



HAL
open science

More on Krichever-Novokov Algebras

Friedrich Wagemann

► **To cite this version:**

Friedrich Wagemann. More on Krichever-Novokov Algebras. *Africa Matematica*, 2012, 23 (1), pp.41-52. 10.1007/s13370-011-0017-y . hal-01057771

HAL Id: hal-01057771

<https://hal.science/hal-01057771>

Submitted on 25 Aug 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

More Remarks on the Cohomology of Krichever-Novikov Algebras

Friedrich Wagemann

Received: date / Accepted: date

AMS classification:

primary: 17B56, 46E25, 32J25, 14C30

secondary: 17B66, 17B68, 14F05, 32J35, 32L10, 41A30

Abstract This article is a continuation of previous work on the cohomology of Krichever-Novikov algebras [18]. We extend here the main result of [18] to the non-algebraically trivialisable case and generalize it to higher dimensions. We note some consequences and examples, and survey related topics.

Introduction

0.1 In *string theory*, elementary objects are strings evolving in space, sweeping out real surfaces. Two dimensional *conformal field theory* describes the physics of such strings by an action with conformal symmetry, leading to the realm of complex geometry. The symmetry Lie algebra admits a central extension (corresponding to an anomaly on quantum level), called the *Virasoro algebra*. It is a central extension of the Lie algebra of vector fields W (like *Witt algebra*) whose coefficient functions are Laurent series $\mathbb{C}[z, z^{-1}]$. Questions about the existence and uniqueness of central extensions can be answered studying the cohomology of the given Lie algebra.

The *local* point of view of interpreting the apparition of W in conformal field theory is as the Lie algebra of symmetry of the circle (string). The *global* point of view looks at it as the Lie algebra of vector fields on the punctured Riemann surface $\mathbb{CP}^1 \setminus \{0, \infty\}$. It is in this global interpretation that the Lie algebra (and also its central extension) is called a *Krichever-Novikov algebra*. This point of view is easily generalized to other

Friedrich Wagemann
Laboratoire de Mathématiques
Faculté des Sciences et Techniques
Université de Nantes
2, rue de la Houssinière
44322 Nantes cedex 3
France
tel.: ++33.2.51.12.59.57
E-mail: wagemann@math.univ-nantes.fr

compact Riemann surfaces instead of \mathbb{CP}^1 and more points. After seminal papers of Krichever-Novikov [8] [9], Schlichenmaier and Schlichenmaier-Sheinman [14] [15] (more references can be found in the latter articles) studied the cases of higher genus and more punctures. Results in this direction include a Sugawara construction, a construction of the sheaf of conformal blocks on moduli space and of a flat projective connection. A nice introduction to these matters can be found in [13]. In the present article, our goal is to study the generalization of this approach to conformal field theory to higher dimensions, the first step being obviously the study of the Lie algebras of meromorphic vector fields on complex manifolds X of dimension $\dim X > 1$. It follows a more technical introduction.

0.2 Let Σ be a compact connected Riemann surface of genus g , or alternatively a complex smooth projective algebraic curve. Fix some distinct points $\{p_1, \dots, p_k\}$ on Σ , and denote by Σ_k the open Riemann surface or affine algebraic curve $\Sigma \setminus \{p_1, \dots, p_k\}$.

The holomorphic tangent bundle of Σ_k is trivial, because on an open Riemann surface, all holomorphic vector bundles are holomorphically trivial. The algebraic tangent bundle on the affine curve Σ_k is in general (even generically) not trivial.

Let $Mer_k(\Sigma)$ denote the space of meromorphic vector fields on Σ , having possibly poles of an arbitrary order in $\{p_1, \dots, p_k\}$. Let $Hol(\Sigma_k)$ be the space of holomorphic vector fields on Σ_k . We regard $Hol(\Sigma_k)$ as space of vector fields having possibly poles of arbitrary order or even essential singularities in $\{p_1, \dots, p_k\}$. As Σ_k is holomorphically trivializable, $Hol(\Sigma_k)$ can be holomorphically identified with the space of holomorphic functions on Σ_k , but $Mer_k(\Sigma)$ cannot in general be identified with the subspace of meromorphic functions.

Being a space of sections of a coherent sheaf on a Stein manifold, $Hol(\Sigma_k)$ carries a canonical Fréchet, i.e. locally convex, complete, metrizable topology. This topology can be identified with the topology of uniform convergence on compact sets by identification of holomorphic fields and holomorphic functions. The subspace $Mer_k(\Sigma)$ can be equipped with the induced topology, and is certainly a locally convex metrizable topological vector space. It is not complete.

The main goal of the present article is to show that $Mer_k(\Sigma)$ is dense in $Hol(\Sigma_k)$ and to study its consequences.

Theorem 1 *$Mer_k(\Sigma)$ is dense in $Hol(\Sigma_k)$ in its induced topology.*

0.3 In case $Mer_k(\Sigma)$ can be identified with the subspace of meromorphic functions in $Hol(\Sigma_k)$ (this occurs for some special configurations of points $\{p_1, \dots, p_k\}$, namely for the associated divisor being a multiple of the divisor of the tangent line bundle), the proof of the Theorem is fairly easy. The density result for functions instead of vector fields is known since work of Behnke and Stein in 1943 [1] Satz 13, see e.g. [5] p. 245 or [18] for a modern account. Whereas in [18] the special case of an algebraically trivializable Σ_k is treated, the present article treats the general case.

As a corollary of this Theorem, the continuous cohomology of the two Lie algebras are isomorphic. This establishes a weak (because continuous) version of the Feigin-Novikov conjecture stating that $Mer_k(\Sigma)$ has $H_{\text{sing}}^*(Map(\Sigma_k, S^3))$ as its algebraic cohomology. Here, $Map(\Sigma_k, S^3)$ is the topological space of continuous maps from Σ_k to the 3-sphere S^3 in the compact-open topology and $H_{\text{sing}}^*(Map(\Sigma_k, S^3))$ is its singular cohomology. It is known since work of Kawazumi [7] that $Hol(\Sigma_k)$ has $H_{\text{sing}}^*(Map(\Sigma_k, S^3))$ as its continuous cohomology.

0.4 Furthermore, we generalize the above Theorem in three directions: inspection of the proof shows that the Theorem generalizes to manifolds X where the divisor,

given by a union of codimension one subvarieties Y , where the meromorphic fields are supposed to have poles is ample, the complex dimension 1 of Riemann surfaces being inessential. But in dimension $n > 1$, the Lie algebra of holomorphic vector fields on a Stein manifold $X \setminus Y$ does not have any non-trivial (continuous) central extensions cf. [19], so this also holds for the Lie algebra of meromorphic vector fields on X with possible poles along Y .

Also, further inspection of the proof shows that any holomorphic vector bundle having a short exact sequence like the one linking tangent and normal bundle of a subvariety of \mathbb{C}^N can be treated in exactly the same way. Furthermore, the algebraic structure (here Lie algebra structure for the tangent sheaves) does not play a rôle: the cohomological isomorphy result about dense subobjects applies equally well to continuous Hochschild, continuous cyclic or any other continuous cohomology.

For example, as continuous cyclic cohomologies of the algebra A of holomorphic functions on $X \setminus Y$ and meromorphic functions on X having possibly poles on Y coincide, one is led to conjecture that the holomorphic current algebra $A \otimes \mathfrak{g}$ where \mathfrak{g} is a simple, finite dimensional Lie algebra, should have an universal central extension by $\Omega^{\text{an},1}(X \setminus Y)/\overline{dA}$ (where $\Omega^{\text{an},1}(X \setminus Y)$ denotes the Fréchet space of holomorphic 1-forms on $X \setminus Y$) while the case of the meromorphic current algebra reduces to the algebraic case. This is indeed the case as discussed in work with K.-H. Neeb [12].

Acknowledgements The author thanks K. H. Neeb, S. Nemirovski, O. Mathieu, L. Menichi, A. Reiman and C. Sorger for discussion at different levels of progression of the present article.

Notations - conventions

We will benefit from viewing an algebraic variety X over the complex numbers in some situations as a complex analytic variety X^{an} , i.e. instead of considering the Zariski topology on the set X , we will consider the complex topology. Starting from these two topologies, one can associate to X many interesting geometrical tools both in the algebraic and in the analytic category. As all our algebraic varieties will be smooth, X^{an} will be a complex manifold in the usual sense. This change of point of view is sometimes called transcendental methods or GAGA theory after Serre's famous paper entitled "Géométrie algébrique et géométrie analytique".

We will adopt the following notations in order to emphasize these different points of view: for an algebraic variety X over the complex numbers, we will denote by \mathcal{O}_X the algebraic structure sheaf, $\mathcal{O}_X(X)$ its space of global sections on X , and more generally \mathcal{F} a coherent sheaf of \mathcal{O}_X -modules. For example, Ω_X^1 will denote the sheaf of regular (i.e. Kähler) 1-forms.

Concerning the corresponding complex analytic variety X^{an} , we will denote by $\mathcal{O}_X^{\text{an}}$ the structure sheaf of the complex analytic variety, i.e. the sheaf of holomorphic functions on X . More generally, \mathcal{F}^{an} will be a coherent sheaf of $\mathcal{O}_X^{\text{an}}$ -modules. For example, Ω_X^{an} will denote the sheaf of holomorphic 1-forms on X .

We will feel free to suppress the superscript "an" for the manifolds, i.e. for example to write Σ (and $\Sigma_k := \Sigma \setminus \{p_1, \dots, p_k\}$) instead of Σ^{an} (resp. Σ_k^{an}) for the projective complex curve Σ viewed as a compact Riemann surface (resp. the affine complex curve Σ_k viewed as an open Riemann surface).

The other exception from our convention concerns the tangent sheaf viewed as a sheaf of (Lie algebras of) vector fields: along with \mathcal{T}_X , we will also denote the algebraic tangent sheaf of X as Mer in case X is an affine variety obtained from a projective

variety by extracting some subvariety. Furthermore, along with $\mathcal{T}_X^{\text{an}}$, we will also denote the complex analytic tangent sheaf of X^{an} as Hol .

1 Continuous Lie algebra cohomology

In this Section, we show the fact that a dense Lie subalgebra of a topological Lie algebra has isomorphic continuous cohomology. The following notions are understood to be linear over the reals or the complex numbers.

1.1 Let \mathfrak{g} be a *topological Fréchet nuclear Lie algebra*. This means that the vector space underlying the Lie algebra structure of \mathfrak{g} carries a topology (in which the bracket is continuous), and that with respect to the latter it is a topological vector space which is locally convex, metrizable and complete (the last three properties constitute what we call *Fréchet*).

Furthermore, the topological vector space underlying \mathfrak{g} is supposed to be *nuclear* which means that two rather natural topologies on the algebraic tensor of \mathfrak{g} with another separated locally convex space (namely the ϵ and the π topology) coincide. In this sense, the algebraic tensor powers of \mathfrak{g} carry a canonical topology. We write $\mathfrak{g} \otimes \mathfrak{g}$ for the algebraic tensor product and for the topological tensor product (using the canonical topology); the context decides which one we consider. We denote $\mathfrak{g} \hat{\otimes} \mathfrak{g}$ the completion of $\mathfrak{g} \otimes \mathfrak{g}$ (with respect to the canonical topology).

The space of continuous p -cochains $C^p(\mathfrak{g})$ of \mathfrak{g} (with values in the trivial \mathfrak{g} -module) is the continuous dual $(\Lambda^p(\mathfrak{g}))^{*\text{cont}}$ of the p th exterior power $\Lambda^p(\mathfrak{g})$ of \mathfrak{g} . $\Lambda^p(\mathfrak{g})$ is a closed subspace of $\underbrace{\mathfrak{g} \hat{\otimes} \dots \hat{\otimes} \mathfrak{g}}_{p\text{-times}}$ and carries the subspace topology.

For $c \in C^p(\mathfrak{g})$, we define the usual coboundary operator

$$dc(\xi_1, \dots, \xi_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} c([\xi_i, \xi_j], \xi_1, \dots, \hat{\xi}_i, \dots, \hat{\xi}_j, \dots, \xi_{p+1}),$$

where $\hat{\xi}_i$ means that ξ_i is omitted. d is surely continuous, but has no reason to be a closed operator.

Finally, the p th *continuous cohomology space* of \mathfrak{g} is defined as usual to be

$$H^p(\mathfrak{g}) = \frac{\ker d : C^p(\mathfrak{g}) \rightarrow C^{p+1}(\mathfrak{g})}{\text{im } d : C^{p-1}(\mathfrak{g}) \rightarrow C^p(\mathfrak{g})}.$$

As the image of d is in general not closed, the quotient topology on $H^p(\mathfrak{g})$ is in general not separated and $H^p(\mathfrak{g})$ is here regarded only as a vector space.

1.2 Now let $\mathfrak{h} \hookrightarrow \mathfrak{g}$ be a Lie subalgebra. We look at \mathfrak{h} as a topological Lie algebra with its subspace topology. Thus, \mathfrak{h} is a metrizable, locally convex, nuclear topological Lie algebra, but in general not complete.

As \mathfrak{h} is not complete, we change slightly the definition of continuous cochains for \mathfrak{h} . Thanks to nuclearity, we still have a canonical topology on the algebraic tensor powers of \mathfrak{h} , and we simply define the space of continuous p -cochains $C^p(\mathfrak{h})$ to be the continuous dual $(\Lambda^p(\mathfrak{h}))^{*\text{cont}}$ of the p th exterior power $\Lambda^p(\mathfrak{h})$ of \mathfrak{h} , where here $\Lambda^p(\mathfrak{h})$ is a closed subspace of $\underbrace{\mathfrak{h} \otimes \dots \otimes \mathfrak{h}}_{p\text{-times}}$ and still carries the subspace topology.

With these cochains, we define cohomology as above. As a remark after the main lemma of the Section, we shall see that it does not matter whether to take completed or non-completed tensor products for \mathfrak{h} in case \mathfrak{h} is dense in \mathfrak{g} . This is the situation which is most important here.

1.3 We now come to the main lemma of this Section. Suppose given a Fréchet nuclear Lie algebra \mathfrak{g} and a subalgebra \mathfrak{h} as above.

Lemma 1 *If $i : \mathfrak{h} \hookrightarrow \mathfrak{g}$ is dense in the subspace topology, then we have an isomorphism of graded vector spaces*

$$C^*(\mathfrak{g}) \rightarrow C^*(\mathfrak{h}),$$

induced by i , and consequently an isomorphism of their continuous cohomologies.

Proof:

1) First let us consider the case $p = 1$. Here, the restriction map

$$i^{*\text{cont}} : C^1(\mathfrak{g}) = \mathfrak{g}^{*\text{cont}} \rightarrow C^1(\mathfrak{h}) = \mathfrak{h}^{*\text{cont}}$$

is surjective by the Hahn-Banach Theorem on extension of continuous linear forms and injective by density.

2) $\mathfrak{h} \otimes \mathfrak{h} \subset \mathfrak{g} \hat{\otimes} \mathfrak{g}$ is easily seen to be dense, thus the reasoning in 1) also applies for $p > 1$. Reference for the used facts is found in [17], prop. 43.9, p. 441 and Ex. 43.2, p. 445. \square

Remark: 1) This lemma handles classical cases like

$$\text{Vect}^{\text{pol}}(S^1) = \text{span}\{e_n \mid n \in \mathbb{Z}, [e_n, e_m] = (m - n)e_{n+m}\} \subset \text{Vect}(S^1),$$

$\text{Vect}(S^1)$ being the Lie algebra of smooth vector fields on the circle S^1 , and shows thus that their continuous cohomologies coincide. The same is true for $\mathbb{C}[z] \frac{d}{dz} \subset \mathbb{C}[[z]] \frac{d}{dz}$, etc.

2) In case $\mathfrak{h} \subset \mathfrak{g}$ is dense, it does not matter whether to take completed tensor products or ordinary tensor products in the definition of continuous cochains (for \mathfrak{h} and \mathfrak{g} !). Indeed, in any case the non-completed products are dense in the completed one's, and passing by the completed versions, one establishes the isomorphism of vector spaces as in the lemma. This is an important point, because it means that *continuous cohomology* is independent on the completion (which is purely topological), whereas *homology* is dependent on whether to take completed or ordinary tensor products.

2 Density of meromorphic in holomorphic vector fields

In this Section, we show the density of meromorphic vector fields in holomorphic vector fields, showing in conjunction with the previous Section that their continuous Lie algebra cohomologies are isomorphic. This gives an affirmative answer to Feigin-Novikov's conjecture in the framework of continuous cohomology.

2.1 Let X be a connected, compact, complex manifold and $Y \subset X$ a complex, codimension 1 submanifold which is not necessarily connected. Suppose Y is the support of an effective ample divisor D_Y . Let \mathcal{L}_{D_Y} be its associated line bundle. By definition, \mathcal{L}_{D_Y} is an ample line bundle, and therefore $\mathcal{L}_{D_Y}^m$ is very ample and can be used to embed X into projective space. But for the bundle $\mathcal{L}_{D_Y}^m = \mathcal{L}_{mD_Y}$ and the

divisor mD_Y also has support Y . Hence we may assume that our divisor is already effective and very ample.

Then X embeds into some projective space \mathbb{P}^N via an embedding $i : X \hookrightarrow \mathbb{P}^N$. One way to give the embedding i is by fixing a basis of the global sections of the bundle and evaluating them at the points. We can take as first element of the basis the section which has as divisor exactly D_Y (recall D_Y is very ample). Then H is the hypersurface defined by the vanishing of the first coordinate. Then $i(X) \cap H = i(Y)$, and we obtain that $X \setminus Y$ is embedded as an affine subvariety in $\mathbb{C}^n = \mathbb{P}^n \setminus H$. X is thus a projective, complex algebraic variety and the affine subvariety $X \setminus Y \subset \mathbb{C}^n$ is Stein.

An example for this situation is given by a compact connected Riemann surface Σ as X , $p_1 + \dots + p_k = D_Y$ meaning that $Y = \{p_1, \dots, p_k\}$ for some distinct points $p_1, \dots, p_k \in \Sigma$ ($k \geq 1$), and the associated affine algebraic curve $X \setminus Y = \Sigma \setminus \{p_1, \dots, p_k\} =: \Sigma_k$.

2.2 Now, denote by $Hol(X \setminus Y)$ the Lie algebra of holomorphic vector fields on the Stein manifold $X \setminus Y$, and $Mer_Y(X)$ the Lie algebra of meromorphic vector fields on X with possible poles along Y . General GAGA theory shows that $Mer_Y(X)$ can be identified with the Lie algebra of rational vector fields on X , regular on $X \setminus Y$.

In the example, let $Mer_k(\Sigma)$ denote the space of meromorphic vector fields on Σ , having possibly poles of an arbitrary order in $\{p_1, \dots, p_k\}$. Let $Hol(\Sigma_k)$ be the space of holomorphic vector fields on Σ_k . As Σ_k is holomorphically trivializable, $Hol(\Sigma_k)$ can be holomorphically identified with the space of holomorphic functions on Σ_k and carries the topology of uniform convergence on compact sets which is Fréchet nuclear. The subspace $Mer_k(\Sigma)$ can be equipped with the induced topology. In general, any space of sections of a holomorphic vector bundle carries a canonical Fréchet topology given locally componentwise by uniform convergence on compact sets.

Theorem 1 *$Mer_Y(X)$ is dense in $Hol(X \setminus Y)$ in its induced topology.*

The following corollary is the main result of this paper:

Corollary 2

$$H^*(Mer_Y(X)) \cong H^*(Hol(X \setminus Y)),$$

in particular

$$H^*(Mer_k(\Sigma)) \cong H^*(Hol(\Sigma_k)).$$

Remark: 1) The corollary permits to give a detailed description of $H^*(Mer_k(\Sigma))$, because Kawazumi [7] showed a vector space isomorphism

$$H^*(Hol(\Sigma_k)) \cong H_{\text{sing}}^*(Map(\Sigma_k, S^3))$$

and the latter space is a Hopf algebra which will be described more explicitly in §3.1. Here, $Map(\Sigma_k, S^3)$ denotes the topological space of continuous maps from Σ_k to the 3-sphere S^3 with the compact-open topology, and $H_{\text{sing}}^*(Map(\Sigma_k, S^3))$ its singular cohomology.

2) This corollary answers Feigin and Novikov's conjecture cf. [11] in the positive, in the setting of continuous cohomology. They conjectured the description of the cohomology algebra of $Mer_k(\Sigma)$ given above. The H^2 -part of the conjecture (in the algebraic setting) follows from Skryabin's article [16].

2.3 The **proof** of the Theorem proceeds with two lemmas:

Lemma 2 *A holomorphic field X on a Stein manifold M can be lifted to a holomorphic field \tilde{X} on \mathbb{C}^N for some $N \in \mathbb{N}$. In particular, this is true for $M = X \setminus Y$ or $M = \Sigma_k$ in the above notations.*

Proof:

In this lemma, all notions refer to the holomorphic category.

It is known that a Stein manifold M can be embedded as a closed submanifold $i : M \subset \mathbb{C}^N$ for some N , [4] Ch. V, §1.1. The lemma follows now from the surjectivity of the restriction map $\mathcal{O}_{\mathbb{C}^N}^{\text{an}}(\mathbb{C}^N) \rightarrow \mathcal{O}_M^{\text{an}}(M)$ in case of a closed subvariety $M \subset \mathbb{C}^N$ [4] thm. 4, Ch. V, §4, because the components of the vector field can thus be extended to \mathbb{C}^N . \square

By this first lemma, we can lift our holomorphic field X from Σ_k to a holomorphic field \tilde{X} on \mathbb{C}^N . There, it is just a holomorphic function, and can be approximated uniformly on compact sets by polynomial functions, or vector fields on \mathbb{C}^N .

Using the map

$$H^0(\mathbb{C}^N, \mathcal{T}_{\mathbb{C}^N}) \rightarrow H^0(\mathbb{C}^N, i_*(i^*\mathcal{T}_{\mathbb{C}^N})) \cong H^0(X \setminus Y, i^*\mathcal{T}_{\mathbb{C}^N}),$$

where $i : X \setminus Y \hookrightarrow \mathbb{C}^N$ is the embedding, we can restrict these polynomial vector fields on \mathbb{C}^N to meromorphic vector fields on $X \setminus Y$, but still having values in the tangent space of \mathbb{C}^N restricted to $X \setminus Y$, i.e. to elements of $i^*(\mathcal{T}_{\mathbb{C}^N})$. Let us show in a second lemma that one can project these fields from the tangent space of \mathbb{C}^N to the tangent space of $X \setminus Y$:

Lemma 3 *The sequence*

$$0 \rightarrow \mathcal{T}_{X \setminus Y} \rightarrow i^*\mathcal{T}_{\mathbb{C}^N} \rightarrow \mathcal{N}_{\mathbb{C}^N/(X \setminus Y)} \rightarrow 0$$

defining the normal bundle of $X \setminus Y$ splits (in the algebraic category), and thus there is a way to project an algebraic field with values in $i^\mathcal{T}_{\mathbb{C}^N}$ to an algebraic vector field on $X \setminus Y$.*

Proof:

As indicated in parenthesis, the proof of this lemma takes place entirely in the algebraic category, and all notions refer to this category.

The reason for the sequence of sheaves of $\mathcal{O}_{X \setminus Y}$ -modules

$$0 \rightarrow \mathcal{T}_{X \setminus Y} \rightarrow i^*\mathcal{T}_{\mathbb{C}^N} \rightarrow \mathcal{N}_{\mathbb{C}^N/(X \setminus Y)} \rightarrow 0$$

to be split is clearly that $X \setminus Y$ is affine. Indeed, this sequence is equivalent to a sequence of B -modules, where $\text{Spec}(B) = X \setminus Y$,

$$0 \rightarrow T_{X \setminus Y} \rightarrow T_{\mathbb{C}^N} \otimes_A B \rightarrow N_{\mathbb{C}^N/(X \setminus Y)} \rightarrow 0.$$

where $A = \mathbb{C}[t_1, \dots, t_N]$. This last sequence defines an element in

$$\text{Ext}_B^1(N_{\mathbb{C}^N/(X \setminus Y)}, T_{X \setminus Y}).$$

But as $N_{\mathbb{C}^N/(X \setminus Y)}$ and $T_{X \setminus Y}$ are finitely generated projective B -modules¹, this Ext -group must be zero, hence the splitting result. \square

¹ This is easily proved directly in the sheaf setting from the definition of projectivity by passing to stalks and using the fact that the modules are locally free, see for example (for this “local criterion of projectivity” in a general commutative algebra setting) [2] p. 109, §5.2, Theorem 1

This completes the proof of the Theorem. \square

2.4 The above Theorem and corollary can be seen as a GAGA Theorem stating that the continuous cohomology of a Lie algebra defined on the algebraic variety (i.e. the Lie algebra $Mer_Y(X)$ of regular vector fields on $X \setminus Y$) is isomorphic to the one of a Lie algebra defined on the corresponding complex analytic manifold (i.e. the Lie algebra $Hol(X \setminus Y)$ of holomorphic vector fields on $X \setminus Y$). The Lie algebras $Mer_Y(X)$ constitute from our point of view the natural generalization of Krichever-Novikov algebras to higher dimensions. Notice that the continuous Lie algebra cohomology of $Hol(X \setminus Y)$ is known in principle (to the same extent as in the case of differentiable vector fields) thanks to [19], cf. §3.4 where we compute an example explicitly.

3 Consequences and examples

As a consequence of corollary 2, we will discuss the cohomology of Krichever-Novikov algebras and central extensions of higher dimensional analogues of Krichever-Novikov algebras in this Section.

3.1 As indicated in the introduction, Krichever-Novikov, generalized later by Schlichenmaier, Schlichenmaier-Sheinman, invented a global approach to conformal field theory by defining central extensions on the one hand of the Lie algebra of meromorphic vector fields $Mer_k(\Sigma)$ on a compact Riemann surface Σ with possible poles in k distinct points $p_1, \dots, p_k \in \Sigma$, and of the corresponding current algebra of meromorphic functions $Mer(\Sigma_k, \mathfrak{g})$ on Σ_k with values in a simple complex Lie algebra \mathfrak{g} on the other hand. The central extension of $Mer_k(\Sigma)$ is given by a cocycle of the type

$$c_0^\omega(f, g) = \frac{c}{24\pi i} \int_{\Sigma_k} \left(\frac{1}{2} \left| \begin{array}{cc} f & g \\ f''' & g''' \end{array} \right| - 2R \left| \begin{array}{cc} f & g \\ f' & g' \end{array} \right| \right) dz \wedge \bar{\omega}.$$

Here ω is a holomorphic 1-form on Σ_k , f, g are coefficient functions of meromorphic vector fields in $Mer_k(\Sigma)$ and R is a projective connection assuring by its transformation behaviour that the term in parenthesis is a globally defined holomorphic 1-form. These cocycles (for ω running through $H^1(\Sigma_k)$) form a basis for the second cohomology of $Hol(\Sigma_k)$ [7], so by corollary 2 this is also true for $Mer_k(\Sigma)$. Actually $H_{\text{sing}}^*(Map(\Sigma_k, S^3))$, the singular cohomology algebra of the space of smooth maps from Σ_k to the 3-sphere S^3 , is a Hopf algebra generated by $\dim(H^1(\Sigma_k))$ classes in degree two and one class (called θ) in degree three. Let us describe this latter class in the next paragraph.

3.2 Three dimensional cohomology corresponds to (equivalence classes of) *crossed modules*, see for example [20] [22], so here we present a crossed module representing the cohomology class $\theta \in H^3(Hol(\Sigma_k))$. Indeed, the Lie algebra $Hol(\Sigma_k)$ acts naturally by Lie derivative on differential forms on the universal covering U of Σ_k (i.e. by the uniformization Theorem either the complex plane \mathbb{C} or the hyperbolic half-space \mathbb{H}). We can define a crossed module by its associated 4-term exact sequence and the latter is obtained by splicing together an abelian extension of $Hol(\Sigma_k)$ by the space of holomorphic 1-forms $\Omega^{\text{an}, 1}(U)$ and a short exact sequence of $Hol(\Sigma_k)$ -modules involving $\Omega^{\text{an}, 1}(U)$: the 2-cocycle [21], evaluated on $f, g \in Hol(\Sigma_k)$,

$$c_1(f, g) = \left| \begin{array}{cc} f' & g' \\ f'' & g'' \end{array} \right| - T \left| \begin{array}{cc} f & g \\ f'' & g'' \end{array} \right| + \left(R - \frac{1}{2}T^2 \right) \left| \begin{array}{cc} f & g \\ f' & g' \end{array} \right|$$

with values in $\Omega^{\text{an},1}(U)$, where R is a projective and T an affine connection on the open Riemann surface U (and where the second and third term serve to make the formula a globally defined 1-form), defines an abelian extension

$$0 \rightarrow \Omega^{\text{an},1}(U) \rightarrow \Omega^{\text{an},1}(U) \times_{c_1} \text{Hol}(\Sigma_k) \rightarrow \text{Hol}(\Sigma_k) \rightarrow 0.$$

The de Rham sequence

$$0 \rightarrow \mathbb{C} \rightarrow \Omega^{\text{an},0}(U) \xrightarrow{d_{\text{dR}}} \Omega^{\text{an},1}(U) \rightarrow 0$$

is an exact sequence of $\text{Hol}(\Sigma_k)$ -modules. The splicing together of the two sequences gives

$$0 \rightarrow \mathbb{C} \rightarrow \Omega^{\text{an},0}(U) \xrightarrow{d_{\text{dR}} \times 0} \Omega^{\text{an},1}(U) \times_{c_1} \text{Hol}(\Sigma_k) \rightarrow \text{Hol}(\Sigma_k) \rightarrow 0$$

where $\Omega^{\text{an},1}(U) \times_{c_1} \text{Hol}(\Sigma_k)$ acts on $\Omega^{\text{an},0}(U)$ via the Lie derivative of the second factor and $\Omega^{\text{an},0}(U)$, $\Omega^{\text{an},1}(U)$ are abelian Lie algebras. The compatibility conditions for a crossed module (cf [22]) are easily checked. The crossed module evidently works - mutatis mutandis - for meromorphic vector fields.

3.3 The Lie algebras $\text{Mer}_Y(X)$ in the setting of §2.1 with $\dim(X) > 1$ are natural analogues of Krichever-Novikov algebras in higher dimensions. Unfortunately, they do not possess non-trivial continuous central extensions:

Theorem 3 *Let X and Y be as in §2.1 with $\dim(X) > 1$. Then the Lie algebra $\text{Mer}_Y(X)$ does not possess any non-trivial central extension given by a continuous cocycle.*

Proof:

This follows from corollary 2 in the following way: indeed, first of all

$$H^*(\text{Mer}_Y(X)) \cong H^*(\text{Hol}(X \setminus Y)).$$

Then the main result of [19] implies that the continuous cohomology of $H^*(\text{Hol}(X \setminus Y))$ is isomorphic to the singular cohomology $H_{\text{sing}}^*(\Gamma(B))$ of the topological space (with respect to the compact-open topology) $\Gamma(B)$ of continuous sections of some bundle B on $X \setminus Y$ with typical fiber X_n . X_n is a simply connected manifold which depends only on the dimension n of X and which incarns just some fixed rational homotopy type. We won't give the definition (see [3] p. 79) of X_n here. For example, $X_1 = S^3$ cf. §2.2, and in general, X_n is known to have no rational cohomology (strictly) below $2n + 1$, see [3] p. 89. From a rational model for (the singular cohomology of) $\Gamma(B)$ it is known (see [6] §3.1) that the degrees from $H_{\text{sing}}^*(X_n)$ and $H_{\text{sing}}^*(X \setminus Y)$ subtract to give the degree of $H_{\text{sing}}^*(\Gamma(B))$, but in order to descend as much as possible, the lowest degree in $H_{\text{sing}}^*(X \setminus Y)$ is $2n + 1$ and the highest degree in $H_{\text{sing}}^*(X \setminus Y)$ is n , so that one can descend at most to $n + 1$. When $n > 1$, this is strictly greater than 2, meaning that there is no continuous 2 cohomology. \square

3.4 As a last application, let us compute the cohomology of $\text{Mer}_Y(X)$ in one easy example with $\dim(X) = n = 2$.

Let X be the projective algebraic variety $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$. Its projectivity is most easily seen by the *Segré embedding* $X \hookrightarrow \mathbb{C}\mathbb{P}^3$ given in homogeneous coordinates by forming all possible products of the coordinates on the two factors:

$$([z_1 : z_2], [w_1 : w_2]) \mapsto [z_1 w_1 : z_1 w_2 : z_2 w_1 : z_2 w_2]$$

Let Y be the divisor $Y := \{[0 : 1]\} \times \mathbb{C}\mathbb{P}^1 \cup \{[1 : 0]\} \times \mathbb{C}\mathbb{P}^1 \cup \mathbb{C}\mathbb{P}^1 \times \{[0 : 1]\}$. Then $X \setminus Y$ is the Stein manifold $X \setminus Y = \mathbb{C}^* \times \mathbb{C} = \{(z, w) \in \mathbb{C}^2 \mid z \neq 0\}$. Thus $\text{Mer}_Y(X)$ is the Lie algebra of vector fields on $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ having possible poles along Y and $\text{Hol}(X \setminus Y)$ is the one of holomorphic vector fields on $\mathbb{C}^* \times \mathbb{C}$. Obviously, $X \setminus Y$ is homotopically equivalent to the unit circle S^1 , so that we have

$$H^*(\text{Mer}_Y(X)) \cong H^*(\text{Hol}(X \setminus Y)) \cong H_{\text{sing}}^*(\text{Map}(S^1, X_2)).$$

Here X_2 is the simply connected manifold of dimension 8 from §3.3 defined in [3] p. 79 having the following singular cohomology spaces [3] p. 89

$$H_{\text{sing}}^*(X_2) = \begin{cases} 0 & \text{for } * \leq 4 \\ \mathbb{C}^2 & \text{for } * = 5 \\ 0 & \text{for } * = 6 \\ \mathbb{C} & \text{for } * = 7 \\ \mathbb{C}^2 & \text{for } * = 8 \\ 0 & \text{for } * \geq 9 \end{cases}$$

All products and Massey products on $H_{\text{sing}}^*(X_2)$ are trivial. As X_2 is homotopically equivalent to a reduced suspension, Theorem 27 of [10] (which is due to Sullivan and Vigué in our framework) shows that the graded vector space $H_{\text{sing}}^*(\text{Map}(S^1, X_2))$ is isomorphic to the graded vector space of tensor powers on $H_{\text{sing}}^*(X_2)$ (i.e. the Hochschild homology of an algebra with zero product). This gives a clear picture of the continuous cohomology of $H^*(\text{Mer}_Y(X))$.

4 Generalizations

In this Section, we investigate generalizations of Theorem 1, namely the density of meromorphic sections of a vector bundle E on a complex manifold X , having poles on a submanifold Y given by an ample divisor, in the holomorphic sections of E on $X \setminus Y$.

4.1 Now let us generalize from vector fields to sections in an arbitrary vector bundle. For this, we maintain the above setting from §2.2 for $Y \subset X$.

Let E^{an} be a holomorphic vector bundle on X . As X is projective, general GAGA theory affirms that E^{an} can be identified with an algebraic vector bundle E on the algebraic variety X . We assume that there is an algebraic vector bundle F on \mathbb{P}^N , on \mathbb{C}^N algebraically trivial², such that under the embedding $i : X \rightarrow \mathbb{P}^N$, E becomes a subbundle of F . This gives rise to the short exact sequence

$$0 \rightarrow E^{\text{an}} \rightarrow i^* F^{\text{an}} \rightarrow Q^{\text{an}} \rightarrow 0 \quad (1)$$

of sheaves of sections on $X \setminus Y$ (Q^{an} denotes the sheaf of sections of the quotient bundle). Note that it is not important whether E^{an} is a subbundle or a quotient bundle of some holomorphic vector bundle on \mathbb{P}^N , i.e. the condition of the existence of

$$0 \rightarrow I^{\text{an}} \rightarrow i^* F^{\text{an}} \rightarrow E^{\text{an}} \rightarrow 0 \quad (2)$$

instead of the sequence (1) is also sufficient for the application of our methods.

² This is automatic: indeed, by the celebrated Theorem of Quillen and Suslin (1976), any finitely generated projective module over $\mathbb{C}[t_1, \dots, t_N]$ is free, meaning that any algebraic vector bundle over \mathbb{C}^N is algebraically (globally) trivial.

Indeed, as $X \setminus Y$ is affine, the short exact sequence of sheaves gives rise to a short exact sequence of global sections, and in any case, we can either include or lift a section of E^{an} on $X \setminus Y$ to one of i^*F^{an} on $X \setminus Y$.

The second lemma from Section 2 stays the same - mutatis mutandis - and hence the proof of Theorem 1 remains unchanged. Thus, denoting $E^{\text{an}}(X \setminus Y)$ the space of sections of E^{an} (i.e. of holomorphic sections of E) on the affine variety $X \setminus Y$, and $E(X \setminus Y)$ the space of sections of E on $X \setminus Y$ (which is the space of regular sections of E on $X \setminus Y$ or meromorphic sections of E^{an} on X with possible poles along Y), we get:

Theorem 4 *Let $Y \subset X$ be as in §2.1, and let E denote an algebraic vector bundle the affine algebraic variety $X \setminus Y$. Assume that there is an algebraic vector bundle F on \mathbb{P}^N such that one of the sequences (1) or (2) holds.*

Then, $E(X \setminus Y) \subset E^{\text{an}}(X \setminus Y)$ is dense (in the induced topology).

Corollary 5 *Let Σ and Σ_k be as in Section 2.*

The subspace of meromorphic 1-forms on Σ with possible poles on $\{p_1, \dots, p_k\}$ (or Kähler differentials on the algebra $\mathcal{O}(\Sigma_k)$ of regular functions on Σ_k) are dense in the holomorphic 1-forms on Σ_k (in the subspace topology).

Indeed, we can take in this case the dual sequence of the sequence defining the normal bundle.

There is a natural generalization of the above corollary to higher dimensions in the setting of §2.1.

4.2 We won't elaborate the third direction of generalization of the density result, namely, the application to other continuous cohomologies (with trivial coefficients as in our Lie algebra cohomology), like the continuous Hochschild cohomology of the algebra of meromorphic resp. holomorphic functions in the above setting. It is only important in this perspective to deal with cohomology and to have an explicit resolving functor of the derived functor in question. The resolving functor should be built on tensor products, and the continuous cohomology should consist in changing from algebraic cochains to continuous cochains. In this framework, the necessary modifications to be carried out seem clear to us. In particular, in view of the density Theorem for meromorphic in holomorphic functions, the foregoing considerations show for example the following Theorem:

Theorem 6 *Continuous Hochschild and cyclic cohomology of $\mathcal{O}(\Sigma_k)$ (i.e. the algebra of regular functions on Σ_k) and of $\mathcal{O}^{\text{an}}(\Sigma_k)$ (i.e. the algebra of holomorphic functions on Σ_k) are isomorphic.*

References

1. Behnke H. and Stein K., Entwicklung analytischer Funktionen auf Riemannschen Flächen, *Math. Ann.*, 1949, V.120, 430–461
2. Bourbaki N., *Éléments de mathématiques, Algèbre commutative, Chapitre 2*, Hermann, Paris, 1961
3. Fuks D. B., *Cohomology of Infinite Dimensional Lie algebras*, Consultants Bureau, New York and London, 1986
4. Grauert H. and Remmert R., *Theory of Stein spaces*, Grundlehren der mathematischen Wissenschaften 236, Springer Verlag, 1977
5. Gunning R. C. and Rossi H., *Analytic Functions of Several Complex Variables*, Prentice Hall, 1965

6. Haefliger A., Sur la cohomologie de l'algèbre de Lie des champs de vecteurs, *Ann. Scient. Éc. Norm. Sup.*, 1976, 4^e série, V.9, 503–532
7. Kawazumi N., On the Complex Analytic Gelfand–Fuks Cohomology of open Riemann Surfaces, *Ann. Inst. Fourier, Grenoble*, 1983, V.43, N.3, 655–712
8. Krichever I. M. and Novikov S. P., Algebras of Virasoro Type, Riemann Surfaces and Structures of the Theory of Solitons, *Funkts. Anal. i. Pril.*, 1987, V.21, N.2, 46–63
9. Krichever I. M. and Novikov S. P., Virasoro Type Algebras, Riemann Surfaces and Strings in Minkowski Space, *Funkts. Anal. i. Pril.*, 1987, V.21, N.4, 47–61
10. Menichi L., The cohomology ring of free loop spaces, *Homology, Homotopy Appl.*, 2001, V.3, N.9, 193–224
11. Millionshchikov D. V., Krichever-Novikov Algebras and the Cohomology of the Algebra of Meromorphic Vector Fields, *Amer. Math. Soc. Transl.*, 1997, V.2, N.179, 101–108
12. Neeb K.-H. and Wagemann F., The universal central extension of the holomorphic current algebra, *Manuscripta Math.*, 2003, V.112, 441–458
13. Schlichenmaier M., An introduction to Riemann Surfaces, Algebraic Curves and Moduli Spaces, Lecture Notes in Physics 322, Springer Verlag, 1989
14. Schlichenmaier M., Krichever-Novikov algebras for more than two points, *Lett. Math. Phys.*, 1990, V.19, 151–165
15. Schlichenmaier M. and Sheinman O., The Wess-Zumino-Witten-Novikov Theory, Knizhnik-Zamolodchikov Equations, and Krichever-Novikov Algebras I and II, *Russian Math. Surv.*, 1999, V.54, 213–250 and preprint
16. Skryabin S., Degree one cohomology for the Lie algebras of derivations, *Lobachevskii J. Math.*, 2004, V.14, 86–123
17. Trèves F., Topological Vector Spaces, Distributions and Kernels, Academic Press, 1967
18. Wagemann F., Some Remarks on Krichever-Novikov Algebras, *Lett. Math. Phys.*, 1999, V.47, 173–177
19. Wagemann F., Differential Graded Cohomology and Lie Algebras of Holomorphic Vector Fields, *Commun. Math. Phys.*, 1999, V.208, 521–540
20. Wagemann F., A Crossed Module Representing the Godbillon-Vey Cocycle, *Lett. Math. Phys.*, 2000, V.51, 293–299
21. Wagemann F., Explicit formulae for cocycles of holomorphic vector fields with values in λ densities, *J. Lie Theory*, 2001, V.11, 173–184
22. Wagemann F., On crossed modules, *Comm. Algebra* **34** (2006), no. 5, 1699–1722