligned} B &:= \frac{n(n-1)}{2} \frac{1}{\binom{n+m}{n}} \binom{n+m-2}{m-2} \int_{[0,L]^{m-2}} \beta(x^1, x^2, \hat{x}^m) \beta^*(y^1, y^2, \hat{x}^m) d\hat{x}^m \\ &= \gamma_\beta^{(2)}(x^1, x^2, y^1, y^2) \end{aligned}$$

and

$$\begin{aligned} C &:= \frac{n(n-1)}{2} \frac{1}{\binom{n+m}{n}} \binom{n+m-2}{m-1} \int_{[0,L]^{n+m-2}} d\hat{x}^{n+m} \\ &\quad (\alpha(x^1, \dots) \beta(x^2, \dots) \alpha^*(y^1, \dots) \beta^*(y^2, \dots) \\ &\quad - \alpha(x^1, \dots) \beta(x^2, \dots) \alpha^*(y^2, \dots) \beta^*(y^1, \dots) \\ &\quad - \alpha(x^2, \dots) \beta(x^1, \dots) \alpha^*(y^1, \dots) \beta^*(y^2, \dots) \\ &\quad + \alpha(x^2, \dots) \beta(x^1, \dots) \alpha^*(y^2, \dots) \beta^*(y^1, \dots)) \\ &= \frac{1}{2} (\gamma_\alpha(x^1, y^1) \gamma_\beta(x^2, y^2) - \gamma_\alpha(x^1, y^2) \gamma_\beta(x^2, y^1) \\ &\quad - \gamma_\alpha(x^2, y^1) \gamma_\beta(x^1, y^2) + \gamma_\alpha(x^2, y^2) \gamma_\beta(x^1, y^1)). \end{aligned}$$

This completes the proof of (4.56). The proof for the one-particle density matrix (4.55) is done similarly and is even simpler. This completes the proof of Lemma 4.9. \square

Proof of Proposition 4.8. The identity (4.53) for one-particle density matrix is a direct consequence of (4.55). We prove (4.54) by induction on k .

For $k = 2$, (4.54) is equivalent to (4.56) after noting that

$$A \otimes^s B = \frac{1}{2} ((A + B) \otimes (A + B) - A \otimes A - B \otimes B).$$

This remark also proves that

$$\gamma_\Psi^{(2)} = \sum_{j=1}^k \gamma_{\zeta_j}^{(2)} + (\text{Id} - \text{Ex}) \sum_{i < j} \gamma_{\zeta_i} \otimes^s \gamma_{\zeta_j} \quad (4.58)$$

which is equality (4.54).

Let us prove (4.58) inductively. Suppose now that (4.58) holds true and consider

$$\Psi_{k+1} = \bigwedge_{j=1}^{k+1} \zeta_j = \left(\bigwedge_{j=1}^k \zeta_j \right) \wedge \zeta_{k+1} = \Psi_k \wedge \zeta_{k+1}.$$

By (4.56), we get

$$\begin{aligned}
\gamma_{\Psi_{k+1}}^{(2)} &= \gamma_{\Psi_k}^{(2)} + \gamma_{\zeta_{k+1}}^{(2)} + (\text{Id} - \text{Ex})\gamma_{\Psi_k} \otimes^s \gamma_{\zeta_{k+1}} \\
&= \sum_{j=1}^k \gamma_{\zeta_j}^{(2)} + (\text{Id} - \text{Ex}) \left(\sum_{\substack{i < j \\ i, j=1, \dots, k}} \gamma_{\zeta_i} \otimes^s \gamma_{\zeta_j} \right) + \gamma_{\zeta_{k+1}}^{(2)} \\
&\quad + (\text{Id} - \text{Ex}) \left(\sum_{j=1}^k \gamma_{\zeta_j} \right) \otimes^s \gamma_{\zeta_{k+1}} \\
&= \sum_{j=1}^{k+1} \gamma_{\zeta_j}^{(2)} + (\text{Id} - \text{Ex}) \sum_{\substack{i < j \\ i, j=1, \dots, k+1}} \gamma_{\zeta_i} \otimes^s \gamma_{\zeta_j}.
\end{aligned}$$

This completes the proof of Proposition 4.8. \square

4.2. The proof of Theorem 1.5. The proof of Theorem 1.5 will rely on Theorem 4.2 and the analysis of $\Psi_\omega^U(L, n)$ performed in section 3. The two sums in (4.3) will be analyzed separately and will be split into various components according to the lengths of the pieces coming into play in each component.

As in the beginning of section 4.1 (see (4.2)), write $\Psi_\omega^U(L, n) = \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} a_{\bar{n}}^Q \Phi_{Q, \bar{n}}$. We will first

transform the results on the ground state obtained in section 3 into a statement on the coefficients $((a_{\bar{n}}^Q)_{Q, \bar{n}})$, namely,

Proposition 4.10. *There exists $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0)$ and $\varepsilon \in (0, 1/10)$, ω almost surely, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

- (a) for an occupation $Q \notin \mathcal{Q}_\rho$ (see (3.83)) and any $\bar{n} \in \mathbb{N}^m$, one has $a_{\bar{n}}^Q = 0$;
- (b) let \mathcal{P}_- be the (indices j of the) pieces $(\Delta_j(\omega))_j$ of lengths less than $3\ell_\rho(1 - \varepsilon)$, and, for Q an occupation, let \mathcal{P}_-^Q be the (indices j of the) pieces in \mathcal{P}_- such that $Q_j \leq 3$. Then, for Q , an occupation number of a ground state $\Psi_\omega^U(L, n)$, letting $(a_{\bar{n}}^Q)_{Q, \bar{n}}$ be its coefficients in the decomposition (4.2), one has

$$\sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \#\{j \in \mathcal{P}_-^Q; n_j \geq 2\} \left| a_{\bar{n}}^Q \right|^2 \leq o\left(\frac{n \cdot \rho}{|\log \rho|}\right). \quad (4.59)$$

The second part of Proposition 4.10 controls the excited particles in the ground state $\Psi_\omega^U(L, n)$. Actually, as the proof shows, we shall prove (4.59) not only for a ground state of $H_\omega^U(L, n)$, but, also for any state Ψ satisfying

$$\frac{1}{n} \langle H_\omega^{Up}(L, n) \Psi, \Psi \rangle \leq \mathcal{E}^0(\rho) + \pi^2 \gamma_* \frac{\rho}{|\log \rho|^{-3}} + o\left(\frac{\rho}{|\log \rho|^{-3}}\right). \quad (4.60)$$

Proof of Proposition 4.10. Point (a) is a rephrasing of Corollary 3.32.

Let us prove point (b). Pick an n -state Ψ and decompose it as $\Psi_\omega^U(L, n) = \sum_{Q \in \mathcal{Q}_\rho} \Psi_Q$. Then, if

$E_{Q_j, n_j}^{j, Up}$ denotes the n_j -th eigenvalue of $-\sum_{l=1}^{Q_j} \frac{d^2}{dx_l^2} + \sum_{1 \leq k \leq l \leq Q_j} U^p(x_k - x_l)$ acting on $\bigwedge_{l=1}^{Q_j} L^2(\Delta_j(\omega))$

with Dirichlet boundary conditions (if $Q_j = 0$, we set $E_{Q_j, n_j}^{j, UP} = 0$ for all n_j), as $H^U \geq H^{UP}$ (see (3.17)), by (3.82), one has

$$\begin{aligned} n (\mathcal{E}^0(\rho) + \pi^2 \gamma_* \rho |\log \rho|^{-3} (1 + O(f_Z(|\log \rho|)))) &\geq \langle H^{UP} \Psi^{UP}, \Psi^{UP} \rangle \\ &\geq \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \left(\sum_{\substack{j \in \mathcal{P}_-^Q \\ Q_j \geq 1}} E_{Q_j, n_j}^{j, UP} \right) \left| a_{\bar{n}}^Q \right|^2. \end{aligned} \quad (4.61)$$

One proves

Lemma 4.11. *There exists $C > 0$ such that, for $j \in \mathcal{P}_-^Q$, $Q_j \geq 1$ and $n_j \geq 2$, one has*

$$E_{Q_j, n_j}^{j, UP} \geq E_{Q_j, 1}^{j, UP} + \frac{1}{C \ell_\rho^2}. \quad (4.62)$$

Plugging (4.62) into (4.61) yields

$$\begin{aligned} \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \left(\sum_{\substack{j \in \mathcal{P}_-^Q \\ Q_j \geq 1}} E_{Q_j, 1}^{j, UP} \right) \left| a_{\bar{n}}^Q \right|^2 + \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \frac{\#\{j \in \mathcal{P}_-^Q; n_j \geq 2\}}{C \ell_\rho^2} \left| a_{\bar{n}}^Q \right|^2 \\ \leq n (\mathcal{E}^0(\rho) + \pi^2 \gamma_* \rho |\log \rho|^{-3} (1 + O(f_Z(|\log \rho|)))) \end{aligned} \quad (4.63)$$

We prove

Lemma 4.12. *There exists $\rho_0 > 0$ such that, for $\rho \in (0, \rho_0)$, $\varepsilon \in (0, 1)$ and ω almost surely, for L sufficiently large and $|n/L - \rho|$ sufficiently small, if Q is an occupation such that*

$$\sum_{j \in \mathcal{P}_-} E_{Q_j, 1}^{j, UP} \leq n (\mathcal{E}^0(\rho) + \rho |\log \rho|^{-3} (\pi^2 \gamma_* + \varepsilon)) \quad (4.64)$$

then,

$$\sum_{\substack{j \in \mathcal{P}_-^Q \\ Q_j \geq 1}} E_{Q_j, 1}^{j, UP} \geq n \left(\mathcal{E}^0(\rho) + \rho |\log \rho|^{-3} \left(\pi^2 \gamma_* - \frac{1}{\rho_0} (\varepsilon + f_Z(|\log \rho|)) \right) \right). \quad (4.65)$$

Lemma 4.12 shows that, for low energy states, most of the energy is carried by pieces carrying three particles and less (compare the set \mathcal{P}_- and \mathcal{P}_-^Q).

Let us postpone the proof of this result for a while and complete the proof of Proposition 4.10.

From (4.65) and (4.63), as $\sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \left| a_{\bar{n}}^Q \right|^2 = 1$ and $f_Z(|\log \rho|) = o(1)$, we get that

$$\sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \frac{\#\{j \in \mathcal{P}_-^Q; n_j \geq 2\}}{C \ell_\rho^2} \left| a_{\bar{n}}^Q \right|^2 \leq o(n \rho |\log \rho|^{-3}).$$

As $\ell_\rho \asymp |\log \rho|$, this immediately yields (4.59) and completes the proof of Proposition 4.10. \square

Proof of Lemma 4.12. By Theorem 3.19, for L large and n/L close to ρ , we have

$$\langle H_\omega^{UP} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle \geq n (\mathcal{E}^0(\rho) + \pi^2 \gamma_* \rho |\log \rho|^{-3} (1 + O(f_Z(|\log \rho|)))).$$

Recall that the occupation Q^{opt} of Ψ^{opt} satisfies

$$Q_j^{\text{opt}} = \begin{cases} 0 & \text{if } |\Delta_j(\omega)| \in [0, \ell_\rho - \rho x_*), \\ 1 & \text{if } |\Delta_j(\omega)| \in [\ell_\rho - \rho x_*, 2\ell_\rho + A_*), \\ 2 & \text{if } |\Delta_j(\omega)| \in [2\ell_\rho + A_*, 3\ell_\rho(1 - \varepsilon)). \end{cases} \quad (4.66)$$

Theorem 3.19 shows that

$$\left| \langle H_\omega^{Up} \Psi^{\text{opt}}, \Psi^{\text{opt}} \rangle - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1}} E_{1,1}^{j,Up} - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2}} E_{2,1}^{j,Up} \right| \lesssim n \frac{\rho}{|\log \rho|^3} f_Z(|\log \rho|). \quad (4.67)$$

Let

$$\Delta E := \sum_{j \in \mathcal{P}_-} E_{Q_j,1}^{j,Up} - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1}} E_{1,1}^{j,Up} - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2}} E_{2,1}^{j,Up}. \quad (4.68)$$

Then, (4.67) and assumption (4.64) imply that

$$|\Delta E| \leq \frac{C n \rho}{|\log \rho|^3} (f_Z(|\log \rho|) + \varepsilon). \quad (4.69)$$

Moreover, one has

$$\begin{aligned} \Delta E &\geq \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0}} E_{Q_j,1}^{j,Up} + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1}} (E_{Q_j,1}^{j,Up} - E_{1,1}^{j,Up}) + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2}} (E_{Q_j,1}^{j,Up} - E_{2,1}^{j,Up}) \\ &= \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0 \\ Q_j \geq 1}} E_{Q_j,1}^{j,Up} + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j \geq 2}} (E_{Q_j,1}^{j,Up} - E_{1,1}^{j,Up}) + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \geq 3}} (E_{Q_j,1}^{j,Up} - E_{2,1}^{j,Up}) \\ &\quad - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j=0}} E_{1,1}^{j,Up} - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \leq 1}} (E_{2,1}^{j,Up} - E_{Q_j,1}^{j,Up}). \end{aligned} \quad (4.70)$$

On the other hand, as $|Q| = n = |Q^{\text{opt}}|$, using Lemma 3.23 as $\Psi_\omega^U(L, n)$ satisfies (4.60), we know that

$$\sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j=0}} 1 + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \leq 1}} (2 - Q_j) = \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0 \\ Q_j \geq 1}} Q_j + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j \geq 2}} (Q_j - 1) + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \geq 3}} (Q_j - 2) + O(n\rho^{1+\eta}). \quad (4.71)$$

Define

$$B := \max \left(\max_{\substack{j: Q_j=0 \\ Q_j^{\text{opt}}=1}} E_{1,1}^{j,Up}, \max_{\substack{j: Q_j^{\text{opt}}=2 \\ 0 \leq Q_j \leq 1}} \frac{E_{2,1}^{j,Up} - E_{Q_j,1}^{j,Up}}{2 - Q_j} \right).$$

Then, (4.70) implies that

$$\begin{aligned} \Delta E \geq & \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0 \\ Q_j \geq 1}} E_{Q_j,1}^{j,UP} + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j \geq 2}} (E_{Q_j,1}^{j,UP} - E_{1,1}^{j,UP}) + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \geq 3}} (E_{Q_j,1}^{j,UP} - E_{2,1}^{j,UP}) \\ & - B \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j=0}} 1 - B \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \leq 1}} (2 - Q_j). \end{aligned}$$

Hence, (4.71) implies that, for some $C > 0$, for ρ sufficiently small, one has

$$\begin{aligned} \Delta E + C n \rho^{1+\eta} \geq & \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0 \\ Q_j \geq 1}} (E_{Q_j,1}^{j,UP} - B) + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j \geq 2}} (E_{Q_j,1}^{j,UP} - E_{1,1}^{j,UP} - B(Q_j - 1)) \\ & + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \geq 3}} (E_{Q_j,1}^{j,UP} - E_{2,1}^{j,UP} - B(Q_j - 2)). \end{aligned} \quad (4.72)$$

Let us upper bound B . Recalling that for a single particle in a piece there is no interaction, a direct computation and (4.66) show that

$$\max_{\substack{j; Q_j=0 \\ Q_j^{\text{opt}}=1}} E_{1,1}^{j,UP} \leq \frac{\pi^2}{(\ell_\rho - \rho x_*)^2}. \quad (4.73)$$

Proposition 1.4 and (4.66) show that, for ρ sufficiently small, one has

$$\begin{aligned} \max_{\substack{j; Q_j=0 \\ Q_j^{\text{opt}}=2}} \frac{E_{2,1}^{j,UP} - E_{Q_j,1}^{j,UP}}{2 - Q_j} & \leq \frac{5\pi^2}{2(2\ell_\rho + A^*)^2} + \frac{2\gamma}{(2\ell_\rho + A^*)^3} \leq \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \\ \max_{\substack{j; Q_j=1 \\ Q_j^{\text{opt}}=2}} \frac{E_{2,1}^{j,UP} - E_{Q_j,1}^{j,UP}}{2 - Q_j} & \leq \frac{4\pi^2}{(2\ell_\rho + A^*)^2} + \frac{2\gamma}{(2\ell_\rho + A^*)^3} \leq \frac{\pi^2}{(\ell_\rho - \rho x_*)^2}. \end{aligned}$$

Thus,

$$B \leq \frac{\pi^2}{(\ell_\rho - \rho x_*)^2}. \quad (4.74)$$

Now, notice that

- for j s.t. $Q_j^{\text{opt}} = 0$ (see (4.66)):
 - if $Q_j = 1$, one has

$$E_{Q_j,1}^{j,UP} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \geq \frac{\pi^2}{|\Delta_j(\omega)|^2} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \geq 0;$$

- if $Q_j \geq 2$, one has

$$E_{Q_j,1}^{j,UP} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \geq \frac{1}{2} E_{Q_j,1}^{j,UP} + \frac{5\pi^2}{2|\Delta_j(\omega)|^2} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \geq \frac{1}{2} E_{Q_j,1}^{j,UP};$$

- for j s.t. $Q_j^{\text{opt}} = 1$ (see (4.66)):

– if $Q_j = 2$, one has

$$\begin{aligned}
E_{Q_j,1}^{j,UP} - E_{1,1}^{j,UP} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} &\geq \frac{4\pi^2}{|\Delta_j(\omega)|^2} + \frac{\gamma}{|\Delta_j(\omega)|^3} + o(\ell_\rho^{-3}) - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \\
&\geq \frac{4\pi^2}{|2\ell_\rho + A_* + \varepsilon_\rho|^2} + \frac{\gamma}{|2\ell_\rho + A_* + \varepsilon_\rho|^3} + o(\ell_\rho^{-3}) - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \\
&\geq \frac{\pi^2}{\ell_\rho^2} - \frac{A_*\pi^2}{2\ell_\rho^3} + \frac{\gamma}{4\ell_\rho^3} + \frac{\pi^2\varepsilon_\rho}{2\ell_\rho^3} + o(\ell_\rho^{-3}) - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \\
&\geq \frac{\pi^2\varepsilon_\rho}{2\ell_\rho^3} + o(\ell_\rho^{-3}) \geq 0
\end{aligned}$$

if ρ sufficiently small (see (3.10)) and $|\Delta_j(\omega)| \leq 2\ell_\rho + A_* - \varepsilon_\rho$; here $\varepsilon_\rho \rightarrow 0^+$ (but not too fast) as $\rho \rightarrow 0^+$;

on the other hand, the number of pieces of length in $2\ell_\rho + A_* + [-\varepsilon_\rho, 0]$ is bounded by $C\rho n\varepsilon_\rho$ (see Proposition 2.2) and for such pieces, one has

$$\left| E_{2,1}^{j,UP} - E_{1,1}^{j,UP} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \right| = o(\ell_\rho^{-3}); \quad (4.75)$$

– if $Q_j \geq 3$, one has

$$\begin{aligned}
E_{Q_j,1}^{j,UP} - E_{1,1}^{j,UP} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2}(Q_j - 1) &\geq \frac{1}{2}E_{Q_j,1}^{j,UP} + \frac{1}{2}E_{Q_j,1}^{j,0} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2}(Q_j - 1) \\
&\geq \frac{1}{2}E_{Q_j,1}^{j,UP} + \frac{\pi^2}{4\ell_\rho^2} \frac{5}{12}(Q_j - 1) \geq \frac{1}{2}E_{Q_j,1}^{j,UP}
\end{aligned}$$

• for j s.t. $Q_j^{\text{opt}} = 2$ (see (4.66)):

– if $Q_j \geq 3$, one has

$$\begin{aligned}
E_{Q_j,1}^{j,UP} - E_{2,1}^{j,UP} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2}(Q_j - 2) &\geq \frac{1}{3}E_{Q_j,1}^{j,UP} + \frac{2}{3}E_{Q_j,1}^{j,0} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2}(Q_j - 2) \\
&\geq \frac{1}{3}E_{Q_j,1}^{j,UP} + \frac{\pi^2}{9(1-\varepsilon)^2\ell_\rho^2} \left(\frac{102}{9} - 9 \right) (Q_j - 2) \\
&\geq \frac{1}{3}E_{Q_j,1}^{j,UP}.
\end{aligned}$$

Plugging these estimates and (4.74) into (4.72), we get that, for ρ sufficiently small,

$$\begin{aligned}
\Delta E + \sum_{|\Delta_j(\omega)| \in [2\ell_\rho + A_* + [-\varepsilon_\rho, 0]]} \left| E_{2,1}^{j,UP} - E_{1,1}^{j,UP} - \frac{\pi^2}{(\ell_\rho - \rho x_*)^2} \right| + C n \rho^{1+\eta} \\
\geq \frac{1}{2} \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0 \\ Q_j \geq 2}} E_{Q_j,1}^{j,UP} + \frac{1}{2} \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j \geq 3}} E_{Q_j,1}^{j,UP} + \frac{1}{3} \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \geq 3}} E_{Q_j,1}^{j,UP}.
\end{aligned}$$

Hence, in view of (4.75) and the estimate on the number of terms in the sum in the left hand side, one gets

$$3(\Delta E + o(n\rho\ell_\rho^{-3})) \geq \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0 \\ Q_j \geq 2}} E_{Q_j,1}^{j,UP} + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j \geq 3}} E_{Q_j,1}^{j,UP} + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \geq 3}} E_{Q_j,1}^{j,UP} \geq 0. \quad (4.76)$$

This implies that

$$o(n\rho\ell_\rho^{-3}) \leq \Delta E = \sum_{j \in \mathcal{P}_-} E_{Q_{j,1}}^{j,Up} - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1}} E_{1,1}^{j,Up} - \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2}} E_{2,1}^{j,Up}$$

hence, by (4.67), that, for some $C > 0$ and ρ sufficiently small, one has

$$\sum_{j \in \mathcal{P}_-} E_{Q_{j,1}}^{j,Up} \geq n (\mathcal{E}^0(\rho) + \pi^2 \gamma_* \rho |\log \rho|^{-3} (1 - C f_Z(|\log \rho|))) \quad (4.77)$$

We complete the proof of Lemma 4.12 by noting that, by the definition of \mathcal{P}_-^Q , one has

$$\begin{aligned} \sum_{\substack{j \in \mathcal{P}_-^Q \\ Q_j \geq 1}} E_{Q_{j,1}}^{j,Up} &= \sum_{j \in \mathcal{P}_-} E_{Q_{j,1}}^{j,Up} - \left(\sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=0 \\ Q_j \geq 3}} E_{Q_{j,1}}^{j,Up} + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=1 \\ Q_j \geq 3}} E_{Q_{j,1}}^{j,Up} + \sum_{\substack{j \in \mathcal{P}_- \\ Q_j^{\text{opt}}=2 \\ Q_j \geq 3}} E_{Q_{j,1}}^{j,Up} \right) \\ &\geq n (\mathcal{E}^0(\rho) + \pi^2 \gamma_* \rho |\log \rho|^{-3} (1 - C(\varepsilon + f_Z(|\log \rho|)))) \end{aligned}$$

where the last lower bound follows from (4.69) and (4.76).

This completes the proof of Lemma 4.12. \square

Let us resume the proof of Theorem 1.5. Recall Theorem 4.2; we analyze the two components $\gamma_{\Psi_\omega^U(L,n)}^{(1),d}$ and $\gamma_{\Psi_\omega^U(L,n)}^{(1),o}$ separately.

Let us start with the analysis of $\gamma_{\Psi_\omega^U(L,n)}^{(1),o}$. We prove

Lemma 4.13. *Under the assumptions of Theorem 4.2, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(1),o} \right\|_{tr} \leq 3. \quad (4.78)$$

Proof. We recall (4.4) from Theorem 4.2 and write

$$\gamma_{\Psi_\omega^U(L,n)}^{(1),o} = \sum_{Q \text{ occ.}} \sum_{\substack{i,j=1 \\ i \neq j \\ Q_j \geq 1}}^m C_1(Q, i, j) \sum_{\tilde{n} \in \mathbb{N}^{m-1}} \sum_{\substack{n_i, n_j \geq 1 \\ n'_i, n'_j \geq 1}} a_{\tilde{n}_i, j}^Q \overline{a_{\tilde{n}'_i, j}^{Q'}} \gamma_{\substack{Q_i, Q_j \\ n_i, n'_j \\ n_i, n'_j}}^{(1)}$$

where, by definition, in the above sums, Q' satisfies $Q'_k = Q_k$ if $k \notin \{i, j\}$, $Q'_i = Q_i + 1$ and $Q'_j = Q_j - 1$.

Note that, by point (a) of Proposition 4.10, here and in the sequel when summing over the occupations Q , we can always restrict ourselves to the occupations in \mathcal{Q}_ρ .

Decompose

$$\gamma_{\Psi_\omega^U(L,n)}^{(1),o} = \gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,+} + \gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,-} + \gamma_{\Psi_\omega^U(L,n)}^{(1),o,-,+} + \gamma_{\Psi_\omega^U(L,n)}^{(1),o,-,-} \quad (4.79)$$

where (see (4.5), (4.6), (4.7) and (4.8))

$$\begin{aligned} \gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,+} &:= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1} \\ i \neq j \\ Q_j \geq 2 \\ Q_i \geq 1}} C_1(Q, i, j) a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}} \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,+,+}, & \gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,-} &:= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1} \\ i \neq j \\ Q_j \geq 2 \\ Q_i = 0}} C_1(Q, i, j) a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}} \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,+,-} \\ \gamma_{\Psi_\omega^U(L,n)}^{(1),o,-,+} &:= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1} \\ i \neq j \\ Q_j = 1 \\ Q_i \geq 1}} C_1(Q, i, j) a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}} \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,-,+}, & \gamma_{\Psi_\omega^U(L,n)}^{(1),o,-,-} &:= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1} \\ i \neq j \\ Q_j = 1 \\ Q_i = 0}} C_1(Q, i, j) a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^{Q'}} \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,-,-} \end{aligned}$$

and

$$\begin{aligned} \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,+,+}(x, y) &:= \int_{\Delta_i^{Q_i} \times \Delta_j^{Q_j-1}} \left(\sum_{\substack{n_i \geq 1 \\ n_j \geq 1}} a_{\tilde{n}_{i,j}}^Q \varphi_{Q_i, n_i}^i(z) \varphi_{Q_j, n_j}^j(x, z') \right) \left(\overline{\sum_{\substack{n'_j \geq 1 \\ n'_j \geq 1}} a_{\tilde{n}'_{i,j}}^{Q'} \varphi_{Q'_i, n'_i}^i(y, z) \varphi_{Q'_j, n'_j}^j(z')} \right) dz dz', \\ \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,+,-}(x, y) &:= \int_{\Delta_j^{Q_j-1}} \left(\sum_{\substack{n_i = 1 \\ n_j \geq 1}} a_{\tilde{n}_{i,j}}^Q \varphi_{Q_j, n_j}^j(x, z') \right) \left(\overline{\sum_{\substack{n'_j \geq 1 \\ n'_j \geq 1}} a_{\tilde{n}'_{i,j}}^{Q'} \varphi_{1, n'_i}^i(y) \varphi_{Q_j-1, n'_j}^j(z')} \right) dz', \\ \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,-,+}(x, y) &:= \int_{\Delta_i^{Q_i}} \left(\sum_{\substack{n_i \geq 1 \\ n_j \geq 1}} a_{\tilde{n}_{i,j}}^Q \varphi_{1, n_j}^j(x) \varphi_{Q_i, n_i}^i(z) \right) \left(\overline{\sum_{\substack{n'_j = 1 \\ n'_j \geq 1}} a_{\tilde{n}'_{i,j}}^{Q'} \varphi_{Q_i+1, n'_i}^i(y, z)} \right) dz, \\ \text{and } \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,-,-}(x, y) &:= \left(\sum_{\substack{n_i = 1 \\ n_j \geq 1}} a_{\tilde{n}_{i,j}}^Q \varphi_{1, n_j}^j(x) \right) \left(\overline{\sum_{\substack{n'_j = 1 \\ n'_j \geq 1}} a_{\tilde{n}'_{i,j}}^{Q'} \varphi_{1, n'_i}^i(y)} \right). \end{aligned}$$

Let us first analyze $\gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,+}$. By Lemma B.1, using the orthonormality of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$ (see the beginning of section 4.1), we know that

$$\begin{aligned} \left\| \gamma_{Q,Q',i,j,\tilde{n}}^{(1),1,+,+} \right\|_{\text{tr}} &\leq \left\| \sum_{n_i, n_j} a_{\tilde{n}_{i,j}}^Q \varphi_{Q_i, n_i}^i \otimes \varphi_{Q_j, n_j}^j \right\| \cdot \left\| \sum_{n'_i, n'_j} a_{\tilde{n}'_{i,j}}^{Q'} \varphi_{Q'_i, n'_i}^i \otimes \varphi_{Q'_j, n'_j}^j \right\| \\ &\leq \frac{1}{2} \left(\sum_{n_i, n_j} |a_{\tilde{n}_{i,j}}^Q|^2 + \sum_{n_i, n_j} |a_{\tilde{n}'_{i,j}}^{Q'}|^2 \right). \end{aligned}$$

Hence, by definition (see the formula following (4.79)) and the symmetry of $C_1(Q, i, j)$ in i and j , we have

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,+} \right\|_{\text{tr}} \leq \sum_{\substack{i,j=1 \\ i \neq j}}^m \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2 \\ Q_i \geq 1}} C_1(Q, i, j) \sum_{\tilde{n} \in \mathbb{N}^m} |a_{\tilde{n}}^Q|^2.$$

Now, by definition (see Theorem 4.2), for $Q_j \geq 2$ and $Q_i \geq 1$, one has

$$C_1(Q, i, j) \leq \frac{Q_i Q_j}{(n-1)(n-2)(n-3)}.$$

Thus,

$$\begin{aligned} \left\| \gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,+} \right\|_{\text{tr}} &\leq \frac{1}{(n-1)(n-2)(n-3)} \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2 \\ Q_i \geq 1}} \left(\sum_j Q_j \right)^2 \sum_{\bar{n} \in \mathbb{N}^m} \left| a_{\bar{n}}^Q \right|^2 \\ &\leq \frac{n^2}{(n-1)(n-2)(n-3)} \sum_{Q, \bar{n} \in \mathbb{N}^m} \left| a_{\bar{n}}^Q \right|^2 = \frac{n^2}{(n-1)(n-2)(n-3)}. \end{aligned} \quad (4.80)$$

Let us now analyze $\gamma_{\Psi_\omega^U(L,n)}^{(1),o,-,-}$. By the definition of $C_1(Q, i, j)$, we write

$$\gamma_{\Psi_\omega^U(L,n)}^{(1),o,-,-}(x, y) = \frac{1}{n-1} \sum_{\substack{\bar{n} \in \mathbb{N}^{m-1} \\ Q \text{ occ.}}} \sum_{\substack{i,j=1 \\ i \neq j \\ Q_j=1 \\ Q_i=0}}^m \left(\sum_{n_i=1, n_j} a_{\bar{n}_{i,j}}^Q \varphi_{1, n_j}^j(x) \right) \left(\overline{\sum_{n'_j=1, n'_i} a_{\bar{n}'_{i,j}}^{Q'} \varphi_{1, n'_i}^i(y)} \right).$$

Thus, by Lemma B.1, one has

$$\begin{aligned} \left\| \gamma_{\Psi_\omega^U(L,n)}^{(1),o,-,-} \right\|_{\text{tr}} &\leq \frac{1}{n-1} \sum_{\substack{\bar{n} \in \mathbb{N}^{m-1} \\ Q \text{ occ.}}} \left\| \sum_j \sum_{Q_j=1} \sum_{n_i=1, n_j} a_{\bar{n}_{i,j}}^Q \varphi_{1, n_j}^j \right\| \cdot \left\| \sum_i \sum_{Q_i=0}^m \sum_{n'_j=1, n'_i} a_{\bar{n}'_{i,j}}^{Q'} \varphi_{1, n'_i}^i \right\| \\ &\leq \frac{1}{2n-2} \sum_{\substack{\bar{n} \in \mathbb{N}^{m-1} \\ Q \text{ occ.}}} \left\| \sum_j \sum_{Q_j=1} \sum_{n_i=1, n_j} a_{\bar{n}_{i,j}}^Q \varphi_{1, n_j}^j \right\|^2 + \left\| \sum_i \sum_{Q_i=0} \sum_{n'_j=1, n'_i} a_{\bar{n}'_{i,j}}^{Q'} \varphi_{1, n'_i}^i \right\|^2 \\ &\leq \frac{1}{2n-2} \sum_{\substack{n \in \mathbb{N}^m \\ Q \text{ occ.}}} \left| a_{\bar{n}}^Q \right|^2 = \frac{1}{2n-2}. \end{aligned} \quad (4.81)$$

Let us now analyze $\gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,-}$. One has

$$\begin{aligned} \gamma_{\Psi_\omega^U(L,n)}^{(1),o,+,-} &= \sum_{\substack{\bar{n} \in \mathbb{N}^{m-1} \\ Q \text{ occ.}}} \sum_{\substack{i \neq j \\ Q_j \geq 2 \\ Q_i = 0}} \frac{(n - Q_j - 1)! Q_j!}{(n-1)!} \int_{\Delta_j^{Q_j-1}} \left(\sum_{n_i=1, n_j} a_{\bar{n}_{i,j}}^Q \varphi_{Q_j, n_j}^j(x, z') \right) \times \\ &\quad \left(\overline{\sum_{n'_i, n'_j} a_{\bar{n}'_{i,j}}^Q \varphi_{1, n'_i}^i(y) \varphi_{Q_j-1, n'_j}^j(z')} \right) dz' \\ &= \sum_{\substack{\bar{n} \in \mathbb{N}^{m-1} \\ Q \text{ occ.}}} \sum_{j; Q_j \geq 2} \frac{(n - Q_j - 1)! Q_j!}{(n-1)!} \int_{\Delta_j^{Q_j-1}} \left(\sum_{n_i=1, n_j} a_{\bar{n}_{i,j}}^Q \varphi_{Q_j, n_j}^j(x, z') \right) \times \\ &\quad \left(\overline{\sum_i \sum_{Q_i=0} \sum_{n'_i, n'_j} a_{\bar{n}'_{i,j}}^Q \varphi_{1, n'_i}^i(y) \varphi_{Q_j-1, n'_j}^j(z')} \right) dz'. \end{aligned}$$

Thus, using Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, as $(n - Q_j)! Q_j! \leq n!$ and $\sum_j Q_j = n$, we get

$$\left\| \gamma_{\Psi_{\omega}^U(L, n)}^{(1), o, +, -} \right\|_{\text{tr}} \leq \frac{1}{n-1} \sum_{\substack{\tilde{n} \in \mathbb{N}^{m-1} \\ Q \text{ occ.}}} \sum_{j=1}^m Q_j \sum_{n_i=1, n_j} |a_{\tilde{n}_i, j}^Q|^2 \leq \frac{n}{n-1} \sum_{\substack{\tilde{n} \in \mathbb{N}^m \\ Q \text{ occ.}}} |a_{\tilde{n}}^Q|^2. \quad (4.82)$$

The term $\gamma_{\Psi_{\omega}^U(L, n)}^{(1), o, -, +}$ is analyzed in the same way. Gathering (4.80), (4.81), (4.82) and using (4.79), we obtain (4.78) and, thus, complete the proof of Lemma 4.13. \square

Let us now turn to the analysis of $\gamma_{\Psi_{\omega}^U(L, n)}^{(1), d}$. Therefore, we write

$$\gamma_{\Psi_{\omega}^U(L, n)}^{(1), d} = \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, -} + \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +} \quad \text{where} \quad \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, -} := \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_-^Q} \sum_{\substack{n_j \geq 1 \\ n'_j \geq 1}} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^Q} \gamma_{Q_j}^{(1)}. \quad (4.83)$$

We prove

Lemma 4.14. *Under the assumptions of Theorem 4.2, for $\eta \in (0, 1)$, there exists $\varepsilon_0 > 0$ and $C > 1$ such that, for $\varepsilon \in (0, \varepsilon_0)$, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +} \right\|_{\text{tr}} \leq Cn \frac{\rho}{\ell_{\rho}}. \quad (4.84)$$

Proof. Define

$$\gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +} = \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +, +} + \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +, 0} \quad (4.85)$$

where

$$\gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +, +} = \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{\substack{j \notin \mathcal{P}_- \\ n_j \geq 1 \\ n'_j \geq 1}} \sum_{n_j, n'_j} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^Q} \gamma_{Q_j}^{(1)} \quad \text{and} \quad \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +, 0} = \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{\substack{j \in \mathcal{P}_-^Q \\ Q_j \geq 4}} \sum_{\substack{n_j \geq 1 \\ n'_j \geq 1}} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^Q} \gamma_{Q_j}^{(1)}.$$

One computes

$$\begin{aligned} \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +, +}(x, y) &= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{\substack{j \notin \mathcal{P}_- \\ n_j \geq 1 \\ n'_j \geq 1}} \sum_{n_j, n'_j} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^Q} \gamma_{Q_j}^{(1)}(x, y) \\ &= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \notin \mathcal{P}_-} Q_j \int_{\Delta_j^{Q_j-1}} \left(\sum_{n_j=1}^{+\infty} a_{\tilde{n}_j}^Q \varphi_{Q_j, n_j}^j(x, z) \right) \overline{\left(\sum_{n'_j=1}^{+\infty} a_{\tilde{n}'_j}^Q \varphi_{Q_j, n'_j}^j(y, z) \right)} dz. \end{aligned}$$

Thus, by Lemma B.1, we get

$$\begin{aligned} \left\| \gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +, +} \right\|_{\text{tr}} &\leq \sum_{\substack{Q \text{ occ.} \\ Q \in \mathcal{Q}_{\rho} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \notin \mathcal{P}_-} Q_j \sum_{n_j=1}^{+\infty} |a_{\tilde{n}_j}^Q|^2 \leq \sum_{Q \text{ occ. in } \mathcal{Q}_{\rho}} \left(\sum_{j \notin \mathcal{P}_-} Q_j \right) \sum_{\tilde{n} \in \mathbb{N}^m} |a_{\tilde{n}}^Q|^2 \\ &\leq \max_{Q \text{ occ. in } \mathcal{Q}_{\rho}} \left(\sum_{j \notin \mathcal{P}_-} Q_j \right) \leq Cn \rho^{1+\eta} \end{aligned} \quad (4.86)$$

by Lemma 3.23.

Finally, one has

$$\gamma_{\Psi_{\omega}^U(L, n)}^{(1), d, +, 0} = \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 4}} \sum_{\substack{j \in \mathcal{P}_- \\ n_j \geq 1 \\ n'_j \geq 1}} \sum_{\tilde{n} \in \mathbb{N}^{m-1}} \sum_{n_j, n'_j} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^Q} \gamma_{Q_j}^{(1)}.$$

Thus, the same computation as above yields

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(1),d,+0} \right\|_{\text{tr}} \leq \sum_{\substack{Q \text{ occ.} \\ Q \in \mathcal{Q}_\rho \\ \tilde{n} \in \mathbb{N}^{m-1}}} \left(\sum_{\substack{j, |\Delta_j(\omega)| \leq 3\ell_\rho(1-\varepsilon) \\ Q_j \geq 4}} Q_j \right) \sum_{n_j=1}^{+\infty} |a_{\tilde{n}_j}^Q|^2 \leq Cn \frac{\rho}{\ell_\rho}$$

by Lemma 3.24.

This completes the proof of Lemma 4.14. \square

Let us now analyze $\gamma_{\Psi_\omega^U(L,n)}^{(1),d,-}$. We recall and compute

$$\gamma_{\Psi_\omega^U(L,n)}^{(1),d,-} := \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_-^Q} \sum_{\substack{n_j \geq 1 \\ n'_j \geq 1}} a_{\tilde{n}_j}^Q \overline{a_{\tilde{n}'_j}^Q} \gamma_{Q_j}^{(1)} = \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_-^Q} Q_j |\varphi_j^{\tilde{n}}\rangle \langle \varphi_j^{\tilde{n}}|$$

where $\varphi_j^{\tilde{n}} = \sum_{n_j \geq 1} a_{\tilde{n}_j}^Q \varphi_{Q_j, n_j}^j$.

For \tilde{n} and Q given, define the two sets

$$\mathcal{P}_{-,+}^{Q,\tilde{n}} := \left\{ j \in \mathcal{P}_-^Q; a_{\tilde{n}_j}^Q = 0 \text{ if } n_j \geq 2 \right\} \text{ and } \mathcal{P}_{-,-}^{Q,\tilde{n}} := \left\{ j \in \mathcal{P}_-^Q; \exists n_j \geq 2 \text{ s.t. } a_{\tilde{n}_j}^Q \neq 0 \right\}. \quad (4.87)$$

Define also

$$\tilde{\varphi}_j^{\tilde{n}} = \begin{cases} \varphi_j^{\tilde{n}} & \text{if } n_j = 1, \\ \|\varphi_j^{\tilde{n}}\| \varphi_{Q_j,1}^j & \text{if } n_j \geq 2. \end{cases} \quad (4.88)$$

Then, we compute

$$\begin{aligned} \gamma_{\Psi_\omega^U(L,n)}^{(1),d,-} &= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_{-,+}^{Q,\tilde{n}}} Q_j |\varphi_j^{\tilde{n}}\rangle \langle \varphi_j^{\tilde{n}}| + \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_{-,-}^{Q,\tilde{n}}} Q_j |\varphi_j^{\tilde{n}}\rangle \langle \varphi_j^{\tilde{n}}| \\ &= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_-^Q} Q_j |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}| + \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_{-,-}^{Q,\tilde{n}}} Q_j (|\varphi_j^{\tilde{n}}\rangle \langle \varphi_j^{\tilde{n}}| - |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}|). \end{aligned} \quad (4.89)$$

The second term in the sum above we estimate by

$$\begin{aligned} \left\| \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_{-,-}^{Q,\tilde{n}}} Q_j (|\varphi_j^{\tilde{n}}\rangle \langle \varphi_j^{\tilde{n}}| - |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}|) \right\|_{\text{tr}} &\leq \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_{-,-}^{Q,\tilde{n}}} Q_j (\|\varphi_j^{\tilde{n}}\|^2 + \|\tilde{\varphi}_j^{\tilde{n}}\|^2) \\ &\leq \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^m}} \sum_{j \in \mathcal{P}_-^Q} \#\{j; n_j \geq 2\} |a_{\tilde{n}}^Q|^2 \\ &\leq n \frac{\rho}{\rho_0 |\log \rho|} f_Z(|\log \rho|). \end{aligned} \quad (4.90)$$

by Lemma 4.11.

As for the first term in the second equality in (4.89), letting \mathcal{P}_{opt} be the pieces of length less than $3\ell_\rho(1-\varepsilon)$ where Ψ^{opt} puts at least one particle, we write

$$\sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_-^Q} Q_j |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}| = \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \left(\sum_{j \in \mathcal{P}_{\text{opt}}} + \sum_{j \in \mathcal{P}_-^Q \setminus \mathcal{P}_{\text{opt}}} - \sum_{j \in \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_-^Q} \right) Q_j |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}| \quad (4.91)$$

One computes

$$\begin{aligned} \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \sum_{j \in \mathcal{P}_{\text{opt}}} Q_j |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}| &= \sum_{j \in \mathcal{P}_{\text{opt}}} Q_j \left(\sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} |a_{\bar{n}}^Q|^2 \right) |\varphi_{Q_j,1}^j\rangle \langle \varphi_{Q_j,1}^j| \\ &= \sum_{j \in \mathcal{P}_{\text{opt}}} Q_j |\varphi_{Q_j,1}^j\rangle \langle \varphi_{Q_j,1}^j| = \gamma_{\Psi^{\text{opt}}} + R \end{aligned} \quad (4.92)$$

where $\|R\|_{\text{tr}} \leq Cn\rho^{1+\eta}$.

By Corollary 3.32, we know that

$$\begin{aligned} &\left\| \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \left(\sum_{\substack{j \in \mathcal{P}_{\text{opt}}^Q \\ |\Delta_j(\omega)| \geq \ell_\rho + C}} - \sum_{\substack{j \in \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_{\text{opt}}^Q \\ |\Delta_j(\omega)| \geq \ell_\rho + C}} \right) Q_j |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}| \right\|_{\text{tr}} \\ &\leq \sum_{Q \text{ occ.}} \left(\sum_{\substack{j \in \mathcal{P}_{\text{opt}}^Q \\ |\Delta_j(\omega)| \geq \ell_\rho + C}} + \sum_{\substack{j \in \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_{\text{opt}}^Q \\ |\Delta_j(\omega)| \geq \ell_\rho + C}} \right) Q_j \sum_{\bar{n} \in \mathbb{N}^m} |a_{\bar{n}}^Q|^2 \\ &\leq Cn\rho \max\left(\sqrt{Z(2|\log \rho|)}, \ell_\rho^{-1}\right) \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} |a_{\bar{n}}^Q|^2 = Cn\rho \max\left(\sqrt{Z(2|\log \rho|)}, \ell_\rho^{-1}\right) \end{aligned}$$

and, in the same way,

$$\left\| \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-1}}} \left(\sum_{\substack{j \in \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_{\text{opt}}^Q \\ |\Delta_j(\omega)| < \ell_\rho + C}} - \sum_{\substack{j \in \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_{\text{opt}}^Q \\ |\Delta_j(\omega)| < \ell_\rho + C}} \right) Q_j |\tilde{\varphi}_j^{\tilde{n}}\rangle \langle \tilde{\varphi}_j^{\tilde{n}}| \right\|_{\text{tr}} \leq Cn \max\left(\sqrt{\rho Z(2|\log \rho|)}, \rho |\log \rho|^{-1}\right).$$

Plugging this and (4.92) into (4.91) and then into (4.89), using (4.90), we obtain

$$\begin{aligned} \left\| \gamma_{\Psi_\omega^U(L,n)}^{(1),d,-} - \gamma_{\Psi^{\text{opt}}}^{(1)} \right\|_{\text{tr}, < \ell_\rho + C} &\leq Cn \max\left(\sqrt{\rho Z(2|\log \rho|)}, \rho |\log \rho|^{-1}\right) \\ \left\| \gamma_{\Psi_\omega^U(L,n)}^{(1),d,-} - \gamma_{\Psi^{\text{opt}}}^{(1)} \right\|_{\text{tr}, \geq \ell_\rho + C} &\leq Cn\rho \max\left(\sqrt{Z(2|\log \rho|)}, \ell_\rho^{-1}\right). \end{aligned}$$

Taking into account the decomposition (4.83), Theorem 4.2 and Lemmas 4.13 and 4.14 then completes the proof of Theorem 1.5. \square

4.3. The proof of Theorem 1.6. We proceed as in the proof of Theorem 1.5: for $\Psi_\omega^U(L,n)$ a ground state of the Hamiltonian $H_\omega^U(L,n)$, we analyze each of the components of the decomposition (4.17) separately.

We prove

Lemma 4.15. *Under the assumptions of Theorem 4.2, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),d,d} \right\|_{\text{tr}} \lesssim n \log n \cdot \log \log n.$$

Proof. Using Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\begin{aligned} \left\| \gamma_{\Psi_\omega^U(L, n)}^{(2), d, d} \right\|_{\text{tr}} &\leq \sum_{Q \text{ occ. for } \Psi_\omega^U(L, n)} \sum_{\substack{j=1 \\ Q_j \geq 2}}^m \frac{Q_j(Q_j - 1)}{2} \sum_{\tilde{n} \in \mathbb{N}^{m-1}} \sum_{n_j \geq 1} \left| a_{\tilde{n}_j}^Q \right|^2 \\ &\leq \sum_{Q \text{ occ. for } \Psi_\omega^U(L, n)} \sum_{\substack{j=1 \\ Q_j \geq 2}}^m \frac{Q_j(Q_j - 1)}{2} \sum_{\tilde{n} \in \mathbb{N}^m} \left| a_{\tilde{n}}^Q \right|^2. \end{aligned}$$

Applying Lemmas 3.23 and 3.24 yields that, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has

$$\max_{Q \text{ occ. for } \Psi_\omega^U(L, n)} \sum_{\substack{j=1 \\ Q_j \geq 2}}^m \frac{Q_j(Q_j - 1)}{2} \lesssim n \log n \cdot \log \log n.$$

This completes the proof of Lemma 4.15 as $\sum_{Q, \tilde{n} \in \mathbb{N}^m} \left| a_{\tilde{n}}^Q \right|^2 = 1$. \square

Lemma 4.16. *Under the assumptions of Theorem 4.2, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_\omega^U(L, n)}^{(2), 2} \right\|_{\text{tr}} \leq 2.$$

Proof. Using Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\left\| \gamma_{\Psi_\omega^U(L, n)}^{(2), 2} \right\|_{\text{tr}} \leq \sum_{i \neq j} \left(\sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2}} + \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1}} \right) \sum_{\tilde{n} \in \mathbb{N}^{m-2}} C_2(Q, i, j) \sum_{n_i, n_j \geq 1} \left| a_{\tilde{n}_i, j}^Q \right|^2.$$

For $Q_j \geq 1$ and $Q_i \geq 1$, one has

$$\begin{aligned} C_2(Q, i, j) &= \frac{(n - Q_j - Q_i - 2)! Q_i! Q_j!}{2(n-2)!} \\ &= \frac{(Q_i + Q_j - 2)! (n - (Q_j + Q_i - 2) - 4)! (Q_i - 1)! (Q_j - 1)!}{(n-4)!} \frac{Q_i Q_j}{(Q_i + Q_j - 2)! 2(n-2)(n-3)} \\ &\leq \frac{Q_i Q_j}{2(n-2)(n-3)}. \end{aligned}$$

For $Q_j \geq 2$, one has

$$\begin{aligned} C_2(Q, i, j) &= \frac{Q_i! (Q_j - 2)! (n - 4 - (Q_j + Q_i - 2))!}{(n-4)!} \frac{Q_j(Q_j - 1)}{2(n-2)(n-3)} \\ &\leq \frac{Q_j(Q_j - 1)}{2(n-2)(n-3)}. \end{aligned} \tag{4.93}$$

Thus, as $\sum_j Q_j = n$, one estimates

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),2} \right\|_{\text{tr}} \leq \frac{2}{2(n-2)(n-3)} \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \left(\sum_j Q_j \right)^2 \left| a_{\bar{n}}^Q \right|^2 \leq \frac{n^2}{(n-2)(n-3)}.$$

This proves Lemma 4.16. \square

Lemma 4.17. *Under the assumptions of Theorem 4.2, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,2} \right\|_{\text{tr}} \leq 1.$$

Proof. Using Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,2} \right\|_{\text{tr}} \leq \sum_{i \neq j} \sum_{\bar{n} \in \mathbb{N}^{m-2}} \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2 \\ Q': Q'_k = Q_k \text{ if } k \notin \{i, j\} \\ Q'_i = Q_i + 2 \\ Q'_j = Q_j - 2}} C_2(Q, i, j) \sum_{n_i, n_j \geq 1} \left| a_{\bar{n}_{i,j}}^Q \right|^2.$$

The bound (4.93) then yields

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,2} \right\|_{\text{tr}} \leq \frac{2}{2(n-2)(n-3)} \sum_{\substack{Q \text{ occ.} \\ \bar{n} \in \mathbb{N}^m}} \left(\sum_j Q_j \right)^2 \left| a_{\bar{n}}^Q \right|^2 \leq \frac{n^2}{2(n-2)(n-3)}.$$

This proves Lemma 4.17. \square

Lemma 4.18. *Under the assumptions of Theorem 4.2, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,3} \right\|_{\text{tr}} + \left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,3'} \right\|_{\text{tr}} \leq \frac{2n}{\rho}.$$

Proof. Using Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,3} \right\|_{\text{tr}} \leq \sum_{\substack{i, j, k \\ \text{distinct}}} \sum_{\bar{n} \in \mathbb{N}^{m-3}} \sum_{\substack{Q \text{ occ.} \\ Q_j \geq 2 \\ Q': Q'_l = Q_l \text{ if } l \notin \{i, j, k\} \\ Q'_i = Q_i + 1 \\ Q'_j = Q_j - 2 \\ Q'_k = Q_k + 1}} C_3(Q, i, j, k) \sum_{n_i, n_j, n_k \geq 1} \left| a_{\bar{n}_{i,j,k}}^Q \right|^2.$$

For $Q_j \geq 2$, one has

$$\begin{aligned} C_3(Q, i, j, k) &= \frac{Q_k! Q_i! (Q_j - 2)! (n - (Q_k + Q_i + Q_j - 2) - 4)!}{(n - 4)!} \frac{Q_j (Q_j - 1)}{2(n-2)(n-3)} \\ &\leq \frac{Q_j (Q_j - 1)}{2(n-2)(n-3)}. \end{aligned} \quad (4.94)$$

Hence, by Proposition 2.2, one has

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,3} \right\|_{\text{tr}} \leq \frac{1}{2(n-2)(n-3)} \sum_{\substack{\bar{n} \in \mathbb{N}^m \\ Q \text{ occ.}}} \left(\sum_j 1 \right) \left(\sum_j Q_j \right)^2 \left| a_{\bar{n}}^Q \right|^2 \leq \frac{Ln^2}{2(n-2)(n-3)} \leq \frac{n}{\rho}.$$

The computation for $\gamma_{\Psi_\omega^U(L,n)}^{(2),4,3'}$ is the same except that, instead of (4.94), one uses, for $Q_k \geq 1$ and $Q_i \geq 1$,

$$\begin{aligned} C_3(Q, i, j, k) &= \frac{(Q_k - 1)!(Q_i - 1)!(Q_j)!(n - (Q_j + Q_i + Q_k - 2) - 4)!}{(n - 4)!} \frac{Q_k Q_i}{2(n - 2)(n - 3)} \\ &\leq \frac{Q_k Q_i}{2(n - 2)(n - 3)}. \end{aligned}$$

This proves Lemma 4.17. \square

Lemma 4.19. *Under the assumptions of Theorem 4.2, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,4} \right\|_{tr} \leq n^{-1}.$$

Proof. As in the proof of Lemma 4.13, we will have to deal with the degenerate cases separately (see Remarks 4.3 and 4.5).

Recall (4.24) and write

$$\gamma_{\Psi_\omega^U(L,n)}^{(2),4,4} = \sum_{\sigma \in \{\pm 1\}^4} \gamma_{\Psi_\omega^U(L,n)}^{(2),4,\sigma} \quad (4.95)$$

where $\sigma = (\sigma_i, \sigma_j, \sigma_k, \sigma_l) \in \{\pm 1\}^4$,

$$\begin{aligned} \gamma_{\Psi_\omega^U(L,n)}^{(2),4,\sigma} &= \sum_{\substack{i,j,k,l \\ \text{distinct}}} \sum_{\tilde{n} \in \mathbb{N}^{m-4}} \sum_{\substack{Q \text{ occ.} \\ (Q_i, Q_j, Q_k, Q_l) \in \mathcal{Q}_\sigma \\ Q': Q'_o = Q_o \text{ if } o \notin \{i, j, k, l\} \\ Q'_i = Q_i - 1, Q'_j = Q_j - 1 \\ Q'_k = Q_k + 1, Q'_l = Q_l + 1}} C_4(Q, i, j, k, l) \sum_{\substack{n_i, n_j, n_k, n_l \geq 1 \\ n'_i, n'_j, n'_k, n'_l \geq 1}} a_{\tilde{n}_{i,j,k,l}}^Q \overline{a_{\tilde{n}'_{i,j,k,l}}^{Q'}} \gamma_{\substack{Q_i, Q_j, Q_k, Q_l \\ n_i, n_j, n_k, n_l \\ n'_i, n'_j, n'_k, n'_l}}^{(2),4,4}, \end{aligned} \quad (4.96)$$

and

$$\begin{aligned} \mathcal{Q}_\sigma &= \left\{ Q_i \geq 1 \text{ and } \sigma_i(Q_i - 1) \geq \frac{\sigma_k + 1}{2} \right\} \cap \left\{ Q_j \geq 1 \text{ and } \sigma_j(Q_j - 1) \geq \frac{\sigma_j + 1}{2} \right\} \\ &\quad \cap \left\{ Q_k \geq 0 \text{ and } \sigma_k Q_k \geq \frac{\sigma_k + 1}{2} \right\} \cap \left\{ Q_l \geq 0 \text{ and } \sigma_l Q_l \geq \frac{\sigma_l + 1}{2} \right\}. \end{aligned}$$

A term in the right hand side of (4.95) degenerates if some σ_\bullet takes the value -1 .

Assume now $\sigma = (1, 1, 1, 1)$. Then,

$$\begin{aligned} \gamma_{\Psi_\omega^U(L,n)}^{(2),4,(1,1,1,1)} &= \sum_{\substack{i,j,k,l \\ \text{distinct}}} \sum_{\tilde{n} \in \mathbb{N}^{m-4}} \sum_{\substack{Q \text{ occ.} \\ Q_i, Q_j \geq 2, Q_k, Q_l \geq 1 \\ Q': Q'_o = Q_o \text{ if } o \notin \{i, j, k, l\} \\ Q'_i = Q_i - 1, Q'_j = Q_j - 1 \\ Q'_k = Q_k + 1, Q'_l = Q_l + 1}} C_4(Q, i, j, k, l) \sum_{\substack{n_i, n_j, n_k, n_l \geq 1 \\ n'_i, n'_j, n'_k, n'_l \geq 1}} a_{\tilde{n}_{i,j,k,l}}^Q \overline{a_{\tilde{n}'_{i,j,k,l}}^{Q'}} \gamma_{\substack{Q_i, Q_j, Q_k, Q_l \\ n_i, n_j, n_k, n_l \\ n'_i, n'_j, n'_k, n'_l}}^{(2),4,4}. \end{aligned}$$

Using Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\left\| \gamma_{\Psi_\omega^U(L,n)}^{(2),4,(+,+,+,+)} \right\|_{tr} \leq 4 \sum_{\substack{i,j,k,l \\ \text{distinct}}} \sum_{\tilde{n} \in \mathbb{N}^{m-4}} \sum_{\substack{Q \text{ occ.} \\ Q_i, Q_j \geq 2, Q_k, Q_l \geq 1 \\ Q': Q'_o = Q_o \text{ if } o \notin \{i, j, k, l\} \\ Q'_i = Q_i - 1, Q'_j = Q_j - 1 \\ Q'_k = Q_k + 1, Q'_l = Q_l + 1}} C_4(Q, i, j, k, l) \sum_{n_i, n_j, n_k, n_l \geq 1} \left| a_{\tilde{n}_{i,j,k,l}}^Q \right|^2.$$

When $Q_i \geq 2, Q_j \geq 2, Q_k \geq 1$ and $Q_l \geq 1$ one has

$$C_4(Q, i, j, k, l) \leq \frac{Q_i(Q_i - 1)Q_j(Q_j - 1)Q_k Q_l}{2n(n - 2)(n - 3)(n - 4)(n - 5)(n - 6)(n - 7)}.$$

Thus, by Lemma 3.23, we obtain

$$\begin{aligned} \left\| \gamma_{\Psi_{\omega}^U(L,n)}^{(2),4,(1,1,1,1)} \right\|_{\text{tr}} &\leq \frac{2}{(n-5)^6} \sum_{\bar{n} \in \mathbb{N}^m} \sum_{Q \text{ occ.}} \left(\sum_j Q_j \right)^2 \left(\sum_j Q_j^2 \right)^2 \left| a_{\bar{n}}^Q \right|^2 \leq \frac{n^4 (\log n)^4}{2(n-7)^6} \\ &\leq n^{-1} \end{aligned} \quad (4.97)$$

for n large.

Assume now $\sigma = (-1, -1, -1, -1)$. Then,

$$\gamma_{\Psi_{\omega}^U(L,n)}^{(2),4,(-1,-1,-1,-1)} = \sum_{\substack{i,j,k,l \\ \text{distinct}}} \sum_{\bar{n} \in \mathbb{N}^{m-4}} \sum_{\substack{Q \text{ occ.} \\ Q_i=Q_j=1, Q_k=Q_l=0 \\ Q'_o=Q_o \text{ if } o \notin \{i,j,k,l\} \\ Q'_i=Q_i-1, Q'_j=Q_j-1 \\ Q'_k=Q_k+1, Q'_l=Q_l+1}} C_4(Q, i, j, k, l) \sum_{\substack{n_i, n_j \geq 1 \\ n_k = n_l = 1 \\ n'_i = n'_j = 1 \\ n'_k, n'_l \geq 1}} a_{\bar{n}_{i,j,k,l}}^Q \overline{a_{\bar{n}'_{i,j,k,l}}^{Q'}} \gamma_{\substack{(2),4,4 \\ 1,1,0,0 \\ n_i, n_j, 1, 1 \\ 1, 1, n'_k, n'_l}}$$

where

$$\begin{aligned} \gamma_{\substack{(2),4,4 \\ 1,1,0,0 \\ n_i, n_j, 1, 1 \\ 1, 1, n'_k, n'_l}}^{(2),4,4}(x, x', y, y') &= \varphi_{1, n_i}^i(x) \varphi_{1, n_j}^j(x') \overline{\varphi_{1, n'_k}^k(y) \varphi_{1, n'_l}^l(y')} + \varphi_{1, n_i}^i(x') \varphi_{1, n_j}^j(x) \overline{\varphi_{1, n'_k}^k(y) \varphi_{1, n'_l}^l(y')} \\ &\quad + \varphi_{1, n_i}^i(x) \varphi_{1, n_j}^j(x') \overline{\varphi_{1, n'_k}^k(y') \varphi_{1, n'_l}^l(y)} + \varphi_{1, n_i}^i(x') \varphi_{1, n_j}^j(x) \overline{\varphi_{1, n'_k}^k(y') \varphi_{1, n'_l}^l(y)}. \end{aligned}$$

As in the derivation of (4.81), using Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\begin{aligned} \left\| \gamma_{\Psi_{\omega}^U(L,n)}^{(2),4,(-1,-1,-1,-1)} \right\|_{\text{tr}} &\leq \frac{2}{(n-2)(n-3)} \sum_{\substack{\bar{n} \in \mathbb{N}^{m-4} \\ Q \text{ occ.}}} \left\| \sum_{\substack{(i,j) \\ Q_i=Q_j=1}} \sum_{\substack{n_i=1 \\ n_j=1 \\ n_k, n_l}} a_{\bar{n}_{i,j,k,l}}^Q \varphi_{1, n_k}^k \otimes \varphi_{1, n_l}^l \right\|^2 \\ &\quad + \left\| \sum_{\substack{(k,l) \\ Q_k=Q_l=0}} \sum_{\substack{n_k=1 \\ n_l=1 \\ n_i, n_j}} a_{\bar{n}_{i,j,k,l}}^Q \varphi_{1, n_i}^i \otimes \varphi_{1, n_j}^j \right\|^2 \\ &\leq \frac{4}{(n-3)^2} \sum_{\substack{\bar{n} \in \mathbb{N}^m \\ Q \text{ occ.}}} \left| a_{\bar{n}}^Q \right|^2 = \frac{4}{(n-3)^2}. \end{aligned}$$

Assume now $\sigma = (-1, 1, 1, 1)$. Then,

$$\gamma_{\Psi_{\omega}^U(L,n)}^{(2),4,(-1,1,1,1)} = \sum_{\substack{i,j,k,l \\ \text{distinct}}} \sum_{\bar{n} \in \mathbb{N}^{m-4}} \sum_{\substack{Q \text{ occ.} \\ Q_i=1, Q_j \geq 2 \\ Q_k, Q_l \geq 1 \\ Q'_o=Q_o \text{ if } o \notin \{i,j,k,l\} \\ Q'_i=Q_i-1, Q'_j=Q_j-1 \\ Q'_k=Q_k+1, Q'_l=Q_l+1}} C_4(Q, i, j, k, l) \sum_{\substack{n_i, n_j, n_k, n_l \geq 1 \\ n'_j, n'_k, n'_l \geq 1 \\ n'_i=1}} a_{\bar{n}_{i,j,k,l}}^Q \overline{a_{\bar{n}'_{i,j,k,l}}^{Q'}} \gamma_{\substack{(2),4,4 \\ 1, Q_j, Q_k, Q_l \\ n_i, n_j, n_k, n_l \\ 1, n'_j, n'_k, n'_l}}$$

where

$$\begin{aligned} C_4(Q, i, j, k, l) &= \frac{(n - Q_j - Q_k - Q_l - 3)! Q_j! Q_k! Q_l!}{2(n-2)!} \\ &\leq \frac{Q_j(Q_j-1) Q_k Q_l}{2(n-2)(n-3)(n-4)(n-5)(n-6)}. \end{aligned} \quad (4.98)$$

The operator $\gamma_{1, Q_j, Q_k, Q_l}^{(2), 4, 4}$ is given by (4.31) and

$$\begin{aligned} \sigma(x, x', y, y') &= \varphi_{1, n_i}^i(x) \int_{\Delta_j^{Q_{j-1}}} \varphi_{Q_j, n_j}^j(x', z) \overline{\varphi_{Q_{j-1}, n'_j}^j(z)} dz \\ &\quad \times \int_{\Delta_k^{Q_k}} \varphi_{Q_k, n_k}^k(z) \overline{\varphi_{Q_k+1, n'_k}^k(y, z)} dz \int_{\Delta_l^{Q_l}} \varphi_{Q_l, n_l}^l(z) \overline{\varphi_{Q_l+1, n'_l}^l(y', z)} dz. \end{aligned}$$

Hence, as in the derivation of (4.82), using Lemma B.1, (4.98) and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\begin{aligned} \left\| \gamma_{\Psi_\omega^U(L, n)}^{(2), 4, (-1, 1, 1, 1)} \right\|_{\text{tr}} &\leq \frac{2}{(n-2)(n-3)(n-4)(n-5)(n-6)} \sum_{\substack{\tilde{n} \in \mathbb{N}^m \\ Q \text{ occ.}}} \left(\sum_{j=1}^m Q_j^2 \right) \left(\sum_{j=1}^m Q_j \right)^2 |a_{\tilde{n}, j}^Q|^2 \\ &\leq \frac{n^{10/3} (\log n)^{2/3}}{(n-6)^5} \leq n^{-3/2}. \end{aligned}$$

In the same way, we obtain that, if σ contains a least one -1 then $\left\| \gamma_{\Psi_\omega^U(L, n)}^{(2), 4, \sigma} \right\|_{\text{tr}} \leq n^{-1}$.

This completes the proof of Lemma 4.19. \square

Let us now turn to the analysis of $\gamma_{\Psi_\omega^U(L, n)}^{(2), d, o}$, the main term of $\gamma_{\Psi_\omega^U(L, n)}$. The analysis will be similar of that of $\gamma_{\Psi}^{(1), d}$ in the proof of Theorem 4.2.

Recall that \mathcal{P}_-^Q is defined in Proposition 4.10 and write

$$\gamma_{\Psi_\omega^U(L, n)}^{(2), d, o} = \gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, -} + \gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, +} \quad (4.99)$$

where

$$\gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, -} = \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1}} \sum_{\tilde{n} \in \mathbb{N}^{m-2}} \sum_{\substack{1 \leq i < j \leq m \\ (i, j) \in (\mathcal{P}_-^Q)^2}} \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}, i, j}^Q \overline{a_{\tilde{n}', i, j}^Q} \gamma_{Q_i, Q_j}^{(2), d, o} \Big|_{\substack{n_i, n_j \\ n'_i, n'_j}}}. \quad (4.100)$$

We prove

Lemma 4.20. *Under the assumptions of Theorem 4.4, for $\eta \in (0, 1)$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, in the thermodynamic limit, with probability $1 - O(L^{-\infty})$, one has*

$$\left\| \gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, +} \right\|_{\text{tr}} \leq n^2 \frac{\rho}{\ell_\rho}.$$

Proof. The proof follows that of Lemma 4.14. One estimates

$$\begin{aligned} \left\| \gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, +} \right\|_{\text{tr}} &= \left\| \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1}} \sum_{\substack{1 \leq i < j \leq m \\ (i, j) \notin (\mathcal{P}_-^Q)^2}} \sum_{\tilde{n} \in \mathbb{N}^{m-2}} \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}, i, j}^Q \overline{a_{\tilde{n}', i, j}^Q} \gamma_{Q_i, Q_j}^{(2), d, o} \Big|_{\substack{n_i, n_j \\ n'_i, n'_j}} \right\|_{\text{tr}} \\ &\leq \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1}} \sum_{\tilde{n} \in \mathbb{N}^{m-2}} \left(\sum_{\substack{1 \leq i < j \leq m \\ i \notin \mathcal{P}_-^Q}} + \sum_{\substack{1 \leq i < j \leq m \\ j \notin \mathcal{P}_-^Q}} \right) \left\| \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}, i, j}^Q \overline{a_{\tilde{n}', i, j}^Q} \gamma_{Q_i, Q_j}^{(2), d, o} \Big|_{\substack{n_i, n_j \\ n'_i, n'_j}} \right\|_{\text{tr}}. \end{aligned} \quad (4.101)$$

Let us analyze the first sum in the right hand side above. Using (4.26), Lemma B.1 and the orthonormality properties of the families $(\varphi_{Q_j, n_j}^j)_{n_j \in \mathbb{N}}$, we compute

$$\begin{aligned} \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1}} \sum_{\substack{1 \leq i < j \leq m \\ i \notin \mathcal{P}_-^Q}} \sum_{\tilde{n} \in \mathbb{N}^{m-2}} \left\| \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^Q} \gamma_{Q_i, Q_j}^{(2), d, o} \right\|_{\text{tr}} &\leq \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1}} \sum_{\substack{1 \leq i < j \leq m \\ i \notin \mathcal{P}_-^Q}} \sum_{\tilde{n} \in \mathbb{N}^{m-2}} \frac{Q_i Q_j}{2} \sum_{n_i, n_j \geq 1} |a_{\tilde{n}_{i,j}}^Q|^2 \\ &\leq \frac{1}{2} \sum_{\substack{\tilde{n} \in \mathbb{N}^m \\ Q \text{ occ.}}} \left(\sum_{i \notin \mathcal{P}_-^Q} Q_i \right) \left(\sum_j Q_j \right) |a_{\tilde{n}}^Q|^2 \\ &\leq C n^2 \frac{\rho}{\ell_\rho} \end{aligned}$$

as in the proof of Lemma 4.14 by Lemma 3.23 and 3.24.

The other sum in the right hand side of (4.101) is analyzed in the same way. This completes the proof of Lemma 4.20. \square

Let us now analyze $\gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, -}$. We proceed as in the analysis of $\gamma_{\Psi_\omega^U(L, n)}^{(1), d}$ (see (4.83) and Lemma 4.14). We recall and compute

$$\gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, -} = \sum_{\substack{Q \text{ occ.} \\ Q_i \geq 1 \\ Q_j \geq 1 \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ (i, j) \in (\mathcal{P}_-^Q)^2}} \sum_{\substack{n_j, n'_j \geq 1 \\ n_i, n'_i \geq 1}} a_{\tilde{n}_{i,j}}^Q \overline{a_{\tilde{n}'_{i,j}}^Q} \gamma_{Q_i, Q_j}^{(2), d, o} = \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ (i, j) \in (\mathcal{P}_-^Q)^2}} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \varphi_{i,j}^{\tilde{n}} \otimes^s \varphi_{i,j}^{\tilde{n}}.$$

where $\varphi_{i,j}^{\tilde{n}} := \sum_{\substack{n_i \geq 1 \\ n_j \geq 1}} a_{\tilde{n}_{i,j}}^Q \varphi_{Q_i, n_i}^i \wedge \varphi_{Q_j, n_j}^j$ and the operators Ex and \otimes^s are defined in Proposition 4.8.

Define also

$$\tilde{\varphi}_{i,j}^{\tilde{n}} = \begin{cases} \varphi_{i,j}^{\tilde{n}} & \text{if } n_i + n_j = 2 \\ \|\varphi_{i,j}^{\tilde{n}}\| \varphi_{Q_i, 1}^i \wedge \varphi_{Q_j, 1}^j & \text{if } n_i + n_j \geq 3. \end{cases} \quad (4.102)$$

Then, recalling (4.87), we compute

$$\begin{aligned} \gamma_{\Psi_\omega^U(L, n)}^{(2), d, o, -} &= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ (i, j) \in (\mathcal{P}_-^Q)^2}} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \varphi_{i,j}^{\tilde{n}} \otimes^s \varphi_{i,j}^{\tilde{n}} \\ &\quad + \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}_-^Q \\ \text{or } j \in \mathcal{P}_-^Q}} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \varphi_{i,j}^{\tilde{n}} \otimes^s \varphi_{i,j}^{\tilde{n}} \\ &= \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ (i, j) \in (\mathcal{P}_-^Q)^2}} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}} \\ &\quad + \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}_-^Q \\ \text{or } j \in \mathcal{P}_-^Q}} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) (\varphi_{i,j}^{\tilde{n}} \otimes^s \varphi_{i,j}^{\tilde{n}} - \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}}). \end{aligned} \quad (4.103)$$

The second term in the sum above we estimate by

$$\begin{aligned}
& \left\| \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}_{-,+}^Q \\ \text{or } j \in \mathcal{P}_{-,+}^Q}} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) (\varphi_{i,j}^{\tilde{n}} \otimes^s \varphi_{i,j}^{\tilde{n}} - \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}}) \right\|_{\text{tr}} \\
& \lesssim \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}_{-,+}^Q \\ \text{or } j \in \mathcal{P}_{-,+}^Q}} Q_i Q_j \left(\|\varphi_{i,j}^{\tilde{n}}\|^2 + \|\tilde{\varphi}_{i,j}^{\tilde{n}}\|^2 \right) \\
& \lesssim \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^m}} \left(\sum_{j \in \mathcal{P}_-^Q} \#\{j; n_j \geq 2\} \right) \left(\sum_{j \in \mathcal{P}_-^Q} Q_j \right) \left| a_{\tilde{n}}^Q \right|^2 \\
& \lesssim n^2 \frac{\rho}{\rho_0 |\log \rho|} f_Z(2|\log \rho|).
\end{aligned} \tag{4.104}$$

by Lemma 4.11.

As for the first term in the second equality in (4.103), letting \mathcal{P}_{opt} be the pieces of length less than $3\ell_\rho(1 - \varepsilon)$ where Ψ^{opt} puts at least one particle, we write

$$\begin{aligned}
& \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{(i,j) \in (\mathcal{P}_-^Q)^2} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}} \\
& = \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \left(\sum_{\substack{1 \leq i < j \leq m \\ (i,j) \in (\mathcal{P}_{\text{opt}})^2}} + \sum_{\substack{1 \leq i < j \leq m \\ i \text{ or } j \text{ in } \mathcal{P}_-^Q \setminus \mathcal{P}_{\text{opt}}} } - \sum_{\substack{1 \leq i < j \leq m \\ i \text{ or } j \text{ in } \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_-^Q}} \right) \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}} \tag{4.105}
\end{aligned}$$

For the first of the three sums above, one computes

$$\begin{aligned}
& \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \sum_{(i,j) \in (\mathcal{P}_{\text{opt}})^2} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}} = \sum_{\substack{1 \leq i < j \leq m \\ (i,j) \in (\mathcal{P}_{\text{opt}})^2}} \left(\sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^m}} \left| a_{\tilde{n}}^Q \right|^2 \right) \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}} \\
& = \sum_{\substack{1 \leq i < j \leq m \\ (i,j) \in (\mathcal{P}_{\text{opt}})^2}} \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \gamma_{\varphi_{Q_i,1}^i}^{(1)} \otimes^s \gamma_{\varphi_{Q_j,1}^j}^{(1)} \\
& = \gamma_{\Psi_{\text{opt}}}^{(2)} + R
\end{aligned} \tag{4.106}$$

where $\|R\|_{\text{tr}} \leq Cn^2 \rho^{1+\eta}$.

In the last line of (4.106), we have used Proposition 4.8, the definition of Ψ^{opt} (3.12) and Lemma 3.23 to obtain the bound on R .

To estimate the remaining two sums in (4.104), we split them into sums where the summation over pieces is restricted to pieces either longer than $\ell_\rho + C$ or shorter than $\ell_\rho + C$ (C is given by Corollary 3.32).

By Corollary 3.32, we know that

$$\begin{aligned}
& \left\| \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \left(\sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}^Q \setminus \mathcal{P}_{\text{opt}}} \\ \text{and } |\Delta_i(\omega)| < \ell_\rho + C} - \sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}^Q \setminus \mathcal{P}_{\text{opt}}} \\ \text{and } |\Delta_i(\omega)| < \ell_\rho + C} \right) \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}} \right\|_{\text{tr}} \\
& \leq \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \left(\sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}^Q \setminus \mathcal{P}_{\text{opt}}} \\ \text{and } |\Delta_i(\omega)| < \ell_\rho + C} + \sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}^Q \setminus \mathcal{P}_{\text{opt}}} \\ \text{and } |\Delta_i(\omega)| < \ell_\rho + C} \right) \frac{Q_i Q_j}{2} \|(\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}}\|_{\text{tr}} \\
& \leq C n^2 \rho \max \left(\sqrt{Z(2|\log \rho|)}, \ell_\rho^{-1} \right) \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^m}} \left| a_{\tilde{n}}^Q \right|^2 = C n^2 \max \left(\sqrt{\rho Z(2|\log \rho|)}, \rho |\log \rho|^{-1} \right).
\end{aligned}$$

In the same way, we estimate

$$\begin{aligned}
& \left\| \sum_{\substack{Q \text{ occ.} \\ \tilde{n} \in \mathbb{N}^{m-2}}} \left(\sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_-^Q \\ \text{and } |\Delta_i(\omega)| \geq \ell_\rho + C} - \sum_{\substack{1 \leq i < j \leq m \\ i \in \mathcal{P}_{\text{opt}} \setminus \mathcal{P}_-^Q \\ \text{and } |\Delta_i(\omega)| \geq \ell_\rho + C} \right) \frac{Q_i Q_j}{2} (\text{Id} - \text{Ex}) \tilde{\varphi}_{i,j}^{\tilde{n}} \otimes^s \tilde{\varphi}_{i,j}^{\tilde{n}} \right\|_{\text{tr}} \\
& \leq C n^2 \rho \max \left(\sqrt{Z(2|\log \rho|)}, \ell_\rho^{-1} \right)
\end{aligned}$$

and one has the same estimates when i is replaced by j .

Plugging these estimates, (4.104) and (4.105) into (4.99), recalling (1.29), we obtain

$$\begin{aligned}
& \left\| \left(\gamma_{\Psi_\omega^U(L,n)}^{(2),d,o,-} - \gamma_{\Psi_{\text{opt}}}^{(2)} \right) \mathbf{1}_{<\ell_\rho+C}^2 \right\|_{\text{tr}} \leq C n^2 \max \left(\sqrt{\rho Z(2|\log \rho|)}, \rho |\log \rho|^{-1} \right) \\
& \left\| \left(\gamma_{\Psi_\omega^U(L,n)}^{(2),d,o,-} - \gamma_{\Psi_{\text{opt}}}^{(2)} \right) \left(\mathbf{1} - \mathbf{1}_{<\ell_\rho+C}^2 \right) \right\|_{\text{tr}, \geq \ell_\rho + C} \leq C n^2 \rho \max \left(\sqrt{Z(2|\log \rho|)}, \ell_\rho^{-1} \right).
\end{aligned}$$

Taking into account the decomposition (4.17) and Lemmas 4.15, 4.16, 4.17, 4.18, 4.19 then completes the proof of Theorem 1.5. \square

5. ALMOST SURE CONVERGENCE FOR THE GROUND STATE ENERGY PER PARTICLE

In this section, we prove that, if interactions decay sufficiently fast at infinity, then the convergence in the thermodynamic limit of the ground state energy per particle $E_\omega^U(L, n)/n$ to $\mathcal{E}^U(\rho)$ holds not only in L_ω^2 (see [Ven13, Theorem 3.5]) but also ω -almost surely. From the proof of [Ven13, Theorem 3.5], one clearly sees that it suffices to improve upon the sub-additive estimate given in [Ven13, Lemma 4.1]. We prove

Theorem 5.1. *Assume that the pair potential U be even and such that $U \in L^r(\mathbb{R})$ for some $r > 1$ and that for some $\alpha > 2$, one has $\int_0^{+\infty} x^\alpha U(x) dx < +\infty$.*

In the thermodynamic limit, for disjoint intervals Λ_1 and Λ_2 with n_1 and n_2 electrons respectively, for $\min(|\Lambda_1|, |\Lambda_2|)$ sufficiently large, with probability $1 - O(\min(|\Lambda_1|, |\Lambda_2|)^{-\infty})$, one

has

$$E_\omega^U(\Lambda_1 \cup \Lambda_2, n_1 + n_2) \leq E_\omega^U(\Lambda_1, n_1) + E_\omega^U(\Lambda_2, n_2) + o(n_1 + n_2). \quad (5.1)$$

Here, $E_\omega^U(\Lambda, n)$ denotes the ground state energy of $H_\omega^U(\Lambda, n)$ (see section 1.1).

To apply this result to U satisfying **(HU)**, it suffices to check

Lemma 5.2. *If U satisfies **(HU)** then for any $0 < \alpha < 3$, one has $\int_0^{+\infty} x^\alpha U(x) dx < +\infty$.*

Proof. Clearly, for $n \geq 0$, one has

$$\int_{2^n}^{2^{n+1}} x^\alpha U(x) dx \leq 2^{\alpha(n+1)} \int_{2^n}^{2^{n+1}} U(x) dx \leq 2^{(\alpha-3)n+\alpha} Z(2^n).$$

As Z is bounded, summing over n yields

$$\int_1^{+\infty} x^\alpha U(x) dx \lesssim \sum_{n \geq 1} 2^{(\alpha-3)n+\alpha} < +\infty.$$

This completes the proof of Lemma 5.2. \square

Thus, the sub-additive estimate (5.1) holds for our model and, following the analysis provided in [Ven13], we obtain Theorem 1.2.

Proof of Theorem 5.1. Without loss of generality, let us assume that $\Lambda_1 = [-L_1, 0]$ and $\Lambda_2 = [0, L_2]$. For $i \in \{1, 2\}$, we denote by Ψ_i^U ground states of $H_\omega^U(\Lambda_i, n_i)$. In case of degeneracy, we may additionally choose particular ground states Ψ_i^U , $i \in \{1, 2\}$ such that each of them belongs to a fixed occupation subspace. Thus, occupation is well defined for Ψ_i^U . As usual, we will implicitly suppose that Ψ_i^U is extended by zero outside $\Lambda_i^{n_i}$. Consider now

$$\Psi = \Psi_1^U \wedge \Psi_2^U.$$

Then,

$$\begin{aligned} E_\omega^U(\Lambda_1 \cup \Lambda_2, n_1 + n_2) &\leq \langle H_\omega^U(\Lambda_1 \cup \Lambda_2, n_1 + n_2) \Psi, \Psi \rangle \\ &= E_\omega^U(\Lambda_1, n_1) + E_\omega^U(\Lambda_2, n_2) + \text{Tr}(U \gamma_{\Psi_1^U}^{(1)} \otimes^s \gamma_{\Psi_2^U}^{(2)}) \\ &= E_\omega^U(\Lambda_1, n_1) + E_\omega^U(\Lambda_2, n_2) + \int_{\Lambda_1 \times \Lambda_2} U(x-y) \rho_{\Psi_1^U}(x) \rho_{\Psi_2^U}(y) dx dy \end{aligned}$$

The proof will be accomplished by the following

Lemma 5.3. *Under the assumptions of Theorem 5.1, one has*

$$\int_{\Lambda_1 \times \Lambda_2} U(x-y) \rho_{\Psi_1^U}(x) \rho_{\Psi_2^U}(y) dx dy = o(n_1 + n_2). \quad (5.2)$$

Proof. By Proposition 2.1, with probability $1 - O(\min(|\Lambda_1|, |\Lambda_2|)^{-\infty})$, for $i \in \{1, 2\}$, the largest piece in Λ_i is of length bounded by $\log |\Lambda_i| \cdot \log \log |\Lambda_i|$. This implies that one can partition Λ_i into sub-intervals each containing an integer number of original pieces (i.e., the extremities of these sub-intervals coincide with the extremities of pieces given by the Poisson random process) of length between $\log^2 |\Lambda_i|$ and $2 \log^2 |\Lambda_i|$. Let these new sub-intervals be denoted by Λ_i^j , $j \in \{1, \dots, m_i\}$; we order the intervals in such a way that their distance to 0 increases with j . Thus,

$$\Lambda_i = \bigcup_{j=1}^{m_i} \Lambda_i^j$$

and

$$\log^2 |\Lambda_i| \leq |\Lambda_i^j| \leq 2 \log^2 |\Lambda_i|. \quad (5.3)$$

The last inequalities and the ordering convention imply that

$$\text{dist}(\Lambda_1^{j_1}, \Lambda_2^{j_2}) \geq (j_1 - 1) \cdot \log^2 |\Lambda_1| + (j_2 - 1) \cdot \log^2 |\Lambda_2| \quad (5.4)$$

and

$$\frac{|\Lambda_i|}{2 \log^2 |\Lambda_i|} \leq m_i \leq \frac{|\Lambda_i|}{\log^2 |\Lambda_i|}. \quad (5.5)$$

We now count the number of particles that Ψ_i^U puts in an interval Λ_i^j . Let $\{\Delta_k^i\}_{k=1}^{M_i}$ be the pieces in Λ_i and let Q_k^i be the corresponding occupation numbers. According to the choice of sub-intervals Λ_i^j above, each Λ_i^j is a union of some of the pieces $(\Delta_k^i)_k$. We establish the following natural

Lemma 5.4. *With the above notations, one has*

$$\int_{\Delta_k^i} \rho_{\Psi_i^U}(x) dx = Q_k^i, \quad i \in \{1, 2\}, \quad k \in \{1, \dots, M_i\}.$$

Proof. For convenience, we drop the superscript i in this proof. Recall the decomposition (4.2)

$$\Psi = \sum_{\substack{(n_k)_{1 \leq k \leq M} \\ \forall k, n_k \geq 1}} a_n \bigwedge_{k=1}^M \varphi_{n_k}^k,$$

where $\varphi_{n_k}^k$ are functions of Q_k variables in the piece Δ_k . Keeping the notations, by Theorem 4.2, one has

$$\gamma_{\Psi}^{(1)} = \sum_{k=1}^M \sum_{\substack{n_k \geq 1 \\ n'_k \geq 1}} \sum_{\tilde{n} \in \mathbb{N}^{M-1}} a_{\tilde{n}_k} \overline{a_{\tilde{n}'_k}} \gamma_{n_k, n'_k}^{(1)},$$

where

$$\gamma_{n_k, n'_k}^{(1)}(x, y) = Q_k \int_{(\Delta_k)^{Q_k-1}} \varphi_{n_k}^k(x, z) \overline{\varphi_{n'_k}^k(y, z)} dz.$$

The off-diagonal term $\gamma_{\Psi}^{(1), o}$ vanishes because the functions $\Psi_{1,2}$ were chosen of a fixed occupation. This immediately yields

$$\begin{aligned} \int_{\Delta_k} \rho_{\Psi}(x) dx &= Q_k \sum_{\substack{n_k \geq 1 \\ n'_k \geq 1}} \sum_{\tilde{n} \in \mathbb{N}^{M-1}} a_{\tilde{n}_k} \overline{a_{\tilde{n}'_k}} \int_{(\Delta_k)^{Q_k}} \varphi_{n_k}^k(x) \overline{\varphi_{n'_k}^k(x)} dx \\ &= Q_k \sum_{\tilde{n} \in \mathbb{N}^{M-1}} \int_{(\Delta_k)^{Q_k}} \sum_{n_k \geq 1} |a_{\tilde{n}_k}|^2 |\varphi_{n_k}^k(x)|^2 dx = Q_k, \end{aligned}$$

where, in the second equality, we used the orthogonality of different Q_k -particles levels in the piece Δ_k and, in the third equality, we used the fact that Ψ is normalized.

This completes the proof of Lemma 5.4. \square

Lemma 5.4 immediately entails

Corollary 5.5. *One computes*

$$\int_{\Lambda_i^j} \rho_{\Psi_i^U}(x) dx = \sum_{k | \Delta_k^i \subset \Lambda_i^j} Q_k^i, \quad i \in \{1, 2\}, \quad j \in \{1, \dots, m_i\}.$$

Next, we derive a simple bound on the number of particles in Λ_i^j . The total ground state energy is bounded by

$$E_\omega^U(\Lambda_i, n_i) \leq C \ell_\rho^{-2} n_i.$$

From the other hand, a system of $q = \sum_{k|\Delta_k^i \subset \Lambda_i^j} Q_k^i$ particles in Λ_i^j has non interacting energy at least

$$\sum_{s=1}^q \frac{\pi^2 s^2}{|\Lambda_i^j|^2} \asymp q^3 |\Lambda_i^j|^{-2}.$$

This implies that

$$q^3 |\Lambda_i^j|^{-2} \leq C \ell_\rho^{-2} n_i$$

or, equivalently,

$$\sum_{k|\Delta_k^i \subset \Lambda_i^j} Q_k^i \leq C_1 (|\Lambda_i^j|/\ell_\rho)^{2/3} n_i^{1/3} \leq C_2 n_i^{1/3} \log^{4/3} L_i.$$

Let us now estimate the left hand side of (5.2) using Hölder's inequality ($1/p + 1/q = 1$, $p, q \geq 1$) as

$$\begin{aligned} \int_{\Lambda_1 \times \Lambda_2} U(x-y) \rho_{\Psi_1^U}(x) \rho_{\Psi_2^U}(y) dx dy &= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \int_{\Lambda_1^{j_1} \times \Lambda_2^{j_2}} U(x-y) \rho_{\Psi_1^U}(x) \rho_{\Psi_2^U}(y) dx dy \\ &\leq \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \|U\|_{p, \Lambda_1^{j_1} \times \Lambda_2^{j_2}} \|\rho_{\Psi_1^U}\|_q \|\rho_{\Psi_2^U}\|_q. \end{aligned} \quad (5.6)$$

where we have set

$$\|U\|_{p, \Lambda_1^{j_1} \times \Lambda_2^{j_2}} := \left(\int_{\Lambda_1^{j_1} \times \Lambda_2^{j_2}} U^p(x-y) dx dy \right)^{1/p}. \quad (5.7)$$

Now, recall that by (6.57), for $i \in \{1, 2\}$, on $\Lambda_i^{j_i}$, one has

$$\|\rho_{\Psi_i^U}\|_{\infty, \Lambda_i^{j_i}} \leq 4 \|\Psi_i^U\|_{H^1(\Lambda_i^{j_i})} \|\Psi_i^U\|_{2, \Lambda_i^{j_i}} \leq C \left(\langle H_\omega^U(\Lambda_i^{j_i}, n_i) \Psi_i^U, \Psi_i^U \rangle_{\Lambda_i^{j_i}} \right)^{1/2} \|\Psi_i^U\|_2.$$

Hence, by Corollary 5.5,

$$\|\rho_{\Psi_i^U}\|_q = \left(\int_{\Lambda_i^{j_i}} \rho_{\Psi_i^U}^{q-1} \rho_{\Psi_i^U} \right)^{1/q} \leq (Q_i^{j_i})^{1/q} \left(\langle H_\omega^U(\Lambda_i^{j_i}, n_i) \Psi_i^U, \Psi_i^U \rangle_{\Lambda_i^{j_i}} \right)^{(q-1)/2q} \|\Psi_i^U\|_{2, \Lambda_i^{j_i}}^{(q-1)/q}.$$

Recalling (5.6), as $\|\Psi_i^U\|_{2, \Lambda_i^{j_i}} \leq 1$ for $i \in \{1, 2\}$, we estimate

$$\begin{aligned} &\int_{\Lambda_1 \times \Lambda_2} U(x-y) \rho_{\Psi_1^U}(x) \rho_{\Psi_2^U}(y) dx dy \\ &\leq \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \|U\|_{p, \Lambda_1^{j_1} \times \Lambda_2^{j_2}} (Q_1^{j_1} Q_2^{j_2})^{1/q} \left(\langle H_\omega^U(\Lambda_1^{j_1}, n_1) \Psi_1^U, \Psi_1^U \rangle_{\Lambda_1^{j_1}} \langle H_\omega^U(\Lambda_2^{j_2}, n_2) \Psi_2^U, \Psi_2^U \rangle_{\Lambda_2^{j_2}} \right)^{(q-1)/2q}. \end{aligned} \quad (5.8)$$

Now, as $Q_{\Psi_i^U} \lesssim n_i^{1/3} \log^{4/3} L_i \lesssim n^{1/3} \log^{4/3} n$ and as

$$\langle H_\omega^U(\Lambda_i^{j_i}, n_i) \Psi_i^U, \Psi_i^U \rangle_{\Lambda_i^{j_i}} \leq \langle H_\omega^U(\Lambda_i) \Psi_i^U, \Psi_i^U \rangle \leq C n_i \leq C n,$$

the estimate (5.8) entails

$$\int_{\Lambda_1 \times \Lambda_2} U(x-y) \rho_{\Psi_1^U}(x) \rho_{\Psi_2^U}(y) dx dy \lesssim n^{(3q-1)/3q} (\log n)^{8/(3q)} \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \|U\|_{p, \Lambda_1^{j_1} \times \Lambda_2^{j_2}}. \quad (5.9)$$

Hence, to prove (5.1), it suffices to choose q (recall $q \geq 1$ and $1/p + 1/q = 1$) such that

$$\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \|U\|_{p, \Lambda_1^{j_1} \times \Lambda_2^{j_2}} = o\left(n^{1/3q} (\log n)^{-8/(3q)}\right). \quad (5.10)$$

Therefore, we recall (5.7) and using the definition of the $(\Lambda_i^{j_i})_{i,j}$, in particular (5.4) and (5.5), we estimate

$$\|U\|_{p, \Lambda_1^{j_1} \times \Lambda_2^{j_2}} \lesssim ((j_1 + j_2) |\log L|^2)^{-k/p} \left(\int_{\Lambda_1^{j_1} \times \Lambda_2^{j_2}} (x-y)^k U^p(x-y) dx dy \right)^{1/p}. \quad (5.11)$$

Now, by (5.3), as U is even, we have

$$\left(\int_{\Lambda_1^{j_1} \times \Lambda_2^{j_2}} (x-y)^k U^p(x-y) dx dy \right)^{1/p} \lesssim (\log n)^{2/p} \left(\int_{\mathbb{R}^+} u^k U^p(u) du \right)^{1/p}. \quad (5.12)$$

On the other hand, if $k/p > 1$ and $\max(m_1, m_2) \lesssim L/\log L \lesssim n/\log n$ (with a good probability), one estimates

$$\sum_{\substack{1 \leq j_1 \leq m_1 \\ 1 \leq j_2 \leq m_2}} (j_1 + j_2)^{-k/p} \leq (\log n)^{k/p-2} n^{2-k/p}.$$

Plugging this, (5.12) and (5.11) into the sum in (5.10), we see that (5.10) is a consequence of

$$(\log n)^{2-2/p+8/(3q)} n^{2-k/p-1/(3q)} = (\log n)^{14/3(p-1)/p} n^{5/3-(3k-1)/(3p)} = o(1).$$

as $p^{-1} + q^{-1} = 1$.

Thus, it suffices to find $k > 0$, $p > 1$ such that $u \mapsto u^{k/p} U(u)$ be in $L^p(\mathbb{R}^+)$ and

$$\frac{5}{3} - \frac{3k-1}{3p} < 0.$$

Recall that, by assumption $u \mapsto u^\alpha U(u)$ is integrable (for some $\alpha > 2$) and $U \in L^r(\mathbb{R}^+)$ for some $r > 1$.

We pick $\eta \in (0, 1)$ and pick p and k of the form $p = 1 + \eta(r-1)$ and $k = \frac{5p+1}{3} + \eta$. Thus,

for $r \in (1, \min(\tilde{r}, 2)]$, setting $\tilde{p} := \frac{r-p}{r-1} \in (0, 1)$, we have

$$\frac{5}{3} - \frac{3k-1}{3p} = -\frac{\eta}{p} < 0, \quad \frac{p-\tilde{p}}{1-\tilde{p}} = r \quad \text{and} \quad \frac{k}{\tilde{p}} = k \frac{r-1}{r-p} = \left(2 + \frac{5}{3}\eta(r-1)\right) \frac{1}{1-\eta} = \alpha$$

for $\eta \in (0, 1)$ well chosen.

For this choice of p , \tilde{p} and k , using Hölder's inequality, we then estimate

$$\int_{\mathbb{R}^+} u^k U^p(u) du \leq \left(\int_{\mathbb{R}^+} u^{k/\tilde{p}} U(u) du \right)^{\tilde{p}} \left(\int_{\mathbb{R}^+} U^{(p-\tilde{p})/(1-\tilde{p})}(u) du \right)^{1-\tilde{p}} < +\infty$$

This completes the proof of (5.10) and, thus, of Lemma 5.3. \square

Lemma 5.4 implies that, under the assumption of Theorem 5.1, in the thermodynamic limit, with probability exponentially close to 1, one has

$$\int_{\Lambda_1 \times \Lambda_2} U(x-y) \rho_{\Psi_1^U}(x) \rho_{\Psi_2^U}(y) dx dy = o(n_1 + n_2).$$

This completes the proof of Theorem 5.1. \square

6. MULTIPLE ELECTRONS INTERACTING IN A FIXED NUMBER OF PIECES

The main goal of this section is to study a system of two interacting electrons in the interval $[0, \ell]$ for large ℓ and prove Proposition 1.4; this is the purpose of section 6.1. The two-particles Hamiltonian is given by (1.15). In section 6.2, we study two electrons in two distinct pieces.

We shall also state and prove one result for more than two interacting electrons in a single piece.

6.1. Two electrons in the same piece. We now study two electrons in a large interval interacting through a pair potential U , that is, the Hamiltonian defined in (1.15). We first Proposition 1.4. Next, in section 6.1.3, we compare the ground state of the interacting system with that of the non-interacting system.

Throughout this section, we will assume U is a repulsive, even pair interaction potential. In the present section, our assumptions on U will be weaker than **(HU)**.

6.1.1. *The proof of Proposition 1.4.* Scaling variables to the unit square, the two-particles Hamiltonians $H^U(\ell, 2)$ and $\ell^{-2}H^{U^\ell}(1, 2)$ are unitarily equivalent. Here, we have defined

$$U^\ell(\cdot) := \ell^2 U(\ell \cdot). \quad (6.1)$$

Recall that, for $i \neq j$, $i, j \in \mathbb{N}$, the normalized eigenfunctions of $H^0(1, 2)$ (i.e., of the two-particles free Hamiltonian in a unit square) are given by the determinant

$$\phi_{(i,j)}(x, y) = \sqrt{2} \begin{vmatrix} \sin(\pi i x) & \sin(\pi j x) \\ \sin(\pi i y) & \sin(\pi j y) \end{vmatrix} \text{ for } (x, y) \in [0, 1]^2. \quad (6.2)$$

For a two-component index, we will use the shorthand notation $\bar{i} = (i, j)$. For the non interacting ground state $\phi_{(1,2)}$ we will also use the notation ϕ_0 . The corresponding ground state energy is $5\pi^2$ and the first excited energy level is at $10\pi^2$.

We decompose $L^2([0, 1]) \wedge L^2([0, 1]) = \mathbb{C}\phi_0 \oplus \phi_0^\perp$. By the Schur complement formula, E is the ground state energy of $H^{U^\ell}(1, 2)$ if and only if $E < 10\pi^2$ and E satisfies

$$5\pi^2 + U_{00}^\ell - E = U_{0+}^\ell (H_+ + U_{++}^\ell - E)^{-1} U_{+0}^\ell, \quad (6.3)$$

where Π_+ is the orthogonal projector on ϕ_0^\perp and

$$\begin{aligned} U_{00}^\ell &= \langle \phi_0, U^\ell \phi_0 \rangle, & H_+ &= \Pi_+ H^0(1, 2) \Pi_+, \\ U_{++}^\ell &= \Pi_+ U^\ell \Pi_+, & U_{+0}^\ell &= \Pi_+ U^\ell \phi_0 & U_{0+}^\ell &= (\Pi_+ U^\ell \phi_0)^*. \end{aligned} \quad (6.4)$$

We expand the r.h.s. of (6.3) as

$$\begin{aligned}
U_{0+}^\ell (H_+ + U_{++}^\ell - E)^{-1} U_{+0}^\ell &= \langle U^\ell \phi_0, (H_+ - E)^{-1/2} \\
&\quad \times (\text{Id} + (H_+ - E)^{-1/2} U^\ell (H_+ - E)^{-1/2})^{-1} \\
&\quad \times (H_+ - E)^{-1/2} U^\ell \phi_0 \rangle \\
&= \frac{1}{\ell} \langle \tilde{\phi}_\ell, A_\ell (\text{Id} + A_\ell^* A_\ell)^{-1} A_\ell^* \tilde{\phi}_\ell \rangle \\
&= \frac{1}{\ell} \langle \tilde{\phi}_\ell, A_\ell A_\ell^* (\text{Id} + A_\ell A_\ell^*)^{-1} \tilde{\phi}_\ell \rangle,
\end{aligned} \tag{6.5}$$

where

$$\tilde{\phi}_\ell = \sqrt{\ell} \sqrt{U^\ell} \phi_0 \quad \text{and} \quad A_\ell = A_\ell(E) = \sqrt{U^\ell} (H_+ - E)^{-1/2}. \tag{6.6}$$

To simplify notations we will drop the reference to the energy E . As $\ell \rightarrow +\infty$, the convergence of $\langle \tilde{\phi}_\ell, A_\ell A_\ell^* (\text{Id} + A_\ell A_\ell^*)^{-1} \tilde{\phi}_\ell \rangle$ is locally uniform in $(-\infty, 10\pi^2)$. To compute this limit, we shall transform the expression $\langle \tilde{\phi}_\ell, A_\ell A_\ell^* (\text{Id} + A_\ell A_\ell^*)^{-1} \tilde{\phi}_\ell \rangle$ once more.

Consider the domain $R_\ell = \{(u, y) \in \mathbb{R} \times [0, 1], \text{s.t. } y + \ell^{-1}u \in [0, 1]\}$ and the change of variables

$$\begin{aligned}
t_\ell : R_\ell &\rightarrow [0, 1]^2 \\
(u, y) &\mapsto \left(y + \frac{u}{\ell}, y\right).
\end{aligned}$$

Define the partial isometry

$$\begin{aligned}
T_\ell : L^2([0, 1]^2) &\rightarrow L^2(\mathbb{R} \times [0, 1]) \\
v &\mapsto \ell^{-1/2} \mathbf{1}_{R_\ell} \cdot v \circ t_\ell,
\end{aligned}$$

that is, $(T_\ell v)(u, y) = \frac{1}{\sqrt{\ell}} \mathbf{1}_{R_\ell}(u, y) v\left(y + \frac{u}{\ell}, y\right)$.

One computes its adjoint

$$\begin{aligned}
T_\ell^* : L^2(\mathbb{R} \times [0, 1]) &\rightarrow L^2([0, 1]^2) \\
v &\mapsto \ell^{1/2} (\mathbf{1}_{R_\ell} v) \circ t_\ell^{-1},
\end{aligned}$$

that is, $(T_\ell^* v)(x, y) = \sqrt{\ell} (\mathbf{1}_{R_\ell} \cdot v)(\ell(x - y), y)$.

One easily checks that

$$T_\ell T_\ell^* = \mathbf{1}_{R_\ell} \quad \text{and} \quad T_\ell^* T_\ell = \text{Id}_{L^2([0, 1]^2)} \tag{6.7}$$

where $\mathbf{1}_{R_\ell} : L^2(\mathbb{R} \times [0, 1]) \rightarrow L^2(\mathbb{R} \times [0, 1])$ is the orthogonal projector on the functions supported in R_ℓ .

One then computes

$$\left\langle \tilde{\phi}_\ell, A_\ell A_\ell^* (\text{Id} + A_\ell A_\ell^*)^{-1} \tilde{\phi}_\ell \right\rangle_{L^2([0, 1]^2)} = \left\langle \phi_\ell, K_\ell (\text{Id} + K_\ell)^{-1} \phi_\ell \right\rangle_{L^2(\mathbb{R} \times [0, 1])} \tag{6.8}$$

where we have defined

$$\phi_\ell := T_\ell \tilde{\phi}_\ell \quad \text{and} \quad K_\ell := K_\ell(E) := T_\ell A_\ell A_\ell^* T_\ell^*. \tag{6.9}$$

Define

- the following functions
 - $\phi(u) := u \sqrt{U(u)}$ for $u \in \mathbb{R}$,
 - $\chi_0(y) := \pi \sqrt{2} (\sin(3\pi y) - 3 \sin(\pi y))$ for $y \in [0, 1]$.

- the non negative (see (6.47)) operator K is on $L^2(\mathbb{R})$ by the kernel

$$K(u, u') = \frac{1}{2} \sqrt{U(u)} (|u + u'| - |u - u'|) \sqrt{U(u')}. \quad (6.10)$$

Define also

$$\tilde{\phi} = \phi \otimes \chi_0 \quad \text{and} \quad \tilde{K} = K \otimes \text{Id}. \quad (6.11)$$

We prove

Lemma 6.1. *Assume that U is non negative and even such that $U \in L^p(\mathbb{R})$ for some $p > 1$ and $x \mapsto x^2 U(x)$ is integrable.*

As $\ell \rightarrow +\infty$, one has:

- (a) *in $L^2(\mathbb{R} \times [0, 1])$, ϕ_ℓ converges to $\tilde{\phi}$;*
- (b) *for $\varphi \in \mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$, as $\ell \rightarrow +\infty$, the sequence $(K_\ell \varphi)_\ell$ converges in L^2 -norm to $\tilde{K} \varphi$*

Proposition 1.4 follows from this result as we shall see now. First, we prove

Lemma 6.2. *Under the assumptions of Lemma 6.1, all the operators $(K_\ell)_\ell$ and the operator K are bounded respectively on $L^2(\mathbb{R} \times [0, 1])$ and $L^2(\mathbb{R})$.*

Note however that, depending on U , one may have

$$\|K_\ell\|_{L^2(\mathbb{R} \times [0, 1]) \rightarrow L^2(\mathbb{R} \times [0, 1])} \xrightarrow{\ell \rightarrow +\infty} +\infty.$$

Proof. By (6.9), to show the boundedness of K_ℓ , it suffices to show that $\tilde{K}_\ell := \sqrt{U_\ell}(H_+ - E)^{-1} \sqrt{U_\ell}$ is bounded. Note that, by our assumption on U , U_ℓ is in $L^p([0, 1]^2)$. Using the eigenfunction expansion of $-\Delta$ on $L^2_-([0, 1]^2)$, we write

$$\tilde{K}_\ell = \sum_{\bar{j} \neq (2, 1)} \frac{1}{\pi^2 |\bar{j}|^2 - E} \sqrt{U_\ell} \phi_{\bar{j}} \otimes \phi_{\bar{j}} \sqrt{U_\ell} \quad (6.12)$$

where the sum is over $\bar{j} = (i, j)$ where $(i, j) \in \mathbb{N}$ such that $i > j$.

For $u \in L^2_-([0, 1]^2)$, as $u \sqrt{U_\ell} \in L^{2p/(1+p)}_-([0, 1]^2)$ and as the functions $(\phi_{\bar{j}})_{\bar{j}}$ are uniformly bounded, by the Hausdorff-Young inequality (see e.g. [Rud87]), one has

$$\sum_{\bar{j}} \left| \left\langle \sqrt{U_\ell} \phi_{\bar{j}}, u \right\rangle \right|^{p/(p-1)} \leq C_\ell \|u\|_2^{p/(p-1)}. \quad (6.13)$$

Moreover, for some C_ℓ , one has $\|\sqrt{U_\ell} \phi_{\bar{j}}\|_2 \leq C_\ell$. Thus, by (6.12), as $p > 1$, we obtain

$$\|\tilde{K}_\ell u\|_2 \leq C_\ell \left(\sum_{\bar{j} \neq (2, 1)} \frac{1}{(\pi^2 |\bar{j}|^2 - E)^p} \right)^{1/p} \|u\|_2 \leq C_\ell \|u\|_2.$$

Using the explicit kernel for K given in (6.10), for $u \in L^2(\mathbb{R})$, we compute

$$(Ku)(x) = 2\sqrt{U(x)} \int_{-x}^x x' \sqrt{U(x')} u(x') dx' + 2\sqrt{U(x)} x \int_{-x}^{-x} \sqrt{U(x')} (u(x') - u(-x')) dx'$$

Thus,

$$\|K\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq 4 \sqrt{\|U\|_1 \|(\cdot)^2 U(\cdot)\|_1}.$$

This completes the proof of Lemma 6.2. \square

By Lemma 6.2, $\mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$ is a common core for all K_ℓ and $K \otimes \text{Id}$. Thus, by [RS80, Theorem VIII.25], we know that $K_\ell \xrightarrow{\ell \rightarrow +\infty} K \otimes \text{Id}$ in the strong resolvent sense. Hence, by [RS80, Theorem VIII.20], the sequence $(K_\ell(\text{Id} + K_\ell)^{-1})_\ell$ converges to $K(\text{Id} + K)^{-1} \otimes \text{Id}$ strongly. These operators are all bounded uniformly by 1 (as K_ℓ and K are non-negative). Thus, by point (a) of Lemma 6.1 and (6.8), we obtain

$$\begin{aligned} \left\langle \tilde{\phi}_\ell, A_\ell A_\ell^* (\text{Id} + A_\ell A_\ell^*)^{-1} \tilde{\phi}_\ell \right\rangle &= \langle \phi \otimes \chi_0, [K(\text{Id} + K)^{-1} \otimes \text{Id}] \phi \otimes \chi_0 \rangle + o(1) \\ &= \langle \phi, K(\text{Id} + K)^{-1} \phi \rangle \cdot \int_0^1 \chi_0^2(y) dy + o(1) \\ &= \pi^2 \cdot \langle \phi, K(\text{Id} + K)^{-1} \phi \rangle + o(1). \end{aligned} \quad (6.14)$$

By point (a) of Lemma 6.1, one also computes

$$\begin{aligned} \ell U_{00}^\ell &= \|\phi \otimes \chi_0\|^2 + o(1) = \int_{\mathbb{R}} u^2 U(u) du \int_0^1 \chi_0^2(y) dy + o(1) \\ &= \frac{5}{2} \pi^2 \int_{\mathbb{R}} u^2 U(u) du + o(1) \end{aligned} \quad (6.15)$$

By (6.15), the eigenvalue equation (6.3) yields that, under the assumptions of Lemma 6.2, the ground state energy of $H^{U^\ell}(1, 2)$ satisfies

$$E^{U^\ell}([0, 1], 2) = 5\pi^2 + \frac{\gamma(U)}{\ell} + o\left(\frac{1}{\ell}\right) \quad (6.16)$$

where

$$\gamma(U) = 10\pi^2 [\|\phi\|^2 - \langle \phi, K(\text{Id} + K)^{-1} \phi \rangle] = 10\pi^2 \langle \phi, (\text{Id} + K)^{-1} \phi \rangle. \quad (6.17)$$

By Lemma 5.2 and assumption **(HU)**, we know that the assumptions of Lemma 6.2 are satisfied. This proves the asymptotic expansion announced in Proposition 1.4. To complete the proof of this proposition, we simply note that, as K is bounded by Lemma 6.2, by (6.17), we know that $\gamma(U) = 0$ if and only if $\phi \equiv 0$, i.e., if and only if $U \equiv 0$. \square

Remark 6.3. If one assumes $x \mapsto x^4 U(x)$ to be integrable and U to be in some $L^p(\mathbb{R})$ ($p > 1$) (which is clearly stronger than **(HU)**), one obtains that, $E^U([0, \ell], 2)$, the ground state energy of the Hamiltonian defined in (1.15) admits the following more precise expansion

$$E^{U^\ell}([0, 1], 2) = 5\pi^2 + \frac{\gamma(U)}{\ell} + O(\ell^{-2}). \quad (6.18)$$

6.1.2. *The proof of Lemma 6.1.* We start with a lemma, the result of a computation, that will be used in several parts of the proof.

Lemma 6.4. *For $\bar{j} = (j_1, j_2)$, $j_1 > j_2$, recall that $\phi_{\bar{j}}$, the \bar{j} -th normalized eigenvector of H_0 , is given by (6.2).*

One has

$$\phi_{\bar{j}}\left(y + \frac{u}{\ell}, y\right) = \phi_{\bar{j}}^0\left(\frac{u}{\ell}, y\right) + \phi_{\bar{j}}^+\left(\frac{u}{\ell}, y\right) + \phi_{\bar{j}}^-\left(\frac{u}{\ell}, y\right) \quad (6.19)$$

where

$$\begin{aligned} \phi_{\bar{j}}^0(2x, y) &:= 2\sqrt{2} \sin(\pi(j_1 + j_2)x) \sin(\pi(j_2 - j_1)x) \sin(\pi j_1 y) \sin(\pi j_2 y), \\ \phi_{\bar{j}}^+(2x, y) &:= \sqrt{2} \cos(\pi(j_2 - j_1)x) \sin(\pi(j_2 + j_1)x) \sin(\pi(j_1 - j_2)y) \\ \phi_{\bar{j}}^-(2x, y) &:= \sqrt{2} \cos(\pi(j_2 + j_1)x) \sin(\pi(j_2 - j_1)x) \sin(\pi(j_1 + j_2)y) \end{aligned} \quad (6.20)$$

Proof. Using standard sum and product formulas for the sine and cosine, we compute

$$\begin{aligned}
\frac{1}{\sqrt{2}}\phi_{\bar{j}}\left(y + \frac{u}{\ell}, y\right) &= \left| \begin{array}{cc} \sin\left(\pi j_1\left(y + \frac{u}{\ell}\right)\right) & \sin\left(\pi j_1 y\right) \\ \sin\left(\pi j_2\left(y + \frac{u}{\ell}\right)\right) & \sin\left(\pi j_2 y\right) \end{array} \right| \\
&= \sin\left(\pi j_1 \frac{u}{\ell}\right) \cos\left(\pi j_1 y\right) \sin\left(\pi j_2 y\right) - \sin\left(\pi j_2 \frac{u}{\ell}\right) \cos\left(\pi j_2 y\right) \sin\left(\pi j_1 y\right) \\
&\quad + \left(\cos\left(\pi j_1 \frac{u}{\ell}\right) - \cos\left(\pi j_2 \frac{u}{\ell}\right)\right) \sin\left(\pi j_1 y\right) \sin\left(\pi j_2 y\right) \\
&= \frac{1}{2} \sin\left(\pi j_1 \frac{u}{\ell}\right) \left(\sin\left(\pi\left(j_1 + j_2\right) y\right) - \sin\left(\pi\left(j_1 - j_2\right) y\right)\right) \\
&\quad - \frac{1}{2} \sin\left(\pi j_2 \frac{u}{\ell}\right) \left(\sin\left(\pi\left(j_1 + j_2\right) y\right) + \sin\left(\pi\left(j_1 - j_2\right) y\right)\right) \\
&\quad + \left(\cos\left(\pi j_1 \frac{u}{\ell}\right) - \cos\left(\pi j_2 \frac{u}{\ell}\right)\right) \sin\left(\pi j_1 y\right) \sin\left(\pi j_2 y\right) \\
&= \frac{1}{2} \left(\sin\left(\pi j_1 \frac{u}{\ell}\right) - \sin\left(\pi j_2 \frac{u}{\ell}\right)\right) \sin\left(\pi\left(j_1 + j_2\right) y\right) \\
&\quad - \frac{1}{2} \left(\sin\left(\pi j_1 \frac{u}{\ell}\right) + \sin\left(\pi j_2 \frac{u}{\ell}\right)\right) \sin\left(\pi\left(j_1 - j_2\right) y\right) \\
&\quad + \left(\cos\left(\pi j_1 \frac{u}{\ell}\right) - \cos\left(\pi j_2 \frac{u}{\ell}\right)\right) \sin\left(\pi j_1 y\right) \sin\left(\pi j_2 y\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1}{\sqrt{2}}\phi_{\bar{j}}\left(y + \frac{u}{\ell}, y\right) &= \sin\left(\pi \frac{j_1 - j_2}{2} \frac{u}{\ell}\right) \cos\left(\pi \frac{j_1 + j_2}{2} \frac{u}{\ell}\right) \sin\left(\pi\left(j_1 + j_2\right) y\right) \\
&\quad - \sin\left(\pi \frac{j_1 + j_2}{2} \frac{u}{\ell}\right) \cos\left(\pi \frac{j_1 - j_2}{2} \frac{u}{\ell}\right) \sin\left(\pi\left(j_1 - j_2\right) y\right) \\
&\quad - 2 \sin\left(\pi \frac{j_1 - j_2}{2} \frac{u}{\ell}\right) \sin\left(\pi \frac{j_1 + j_2}{2} \frac{u}{\ell}\right) \sin\left(\pi j_1 y\right) \sin\left(\pi j_2 y\right).
\end{aligned}$$

This completes the proof of Lemma 6.4. \square

We start with the proof of point (a) of Lemma 6.1. As $\phi_0 = \phi_{(2,1)}$, by (6.19) and (6.20), using the Taylor expansion of the sine and cosine near 0, we compute

$$\begin{aligned}
(T_\ell \tilde{\phi}_\ell)(u, y) &= \ell \sqrt{U(u)} \mathbf{1}_{R_\ell}(u, y) \phi_{(2,1)}\left(y + \frac{u}{\ell}, y\right) \\
&= u \sqrt{U(u)} \chi_0(y) \mathbf{1}_{R_\ell}(u, y) + \frac{u^2}{\ell} \sqrt{U(u)} \chi_1\left(\frac{u}{\ell}, y\right) \mathbf{1}_{R_\ell}(u, y)
\end{aligned}$$

where χ_0 is defined in Lemma 6.1 and χ_1 is continuous and bounded on $\mathbb{R} \times [0, 1]$.

We estimate

$$\left\| \frac{(\cdot)^2}{\ell} \sqrt{U(\cdot)} \chi_1\left(\frac{\cdot}{\ell}, \cdot\right) \mathbf{1}_{R_\ell} \right\|_{L^2(\mathbb{R} \times [0, 1])}^2 \lesssim \int_{R_\ell} \frac{u^2}{\ell^2} u^2 U(u) du dy \leq \int_{\mathbb{R}} \frac{u^2 \mathbf{1}_{|u| \leq \ell}}{\ell^2} u^2 U(u) du.$$

The last integral tends to 0 by the dominated convergence theorem as $u \mapsto u^2 U(u)$ is integrable.

This completes the proof of point (a) of Lemma 6.1.

Let us now turn to the analysis of the operator family $(K_\ell)_\ell$. It is easily seen that its kernel (we use the same notations for the operator and its kernel) is given by

$$K_\ell(E; u, y, u', y') = \ell \mathbf{1}_{R_\ell \times R_\ell} \sqrt{U(u)U(u')} \cdot \tilde{K}\left(E; y + \frac{u}{\ell}, y, y' + \frac{u'}{\ell}, y'\right)$$

where $\tilde{K}(E; x, y, x', y')$ is the kernel of $(H_+ - E)^{-1}$. The kernel $\tilde{K}(E)$ is easily expressed in terms of the eigenfunctions of H . Using this and the representation yielded by Lemma 6.4 leads to the following representation for the kernel K_ℓ

$$\begin{aligned} K_\ell(E; u, y, u', y') &= \ell \mathbf{1}_{R_\ell \times R_\ell} \sum_{j \neq (2,1)} \frac{\sqrt{U(u)U(u')}}{\pi^2 |j|^2 - E} \phi_j \left(y + \frac{u}{\ell}, y \right) \phi_j \left(y' + \frac{u'}{\ell}, y' \right) \\ &= K_\ell^-(E; u, y, u', y') + K_\ell^+(E; u, y, u', y') + K_\ell^0(E; u, y, u', y') \end{aligned} \quad (6.21)$$

where, for $\bullet \in \{0, +, -\}$, we have set

$$K_\ell^\bullet(E; u, y, u', y') = \ell \mathbf{1}_{R_\ell \times R_\ell} \sum_{j \neq (2,1)} \frac{\sqrt{U(u)U(u')}}{\pi^2 |j|^2 - E} \phi_j \left(y + \frac{u}{\ell}, y \right) \phi_j^\bullet \left(\frac{u'}{\ell}, y' \right).$$

To prove point (b) of Lemma 6.1, it suffices to prove that, for $v \in \mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$, one has $K_\ell v \rightarrow \tilde{K}v$ in $L^2(\mathbb{R} \times [0, 1])$. We first prove

Lemma 6.5. *For $v \in \mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$, one has*

- (a) $\|K_\ell^- v\|_2 \rightarrow 0$ as $\ell \rightarrow +\infty$,
- (b) $\|K_\ell^0 v\|_2 \rightarrow 0$ as $\ell \rightarrow +\infty$.

Proof. We first study the sequence $K_\ell^+ v$. We compute

$$(K_\ell^+ v)(u, y) = \sqrt{U(u)} \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{C_{j,k}(v)}{\pi^2((j+k)^2 + j^2) - E} \mathbf{1}_{R_\ell}(u, y) \phi_{(j+k,j)} \left(y + \frac{u}{\ell}, y \right) \quad (6.22)$$

where

$$C_{j,k}(v) := \ell \int_{-\ell}^{\ell} \sqrt{U(u')} \sin \left(\pi \frac{(2j+k)u'}{2\ell} \right) \cos \left(\pi \frac{ku'}{\ell} \right) c_{2j+k}(u') du' \quad (6.23)$$

and

$$\begin{aligned} c_j(u') &:= \int_0^1 (\mathbf{1}_{R_\ell} v)(u', y') \sin(\pi j y') dy' = \int_{\max(0, -u'/\ell)}^{\min(1, 1-u'/\ell)} v(u', y') \sin(\pi j y') dy' \\ &= \int_0^1 v(u', y') \sin(\pi j y') dy' \end{aligned} \quad (6.24)$$

for ℓ sufficiently large as $v \in \mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$.

Integrating the last integral in (6.24) by parts, we obtain

$$\|c_j\|_{L^2(\mathbb{R})} = O(j^{-\infty}). \quad (6.25)$$

By (6.24) and (6.23), as $u \mapsto u^2 U(u)$ is summable, we obtain

$$\begin{aligned} |C_{j,k}(v)| &\leq O((2j+k)^{-\infty}) \ell \sqrt{\int_{\mathbb{R}} U(u') \sin^2 \left(\pi \frac{(2j+k)u'}{2\ell} \right) du'} \\ &\leq O((2j+k)^{-\infty}) \min(\ell, 2j+k) \end{aligned} \quad (6.26)$$

Estimating $\|K_\ell^- v\|$ using (6.22) and the triangular inequality, as

$$\begin{aligned} &\int_{\mathbb{R} \times [0,1]} U(u) \mathbf{1}_{R_\ell}(u, y) \phi_{(j+k,j)}^2 \left(y + \frac{u}{\ell}, y \right) dud y \\ &\lesssim \int_{\mathbb{R}} U(u) \sin^2 \left(\pi k \frac{u}{\ell} \right) du + \int_{\mathbb{R}} U(u) \sin^2 \left(\pi (2j+k) \frac{u}{\ell} \right) du \\ &\lesssim \frac{\min^2(2j+k, \ell) + \min^2(k, \ell)}{\ell^2}, \end{aligned} \quad (6.27)$$

for $p \geq 4$, we get

$$\|K_\ell^- v\| \lesssim \frac{1}{\ell} \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{1}{(j+k)^p}.$$

Thus, one gets that $\|K_\ell^- v\| \rightarrow 0$ as $\ell \rightarrow +\infty$. This completes the proof of point (a) of Lemma 6.5.

To prove point (b), as $2 \sin a \sin b = \cos(a-b) - \cos(a+b)$, we compute

$$(K_\ell^0 v)(u, y) = \sqrt{U(u)} \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{A_{j,k}^-(v) - A_{j,k}^+(v)}{\pi^2((j+k)^2 + j^2) - E} \mathbf{1}_{R_\ell}(u, y) \phi_{(j+k,j)} \left(y + \frac{u}{\ell}, y \right)$$

where

$$A_{j,k}^+(v) := \ell \int_{-\ell}^{\ell} \sqrt{U(u')} \sin \left(\pi \frac{(2j+k)u'}{2\ell} \right) \sin \left(\pi \frac{ku'}{\ell} \right) a_{2j+k}(u') du',$$

$$A_{j,k}^-(v) := \ell \int_{-\ell}^{\ell} \sqrt{U(u')} \sin \left(\pi \frac{(2j+k)u'}{2\ell} \right) \sin \left(\pi \frac{ku'}{\ell} \right) a_k(u') du'$$

and

$$a_k(u') := \int_0^1 (\mathbf{1}_{R_\ell} v)(u', y') \cos(\pi k y') dy'.$$

As in (6.24), we obtain

$$\|a_k\|_{L^2(\mathbb{R})} = O(k^{-\infty}).$$

As in (6.26), we obtain

$$|A_{j,k}^\pm(v)| \leq O(k^{-\infty}) \min(\ell, k).$$

By (6.27), for $p \geq 2$, we then get

$$\|K_\ell^0 v\| \lesssim \frac{1}{\ell} \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{\min(\ell, k)(\min(\ell, k) + \min(\ell, j+k))}{k^{-p}((j+k)^2 + j^2)} \lesssim \frac{1}{\ell} + \sum_{j \geq 1} \frac{\min(1, j/\ell)}{j^2} \quad (6.28)$$

The last term converges to 0 by the dominated convergence theorem. This completes the proof of point (b) of Lemma 6.5, thus, of Lemma 6.5. \square

Next, we decompose K_ℓ^+ expanding $\phi_{\bar{j}}(y + u/\ell, y)$ according to (6.19). This gives

$$K_\ell^+ = K_\ell^{+,+} + K_\ell^{+,-} + K_\ell^{+,0},$$

where, for $\bullet \in \{0, +, -\}$, we have set

$$K_\ell^{+,\bullet}(E; u, y, u', y') = \ell \mathbf{1}_{R_\ell \times R_\ell} \sum_{\bar{j} \neq (2,1)} \frac{\sqrt{U(u)U(u')}}{\pi^2|\bar{j}|^2 - E} \phi_{\bar{j}}^\bullet \left(y + \frac{u}{\ell}, y \right) \times \phi_{\bar{j}}^+ \left(\frac{u'}{\ell}, y \right).$$

We now prove

Lemma 6.6. *For $v \in C_0^\infty(\mathbb{R} \times (0, 1))$, one has*

- (a) $\|K_\ell^{-,+} v\| \rightarrow 0$ as $\ell \rightarrow +\infty$,
- (b) $\|K_\ell^{0,+} v\| \rightarrow 0$ as $\ell \rightarrow +\infty$.

Proof. As in the proof of Lemma 6.5, the two points in Lemma 6.6 are proved in very similar ways. We will only detail the proof of point (a).

We compute

$$(K_\ell^{-,+} v)(u, y) = \sqrt{U(u)} \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{C_{j,k}(v)}{\pi^2((j+k)^2 + j^2) - E} \mathbf{1}_{R_\ell}(u, y) \phi_{(j+k,j)}^- \left(y + \frac{u}{\ell}, y \right) \quad (6.29)$$

where

$$C_{j,k}(v) := \ell \int_{-\ell}^{\ell} \sqrt{U(u')} \sin\left(\pi \frac{(2j+k)u'}{2\ell}\right) \cos\left(\pi \frac{ku'}{2\ell}\right) c_k(u') du' \quad (6.30)$$

and

$$c_k(u') := \int_0^1 (\mathbf{1}_{R_\ell} v)(u', y') \sin(\pi k y') dy' = \int_0^1 v(u', y') \sin(\pi k y') dy'$$

for ℓ sufficiently large as $v \in \mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$.

Integrating the last integral in (6.24) by parts, we obtain

$$\|c_k\|_{L^2(\mathbb{R})} = O(k^{-\infty}). \quad (6.31)$$

As in (6.26), we obtain

$$|C_{j,k}(v)| \leq O(k^{-\infty}) \min(\ell, 2j+k). \quad (6.32)$$

Using (6.20), one estimates

$$\sqrt{\int_{\mathbb{R} \times [0,1]} U(u) \mathbf{1}_{R_\ell}(u, y) \left| \phi_{(j+k,j)}^-\left(y + \frac{u}{\ell}, y\right) \right|^2 dudy} \lesssim \frac{\min(k, \ell)}{\ell}. \quad (6.33)$$

Thus, for $p \geq 2$, we get

$$\|K_\ell^{-,+} v\| \lesssim \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{\min(k, \ell)}{\ell} \frac{\min(2j+k, \ell)}{k^p ((j+k)^2 + j^2)}. \quad (6.34)$$

Thus, by the dominated convergence theorem, as in (6.28), one gets that $\|K_\ell^{-,+} v\| \rightarrow 0$ as $\ell \rightarrow +\infty$. This completes the proof of point (a) of Lemma 6.6.

Point (b) is proved similarly except that estimate (6.33) is replaced with

$$\sqrt{\int_{\mathbb{R} \times [0,1]} U(u) \mathbf{1}_{R_\ell}(u, y) \left| \phi_{(j+k,j)}^0\left(y + \frac{u}{\ell}, y\right) \right|^2 dudy} \lesssim \frac{\min(k, \ell) \min(2j+k, \ell)}{\ell^2}.$$

Thus, taking $p > 3$, estimate (6.34) in this case becomes

$$\|K_\ell^{0,+} v\| \lesssim \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{\min(k, \ell)}{k^p} \frac{\min^2(2j+k, \ell)}{\ell^2 ((j+k)^2 + j^2)} \lesssim \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{1}{k^{p-2}} \frac{\min^2(j, \ell)}{\ell^2 j^2} \lesssim \sum_{j \geq 1} \frac{\min^2(j, \ell)}{\ell^2} \frac{1}{j^2}$$

which converges to 0 as $\ell \rightarrow +\infty$.

This completes the proof of Lemma 6.6. \square

We are now left with computing the limit of $K_\ell^{+,+}$ where

$$K_\ell^{+,+}(u, y, u', y') = \sum_{\substack{j \geq 1, k \geq 1 \\ (j,k) \neq (1,1)}} \frac{\ell \sqrt{U(u)U(u')}}{\pi^2 ((j+k)^2 + j^2) - E} \times \phi_{(j+k,j)}^+\left(y + \frac{u}{\ell}, y\right) \phi_{(j,j+k)}^+\left(y' + \frac{u'}{\ell}, y'\right). \quad (6.35)$$

We prove

Lemma 6.7. *In the strong topology, one has*

$$K_\ell^{+,+} \rightarrow K \otimes \text{Id} \quad \text{as} \quad \ell \rightarrow +\infty. \quad (6.36)$$

where K is defined in (6.10).

Proof. To simplify the computations, we note that it suffices to show the convergence of $K_\ell^{+,+}v$ for $v \in \mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$. For ℓ sufficiently large, compute

$$(K_\ell^{+,+}v)(u, y) = \sum_{k \geq 1} \sin(\pi ky) \mathbf{1}_{R_\ell}(u, y) c(K_k^\ell, u)$$

where

$$c(K_k^\ell, u) := \frac{1}{2} \sqrt{U(u)} \int_{\mathbb{R}} K_k^\ell(u, u') \sqrt{U(u')} c_k(u') du', \quad (6.37)$$

$u \mapsto c_k(u)$ being defined by (6.24), and

$$K_k^\ell(u, u') := \ell \sum_{\substack{j \in \mathbb{N} \\ (j,k) \neq (1,1)}} \frac{\sin\left(\pi \frac{2j+k}{2\ell} u\right) \sin\left(\pi \frac{2j+k}{2\ell} u'\right) \cos\left(\pi \frac{k}{2\ell} u\right) \cos\left(\pi \frac{2k}{2\ell} u'\right)}{\pi^2 (j+k/2)^2 + (\pi k/2)^2 - E} \quad (6.38)$$

Define

$$\begin{aligned} L_k^\ell(u, u') &:= \ell \sum_{\substack{j \in \mathbb{N} \\ (j,k) \neq (1,1)}} \frac{\sin\left(\pi \frac{2j+k}{2\ell} u\right) \sin\left(\pi \frac{2j+k}{2\ell} u'\right)}{\pi^2 (j+k/2)^2}, \\ M_k^\ell(u, u') &:= K_k^\ell(u, u') - L_k^\ell(u, u'), \\ (L_\ell^{+,+}v)(u, y) &:= \sum_{k \geq 1} \sin(\pi ky) \mathbf{1}_{R_\ell}(u, y) c(L_k^\ell, u), \\ (M_\ell^{+,+}v)(u, y) &:= \sum_{k \geq 1} \sin(\pi ky) \mathbf{1}_{R_\ell}(u, y) c(M_k^\ell, u). \end{aligned} \quad (6.39)$$

Here and in the sequel, $c(L_k^\ell, u)$ and $c(M_k^\ell, u)$ are defined as $c(K_k^\ell, u)$ in (6.37) with K_k^ℓ replaced respectively by L_k^ℓ and M_k^ℓ .

Note that

$$\|L_\ell^{+,+}v\|_{L^2(\mathbb{R} \times [0,1])}^2 := \frac{1}{2} \sum_{k \geq 1} \int_0^1 \|\mathbf{1}_{R_\ell}(y, \cdot) c(L_k^\ell, \cdot)\|_{L^2(\mathbb{R})}^2 dy \leq \frac{1}{2} \sum_{k \geq 1} \|c(L_k^\ell, \cdot)\|_{L^2(\mathbb{R})}^2 \quad (6.40)$$

We prove

Lemma 6.8. *As $\ell \rightarrow +\infty$, $\|M_\ell^{+,+}v\|_{L^2(\mathbb{R} \times [0,1])} \rightarrow 0$*

Proof. The proof is similar to those of Lemmas 6.5 and 6.6. We write

$$M_k^\ell(u, u') = M_k^{1,\ell}(u, u') + M_k^{2,\ell}(u, u') + M_k^{3,\ell}(u, u')$$

where

$$\begin{aligned} M_k^{1,\ell}(u, u') &= \ell \sum_{\substack{j \in \mathbb{N} \\ (j,k) \neq (1,1)}} \frac{\sin\left(\pi \frac{2j+k}{2\ell} u\right) \sin\left(\pi \frac{2j+k}{2\ell} u'\right) \cos\left(\pi \frac{k}{2\ell} u\right) \cos\left(\pi \frac{k}{2\ell} u'\right) ((\pi k/2)^2 - E)}{\pi^4 (j+k/2)^2 ((j+k/2)^2 + (\pi k/2)^2 - E)} \\ M_k^{2,\ell}(u, u') &:= \ell \sum_{\substack{j \in \mathbb{N} \\ (j,k) \neq (1,1)}} \frac{\sin\left(\pi \frac{2j+k}{2\ell} u\right) \sin\left(\pi \frac{2j+k}{2\ell} u'\right) \cos\left(\pi \frac{k}{2\ell} u\right) (\cos\left(\pi \frac{k}{2\ell} u'\right) - 1)}{\pi^2 (j+k/2)^2} \\ M_k^{3,\ell}(u, u') &:= \ell \sum_{\substack{j \in \mathbb{N} \\ (j,k) \neq (1,1)}} \frac{\sin\left(\pi \frac{2j+k}{2\ell} u\right) \sin\left(\pi \frac{2j+k}{2\ell} u'\right) (\cos\left(\pi \frac{k}{2\ell} u\right) - 1)}{\pi^2 (j+k/2)^2}. \end{aligned}$$

Following the definitions (6.38) and using (6.40), we estimate

$$\begin{aligned} \|M_\ell^{1,+,+}v\|_{L^2(\mathbb{R}\times[0,1])}^2 &\leq \frac{1}{2} \sum_{k\geq 1} \left\| c\left(M_k^{1,\ell}, \cdot\right) \right\|_{L^2(\mathbb{R})}^2 \\ &\lesssim \sum_{k\geq 1} k^2 \|c_k\|_{L^2(\mathbb{R})}^2 \sum_{\substack{j\geq 1 \\ (j,k)\neq(1,1)}} \frac{(\min(2j+k, \ell))^2}{\ell(2j+k)^4} \\ &\lesssim \frac{1}{\ell} \sum_{k\geq 1} k^2 \|c_k\|_{L^2(\mathbb{R})}^2 \end{aligned}$$

which, by (6.31), converges to 0 as ℓ goes to $+\infty$.

That the term coming from $M_\ell^{2,+,+}$ (resp. $M_\ell^{3,+,+}$) also vanishes as $\ell \rightarrow +\infty$ follows from computations similar to those done in Lemma 6.5 (resp. Lemma 6.6). This completes the proof of Lemma 6.8. \square

Note that

$$L_k^\ell(u, u') := \frac{1}{\ell} \sum_{\substack{j\in\mathbb{N} \\ (j,k)\neq(1,1)}} \frac{\sin\left(\pi\frac{2j+k}{2\ell}u\right) \sin\left(\pi\frac{2j+k}{2\ell}u'\right)}{\pi^2\left(\frac{2j+k}{2\ell}\right)^2} \quad (6.41)$$

Define

$$a(L^+, u) := \frac{1}{2} \sqrt{U(u)} \int_{\mathbb{R}} L^+(u, u') \sqrt{U(u')} c_k(u') du' \quad (6.42)$$

where

$$L^+(u, u') = \int_0^{+\infty} \frac{\sin(\pi x u) \sin(\pi x u')}{\pi^2 x^2} dx. \quad (6.43)$$

We prove

Lemma 6.9. *For any $k \geq 1$, one has*

$$\sup_{(u,u')\in[-\ell,\ell]^2} \frac{|L_k^\ell(u, u') - L^+(u, u')|}{|u||u'|} \lesssim \frac{k}{\ell}. \quad (6.44)$$

Proof. Define

$$l(x, u, u') := \frac{\sin(\pi x u) \sin(\pi x u')}{\pi^2 x^2}.$$

Assume first $k \neq 1$. As l is an even function of x , write

$$L_k^\ell(u, u') = \frac{1}{2\ell} \sum_{j\in\mathbb{Z}} l\left(\frac{j+k/2}{\ell}, u, u'\right) - \frac{1}{2\ell} \sum_{j=-k}^0 l\left(\frac{j+k/2}{\ell}, u, u'\right). \quad (6.45)$$

Using the Poisson formula, one computes

$$\frac{1}{2\ell} \sum_{j\in\mathbb{Z}} l\left(\frac{j+k/2}{\ell}, u, u'\right) = \frac{1}{2} \sum_{j\in\mathbb{Z}} e^{i\pi k j} \cdot \hat{l}(j, u, u') \quad (6.46)$$

where $\hat{l}(\cdot, u, u')$ is the Fourier transform of $x \mapsto l(x, u, u')$.

By the Paley-Wiener Theorem (or by a direct computation of the Fourier transform), one checks that $\hat{l}(\cdot, u, u')$ is supported in $[-\pi(|u| + |u'|), \pi(|u| + |u'|)]$. Thus, for $-\ell \leq u, u' \leq \ell$, all the terms in right hand side of (6.46) vanish except the term for $j = 0$. That is, for $-\ell \leq u, u' \leq \ell$, one has

$$\frac{1}{2\ell} \sum_{j\in\mathbb{Z}} l\left(\frac{j+k/2}{\ell}, u, u'\right) = \frac{1}{2} \hat{l}(0, u, u') = L^+(u, u').$$

This and (6.46) then yields that, for $-\ell \leq u, u' \leq \ell$,

$$L_k^\ell(u, u') - L^+(u, u') = -\frac{u u'}{2\ell} \sum_{j=-k}^0 \frac{l\left(\frac{j+k/2}{\ell}, u, u'\right)}{u u'}.$$

Now, as

$$\sup_{(x, u, u') \in \mathbb{R}^3} \left| \frac{l(x, u, u')}{u u'} \right| < +\infty,$$

we immediately obtain (6.44) and complete the proof of Lemma 6.9 when $k \neq 1$.

When $k = 1$, the proof is done in the same way up to a shift in the index j . This completes the proof of Lemma 6.9. \square

As $v \in \mathcal{C}_0^\infty(\mathbb{R} \times (0, 1))$, one has

$$\forall N \geq 0, \quad \exists C_N > 0, \quad \forall k \in \mathbb{Z}, \quad \|c_k\|_{L^2(\mathbb{R})} \leq C_N \frac{1}{1 + |k|^N}.$$

Thus, as $x \mapsto x\sqrt{U(x)}$ is square integrable, the bound (6.44) yields that, for some $C_2 > 0$, one has

$$\forall k \in \mathbb{Z}, \quad \|c(K_k^\ell, \cdot) - c(L^+, \cdot)\|_{L^2([- \ell, \ell])} \leq \frac{1}{\ell} \frac{C_2}{1 + |k|^2}.$$

Thus, taking into account the following computation

$$\begin{aligned} L^+(u, u') &= \int_{\mathbb{R}} \frac{\sin(\pi x u) \sin(\pi x u')}{\pi^2 x^2} dx \\ &= \frac{1}{2\pi^2} \left[\int_{\mathbb{R}} \frac{\cos(\pi x(u - u')) - 1}{x^2} dx + \int_{\mathbb{R}} \frac{1 - \cos(\pi x(u + u'))}{x^2} dx \right] \\ &= \frac{1}{2\pi^2} \left[|u - u'| \int_{\mathbb{R}} \frac{\cos(\pi x) - 1}{x^2} dx + |u + u'| \int_{\mathbb{R}} \frac{1 - \cos(\pi x)}{x^2} dx \right] \\ &= \frac{1}{2} (|u + u'| - |u - u'|), \end{aligned} \tag{6.47}$$

the definition of K , (6.10) and (6.40), we obtain that

$$\|L_\ell^{+,+} v - (K \otimes \mathbf{1})v\|_{L^2(\mathbb{R} \times [0, 1])} \xrightarrow{\ell \rightarrow +\infty} 0$$

Thus, Lemma 6.7 is proved. \square

Clearly, the proof of Lemma 6.1 generalizes to arbitrary $\phi_{(i,j)}$, a normalized eigenfunction of $H^0(1, 2)$; one thus proves

Corollary 6.10. *Consider two particles on i -th and j -th energy levels in an interval of length ℓ . Their interaction amplitude is given by*

$$\langle U \phi_{(i,j)}, \phi_{(i,j)} \rangle = 2\pi^2 (i^2 + j^2) \cdot \int_{\mathbb{R}} u^2 U(u) du \cdot \ell^{-3} (1 + O(\ell^{-1})). \tag{6.48}$$

6.1.3. *The ground state of two interacting electrons and its density matrices.* Recall that $\varphi_{[0,\ell]}^j$ denotes the j -th normalized eigenvector of $-\Delta_{[0,\ell]}^D$ and $\zeta_{[0,\ell]}^j$ the j -th normalized eigenvector of (1.15). In the sequel, we drop the subscript $[0, \ell]$ as we always work on the interval $[0, \ell]$. We remark that, when the interactions are absent, one has

$$\zeta^{1,0} = \varphi^1 \wedge \varphi^2. \tag{6.49}$$

The next proposition estimates the difference $\zeta^{1,U} - \zeta^{1,0}$ induced by the presence of interactions.

Proposition 6.11. *For $\ell \geq 1$, one has*

$$\|\zeta^{1,U} - \zeta^{1,0}\|_{L^2([0,\ell]^2)} \lesssim \ell^{-1/2}. \quad (6.50)$$

Proof. Scaling the variables to the unit square (see section 6.1.1), it suffices to show that the normalized ground state of $H^{U^\ell}(1,2)$ (see (6.1)), say, $\phi_0^{U^\ell}$ satisfies

$$\|\phi_0^{U^\ell} - \phi_0\|_{L^2([0,1]^2)} \lesssim \ell^{-1/2}. \quad (6.51)$$

where we recall that $\phi_0 = \phi_{(1,2)}$ (see (6.2)).

Decomposing $L^2([0,1]) \wedge L^2([0,1]) = \mathbb{C}\phi_0 \oplus \phi_0^\perp$ and defining $E_0^{U^\ell}$ to be the ground state energy of $H^{U^\ell}(1,2)$, we rewrite $\phi_0^{U^\ell}$ as

$$\phi_0^{U^\ell} = \alpha\phi_0 + \tilde{\phi}, \quad \tilde{\phi} \perp \phi_0, \quad \alpha \in \mathbb{R}^+$$

and the eigenvalue equation it satisfies as

$$\begin{pmatrix} 5\pi^2 + U_{00}^\ell - E_0^{U^\ell} & U_{0+}^\ell \\ U_{+0}^\ell & H_+ + U_{++}^\ell - E_0^{U^\ell} \end{pmatrix} \begin{pmatrix} \alpha \\ \tilde{\phi} \end{pmatrix} = 0. \quad (6.52)$$

where the terms in the matrix are defined in (6.4).

Thus, to prove (6.51) it suffices to prove that

$$\|\tilde{\phi}\|_{L^2([0,1]) \wedge L^2([0,1])} \leq C\ell^{-1/2}.$$

By (6.52), as $\phi_0^{U^\ell}$ is normalized, as $10\pi^2 \leq H_+ + U_{++}^\ell$ and as $E_0^{U^\ell} \xrightarrow{\ell \rightarrow +\infty} 5\pi^2$, using (6.4) and (6.8), one computes

$$\|\tilde{\phi}\|_{L^2([0,1]) \wedge L^2([0,1])}^2 \leq U_{0+}^\ell \left(H_+ + U_{++}^\ell - E_0^{U^\ell} \right)^{-2} U_{+0}^\ell \leq \frac{C}{\ell} \langle \phi_\ell, K_\ell(\text{Id} + K_\ell)^{-1} \phi_\ell \rangle_{L^2(\mathbb{R} \times [0,1])}.$$

Thus, (6.51) is an immediate consequence of Lemma 6.1. This completes the proof of Proposition 6.11. \square

We obtain the following corollary for the one particle density matrices of $\zeta^{1,U}$.

Corollary 6.12. *Under assumptions of Proposition 6.11, one has*

$$\|\gamma_{\zeta^{1,U}} - \gamma_{\varphi^1} - \gamma_{\varphi^2}\|_1 = O(\ell^{-1}).$$

Corollary 6.12 is an immediate consequence of (6.50) and

Lemma 6.13. *Let $\psi, \phi \in L^2([0,\ell]) \wedge L^2([0,\ell])$ be two normalized two-particles states. Then*

$$\|\gamma_\psi - \gamma_\phi\|_1 \leq 4\|\psi - \phi\|.$$

Proof of Lemma 6.13. For $\varphi \in L^2([0,\ell]) \wedge L^2([0,\ell])$, consider the operator A_φ defined as

$$(A_\varphi f)(x) = \int_0^\ell \varphi(x,y) f(y) dy.$$

Note that A_φ is a Hilbert-Schmidt operator and $\|A_\varphi\|_2 = \|\varphi\|$ and the one-particle density matrix of φ satisfies $\gamma_\varphi = 2A_\varphi^* A_\varphi$. Thus, for ψ, ϕ as in Lemma 6.13, we obtain

$$\|\gamma_\psi - \gamma_\phi\|_1 = 2\|A_\psi^* A_\psi - A_\phi^* A_\phi\|_1 \leq 2(\|A_\psi^*\|_2 \|A_\psi - A_\phi\|_2 + \|A_\psi^* - A_\phi^*\|_2 \|A_\phi\|_2) \leq 4\|\psi - \phi\|.$$

This completes the proof of Lemma 6.13. \square

6.2. Electrons in distinct pieces. In the present section, we assume that U satisfies **(HU)** (see section 1.1); thus, it decreases sufficiently fast at infinity (roughly better than x^{-4}) and is in L^p for some $p > 1$.

Let the first piece be $\Delta_1 = [-\ell_1, 0]$ and the second be $\Delta_2 = [a, a + \ell_2]$; so, the pieces' lengths are ℓ_1 and ℓ_2 , while the distance between them is denoted by a . As for the one-particle systems living in each of these pieces, we will primarily be interested in the following three cases:

- (a) the interaction of two eigenstates of the one-particle Hamiltonian on each piece, i.e., following the notations of section 6.1, of $\varphi_{\Delta_1}^i$ and $\varphi_{\Delta_2}^j$,
- (b) the interaction of a one-particle eigenstate with a one-particle reduced density matrix of a two-particle ground state, i.e., $\varphi_{\Delta_1}^i$ with $\gamma_{\zeta_{\Delta_2}^1}$,
- (c) the interaction of two one-particle density matrices, i.e., $\gamma_{\zeta_{\Delta_1}^1}$ and $\gamma_{\zeta_{\Delta_2}^1}$.

We observe that for a one-particle eigenstate in a piece of length ℓ , the following uniform pointwise bound holds true:

$$\|\varphi_{[0,\ell]}^i\|_{L^\infty} \leq \sqrt{\frac{2}{\ell}}. \quad (6.53)$$

For the one-particle reduced density matrix we establish the following estimates.

Lemma 6.14. *Let $\zeta \in L^2([0, \ell]) \wedge L^2([0, \ell])$ be a two particle state and $\gamma_\zeta(x, y)$ the kernel of the corresponding one-particle density matrix. Let $p \in \mathbb{N}$. Then, $\zeta \in H^p([0, \ell]^2)$ implies $\gamma_\zeta \in H^p([0, \ell]^2)$ and*

$$\|\gamma_\zeta\|_{H^p} \leq 4\|\zeta\|_{H^p}. \quad (6.54)$$

In particular, unconditionally $\|\gamma_\zeta\|_{L^2} \leq 4$.

Proof. First recall that

$$\gamma_\zeta(x, y) = 2 \int_0^\ell \zeta(x, z)\zeta^*(y, z)dz.$$

Then, one differentiates under the integration sign to get

$$\frac{\partial^p}{\partial x^p} \gamma_\zeta(x, y) = 2 \int_0^\ell \partial_x^p \zeta(x, z)\zeta^*(y, z)dz.$$

This in turn implies by the Cauchy-Schwarz inequality that

$$\begin{aligned} \left\| \frac{\partial^p}{\partial x^p} \gamma_\zeta \right\|_{L^2}^2 &= 4 \int_{[0,\ell]^2} \left| \int_0^\ell \partial_x^p \zeta(x, z)\zeta^*(y, z)dz \right|^2 dx dy \\ &\leq 4 \int_{[0,\ell]^4} |\partial_x^p \zeta(x, z)|^2 \cdot |\zeta(y, z')|^2 dx dy dz dz' = 4 \|\partial_x^p \zeta\|_{L^2}^2, \end{aligned}$$

which proves (6.54). □

Lemma 6.15. *Let $\zeta = \zeta_{[0,\ell]}^{1,U}$ be the ground state of a system of two interacting electrons in $[0, \ell]$. Then, $\zeta \in H^1([0, \ell]^2)$ and there exists a constant $C > 0$ independent of ℓ such that*

$$\|\zeta\|_{H^1} \leq C/\sqrt{\ell}. \quad (6.55)$$

Proof. We use the construction of the proof of Proposition 6.11. Then, employing the same notations, for the problem scaled to the unit square one has

$$\phi_0^{U^\ell} = \alpha\phi_0 + \tilde{\phi},$$

where ϕ_0 is the ground state for a system of two non-interacting electrons, $|\alpha| \leq 1$ and $\tilde{\phi} \perp \phi_0$. Obviously, $\phi_0 \in H^p$ for all $p \in \mathbb{N}$. Moreover, according to (6.52),

$$\begin{aligned} \|\tilde{\phi}\|_{H^1} &= \left\| (H_+ + U_{++}^\ell - E_0^{U^\ell})^{-1} U_{+0}^\ell \alpha \phi_0 \right\|_{H^1} \leq \left\| (H_+ + U_{++}^\ell - E_0^{U^\ell})^{-1} \right\|_{L^2 \rightarrow H^1} \cdot \|U_{+0}^\ell \phi_0\|_{L^2} \\ &\leq \left\| (H_+ - E_0^{U^\ell})^{-1} \right\|_{L^2 \rightarrow H^1} \cdot \|U_{+0}^\ell \phi_0\|_{L^2}. \end{aligned}$$

Arguing as in section 6.1, one can prove that

$$\|U_{+0}^\ell \phi_0\|_{L^2} \leq \|U^\ell \phi_0\|_{L^2} \leq C\sqrt{\ell}$$

and $(H_+ - E_0^{U^\ell})^{-1}$ is a bounded operator from $L^2([0, 1]^2)$ to $H^1([0, 1]^2)$ because H_+ is just a part of $-\Delta_2$ acting in a subspace of functions orthogonal to ϕ_0 and the bottom of its spectrum is separated from $E_0^{U^\ell}$. Thus, we proved that

$$\|\tilde{\phi}\|_{H^1} \leq C\sqrt{\ell}$$

which immediately implies

$$\|\phi_0^{U^\ell}\|_{H^1} \leq C\sqrt{\ell}.$$

Scaling back to the original domain $[0, \ell]^2$ yields (6.55) and completes the proof of Lemma 6.15. \square

Corollary 6.16. *Restricted to the diagonal, the kernel of the ground state one-particle density matrix $x \in [0, \ell] \mapsto \gamma_\zeta(x, x)$ is a bounded function; more precisely, there exists a constant $C > 0$ such that*

$$\|\gamma_\zeta\|_{L^\infty([0, \ell])} \leq C/\ell. \quad (6.56)$$

Proof. Remark first that, as ζ satisfies Dirichlet boundary conditions, so does the kernel $(x, y) \mapsto \gamma_\zeta(x, y)$. Using anti-symmetry, we compute

$$\begin{aligned} |\gamma_\zeta(x, x)| &= 2 \left| \int_0^x \frac{d}{dt} [\gamma_\zeta(t, t)] \right| = 4 \left| \operatorname{Im} \left(\int_0^x \int_0^\ell \partial_t \zeta(t, x) \overline{\zeta(t, x)} dx dt \right) \right| \\ &\leq 4 \|\partial_t \zeta\|_{L^2} \cdot \|\zeta\|_{L^2} \leq 4 \|\zeta\|_{H^1}^2 \end{aligned} \quad (6.57)$$

Combining this with (6.55) gives (6.56) and completes the proof of Corollary 6.16. \square

Having now pointwise bounds (6.53) and (6.56), we estimate the interactions in each of the three cases described in the beginning of the current section. We will also obtain different bounds for close enough and distant pieces $\Delta_1 = [-\ell_1, 0]$ and $\Delta_2 = [a, a + \ell_2]$, i.e., we will discuss different bounds depending on whether a is large or small.

For the case (a) of two interacting one-particle eigenstates we prove the following two estimates. For long distance interactions, i.e., when a is large, we will use

Lemma 6.17. *Suppose U satisfies (HU). Then, for $\Delta_1 = [-\ell_1, 0]$ and $\Delta_2 = [a, a + \ell_2]$, one has*

$$\sup_{i,j} \int_{\Delta_1 \times \Delta_2} U(x-y) |\varphi_{\Delta_1}^i(x)|^2 \cdot |\varphi_{\Delta_2}^j(y)|^2 dx dy \leq \frac{2a^{-3} Z(a)}{\max(\ell_1, \ell_2)} \quad (6.58)$$

where Z is defined in (1.26).

Proof. Let us suppose without loss of generality that Δ_1 is the larger piece, i.e., $\ell_1 \geq \ell_2$. Then, using (6.53) and the fact that the functions $(\varphi_{\Delta_j}^i)_{i,j}$ are normalized, we compute

$$\begin{aligned} \int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) |\varphi_{\Delta_1}^i(x)|^2 \cdot |\varphi_{\Delta_2}^j(y)|^2 dx dy &\leq \frac{2}{\ell_1} \int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) |\varphi_{\Delta_2}^j(y)|^2 dx dy \\ &\leq \frac{2}{\ell_1} \sup_{y \in [0, \ell_2]} \int_0^{\ell_1} U(x+y+a) dx \\ &\leq \frac{2}{\ell_1} \int_0^{+\infty} U(x+a) dx \\ &= \frac{2}{\ell_1} a^{-3} Z(a), \quad a \rightarrow +\infty. \end{aligned}$$

This completes the proof of Lemma 6.17. \square

On the other hand, for close by interactions, i.e., a small and low-lying one-particle energy levels the following lemma gives a more precise estimate.

Lemma 6.18. *Suppose U satisfies (HU). Let $(i, j) \in \{1, 2\}^2$. Then, for any $\varepsilon \in (0, 2)$ and $\Delta_1 = [-\ell_1, 0]$ and $\Delta_2 = [a, a + \ell_2]$, one has*

$$\int_{\Delta_1 \times \Delta_2} U(x-y) |\varphi_{\Delta_1}^i(x)|^2 \cdot |\varphi_{\Delta_2}^j(y)|^2 dx dy = O\left(\frac{a^{-\varepsilon} Z(a)}{\max(\ell_1, \ell_2)^2 \min(\ell_1, \ell_2)^{2-\varepsilon}}\right). \quad (6.59)$$

If $Z(a) = O(a^{-0})$, $a \rightarrow +\infty$, then ε can be taken to zero.

Proof. As in the proof of the previous lemma we suppose that $\ell_1 \geq \ell_2$. If $j \in \{1, 2\}$ then

$$|\varphi_{\Delta_1}^j(x)| = \left| \sqrt{\frac{2}{\ell_1}} \sin\left(\frac{\pi i x}{\ell_1}\right) \right| \leq \sqrt{\frac{2}{\ell_1}} \frac{\pi |x|}{\ell_1} \quad (6.60)$$

and the same type inequality holds for $\varphi_{\Delta_2}^j(y)$. Then, using (6.60) and (6.53), we compute

$$\begin{aligned} \int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) |\varphi_{\Delta_1}^i(x)|^2 \cdot |\varphi_{\Delta_2}^j(y)|^2 dx dy &\leq \frac{C_1}{\ell_1^2 \ell_2^{2-\varepsilon}} \int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) xy^{1-\varepsilon} dx dy \\ &\leq \frac{C_1}{\ell_1^2 \ell_2^{2-\varepsilon}} \int_{\mathbb{R}_+^2} U(x+y+a) xy^{1-\varepsilon} dx dy \\ &= \frac{C_2}{\ell_1^2 \ell_2^{2-\varepsilon}} \int_0^{+\infty} \int_{-s}^s U(s+a)(s+t)(s-t)^{1-\varepsilon} dt ds \\ &\leq \frac{C_3}{\ell_1^2 \ell_2^{2-\varepsilon}} \int_a^{+\infty} U(s) s^{3-\varepsilon} ds. \end{aligned}$$

It is now only left to prove that (HU) and (1.26) imply that the last integral converges and is $O(a^{-\varepsilon} Z(a))$. Therefore, we note that

$$\begin{aligned} \int_a^{+\infty} U(s) s^{3-\varepsilon} ds &= \sum_{n=0}^{+\infty} \int_{2^n a}^{2^{n+1} a} U(s) s^{3-\varepsilon} ds \leq \sum_{n=0}^{+\infty} (2^{n+1} a)^{3-\varepsilon} \int_{2^n a}^{2^{n+1} a} U(s) ds \\ &\leq 2^{3-\varepsilon} a^{-\varepsilon} \sum_{n=0}^{+\infty} 2^{-\varepsilon n} (2^n a)^3 \int_{2^n a}^{+\infty} U(s) ds = 2^{3-\varepsilon} a^{-\varepsilon} \sum_{n=0}^{+\infty} 2^{-\varepsilon n} Z(2^n a) \\ &\leq C a^{-\varepsilon} Z(a). \end{aligned} \quad (6.61)$$

If $Z(a) = O(a^{-0})$, i.e., if there exists $\delta > 0$ s.t. $Z(a) = O(a^{-\delta})$ for $a \rightarrow +\infty$, then, the sum in the second line of (6.61) converges for $\varepsilon = 0$.

This concludes the proof of (6.59). \square

Let us now pass to the case (b) of one-particle eigenstate interacting with a one-particle density matrix of a two-particle eigenstate. For large a , we prove

Lemma 6.19. *Suppose U satisfies (HU). Then, for a sufficiently large, one has*

$$\sup_{i,j} \int_{\Delta_1 \times \Delta_2} U(x-y) |\varphi_{\Delta_1}^i(x)|^2 \cdot \gamma_{\zeta_{\Delta_2}^j}(y,y) dx dy \leq \frac{4a^{-3}Z(a)}{\ell_1}. \quad (6.62)$$

Proof. The proof follows that of Lemma 6.17. The only change concerns the replacement of the fact that $\varphi_{\Delta_2}^j$ is normalized, $\int_{\Delta_2} |\varphi_{\Delta_2}^j(y)|^2 dy = 1$, by the fact that the trace of $\gamma_{\zeta_{\Delta_2}^j}$ is equal to 2. \square

For a small, we prove

Lemma 6.20. *Suppose U satisfies (HU). Let $i \in \{1, 2\}$. Then, for any $\varepsilon \in (0, 2)$,*

$$\int_{\Delta_1 \times \Delta_2} U(x-y) |\varphi_{\Delta_1}^i(x)|^2 \cdot \gamma_{\zeta_{\Delta_2}^j}(y,y) dx dy = O\left(\ell_1^{-3+\varepsilon} \ell_2^{-1/2} a^{-\varepsilon} Z(a)\right). \quad (6.63)$$

If $Z(a) = O(a^{-0})$ as $a \rightarrow +\infty$, one can choose $\varepsilon = 0$.

Proof. As in the proof of Lemma 6.18 mixing once more (6.53), (6.56) and (6.60), we obtain

$$\begin{aligned} \int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) |\varphi_{\Delta_1}^i(x)|^2 \gamma_{\zeta_{\Delta_2}^j}(y,y) dx dy &\leq \frac{C_1}{\ell_1^{3-\varepsilon} \ell_2^{1/2}} \int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) x^{2-\varepsilon} dx dy \\ &\leq \frac{C_1}{\ell_1^{3-\varepsilon} \ell_2^{1/2}} \int_0^{+\infty} \int_a^{+\infty} U(x+y) x^{2-\varepsilon} dx dy \\ &= \frac{C_2}{\ell_1^{3-\varepsilon} \ell_2^{1/2}} \int_a^{+\infty} \int_{-s}^s U(s) (s+t)^{2-\varepsilon} dt ds \\ &\leq \frac{C_3}{\ell_1^{3-\varepsilon} \ell_2^{1/2}} \int_a^{+\infty} U(s) s^{3-\varepsilon} ds \\ &\leq \frac{C_4 a^{-\varepsilon} Z(a)}{\ell_1^{3-\varepsilon} \ell_2^{1/2}}. \end{aligned}$$

This completes the proof of Lemma 6.20. \square

We are left with the case (c) of two interacting reduced density matrices. We do not make the difference between close and far away pieces in this case.

Lemma 6.21. *Suppose U satisfies (HU). Then, there exists $C > 0$ such that*

$$\sup_{i,j} \int_{\Delta_1 \times \Delta_2} U(x-y) \gamma_{\zeta_{\Delta_1}^i}(x,x) \cdot \gamma_{\zeta_{\Delta_2}^j}(y,y) dx dy \leq C \ell_1^{-1/2} \ell_2^{-1/2} \min(1, a^{-2} Z(a)) \quad (6.64)$$

Proof. Using (6.56) one obtains

$$\begin{aligned} \int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) \gamma_{\zeta_{\Delta_1}^i}(x,x) \gamma_{\zeta_{\Delta_2}^j}(y,y) dx dy &\leq \frac{C_1}{\sqrt{\ell_1 \ell_2}} \int_{\mathbb{R}_+^2} U(x+y+a) dx dy \\ &\leq \frac{C_2}{\sqrt{\ell_1 \ell_2}} \int_0^{+\infty} U(s+a) s ds \\ &\leq \frac{C_2}{\sqrt{\ell_1 \ell_2}} \int_a^{+\infty} \left(\int_t^{+\infty} U(s) ds \right) dt \end{aligned}$$

Thus,

$$\int_0^{\ell_1} \int_0^{\ell_2} U(x+y+a) \gamma_{\zeta_{\Delta_1}^i}(x,x) \gamma_{\zeta_{\Delta_2}^j}(y,y) dx dy \leq \frac{C_3 \min(C, a^{-2} Z(a))}{\sqrt{\ell_1 \ell_2}}$$

where the last equality is just (6.61) for $\varepsilon = 2$ and $C := \int_0^{+\infty} \left(\int_t^{+\infty} U(s) ds \right) dt < +\infty$.

This completes the proof of Lemma 6.21. \square

Finally, we give estimates for the case of compactly supported interaction potential U . We prove

Lemma 6.22. *Assume that U has a compact support. Then, there exists $C > 0$ such that, for $i \geq 1$ and $j \geq 1$, one has*

$$\langle U \phi_{(i,j)}, \phi_{(i,j)} \rangle \leq C \cdot \frac{[\min(i, \ell_1) \min(j, \ell_2)]^2}{\ell_1^3 \ell_2^3}.$$

Proof. Due to the anti-symmetry of the functions $(\phi_{(i,j)})_{i,j}$, it suffices to compute the scalar product on $[-\ell_1, 0] \times [a, a + \ell_2]$. Thus,

$$\begin{aligned} \langle U \phi_{(i,j)}, \phi_{(i,j)} \rangle &\leq \sup_{|a| \leq \text{diam}(\text{supp}(U))} \frac{1}{2\ell_1 \ell_2} \int_{[0, \ell_1] \times [0, \ell_2]} U(x+y+a) \\ &\quad \times \sin^2\left(\frac{i\pi x}{\ell_1}\right) \sin^2\left(\frac{j\pi y}{\ell_2}\right) dx dy \\ &\leq C(U) \frac{[\min(i, \ell_1) \min(j, \ell_2)]^2}{\ell_1^3 \ell_2^3} \end{aligned}$$

where

$$C(U) := \frac{1}{2} \sup_{0 \leq a \leq \text{diam}(\text{supp}(U))} \int_{\mathbb{R}^+ \times \mathbb{R}^+} U(x+y+a) (1+x^2)(1+y^2) dx dy.$$

This completes the proof of Lemma 6.22. \square

Proposition 6.23. *Consider a system of two interacting electrons, one in $[0, \ell_1]$, another in $[\ell_1 + r, \ell_1 + r + \ell_2]$ with $r \leq R_0$. Then, the ground state energy of this system has the following asymptotic expansion*

$$E((\ell_1, r, \ell_2), (1, 1)) = \frac{\pi^2}{\ell_1^2} + \frac{\pi^2}{\ell_2^2} + O(\ell_1^{-6} + \ell_2^{-6}). \quad (6.65)$$

Proof. Obviously, the energy of this system is greater than the energy of the system without interactions that is given by the main term of (6.65). Taking the ground state of a non-interacting system as a test function and using Lemma 6.22 to estimate the quadratic form of the interaction potential, gives the upper bound and, thus, completes the proof. \square

6.3. The proof of Lemma 4.11. Recall that $E_{q,n}^U$ denotes the n -th eigenvalue of $-\sum_{l=1}^q \frac{d^2}{dx_l^2} +$

$\sum_{1 \leq k < l \leq q} U(x_k - x_l)$ acting on $\bigwedge_{l=1}^q L^2([0, \ell])$. Rescaling as in section 6.1.1, we need to study the case $\ell = 1$ and prove that, in this case, there exists $C > 1$ such that, for $n \geq 2$ and U^ℓ given by (6.1), one has

$$E_{q,n}^{U^\ell} \geq E_{q,1}^{U^\ell} + \frac{1}{C}. \quad (6.66)$$

Indeed in Lemma 4.11, the length ℓ is assumed to be less than $3\ell_\rho$.

As $q \leq 3$, the same computations as in the beginning of section 6.1.1 show that $E_{q,1}^{U^\ell}$ satisfies, for some $C > 1$, for ℓ large,

$$E_{q,1}^{U^\ell} \leq E_{q,1}^0 + \langle \phi_0, U^\ell \phi_0 \rangle \leq E_{q,1}^0 + \frac{C}{\ell}. \quad (6.67)$$

On the other hand, for some $C > 1$, one has

$$E_{q,n}^{U^\ell} \geq E_{q,n}^0 \geq E_{q,1}^0 + \frac{2}{C}.$$

Plugging (6.67) into this immediately yields (6.66) and completes the proof of Lemma 4.11. \square

APPENDIX A. THE STATISTICS OF THE PIECES

In this appendix, we prove most of the results on the statistics of the pieces stated in section 2.2.

A.1. Facts on the Poisson process. Let Π be the support of $d\mu(\omega)$, the Poisson process of intensity 1 on \mathbb{R}_+ (see section 1). Let $\Pi \cap [0, L] = \{x_i; 1 \leq i \leq m(\omega) - 1\}$ (where $x_i < x_{i+1}$). Then,

$$\mathbb{P}(\#\Pi \cap [0, L] = k) = e^{-L} \frac{L^k}{k!}, \quad k \in \mathbb{N}. \quad (A.1)$$

The following large deviation principle is well known (and easily checked): for any $\beta \in (1/2, 1)$, one has

$$\mathbb{P}(|\#\Pi \cap [0, L] - L| \geq L^\beta) = O(L^{-\infty}). \quad (A.2)$$

The points $(x_i)_{1 \leq i \leq m(\omega) - 1}$ partition the interval $[0, L]$ in $m(\omega)$ pieces of lengths Δ_i .

For $L > e^{e^2}$, one has

$$\begin{aligned} \mathbb{P}(\exists i; |\Delta_i| \geq \log L \log \log L) &\leq \mathbb{P}(\exists n \in [0, L] \cap \mathbb{N}; \#\Pi \cap (n + [0, \log L \log \log L/2]) = 0) \\ &\leq L e^{-\log L \log \log L/2} = O(L^{-\infty}). \end{aligned}$$

This proves Proposition 2.1.

A.2. The proof of Proposition 2.2. Consider the partition of $[0, L]$ into pieces (see section 1). For a, b both non negative, let now $X_{[0,L]}$ to be the number of pieces of length in $[a, a + b]$. We first compute the expectation of $X_{[0,L]}/L$, that is, prove

Proposition A.1. *For $L \geq a + b$, one has*

$$\mathbb{E} \left[\frac{X_{[0,L]}}{L} \right] = e^{-a}(1 - e^{-b}) + \frac{e^{-a}((a+b)e^{-b} - a)}{L} = e^{-a} \left(1 - \frac{a}{L} \right) - e^{-a-b} \left(1 - \frac{a+b}{L} \right).$$

Proof. Let Π be the support of the support of $d\mu(\omega)$, the Poisson process of intensity 1 on \mathbb{R}_+ (see section 1). Then, one has

$$X_{[0,L]} = \sum_{X \in \Pi} G(\Pi \cap [0, X])$$

where the set-functions G is defined as

$$G(\Pi \cap [0, X]) = \begin{cases} 1 & \text{if the distance from } X \text{ to the right most point} \\ & \text{in } \{0\} \cup (\Pi \cap [0, X]) \text{ belongs to } [a, a + b], \\ 0 & \text{if not.} \end{cases} \quad (A.3)$$

The Palm formula (see e.g. [Ber06, Lemma 2.3]) yields

$$\mathbb{E}(X_{[0,L]}) = \int_{0 \leq x \leq L} \mathbb{E}[G(\Pi \cap [0, x])] dx.$$

Now, let \mathcal{E} be an exponential random variable with parameter 1. As the Poisson point process has independent increments, one easily checks that

$$\mathbb{E}[G(\Pi \cap [0, x])] = \mathbb{P}(\min(x, \mathcal{E}) \in [a, a+b]) = \begin{cases} e^{-a}(1-e^{-b}) & \text{if } x \geq a+b, \\ e^{-a} & \text{if } x \in [a, a+b], \\ 0 & \text{if } x \leq a, \end{cases} \quad (\text{A.4})$$

Hence,

$$\mathbb{E}(X_{[0,L]}) = e^{-a}(1-e^{-b}) \int_{0 \leq x \leq L} dx + e^{-a-b} \int_a^{a+b} dx - e^{-a}(1-e^{-b}) \int_0^a dx = e^{-a}(1-e^{-b})L - R$$

where

$$R = e^{-a}((a+b)e^{-b} - a). \quad (\text{A.5})$$

This completes the proof of Proposition A.1. \square

Let us now prove Proposition 2.2. Therefore, set $M := e^{-a}(1-e^{-b})$ and partition $[0, L] = \cup_{j=1}^J [j\ell, (j+1)\ell]$ so that $J \asymp L^\nu$ and $\ell \asymp L^{1-\nu}$ for some $\nu \in (0, 1)$ to be fixed. As $(a, b) = (a_L, b_L) \in [0, \log L \cdot \log \log L]^2$, one then has

$$\left| X_{[0,L]} - \sum_{j=1}^J X_{[j\ell, (j+1)\ell]} \right| \leq 2J. \quad (\text{A.6})$$

Moreover, the random variables $(\ell^{-1} X_{[j\ell, (j+1)\ell]})_{1 \leq j \leq J}$ are independent sub-exponential random variables. Indeed, $X_{[0,L]}$ is clearly bounded by $\#\Pi \cap [0, L]$, the number of points the Poisson process puts in $[0, L]$ and $L^{-1}\#\Pi \cap [0, L]$ has a Poisson law with parameter 1. We want to use the Bernstein inequality (see e.g. [Ver12, Proposition 5.16]). To estimate $\|\ell^{-1} X_{[j\ell, (j+1)\ell]}\|_{\Psi_1}$ (see e.g. [Ver12, Definition 5.13]), we use this bound and the Stirling formula to get, for $p \geq 1$,

$$\begin{aligned} \mathbb{E}(|X_{[j\ell, (j+1)\ell]}|^p) &\leq e^{-\ell} \sum_{k \geq 1} \frac{k^p \ell^k}{k!} \leq e^{-\ell} \sum_{k=1}^{2p-1} \frac{k^p \ell^k}{k!} + e^{-\ell} \sum_{k \geq 2p} \frac{k^p \ell^k}{k!} \\ &\leq (2p)^p + e^{-\ell} \sum_{k \geq 2p} \frac{k^p \ell^p}{k \cdots (k-p+1)} \frac{\ell^{k-p}}{(k-p)!} \\ &\leq (2p)^p + \ell^p \max_{k \geq p} \frac{(k+p)^p k!}{(k+p)!} \leq (2p)^p + (e\ell)^p. \end{aligned}$$

Hence, for $\ell \geq 1$,

$$\begin{aligned} \|\ell^{-1} X_{[j\ell, (j+1)\ell]}\|_{\Psi_1} &= \frac{1}{\ell} \|X_{[j\ell, (j+1)\ell]}\|_{\Psi_1} = \frac{1}{\ell} \sup_{p \geq 1} \frac{1}{p} \sqrt[p]{\mathbb{E}(|X_{[j\ell, (j+1)\ell]}|^p)} \\ &\leq \sup_{p \geq 1} \sqrt[p]{\frac{2^p}{\ell^p} + \frac{e^p}{p^p}} \leq \frac{2}{\ell} + e \leq 2e. \end{aligned}$$

Thus, the Bernstein inequality, estimate (A.6) and Proposition A.2 yield that there exists $\kappa > 0$ (independent of a, b) such that, for $\alpha = \alpha(L) \geq 2(R+2)/\ell$ (here, R is given by (A.5)),

one has

$$\begin{aligned} \mathbb{P} \left(\left| \frac{X_{[0,L]}}{L} - M \right| \geq \alpha \right) &\leq \mathbb{P} \left(\left| \sum_{j=1}^J \frac{X_{[j\ell, (j+1)\ell]} - \mathbb{E}[X_{[j\ell, (j+1)\ell]}]}{\ell} \right| \geq J \left(\alpha - \frac{R+2}{\ell} \right) \right) \\ &\leq 2e^{-\kappa\alpha^2 J}. \end{aligned}$$

To obtain Proposition 2.3, it now suffices to take $\alpha = L^{\beta-1}$ and $(\beta, \nu) \in (0, 1)$ such that $1 - \beta < 1 - \nu$ and $2(\beta - 1) + \nu > 0$; this requires $\beta > 2/3$.

The proof of Proposition 2.2 is complete. \square

A.3. The proof of Propositions 2.3 and 2.4. For any a, b, c, d, f, g all non negative, define now $X_{[0,L]}$ to be the number of pairs of pieces such that

- the length of the left most piece is contained in $[a, a + b]$,
- the length of the right most piece is contained in $[c, c + d]$,
- the distance between the two pieces belongs to $[g, g + f]$.

Again, we first compute the expectation of $X_{[0,L]}/L$, that is, prove

Proposition A.2. *For $L \geq a + b + c + d + f + g$, one has*

$$\mathbb{E} \left[\frac{X_{[0,L]}}{L} \right] = f e^{-a-c} (1 - e^{-b})(1 - e^{-d}) + \frac{R_L}{L} \quad \text{where } |R_L| \leq f e^{-a-c}. \quad (\text{A.7})$$

Proof. Recall that Π denotes the support of the support of $d\mu(\omega)$, the Poisson process of intensity 1 on \mathbb{R}_+ . Then, one can rewrite

$$X_{[0,L]} = \sum_{\substack{(X,Y) \in \Pi^2 \\ X < Y}} \mathbf{1}_{g \leq Y - X \leq g+f} G(\Pi \cap [0, X]) H(\Pi \cap (Y, L])$$

where the set-functions G and H have been defined respectively by (A.3) and

$$H(\Pi \cap (Y, L]) = \begin{cases} 1 & \text{if the distance from } Y \text{ to the left most point} \\ & \text{in } \{L\} \cup (\Pi \cap (Y, L]) \text{ belongs to } [c, c + d], \\ 0 & \text{if not.} \end{cases} \quad (\text{A.8})$$

The Palm formula, thus, yields

$$\begin{aligned} \mathbb{E}(X_{[0,L]}) &= \int_{\substack{0 \leq x, y \leq L \\ g \leq y - x \leq g+f}} \mathbb{E} [G(\Pi \cap [0, x]) H(\Pi \cap (y, L])] dx dy \\ &= \int_{\substack{0 \leq x, y \leq L \\ g \leq y - x \leq g+f}} \mathbb{E} [G(\Pi \cap [0, x])] \mathbb{E} [H(\Pi \cap (y, L])] dx dy \end{aligned}$$

as the random sets $\Pi \cap [0, x)$ and $\Pi \cap (y, L]$ are independent.

As in (A.4), one checks that

$$\mathbb{E} [H(\Pi \cap (y, L])] = \mathbb{P}(\min(L - y, \mathcal{E}) \in [c, c + d]) = \begin{cases} e^{-c} (1 - e^{-d}) & \text{if } y \leq L - c - d, \\ e^{-c} & \text{if } y \in L - [c, c + d], \\ 0 & \text{if } y \geq L - c. \end{cases}$$

Hence,

$$\begin{aligned} \mathbb{E}(X_{[0,L]}) &= e^{-a-c} (1 - e^{-d}) (1 - e^{-b}) \int_{\substack{0 \leq x, y \leq L \\ g \leq y - x \leq g+f}} dx dy + R_1 \\ &= f e^{-a-c} (1 - e^{-b})(1 - e^{-d}) L + R_2 \end{aligned}$$

where, respectively, $R_1 \leq e^{-a-c}$ and

$$R_2 \leq R := e^{-a-c}(1 + f^2 + fg). \quad (\text{A.9})$$

This completes the proof of Proposition A.2. \square

Let us now prove Proposition 2.3. We want to go along the same lines as in the proof of Proposition 2.2. Therefore, we set $M := f e^{-a-c}(1 - e^{-b})(1 - e^{-d})$ and partition $[0, L] = \cup_{j=0}^J [j\ell, (j+1)\ell]$ so that $J \asymp L^\nu$ and $\ell \asymp L^{1-\nu}$ for some $\nu \in (0, 1)$ to be fixed. For the same reasons as before, the random variables $(\ell^{-1} X_{[j\ell, (j+1)\ell]})_{1 \leq j \leq J}$ are independent sub-exponential random variables.

We now need a replacement for (A.6). Therefore, we set

$$r := 1 + a + b + c + d + f + g \quad (\text{A.10})$$

and, for $0 \leq j \leq J$, we let

- Y_j be the number of pieces in the interval $(j+1)\ell + [-r, 0]$ of length in $[a, a+b]$,
- Z_j be the number of pieces in the interval $j\ell + [0, r]$ of length in $[c, c+d]$.

Then, we have

$$-K_a \sum_{j=0}^J Y_j - K_c \sum_{j=0}^J Z_j \leq X_{[0, L]} - \sum_{j=0}^J X_{[j\ell, (j+1)\ell]} \leq K_a \sum_{j=0}^J Y_j + K_c \sum_{j=0}^J Z_j \quad (\text{A.11})$$

where we have set

$$K_a := 1 + \frac{f+g}{a} \quad \text{and} \quad K_c = 1 + \frac{f+g}{c}. \quad (\text{A.12})$$

Indeed, if a pair of pieces counted by $X_{[0, L]}$ does not have any of its intervals in any of the $(j\ell + [-r, r])_{1 \leq j \leq J}$, then the convex closure of the pair is inside some $j\ell + [0, \ell]$, thus, the pair is counted by $X_{[j\ell, (j+1)\ell]}$. This yields the upper bound in (A.11) as, any given interval is the left (resp. right) most interval for at most $1 + (f+g)/c$ (resp. $1 + (f+g)/a$) pairs satisfying both the requirements on lengths and distance. The lower bound is obtained in the same way.

For L sufficiently large, the random variables $(Y_j)_{1 \leq j \leq J}$ and $(Z_j)_{1 \leq j \leq J}$ are i.i.d. sub-exponential. Thus, applying the Bernstein inequality as in the proof of Proposition 2.2 yields that, for some constant $\kappa > 0$ (independent of (a, b, c, d, f, g)) and $\beta \in (2/3, 1)$, with probability $1 - O(J^{-\infty}) = 1 - O(L^{-\infty})$, one has

$$\sum_{j=1}^J Y_j \leq \kappa J(e^{-a} + J^{\beta-1})r \quad \text{and} \quad \sum_{j=0}^{J-1} Z_j \leq \kappa J(e^{-c} + J^{\beta-1})r; \quad (\text{A.13})$$

Now, we can estimate $\|\ell^{-1} X_{[j\ell, (j+1)\ell]}\|_{\Psi_1}$ as in the proof of Proposition 2.2. Thus, the Bernstein inequality and Proposition A.2 yield that, for some κ (independent of (a, b, c, d, f, g)), for $\nu \in (2/3, 1)$ and $\ell \asymp L^{1-\nu}$, with probability $1 - O(L^{-\infty})$, one has

$$\left| \sum_{j=0}^J \frac{X_{[j\ell, (j+1)\ell]}}{\ell} - JM \right| \leq \kappa \frac{R_L J}{\ell}.$$

Taking (A.11) and (A.13) into account, we get that, for some $\kappa > 0$ (independent of (a, b, c, d, f, g)), with probability $1 - O(L^{-\infty})$, one has

$$\left| \frac{X_{[0, L]}}{L} - M \right| \leq \kappa \frac{R + (K_a e^{-a} + K_c e^{-c} + (K_a + K_c) J^{\beta-1})r}{\ell}.$$

This proves (2.5) where the constants are given by

$$R(a, b, c, d, f, g) = \kappa r (R + K_a e^{-a} + K_c e^{-c}) \quad \text{and} \quad K(a, c, f, g) = (K_a + K_c)r \quad (\text{A.14})$$

(see (A.9), (A.10) and (A.12).)

The proof of Proposition 2.3 is complete. \square

The proof of Proposition 2.4 is identical to that of Proposition 2.3: it suffices to take $b = d = +\infty$.

A.4. The proofs of Proposition 2.5. This proof is essentially identical to that of Proposition 2.3. Let us just say a word about the differences.

For $\ell, \ell', \ell'', d > 0$, let now $X_{[0,L]}$ to be the number triplets of pieces at most at a distance d from each other such that

- the left most piece longer than ℓ ,
- the middle piece longer than ℓ' ,
- the right most piece longer than ℓ'' .

Then, one has

$$X_{[0,L]} = \sum_{\substack{(X,Y,W,Z) \in \Pi^4 \\ X < Y < W < Z}} \mathbf{1}_{\substack{0 < Y-X \leq d \\ \ell' \leq W-Y \\ 0 < Z-W \leq d}} G(\Pi \cap [0, X)) K(\Pi \cap (Y, W)) H(\Pi \cap (Z, L])$$

where the set-functions G and H have been defined as

$$\begin{aligned} G(\Pi \cap [0, X)) &= \begin{cases} 1 & \text{if the distance from } X \text{ to the right most point} \\ & \text{in } \{0\} \cup (\Pi \cap [0, X)) \text{ belongs to } [l, +\infty), \\ 0 & \text{if not,} \end{cases} \\ K(\Pi \cap (Y, W)) &= \begin{cases} 1 & \text{if } \Pi \cap (Y, W) = \emptyset, \\ 0 & \text{if not,} \end{cases} \\ H(\Pi \cap (Z, L]) &= \begin{cases} 1 & \text{if the distance from } Z \text{ to the left most point} \\ & \text{in } \{L\} \cup (\Pi \cap (Z, L]) \text{ belongs to } [l'', +\infty), \\ 0 & \text{if not.} \end{cases} \end{aligned}$$

Following the proof of Proposition A.2, one proves

Proposition A.3. *For L sufficiently large, one has*

$$\mathbb{E} \left[\frac{X_{[0,L]}}{L} \right] = d^2 e^{-\ell-\ell'-\ell''} + \frac{R_L}{L} \quad \text{where } |R_L| \leq d^2 e^{-\ell-\ell'-\ell''}.$$

One then derives Proposition 2.2 from Proposition A.3 in the same way as Proposition 2.3 was derived from Proposition A.2.

A.5. Proof of Proposition 2.6. First of all, let us note that a piece of length l in $[k\ell_E, (k+1)\ell_E)$ generates exactly k energy levels that do not exceed E . To count the energies less than E , we are only interested in intervals of length l larger than ℓ_E . Other intervals do not generate any energy levels we are interested in. Thus, by Proposition 2.2, for $\beta \in (2/3, 1)$, we obtain that with probability $1 - O(L^{-\infty})$, the number of intervals generating k energy levels below energy E is

$$L(e^{-k\ell_E} - e^{-(k+1)\ell_E}) + L^\beta R_L \quad \text{where } |R_L| \leq 3 \tag{A.15}$$

where $O(\cdot)$ is uniform in k .

Let $m_L = \log L \cdot \log \log L$. By Proposition 2.1, with probability $1 - O(L^{-\infty})$, for L large,

one computes

$$\begin{aligned}
N_L^D(E) &= L^{-1} \sum_{k=1}^{[m_L/\ell_E]} k \cdot L(e^{-k\ell_E} - e^{-(k+1)\ell_E}) + m_L L^{-1+\beta} R_L \quad \text{where} \quad |R_L| \leq \frac{1}{\ell_E} \\
&= \sum_{k=1}^{[m_L/\ell_E]} e^{-k\ell_E} - \frac{[m_L/\ell_E]}{e^{([m_L/\ell_E]+1)\ell_E}} + m_L L^{-1+\beta} R_L \\
&= \sum_{k=1}^{+\infty} e^{-k\ell_E} + m_L L^{-1+\beta} (R_L + 1) = \frac{e^{-\ell_E}}{1 - e^{-\ell_E}} + m_L L^{-1+\beta} (R_L + 2).
\end{aligned}$$

Thus, decreasing β above somewhat, with probability $1 - O(L^{-\infty})$, for L sufficiently large, one has

$$\left| N_L^D(E) - \frac{e^{-\ell_E}}{1 - e^{-\ell_E}} \right| \leq L^{-1+\beta}. \quad (\text{A.16})$$

This proves (2.6). Using the fact that $E \mapsto N_L^D(E)$ is monotonous and the Lipschitz continuity of $E \mapsto N(E)$, (A.16) yields that, for $E_0 > 0$, with probability $1 - O(L^{-\infty})$, for L sufficiently large, one has

$$\sup_{E \in [0, E_0]} \left| N_L^D(E) - \frac{e^{-\ell_E}}{1 - e^{-\ell_E}} \right| \leq L^{-1+\beta}. \quad (\text{A.17})$$

The formulas (2.8) and (2.9) for the Fermi energy and the Fermi length follow trivially. This completes the proof of Proposition 2.6.

APPENDIX B. A SIMPLE LEMMA ON TRACE CLASS OPERATORS

The purpose of the present section is to prove

Lemma B.1. *Pick $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ a separable Hilbert space and (Z, μ) a measured space with μ a positive measure. Consider a weakly measurable mapping $z \in Z \rightarrow T(z) \in \mathfrak{S}_1(\mathcal{H})$. Here, $\mathfrak{S}_1(\mathcal{H})$ denotes the trace class operators in \mathcal{H} , the trace class norm being denoted by $\|\cdot\|_{tr}$. Assume*

$$\int_Z \|T(z)\|_{tr} d\mu(z) < +\infty. \quad (\text{B.1})$$

Then, the integral $T := \int_Z T(z) d\mu(z)$ converges weakly and defines a trace class operator that satisfies

$$\|T\|_{tr} = \left\| \int_Z T(z) d\mu(z) \right\|_{tr} \leq \int_Z \|T(z)\|_{tr} d\mu(z). \quad (\text{B.2})$$

Proof. By assumption, for $(\varphi, \psi) \in \mathcal{H}^2$, one has $z \rightarrow \langle T(z)\varphi, \psi \rangle$ is measurable and bounded by $z \rightarrow \|T(z)\|_{tr} \|\varphi\| \|\psi\|$ which by (B.1) is integrable. It, thus, is integrable and one has

$$\left| \int_Z \langle T(z)\varphi, \psi \rangle d\mu(z) \right| \leq \int_Z |\langle T(z)\varphi, \psi \rangle| d\mu(z) \leq \int_Z \|T(z)\|_{tr} d\mu(z) \|\varphi\| \|\psi\|.$$

Thus, the operator $T := \int_Z T(z) d\mu(z)$ is well defined by

$$\langle T\varphi, \psi \rangle := \int_Z \langle T(z)\varphi, \psi \rangle d\mu(z).$$

and bounded.

Let us prove that it is trace class and satisfies (B.2). Let $(\varphi_n)_{n \geq 1}$ be an orthonormal basis of \mathcal{H} . Then,

$$|\langle T\varphi_n, \varphi_n \rangle| \leq \int_Z |\langle T(z)\varphi_n, \varphi_n \rangle| d\mu(z).$$

Thus,

$$\sum_{n=1}^N |\langle T\varphi_n, \varphi_n \rangle| \leq \int_Z \left(\sum_{n=1}^N |\langle T(z)\varphi_n, \varphi_n \rangle| \right) d\mu(z) \leq \int_Z \|T(z)\|_{\text{tr}} d\mu(z).$$

Taking $N \rightarrow +\infty$ proves that, for any orthonormal basis of \mathcal{H} , say, $(\varphi_n)_{n \geq 1}$, one has

$$\sum_{n=1}^{+\infty} |\langle T\varphi_n, \varphi_n \rangle| \leq \int_Z \|T(z)\|_{\text{tr}} d\mu(z) < +\infty.$$

Thus, T is trace class (see e.g. [RS80]) and satisfies (B.2). This completes the proof of Lemma B.1. \square

APPENDIX C. ANTI-SYMMETRIC TENSORS: THE PROJECTOR ON ANTI-SYMMETRIC FUNCTIONS

Pick $\Psi \in L^2(\Lambda^n)$ and let $\Pi_n^\wedge : L^2(\Lambda^n) \rightarrow \bigwedge^n L^2(\Lambda)$ be the orthogonal projector on totally anti-symmetric function. Then,

$$(\Pi_n^\wedge \Psi)(x) = \frac{1}{n!} \sum_{\substack{\sigma \text{ permutation} \\ \text{of } \{1, \dots, n\}}} \text{sgn } \sigma \cdot \Psi(\sigma x)$$

where, for $x = (x_1, \dots, x_n)$, $\sigma x = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ and $\text{sgn } \sigma$ is the signature of the permutation σ .

Hence, if $n = Q_1 + \dots + Q_m$ and, for $1 \leq j \leq m$, $\varphi_j \in \bigwedge^{Q_j} L^2(\Delta_j)$, we define

$$\left(\prod_{j=1}^m \|\varphi_j\| \right)^{-1} \bigwedge_{j=1}^m \varphi_j := \left\| \Pi_n^\wedge \left(\bigotimes_{j=1}^m \varphi_j \right) \right\|^{-1} \Pi_n^\wedge \left(\bigotimes_{j=1}^m \varphi_j \right) \quad (\text{C.1})$$

and compute

$$\begin{aligned} \Pi_n^\wedge \left(\bigotimes_{j=1}^m \varphi_j \right) &= \frac{1}{n!} \sum_{\substack{\sigma \text{ permutation} \\ \text{of } \{1, \dots, n\}}} \text{sgn } \sigma \left(\bigotimes_{j=1}^m \varphi_j \right) (\sigma x) \\ &= \frac{1}{n!} \sum_{\substack{\sigma \text{ permutation} \\ \text{of } \{1, \dots, n\}}} \text{sgn } \sigma \left(\prod_{j=1}^m \varphi_j(x_{\sigma(Q_j)}) \right) \end{aligned}$$

where

$$\begin{aligned} x_{\sigma(Q_j)} &= (x_{\sigma(Q_1 + \dots + Q_{j-1} + 1)}, \dots, x_{\sigma(Q_1 + \dots + Q_{j-1} + Q_j)}), \\ \mathcal{Q}_j &= \{Q_1 + \dots + Q_{j-1} + 1, \dots, Q_1 + \dots + Q_{j-1} + Q_j\}. \end{aligned}$$

Thus,

$$\begin{aligned}
n! \cdot \Pi_n^\wedge \left(\bigotimes_{j=1}^m \varphi^j \right) &= \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \sum_{\substack{\sigma \text{ permutation} \\ \text{of } \{1, \dots, n\} \\ \text{s.t. } \forall j, \sigma(Q_j) = A_j}} \operatorname{sgn} \sigma \left(\prod_{j=1}^m \varphi^j(x_{\sigma(Q_j)}) \right) \\
&= \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \left(\sum_{\substack{\sigma \text{ permutation} \\ \text{of } \{1, \dots, n\} \\ \text{s.t. } \forall j, \sigma(Q_j) = A_j}} \left(\operatorname{sgn} \sigma \prod_{j=1}^m \operatorname{sgn} \sigma|_{Q_j} \right) \left(\prod_{j=1}^m \varphi^j(x_{A_j}) \right) \right) \\
&= \prod_{j=1}^m Q_j! \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \varepsilon(A_1, \dots, A_m) \left(\prod_{j=1}^m \varphi^j(x_{A_j}) \right)
\end{aligned}$$

where we recall that $\varepsilon(A_1, \dots, A_m)$ is the signature of $\sigma(A_1, \dots, A_m)$ the unique permutation of $\{1, \dots, n\}$ such that, if $A_j = \{a_{ij}; 1 \leq i \leq Q_j, a_{i_1 j} < a_{i_2 j} \text{ for } i_1 < i_2\}$ for $1 \leq j \leq m$ then $\sigma(a_{ij}) = Q_1 + \dots + Q_{j-1} + i$.

As $\Delta_j \cap \Delta_k = \emptyset$ if $j \neq k$, one has

$$\begin{aligned}
\left\| \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \varepsilon(A_1, \dots, A_m) \left(\prod_{j=1}^m \varphi^j(x_{A_j}) \right) \right\|^2 &= \prod_{j=1}^m \|\varphi^j\|^2 \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} 1 \\
&= \frac{n!}{\prod_{j=1}^m Q_j!} \prod_{j=1}^m \|\varphi^j\|^2.
\end{aligned}$$

Hence, by (C.1), we get

$$\left(\bigwedge_{j=1}^m \varphi^j \right) (x) = \sqrt{\frac{\prod_{j=1}^m Q_j!}{n!}} \sum_{\substack{|A_j|=Q_j, \forall 1 \leq j \leq m \\ A_1 \cup \dots \cup A_m = \{1, \dots, n\} \\ A_j \cap A_{j'} = \emptyset \text{ if } j \neq j'}} \varepsilon(A_1, \dots, A_m) \left(\prod_{j=1}^m \varphi^j(x_{A_j}) \right). \quad (\text{C.2})$$

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