HAL
open science

Generalized stochastic flows and applications to incompressible viscous fluids

Alexandra Antoniouk, Marc Arnaudon, Ana Bela Cruzeiro

To cite this version:

Alexandra Antoniouk, Marc Arnaudon, Ana Bela Cruzeiro. Generalized stochastic flows and applications to incompressible viscous fluids. Bulletin des Sciences Mathématiques, 2014, 138 (4), pp.565-584. hal-01057528

HAL Id: hal-01057528

https://hal.science/hal-01057528v1

Submitted on 9 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

GENERALIZED STOCHASTIC FLOWS AND APPLICATIONS TO INCOMPRESSIBLE VISCOUS FLUIDS

ALEXANDRA ANTONIOUK, MARC ARNAUDON, AND ANA BELA CRUZEIRO

Abstract. We introduce a notion of generalized stochastic flows on manifolds, that extends to the viscous case the one defined by Brenier for perfect fluids. Their kinetic energy extends the classical kinetic energy to Brownian flows, defined as the L^2 norm of their drift. We prove that there exists a generalized flow which realizes the infimum of the kinetic energy among all generalized flows with prescribed initial and final configuration. We also construct generalized flows with prescribed drift and kinetic energy smaller than the L^2 norm of the drift.

The results are actually presented for general L^q norms, thus including not only the Navier-Stokes equations but also other equations such as the porous media.

Contents

1. Introduction 1
2. Generalized stochastic flows 3
3. Existence of generalized flow minimizing the energy 9
4. Constructing generalized flows with prescribed drift 14
5. Constructing generalized flows from solutions of finite variation transport equations 15
References 18

1. Introduction

Consider the Euler equations describing the velocity of incompressible non viscous (perfect) fluids,

$$\frac{\partial}{\partial t}u = -(u.\nabla)u - \nabla p, \ \operatorname{div} u = 0. \tag{1.1}$$

V. Arnold showed that the corresponding Lagrangian flows $g(t)$, namely the integral curves for u, solutions of $\frac{d}{dt}g(t)(x) = u(t, g(t)(x))$, $g(0)(x) = x$, can be characterized as geodesics on the infinite dimensional "manifold" of measure preserving diffeomorphisms of the underlying configuration space (c.f. [4], [5]). In particular such solutions $g(t)$ minimize the following energy functional, defined in the time interval $[0, T]$,

$$S[g] = \frac{1}{2}\int_0^T \int \left|\frac{d}{dt}g(t)(x)\right|^2 dx \tag{1.2}$$

and the corresponding Euler-Lagrange equations are precisely equations (1.1).

Ebin and Marsden ([12]) proved that, given a final condition $g(T)$ with some suitable Sobolev regularity (and, in particular, smooth) lying in a small neighborhood of the identity, existence and uniqueness of a local minimal geodesic can be obtained. However, in general, there may be situations where such a geodesic is not defined (c.f.[21]). The main difficulty lies in that the topology induced by the energy is not strong enough to deal with the regularity of the maps.

To overcome such difficulties Y. Brenier introduced in [6] the notion of generalized solutions for the minimal action principle, in the spirit of the Monge-Kantarovich problem. These solutions are probability measures defined on the set of Lagrangian trajectories. This weaker variational approach allows to consider measure-preserving maps $g(t)$ with possible splitting and crossing during the evolution. He proved, in particular, that classical solutions can be regarded as generalized solutions and that there exist generalized solutions which do not correspond to classical flows.

In the viscous case, where the velocity obeys the Navier-Stokes system

$$\frac{\partial}{\partial t}u = -(u.\nabla)u + \nu\Delta u - \nabla p, \ \operatorname{div} u = 0 \tag{1.3}$$

with a viscosity coefficient $\nu > 0$, it is not so clear how to define a corresponding variational principle. Following the initial ideas in [19] and [23] we have considered a stochastic variational principle defined on stochastic Lagrangian flows. The kinetic energy is defined for these flows and we have characterized the stochastic processes which are critical for the energy as processes whose drift satisfies Navier-Stokes equation (c.f. [7] for flows living on the flat torus and [2] for flows in a general compact Riemannian manifold). Formally, when the viscosity coefficient vanishes, the Lagrangian flows become deterministic and we are back to the Arnold characterization of Euler Lagrangian paths.

We mention that another approach to Lagrangian trajectories for the Navier-Stokes equation as geodesics in a different geometric framework was considered in [22] and [16]. There the approach is deterministic: what is deformed to pass from Euler to Navier-Stokes is the geometry. Other related approaches are considered in [8] and [13].

Actually these ideas are more general: replacing power 2 by some $q \geq 2$ in the energy functional, the stochastic processes on the flat torus which are critical for the energy have a drift satisfying the weighted porous media type equation

$$\frac{\partial}{\partial t}u = (-u.\nabla + \nu\Delta)\left(\|u\|^{q-2}u\right) - \nabla p, \ \operatorname{div} \ u = 0 \tag{1.4}$$

(see [10] and [11] for these equations). This was recently proved in [1].

So, in the spirit of Brenier's work, we introduce and study here a concept of generalized Lagrangian flows for the Navier-Stokes problem. We present our results for general L^q norms, thus including other applications (Navier-Stokes corresponding to the case where $q = 2$).

The abovementionned difficulties encountered in the Euler case to prove existence of critical paths remain in this setting.

2. Generalized stochastic flows

Let us consider any compact oriented Riemannian manifold M of dimension $N \geq 2$ and without boundary. We shall use notation dx for integration with respect to the normalized volume measure on M. Let $\mathcal{H}$ be a Hilbert space.

Definition 2.1. An *Itô stochastic flow* $g_t(x)(\omega)$ on M, $x \in M$, $t \in [0,T]$, $T > 0$ is a stochastic process which satisfies

1. $g_0(x)(\omega)=x$;
2. $g_t(x)(\omega)$ is a semimartingale which satisfies the Itô stochastic equation:

$$dg_t(x)(\omega) = \sigma(g_t(x)(\omega))dW_t + u_t(g_t(x)(\omega),\omega)\,dt, \tag{2.1}$$

where $\sigma \in \Gamma(Hom(\mathcal{H}, TM))$ is a C^2-map satisfying for all $x \in M$ $(\sigma\sigma^*)(x) = \mathrm{id}_{T_xM}$; W_t is a cylindric Brownian motion in the Hilbert space $\mathcal{H}$, and $(t,x,\omega) \to u_t(x,\omega) \in T_xM$ is a time dependent adapted drift with locally bounded variation in x.

Frequently we shall drop the probability space parameter in the notations.

Recall that if $P(g(x))_t : T_xM \to T_{g(t)(x)}M$ is the parallel transport along $g_t(x)$, then

$$dg_t(x) = P(g(x))_t d\left(\int_0^{\cdot} P(g(x))_s^{-1} \circ dg_s(x)\right)_t.$$

Let us mention that at this stage we do not ask for any further regularity of $u_t(x,\omega)$. The solution to equation (2.1) is not unique in general, but for any given $x \in M$ the process $u_t(g_t(x),\omega)$ is entirely determined by the process $g_t(x)(\omega)$: in a local chart, after removing the term coming from Christoffel symbols, $u_t(g_t(x),\cdot)$ is the time derivative of the drift of $g_t(x)$.

Definition 2.2. An *incompressible stochastic flow* $g_t(x)(\omega)$ is a stochastic flow such that a.s. ω for all $t \geq 0$, the map $x \mapsto g_t(x)(\omega)$ is a volume preserving diffeomorphism of M.

Proposition 2.3. *Assume that the following properties are satisfied:*

(i) *$u_t(x,\omega)$ is jointly continuous in t and x and C^2 in x with derivatives uniformly bounded in ω;*
(ii) *almost surely for all $t \in [0,T]$:* $\mathrm{div}\ u_t(\cdot,\omega) = 0$*;*
(iii) *tr $\nabla_{\sigma(\cdot)}\sigma(\cdot) = 0$;*
(iv) *for all $v \in \mathcal{H}$: $div\sigma(v)(\cdot) = 0$.*

Then the Itô stochastic flow g_t is incompressible, namely ω a.s., for all $t \in [0,T]$, for all $f \in C(M)$

$$\int_M f\big(g_t(x)(\omega)\big)\,dx = \int_M f(x)\,dx. \tag{2.2}$$

Concerning assumptions i) and (ii) on the regularity of u, we refer to [17] for possible more general conditions (but in this reference u does not depend on ω).

Proof. Under the condition of the proposition almost surely for all t the map $x \mapsto g_t(x)(\omega)$ is a diffeomorphism of M. So for $f \in C(M)$

$$\int_M f\big(g_t(x)(\omega)\big)\,dx = \int_M f(y)|\det T_y g_t^{-1}(\cdot)(\omega)|\,dy \tag{2.3}$$

where $T_y g_t^{-1}(\cdot)(\omega)$ is the tangent map of $y \mapsto g_t^{-1}(y)(\omega)$. On the other hand condition (iii) implies that equation (2.1) is equivalent to the Stratonovich equation

$$dg_t(x)(\omega) = \sigma(g_t(x)(\omega)) \circ dW_t + u_t(g_t(x)(\omega), \omega)\, dt. \tag{2.4}$$

Differentiating this equation with respect to the starting point yields

$$d\left(P(g(x))_t^{-1} T_x g_t\right) = P(g(x))_t^{-1} \left(\nabla_{T_x g_t} \sigma \circ dW_t + \nabla_{T_x g_t} u_t\, dt\right) \tag{2.5}$$

where again $P(g(x))_t$ denotes parallel transport along $g_t(x)$. Then using the fact that

$$T_{g_t(x)} g_t^{-1} P(g(x))_t P(g(x))_t^{-1} T_x g_t \equiv \mathrm{id}_{T_x M}$$

we get

$$\begin{aligned} &d\left(T_{g_t(x)} g_t^{-1} P(g(x))_t\right) \\ &= T_{g_t(x)} g_t^{-1} P(g(x))_t P(g(x))_t^{-1} \left(-\nabla_{P(g(x))_t} \sigma \circ dW_t - \nabla_{P(g(x))_t} u_t\, dt\right) \end{aligned} \tag{2.6}$$

and this yields

$$\begin{aligned} &d\left(\det T_{g_t(x)} g_t^{-1} P(g(x))_t\right) \\ &= \det T_{g_t(x)} g_t^{-1} P(g(x))_t \operatorname{tr}\left\{P(g(x))_t^{-1} \left(-\nabla_{P(g(x))_t} \sigma \circ dW_t - \nabla_{P(g(x))_t} u_t\, dt\right)\right\} \\ &= \det T_{g_t(x)} g_t^{-1} P(g(x))_t \left(-\operatorname{div} \sigma(g_t(x)) \circ dW_t - \operatorname{div} u_t(g_t(x))\, dt\right) \\ &= 0. \end{aligned} \tag{2.7}$$

With the initial condition $\det T_x g_0^{-1} P(g(x))_0 = 1$ and the property $\det P(g(x))_t \equiv 1$ we conclude that

$$\det T_{g_t(x)} g_t^{-1}(\cdot)(\omega) \equiv 1 \tag{2.8}$$

almost surely for all t and $x \in M$. But almost surely the map $x \mapsto g_t(x)(\omega)$ is a diffeomorphism of M, so almost surely for all t and $y \in M$

$$\det T_y g_t^{-1}(\cdot)(\omega) \equiv 1 \tag{2.9}$$

As a conclusion,

$$\int_M f\left(g_t(x)(\omega)\right) dx = \int_M f(y)\, dy.$$

□

Remark 2.4. In the following, when looking for incompressible flows, conditions (iii) and (iv) of Proposition 2.3 will be unavoidable.

Our main example concerns $M = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ the two dimensional torus, H the Hilbert space of real-valued sequences indexed by $\mathbb{Z}^2 \oplus \mathbb{Z}^2$, σ the map defined by

$$\sigma((k_1, k_2) + (0,0))(\theta) =: A_{(k_1,k_2)}(\theta) = (k_2, -k_1) \cos k \cdot \theta \tag{2.10}$$

and

$$\sigma((0,0) + (k_1, k_2))(\theta) =: B_{(k_1,k_2)}(\theta) = (k_2, -k_1) \sin k \cdot \theta. \tag{2.11}$$

Another example is given by any compact semi-simple Lie group G endowed with the metric given by the opposite of the Killing form. Then one can take $H = T_e G$ where e is the identity of G and for $x \in G$, $\sigma(x) = TL_x$ with L_x the left translation.

In the same spirit many examples can be constructed by projection on symmetric spaces of compact type.

Definition 2.5. For an Itô stochastic flow g_t satisfying

$$dg_t = \sigma_t(g_t)dW_t + u_t(g_t,\omega)dt,$$

we define the *drift* as

$$Dg_t(\omega) := u_t(g_t,\omega).$$

Let us define for $q > 1$ the *q-energy functional* by

$$\mathcal{E}_q(g) := \frac{1}{q}\mathbb{E}\left[\int_0^T \int_M \|Dg_t(x)(\omega)\|^q \, dx\, dt\right]. \tag{2.12}$$

When $q = 2$ the 2-energy corresponds to the kinetic energy in hydrodynamics.

As we remarked before, the drift and the energy are uniquely determined by the stochastic flow g_t.

Now we would like to give some explanation and motivation for further definitions. For the Itô incompressible stochastic flow $g_t(x)$, consider the bilinear map Θ_t^g, which to two elements $\varphi, \psi \in L_2(M)$ associates the process:

$$\Theta_t^g(\varphi,\psi) = \int_M \varphi(x)\psi\big(g_t(x)\big)dx. \tag{2.13}$$

Definition 2.6. We call the bilinear map Θ_t^g *a g-flow associated to the Itô stochastic flow* g_t.

It is easy to see that for fixed $\varphi, \psi \in C^\infty(M)$ the flow Θ_t^g is a real valued semi-martingale process which satisfies

$$\begin{aligned}\Theta_t^g(\varphi,\psi) &= (\varphi,\psi)_{L_2(M)} + \sum_{i=1}^\infty \int_0^t \Theta_s^g\big(\varphi, \langle\nabla\psi,\sigma_i\rangle\big)\, dW_s^i + \\ &+ \int_0^t \Theta_s^g\big(\varphi, \langle\nabla\psi, u_s(\cdot,\omega)\rangle\big)\, ds + \frac{1}{2}\int_0^t \Theta_s^g\big(\varphi,\Delta\psi\big)\, ds,\end{aligned} \tag{2.14}$$

where for fixed orthonormal basis $e_i, i \geq 1$ in $\mathcal{H}$ we denote $W_s^i = \langle W_s, e_i\rangle$ and $\sigma_i = \sigma(e_i)$.

When g_t is a flow of diffeomorphisms, then

$$\Theta_t^g(\varphi,\psi) = (\theta^\varphi(t,\cdot),\psi)_{L^2(M)} \tag{2.15}$$

where $x \mapsto \theta^\varphi(t,x)(\omega)$ is the function $M \to \mathbb{R}$ defined by

$$\theta^\varphi(t,x) = \varphi\left((g(t)(\cdot)(\omega))^{-1}(x)\right). \tag{2.16}$$

Notice that for A a Borelian set in M and $\varphi = 1_A$ we have $\theta^\varphi(t,x) = 1_{g(t)(A)}(x)$.

It is not clear how to obtain the existence of critical points for our variational principles within a space of laws of g-flows, which are measures supported on real semimartingales with, in particular, $g(0)(x) = x$. We therefore introduce the so-called *generalized flows which include as particular cases the g-flows* Θ_t^g.

Definition 2.7. We call a bilinear map $\Theta_t(\cdot,\cdot)$ *a generalized flow* if to each $\varphi, \psi \in C^\infty(M)$ it associates a continuous semi-martingale $t \mapsto \Theta_t(\varphi,\psi)$ defined in a common filtered probability space for all φ, ψ, and which for all $\varphi, \psi, \varphi_1, \psi_1 \in C^\infty(M)$

satisfies the following list of properties:

(1) $\Theta_t(\varphi, 1) \equiv \int_M \varphi(x)\, dx,$

(2) $\Theta_t(1, \psi) \equiv \int_M \psi(x)\, dx,$ a.s. for all $t \in [0, T]$,

(3) $\Theta_0(\varphi, \psi) = (\varphi, \psi)_{L^2(M)}$;

(4) for $\varphi, \psi \geq 0$ it is true that $\Theta_t(\varphi, \psi) \geq 0$ a.s.;

(5) $|\Theta_t(\varphi, \psi)| \leq \|\varphi\|_{L^2(M)} \|\psi\|_{L^2(M)}$, a.s. for all $t \in [0, T]$;

(6) $d\left[\Theta(\varphi, \psi), \Theta(\varphi_1, \psi_1)\right]_t = \sum_{i \geq 1} \Theta_t\big(\varphi, \langle \nabla\psi, \sigma_i\rangle\big) \cdot \Theta_t\big(\varphi_1, \langle \nabla\psi_1, \sigma_i\rangle\big)\, dt.$

Let us notice that (5) and (6) imply

$$d\left[\Theta(\varphi, \psi), \Theta(\varphi, \psi)\right]_t \leq \|\varphi\|^2_{L^2(M)} \cdot \|\nabla\psi\|^2_{L^2(M)}\, dt. \tag{2.17}$$

Definition 2.8. For $\varphi, \psi \in C^2(M)$, we introduce

$$\widetilde{\Theta}_t(\varphi, \psi) = \Theta_t(\varphi, \psi) - \frac{1}{2} \int_0^t \Theta_s(\varphi, \Delta\psi)\, ds. \tag{2.18}$$

If $\widetilde{\Theta}_t$ has absolutely continuous finite variation part, we denote by $D\tilde{\Theta}(\varphi, \psi)$ its drift in the sense of Definition 2.5, i.e. the time derivative of its finite variation part.

Definition 2.9. Let $q > 1$. We say that the generalized flow $\Theta_t(\cdot, \cdot)$ has *q-finite energy* iff

(i) the semi-martingale $\widetilde{\Theta}_t(\varphi, \psi)$ has absolutely continuous finite variation part;

(ii) the following functional is finite:

$$\mathcal{E}'_q(\Theta) = \frac{1}{q} \sup_{\varphi, \psi, \ell, m} \left\{ \mathbb{E} \int_0^T \sum_{j=1}^m \left[\sum_{k=1}^\ell \frac{\big(D\widetilde{\Theta}_t(\varphi_j, \psi_k)\big)^2}{\Theta_t(\varphi_j, 1)^\alpha} \right]^{q/2} dt \right\}, \tag{2.19}$$

where $\alpha = \dfrac{2(q-1)}{q}$, sup is taken on all vectors $\varphi = (\varphi_1, ..., \varphi_m)$, $\psi = (\psi_1, ..., \psi_\ell)$ for any $m, \ell \geq 1$, such that $\varphi_j, \psi_k \in C^\infty(M)$, $\varphi_j \geq 0$, $\sum_{j=1}^m \varphi_j = 1$ and ψ_k are such that for all $v \in TM$: $\sum_{k=1}^\ell \langle \nabla\psi_k, v\rangle^2 \leq ||v||^2$.

We call the functional $\mathcal{E}'_q(\Theta)$ the *q-energy functional* of the generalized flow $\Theta_t(\cdot, \cdot)$.

Let η be a probability measure on $M \times M$ with marginals equal to dx. In particular it disintegrates as $\eta(dx, dx) = dx\, \eta_x(dy)$.

Definition 2.10. The generalized flow Θ_t is said to *satisfy the endpoints configuration* η if

$$\mathbb{E}\left[\Theta_T(\varphi, \psi)\right] = \int_{M \times M} \varphi(x)\psi(y)\, \eta(dx, dy). \tag{2.20}$$

Proposition 2.11. *Let $g_t(x)$ be an incompressible Itô stochastic flow on M, satisfying the requirements of Proposition 2.3, such that for all $x \in M$, $g_T(x)$ has law η_x. Then the g-flow Θ_t^g associated to the Itô stochastic flow g_t by formula (2.13) is a generalized flow with endpoints configuration η.*

Proof. We need to check the requirements of the Definitions 2.7 and 2.10. The requirements (1) – (5) of Definition 2.7 are trivially fulfilled. For example, to prove (2) it is sufficient to note that

$$\Theta_t^g(1,\psi) = \int\limits_M \psi(g_t(x))\,dx = \int\limits_M \psi(x)\,dx.$$

Property (6) may be easily calculated from (2.14). Finally (2.20) follows from the fact that $g_T(x)$ has law η_x. □

Lemma 2.12. *For a generalized flow Θ_t, the following estimate holds true:*

$$\mathbb{E}\int_0^T \left|D\tilde{\Theta}_t(\varphi,\psi)\right|^q dt \le 2^q q\,\mathcal{E}_q'(\Theta)\|\varphi\|^q_{L^\infty(M)}\,\|\nabla\psi\|^q_{L^\infty(M)}\,.$$

Proof. We can assume that $\|\varphi\|_\infty > 0$ otherwise the left hand side is 0. First assume that $\varphi \ge 0$. Then $\int_M \varphi > 0$. So

$$\begin{aligned}
&\mathbb{E}\int_0^T |D\tilde{\Theta}_t(\varphi,\psi)|^q dt \\
&= \|\varphi\|^q_\infty \cdot \|\nabla\psi\|^q_\infty \cdot \left(\int_M \frac{\varphi(x)}{\|\varphi\|_\infty}\,dx\right)^{\alpha q/2} \cdot \mathbb{E}\int\limits_0^T \frac{\left|D\tilde{\Theta}\left(\frac{\varphi}{\|\varphi\|_\infty},\frac{\psi}{\|\nabla\psi\|_\infty}\right)\right|^q}{\left(\int_M \frac{\varphi}{\|\varphi\|_\infty}\right)^{\alpha q/2}}\,dt \\
&\le \|\varphi\|^q_\infty \cdot \|\nabla\psi\|^q_\infty \cdot q\,\mathcal{E}_q'(\Theta).
\end{aligned}$$

by definition of $\mathcal{E}_q'(\Theta)$.

For general φ we make the splitting into positive and negative parts: $\varphi = \varphi_+ - \varphi_-$. Then we write

$$\begin{aligned}
|D\tilde{\Theta}_t(\varphi,\psi)|^q &= |D\tilde{\Theta}_t(\varphi_+,\psi) - D\tilde{\Theta}_t(\varphi_-,\psi)|^q \\
&\le 2^{q-1}\left(|D\tilde{\Theta}_t(\varphi_+,\psi)|^q + |D\tilde{\Theta}_t(\varphi_-,\psi)|^q\right)
\end{aligned}$$

then we are left to apply the first part of the proof, using the fact that $\|\varphi_+\|_\infty \le \|\varphi\|_\infty$ and $\|\varphi_-\|_\infty \le \|\varphi\|_\infty$. □

Theorem 2.13. *Let $q > 1$ and g_t be an Itô stochastic flow. For the energy functionals $\mathcal{E}_q'(\Theta^g)$ and $\mathcal{E}_q(g)$ the following inequality is true*

$$\mathcal{E}_q'(\Theta^g) \le \mathcal{E}_q(g). \tag{2.21}$$

Proof. If $\mathcal{E}_q(g) = \infty$ then there is nothing to prove. So we assume that $\mathcal{E}_q(g) < \infty$. If we take the generalized flow Θ_t^g, corresponding to the ordinary flow g in sense of (2.13), then

$$D\widetilde{\Theta}_t^g(\varphi_j,\psi_k) = \int\limits_M \varphi_j(x)\big\langle u_t\big(g_t(x)\big), \nabla\psi_k\big(g_t(x)\big)\big\rangle_{T_xM} dx =: \int\limits_M \varphi_j b_k dx$$

where

$$b_k = b_k(t,x) = \big\langle u_t\big(g_t(x)\big), \nabla\psi_k\big(g_t(x)\big)\big\rangle_{T_xM}. \tag{2.22}$$

In this notations we rewrite the kinetic energy functional $\mathcal{E}'_q(\Theta^g)$ in the form:

$$\mathcal{E}'_q(\Theta^g) \quad = \quad \frac{1}{q} \sup_{\varphi,\psi,\ell,m} \left\{ \mathbb{E} \int_0^T \sum_{j=1}^m \Big[\sum_{k=1}^\ell \frac{\big(\int\limits_M \varphi_j(x) b_k(x) dx\big)^2}{\big(\int\limits_M \varphi_j(x) dx\big)^\alpha} \Big]^{q/2} \right\}$$

and denote by $E'_q(\Theta^g)$ the expression under the supremum. Then

$$E'_q(\Theta^g) \quad \leq \quad \mathbb{E} \int_0^T \sum_{j=1}^m \Big[\sum_{k=1}^\ell \Big(\int\limits_M \varphi_j(x) dx \Big)^{(1-\alpha)} \int\limits_M \varphi_j(x) b_k^2(x) dx \Big]^{q/2} dt.$$

Above we used inequality

$$\Big(\int\limits_M \varphi_j b_k dx \Big)^r \leq \Big(\int\limits_M \varphi_j \, dx \Big)^{r-1} \int\limits_M \varphi_j b_k^r dx \tag{2.23}$$

with $r = 2$. Due to the fact that ψ_k are such that for all $u \in TM$: $\sum\limits_{k=1}^\ell \langle \nabla \psi_k, u \rangle^2 =: \sum\limits_{k=1}^\ell b_k^2 \leq ||u||^2$ we have:

$$\begin{aligned} E'_q(\Theta^g) \quad &\leq \quad \mathbb{E} \int_0^T \sum_{j=1}^m \Big[\Big(\int\limits_M \varphi_j dx \Big)^{(1-\alpha)} \int\limits_M \varphi_j ||u||^2 dx \Big]^{q/2} dt \leq \\ &\leq \quad \mathbb{E} \int_0^T \sum_{j=1}^m \Big[\Big(\int\limits_M \varphi_j dx \Big)^{(1-\alpha)q/2} \Big(\int\limits_M \varphi_j dx \Big)^{q/2-1} \int\limits_M \varphi_j ||u||^q dx \Big]^{q/2} dt. \end{aligned}$$

Above we have also applied the inequality (2.23) with $r = q/2$ to the last multiple in brackets: $\int\limits_M \varphi_j ||u||^2 dx$. Since for $\alpha = \frac{2(q-1)}{q}$ we have $\frac{(1-\alpha)q}{2} + \frac{q}{2} - 1 = 0$ we obtain the required estimate (2.21). □

Corollary 2.14. *With the assumptions of Proposition 2.11, if we furthermore assume that the Itô stochastic flow g_t has finite q-energy $\mathcal{E}_q(g)$ then the associated g-flow Θ_t^g also has finite q-energy $\mathcal{E}'_q(\Theta^g)$.*

Theorem 2.15. *Let $q > 1$ an g_t be an Itô stochastic flow. Then*

$$\mathcal{E}'_q(\Theta^g) = \mathcal{E}_q(g).$$

Proof. Because of previous theorem it is sufficient to prove

$$\mathcal{E}'_q(\Theta^g) \geq \mathcal{E}_q(g).$$

First note that for any fixed $\varepsilon > 0$ we may choose functions $\varphi_1^\varepsilon, ..., \varphi_{m_\varepsilon}^\varepsilon$ which satisfy $\sum\limits_{j=1}^m \varphi_j(x) = 1$ on M such that $supp\,(\varphi_j^\varepsilon) \subset B(x_j, \varepsilon)$ and for some fixed δ independent on ε and j:

$$\int\limits_M \varphi_j^\varepsilon(x) dx \geq \delta \varepsilon^d. \tag{2.24}$$

Using notation (2.22) and $b = (b_1, \dots, b_\ell) \in \mathbb{R}^\ell$ together with $\|b\| = \|b\|_{\mathbb{R}^\ell}$ the Euclidean norm, we may write

$$
\begin{aligned}
0 \;\le\; & \mathcal{E}_q(g) - \mathcal{E}'_q(\Theta^g) \le \frac{1}{q}\mathbb{E}\int\limits_0^T \left\{ \int\limits_M \|b(y)\|^q dy - \sum_{j=1}^m \frac{\left\| \int\limits_M \varphi_j(x) b(x) dx \right\|^q}{\left(\int\limits_M \varphi_j(x) dx \right)^{q-1}} \right\} dt = \\
= \; & \frac{1}{q}\mathbb{E}\int\limits_0^T \int\limits_M \sum_{j=1}^m \varphi_j(y) \left\{ \|b(y)\|^q - \frac{\left\| \int\limits_M \varphi_j(x) b(x) dx \right\|^q}{\left(\int\limits_M \varphi_j(x) dx \right)^q} \right\} dy\, dt \le \\
(2.25) \;=\; & \mathbb{E}\int\limits_0^T \int\limits_M \sum_{j=1}^m \varphi_j(y) \|b(y)\|^{q-2} \Big\langle b(y), b(y) - \frac{\int\limits_M \varphi_j(x) b(x) dx}{\int\limits_M \varphi_j(x) dx} \Big\rangle\, dy\, dt.
\end{aligned}
$$

Above we have used the convexity of the function $f(x) = x^q$ and applied the inequality

$$f(x) - f(x_0) \le \langle f'(x), x - x_0 \rangle.$$

Conducting the change of variables we continue

$$
\begin{aligned}
(2.25) = \; & \mathbb{E}\int\limits_0^T \sum_{j=1}^m \frac{1}{\int\limits_M \varphi_j dx} \int\limits_M \varphi_j(y) \|b(y)\|^{q-2} \Big\langle b(y), \int\limits_M b(y)\varphi_j(x) dx - \int\limits_M \varphi_j(x) b(x) dx \Big\rangle\, dy\, dt = \\
= \; & \mathbb{E}\int\limits_0^T \sum_{j=1}^m \frac{1}{\int\limits_M \varphi_j dx} \int\limits_{M\times M} \varphi_j(y) \|b(y)\|^{q-2} \big\langle b(y), b(y)\varphi_j(x) - \varphi_j(x) b(x) \big\rangle\, dx\, dy\, dt = \\
= \; & \mathbb{E}\int\limits_0^T \sum_{j=1}^m \frac{1}{\int\limits_M \varphi_j dx} \int\limits_{B(0,\varepsilon)} \int\limits_M \varphi_j(y) \|b(y)\|^{q-2} \big\langle b(y), \varphi_j(y+r)\big(b(y) - b(y+r)\big) \big\rangle dy\, dr\, dt \le \\
\le \; & \frac{1}{\delta\varepsilon^d} \int\limits_{B(0,\varepsilon)} \sum_{j=1}^m \mathbb{E}\int\limits_0^T \int\limits_M \varphi_j(y) \|b(y)\|^{q-1} \|b(y) - b(y+r)\| dy\, dt\, dr \le \\
\le \; & \frac{1}{\delta\varepsilon^d} \int\limits_{B(0,\varepsilon)} \left(\mathbb{E}\int\limits_0^T \int\limits_M \|b(y)\|^q dy\, dt \right)^{\frac{q-1}{q}} \cdot \left(\mathbb{E}\int\limits_0^T \int\limits_M \|b(y) - b(y+r)\|^q dy\, dt \right)^{1/q} dr \le \\
\le \; & \frac{Vol(B(0,\varepsilon))}{\delta\varepsilon^d}\, q^{\frac{q-1}{q}} \mathcal{E}(g)^{\frac{q-1}{q}} \sup_{r\in B(0,\varepsilon)} \|\tau_r b - b\|_q,
\end{aligned}
$$

where $\|\cdot\|$ is a standard L_q-norm of vector functions on M and $\tau_a b(y) = b(y-a)$. Above we also used the property (2.24) of functions φ_j. Due to the continuity in $L_q(M^\ell)$ of the map $a \mapsto \tau_a b$ for $b \in L_q(M^\ell)$, the above expression tends to zero as $\varepsilon \to 0$, that finishes the proof.

□

3. Existence of generalized flow minimizing the energy

In this section we investigate the conditions under which there exists a generalized flow Θ_t, which minimizes the energy functional $\mathcal{E}'_q(\Theta)$ defined in (2.19).

Definition 3.1. For $q > 1$, let $\mathcal{H}_q(\eta) = \mathcal{H}_q(\sigma, \eta, T)$ the set of laws of incompressible Itô stochastic flows g_t solutions to (2.1), satisfying the requirements of Proposition 2.3, with finite q-energy $\mathcal{E}_q(g)$ and endpoints configuration η.

We let

$$\mathcal{E}_q(\eta) = \mathcal{E}_q(\sigma, \eta, T) := \inf \{\mathcal{E}_q(g), \quad \mathrm{Law}(g) \in \mathcal{H}_q(\eta)\} \tag{3.1}$$

where by convention $\mathcal{E}_q(\eta) = \infty$ if $\mathcal{H}_q(\eta)$ is empty.

Definition 3.2. For $q > 1$, let $\mathcal{H}'_q(\eta) = \mathcal{H}'_q(\sigma, \eta, T)$ the set of laws of generalized flows Θ_t of Definition 2.7, with finite q-energy (2.19) and endpoints configuration η.

We let

$$\mathcal{E}'_q(\eta) = \mathcal{E}'_q(\sigma, \eta, T) := \inf \left\{\mathcal{E}'_q(\Theta), \quad \mathrm{Law}(\Theta) \in \mathcal{H}'_q(\eta)\right\} \tag{3.2}$$

with by convention $\mathcal{E}'(\eta) = \infty$ if there is no generalized flow with endpoints configuration η.

Remark 3.3. From Proposition 2.11 and Theorem 2.15 we have

$$\mathcal{E}'_q(\eta) \leq \mathcal{E}_q(\eta). \tag{3.3}$$

Theorem 3.4. *If $\mathcal{E}'_q(\eta) < \infty$ then there exists a generalized flow Θ with law in $\mathcal{H}'_q(\eta)$, such that $\mathcal{E}'_q(\Theta) = \mathcal{E}'_q(\eta)$. In other words, the infimum of the q-energy functional for given endpoints configuration η is achieved on an element of the set $\mathcal{H}'_q(\eta)$.*

Proof. Assume $\mathcal{E}'_q(\eta) < \infty$. Let $\{\Theta^n\}_{n\geq 1}$ be a sequence of generalized flows with laws in $\mathcal{H}'_q(\eta)$ with energies converging to the infimum value for given endpoints configuration η:

$$\lim_{n\to\infty} \mathcal{E}'_q(\Theta^n) = \mathcal{E}'_q(\eta). \tag{3.4}$$

Consider two sequences $\Phi = \{\Phi^j\}_{j\geq 1}$, $\Psi = \{\Psi^k\}_{k\geq 1}$ of elements of $C^\infty(M)$, where Φ is dense in the topology of uniform convergence, and Ψ is dense in the topology of uniform convergence of functions and their first and second order derivatives.

We prove that there exists a subsequence $\{\Theta^{n_\ell}\}_{\ell\geq 1}$ of generalized flows such that the family of semimartingales $\left\{\Theta^{n_\ell}_t(\Phi^j, \Psi^k)\right\}_{j,k\geq 1}$ converges as ℓ goes to infinity to some family of semimartingales $\left\{\Theta_t(\Phi^j, \Psi^k)\right\}_{j,k\geq 1}$, and all $\Theta_t(\Phi^j, \Psi^k)$, $j, k \geq 1$ are defined in the same probability space.

For this it is sufficient to prove that for fixed $j, k \geq 1$ and $\Phi^j \in \Phi$, $\Psi^k \in \Psi$ the family of semimartingales $\left\{\Theta^n_t(\Phi^j, \Psi^k)\right\}_{n\geq 1}$ is tight. A diagonal extraction of subsequence then gives the result.

Proving that the family $\left\{\Theta^n_t(\Phi^j, \Psi^k)\right\}_{n\geq 1}$ is tight can be done as in [3] by checking the conditions of Theorem 3 in [24]. For any fixed $n \in \mathbb{N}$, define $\{Y^n_i\}_{i\in\mathbb{N}}$ as the family of all $\tilde{\Theta}^n(\Phi^j, \Psi^k)$ and $\frac{1}{2}\int_0^{\cdot} \Theta^n_s(\Phi^j, \Delta\Psi^k)\, ds$, for $j, k \geq 1$ and all their covariations enumerated in proper order. They can be renamed this way because there is a countable number of them. Like in [3] and due to Lemma 2.12 we have:

$$\begin{aligned} \mathbb{E}\int_0^T \left|D\tilde{\Theta}^n_t(\Phi^j, \Psi^k)\right|^q dt &\leq q2^q \left\|\Phi^j\right\|^q_{L^\infty(M)} \left\|\nabla\Psi^k\right\|^q_{L^\infty(M)} \mathcal{E}'_q(\Theta^n) \\ &\leq q2^q \left\|\Phi^j\right\|^q_{L^\infty(M)} \left\|\nabla\Psi^k\right\|^q_{L^\infty(M)} (\mathcal{E}'_q(\eta) + 1) \end{aligned} \tag{3.5}$$

for n sufficiently large. Due to (5) in Definition 2.7 we also have

$$\mathbb{E}\left[\int_0^T \left|\Theta_s^n(\Phi^j,\Delta\Psi^k)\right|^2 ds\right] \le T\left\|\Phi^j\right\|_{L^2(M)}^2 \left\|\Delta\Psi^k\right\|_{L^2(M)}^2. \tag{3.6}$$

On the other hand due to (2.17) the covariation of $\tilde{\Theta}_t^n(\Phi^j,\Psi^k)$ satisfies

$$d\left[\tilde{\Theta}^n(\Phi^j,\Psi^k),\tilde{\Theta}^n(\Phi^j,\Psi^k)\right]_t \le \left\|\Phi^j\right\|_{L^2(M)}^2 \left\|\nabla\Psi^k\right\|_{L^2(M)}^2 dt. \tag{3.7}$$

Therefore the derivative of this covariation is uniformly bounded.

From (3.5), (3.6) and (3.7), by Theorem 3 in [24], for any fixed $d\in\mathbb{N}$, there exists a subsequence $n\mapsto\gamma_d(n)$ such that the sequence of vectors $\left\{(Y_i^{\gamma_d(n)})_{1\le i\le d}\right\}_{n\in\mathbb{N}}$ converges in law to some vector $(Y_i^d)_{1\le i\le d}$. By a diagonal extraction together with Kolmogorov extension theorem, we can find a subsequence $n\mapsto\gamma(n)$ and a family of random variables $(Y_i)_{i\in\mathbb{N}}$ such that for all $d\in\mathbb{N}$, the sequence of finite vectors $\left\{(Y_i^{\gamma(n)})_{1\le i\le d}\right\}_{n\in\mathbb{N}}$ converges to the vector $(Y_i)_{1\le i\le d}$. For simplicity we rename $Y_i^{\gamma(n)}$ by Y_i^n. Denote by

$$\tilde{\Theta}_t(\Phi^j,\Psi^k) = \mathscr{L}\lim_{n\to\infty}\tilde{\Theta}_t^n(\Phi^j,\Psi^k) \tag{3.8}$$

and

$$A_t(\Phi^j,\Psi^k) = \mathscr{L}\lim_{n\to\infty}\frac{1}{2}\int_0^t \Theta_s^n(\Phi^j,\Delta\Psi^k)\,ds, \tag{3.9}$$

where $\mathscr{L}\lim$ denotes limit in law. Then the required limiting process is

$$\Theta_t(\Phi^j,\Psi^k) := \tilde{\Theta}_t(\Phi^j,\Psi^k) + A_t(\Phi^j,\Psi^k). \tag{3.10}$$

Now we are going to prove that the process Θ_t may be extended to generalized flow in the sense of Definition 2.7, which satisfies the endpoints condition η and minimizes the q-energy functional $\mathcal{E}_q'(\Theta)$.

From the procedure above for the construction of the sequence $Y_i^n := Y_i^{\gamma(n)}$ we have that for any fixed d all linear combinations of Y_i^n, $1\le i\le d$ converge to the same linear combination of Y_i, $1\le i\le d$, since linear combinations are continuous functions of finite families.

As for the starting point we have

$$\tilde{\Theta}_0(\Phi^j,\Psi^k) = (\Phi^j,\Psi^k)_{L^2(M)}. \tag{3.11}$$

By Theorem 10 in [18] (which is stated for $q=2$ but extends to any $q>1$ using Hölder inequality), we have

$$\mathbb{E}\left[\int_0^T dt\left(D\tilde{\Theta}_t(\Phi^j,\Psi^k)\right)^q\right] \le \limsup_{n\to\infty}\mathbb{E}\left[\int_0^T dt\left(D\tilde{\Theta}_t^n(\Phi^j,\Psi^k)\right)^q\right]. \tag{3.12}$$

This by Lemma 2.12 implies

$$\mathbb{E}\left[\int_0^T dt\left(D\tilde{\Theta}_t(\Phi^j,\Psi^k)\right)^q\right] \le q2^q\left\|\Phi^j\right\|_{L^\infty(M)}^q\left\|\nabla\Psi^k\right\|_{L^\infty(M)}^q(\mathcal{E}_q'(\eta)+1). \tag{3.13}$$

We have that all covariations of Y_i^n and Y_j^n converge to covariation of Y_i, Y_j, this is due to Theorem 3 in [24]. So inequality (3.7) also extends to the limiting process:

$$d\left[\tilde{\Theta}(\Phi^j,\Psi^k),\tilde{\Theta}(\Phi^j,\Psi^k)\right]_t \le \left\|\Phi^j\right\|_{L^2(M)}^2\left\|\nabla\Psi^k\right\|_{L^2(M)}^2 dt. \tag{3.14}$$

Furthermore, by bilinearity of all the $\tilde{\Theta}_t^n$, statements (3.11), (3.13) and (3.14) are still true with functions φ and ψ which are linear combinations of Φ^j and Ψ^k. Moreover the limit $\tilde{\Theta}_t$ is also bilinear for theses combinations.

Thus by (3.10) we have defined $\Theta_t(\Phi^j, \Psi^k)$ for $\{\Phi^j\}_{j\geq 1}$ and $\{\Psi^k\}_{k\geq 1}$ dense in $C^\infty(M)$ in corresponding topology. Our next task is to define it for any $\varphi, \psi \in C^\infty(M)$. Let us emphasize the fact that for all j, k the $\Theta_t(\Phi^j, \Psi^k)$ are defined on the same filtered probability space which will also serve to define the $\Theta_t(\varphi, \psi)$.

For $\varphi, \psi \in C^\infty(M)$ there exist subsequences $\{\Phi^{j_\ell}\}_{\ell\geq 1}$ and $\{\Psi^{k_\ell}\}_{\ell\geq 1}$ which converge uniformly to φ and ψ (for the second sequence uniform convergence holds for functions and their first order derivatives). From (3.11), (3.13), and (3.14) and the bilinearity of $\tilde{\Theta}_t$ we deduce that $\tilde{\Theta}_t(\Phi^{j_\ell}, \Psi^{k_\ell})$ converges to a semimartingale $\tilde{\Theta}(\varphi, \psi)$ which does not depend on the choice of subsequences $\{\Phi^{j_\ell}\}_{\ell\geq 1}$ and $\{\Psi^{k_\ell}\}_{\ell\geq 1}$. Here the convergence is taken in the topology of L^q convergence of the drift and the convergence of the quadratic variation (the so-called H_q topology). It is also easy to check that $(\varphi, \psi) \mapsto \tilde{\Theta}_t(\varphi, \psi)$ is bilinear map and that for all $\varphi, \psi \in C^\infty(M)$, $\tilde{\Theta}_t(\varphi, \psi)$ is the limit in law of $\{\tilde{\Theta}_t^n(\varphi, \psi)\}_{n\geq 1}$ (the last point is due to the fact that the bounds in the right of (3.5) and (3.7) can be taken indepent of n, and this allows to identify the limit of a subsequence of $\{\tilde{\Theta}_t^n(\varphi, \psi)\}_{n\geq 1}$ with $\tilde{\Theta}_t(\varphi, \psi)$). Similarly, for $\varphi_1, \psi_1, \varphi_2, \psi_2 \in C^\infty(M)$, the process $[\tilde{\Theta}(\varphi_1, \psi_1), \tilde{\Theta}(\varphi_2, \psi_2)]_t$ is the limit in law of $\{[\tilde{\Theta}^n(\varphi_1, \psi_1), \tilde{\Theta}^n(\varphi_2, \psi_2)]_t\}_{n\geq 1}$.

From (3.9), by bilinearity of Θ_t^n and (3.6) we have, by the same subsequence procedure, that for all $\varphi, \psi \in C^\infty(M)$

$$A_t(\varphi, \psi) = \mathscr{L} \lim_{n\to\infty} \frac{1}{2} \int_0^t \Theta_s^n(\varphi, \Delta\psi). \tag{3.15}$$

This together with (3.8) yields

$$\Theta_t(\varphi, \psi) = \mathscr{L} \lim_{n\to\infty} \Theta_t^n(\varphi, \psi) \tag{3.16}$$

and in particular

$$\Theta_t(\varphi, \Delta\psi) = \mathscr{L} \lim_{n\to\infty} \Theta_t^n(\varphi, \Delta\psi). \tag{3.17}$$

So by integration we get

$$\int_0^t \Theta_s(\varphi, \Delta\psi) = \mathscr{L} \lim_{n\to\infty} \int_0^t \Theta_s^n(\varphi, \Delta\psi). \tag{3.18}$$

Finaly by identifying the limits in (3.15) and (3.18) we have

$$A_t(\varphi, \psi) = \frac{1}{2} \int_0^t \Theta_s(\varphi, \Delta\psi)\, ds. \tag{3.19}$$

So for $\varphi, \psi \in C^\infty(M)$

$$\Theta_t(\varphi, \psi) = \tilde{\Theta}_t(\varphi, \psi) + \frac{1}{2} \int_0^t \Theta(\varphi, \Delta\psi)\, ds. \tag{3.20}$$

It remains to prove that $\operatorname{Law} \Theta_t$ is an element of $\mathcal{H}'(\eta)$ and that

$$\mathcal{E}_q'(\Theta) \leq \mathcal{E}_q'(\eta). \tag{3.21}$$

By passing to the limit in (1)-(6), of Definition 2.7 for Θ_t^n we get them for Θ_t .

We are left to prove (3.21). For this we need to improve (3.13).

For $\ell \geq 1$, $\varphi, \psi_1, \ldots, \psi_\ell \in C^\infty(M)$, $\psi = (\psi_1, \ldots, \psi_\ell)$, let

$$|D\tilde{\Theta}_t^n(\varphi, \psi)| := \left(\sum_{k=1}^{\ell} D\tilde{\Theta}_t^n(\varphi, \psi_k)^2 \right)^{1/2}.$$

Theorem 10 in [18] (extended to any $q > 1$ says that for any $\varphi, \psi \in C^\infty(M)$ and any $K > 0$, if for all $n \geq 1$

$$\mathbb{E}\left[\int_0^T \left| D\tilde{\Theta}_t^n(\varphi, \psi) \right|^q dt \right] \leq K$$

then

$$\mathbb{E}\left[\int_0^T \left| D\tilde{\Theta}_t(\varphi, \psi) \right|^q dt \right] \leq K.$$

Let

$$K = \liminf_{n \to \infty} \mathbb{E}\left[\int_0^T \left| D\tilde{\Theta}_t^n(\varphi, \psi) \right|^q dt \right].$$

Consider a subsequence Θ^{n_ℓ} such that

$$\lim_{\ell \to \infty} \mathbb{E}\left[\int_0^T \left| D\tilde{\Theta}_t^{n_\ell}(\varphi, \psi) \right|^q dt \right] = K.$$

Then fixing $\varepsilon > 0$ and applying the above result to the sequence $\tilde{\Theta}^{n_\ell}$ for sufficiently large ℓ we obtain

$$\mathbb{E}\left[\int_0^T \left| D\tilde{\Theta}_t(\varphi, \psi) \right|^q dt \right] \leq K + \varepsilon.$$

Letting $\varepsilon \to 0$ we obtain

$$\mathbb{E}\left[\int_0^T \left| D\tilde{\Theta}_t(\varphi, \psi) \right|^q dt \right] \leq \liminf_{n \to \infty} \mathbb{E}\left[\int_0^T \left| D\tilde{\Theta}_t^n(\varphi, \psi) \right|^q dt \right]. \tag{3.22}$$

Letting φ^j, ψ^k as in (2.19) with $\psi = (\psi_1, ..., \psi_\ell)$, we have

$$\begin{aligned}
\sum_{j=1}^m \mathbb{E}\left[\int_0^T \frac{\left| D\tilde{\Theta}_t(\varphi^j, \psi) \right|^q}{\Theta_t(\varphi^j, 1)^\alpha} dt \right] &\leq \sum_{j=1}^m \liminf_{n \to \infty} \mathbb{E}\left[\int_0^T \frac{\left| D\tilde{\Theta}_t^n(\varphi^j, \psi) \right|^q}{\Theta_t^n(\varphi^j, 1)^\alpha} dt \right] \\
&\leq \liminf_{n \to \infty} \sum_{j=1}^m \mathbb{E}\left[\int_0^T \frac{\left| D\tilde{\Theta}_t^n(\varphi^j, \psi) \right|^q}{\Theta_t^n(\varphi^j, 1)^\alpha} dt \right] \\
&\leq \mathcal{E}_q'(\eta).
\end{aligned}$$

Finally taking the supremum in the left as in (2.19) yields

$$\mathcal{E}_q'(\Theta) \leq \mathcal{E}_q'(\eta) \tag{3.23}$$

and this achieves the proof. □

4. Constructing generalized flows with prescribed drift

We have shown in section 2 that semimartingale flows can be regarded as generalized ones. In particular (considering $q = 2$) a smooth solution of Navier-Stokes equation will thus give rise to a generalized solution of the corresponding variational problem.We shall now show that these are not the only possible generalized flows: indeed, we can define weaker solutions (which, in particular, will not necessarily correspond to semimartingale flows) of the Navier-Stokes variational problem built upon weak solutions of some transport equations.

Consider a deterministic drift $b(t,x)$ such that $b \in L^1([0,T], L^q(TM))$ for some $q > 1$, and $\operatorname{div} b \equiv 0$ in the weak sense. The following result states existence of generalized flows with drift b (see [14], [15] and [17] for related results).

Theorem 4.1. *There exists a generalized flow Θ_t with drift $b(t,x)$ i.e. such that for all $\varphi, \psi \in C^\infty(M)$, $(t,\omega) \in [0,T] \times \Omega$ almost everywhere*

$$D\tilde{\Theta}_t(\varphi,\psi) = \Theta_t(\varphi, \operatorname{div}(\psi b)), \tag{4.1}$$

and with kinetic energy smaller than or equal to $\frac{1}{q}\int_0^T \left(\int_M \|b(t,x)\|^q \, dx\right) dt$.

Proof. For $\varepsilon > 0$, as in [17] Section 4.4 we regularize b by using the de Rham-Hodge semi-group on differential forms $e^{\varepsilon\Box}$ with $\Box = -(d\delta + \delta d)$, δ the codifferential form of d. For a differential 1-form α on M we denote by $\alpha^\sharp$ the vector field on M associated to α by the metric, and for a vector field A on M we denote by $A^\flat$ the differential 1-form associated to A by the metric: we have

$$\langle \alpha, A\rangle = \langle \alpha^\sharp, A\rangle = \langle \alpha, A^\flat\rangle$$

where the first bracket is for duality, the second one is the scalar product in TM, the third one the scalar product in T^*M. By letting

$$(b')^\varepsilon = \left(e^{\varepsilon\Box}(b^\flat)\right)^\sharp \tag{4.2}$$

we get a smooth time-dependent vector field satisfying $\operatorname{div}(b')^\varepsilon = 0$ (see [17] Proposition 4.4.1). Then we regularize $(b')^\varepsilon$ in time by convolution with a smooth kernel with support $[-\varepsilon/2, \varepsilon/2]$ (for this we need to extend b by letting $b(t,x) = 0$ for $t < 0$ and for $t > T$). Let us call $b^\varepsilon(t,x)$ the regularized vector field. It is also divergence free since the divergence operator commutes with the time integration, and it approximates b in $(L^q([0,T] \times M, TM))$. For each $\varepsilon > 0$ we can construct a semimartingale flow as a strong solution to (2.1) where $u(t, g(t)(x), \omega)$ has been replaced by $b^\varepsilon(t, g(t)(x))$. Let us denote by g^ε the solution. Letting $(\varepsilon_n)_{n\geq 0}$ a sequence of positive numbers decreasing to 0 we let $\Theta^n = \Theta^{g^{\varepsilon_n}}$. Now using the fact that

$$\mathscr{E}(g^\varepsilon) = \frac{1}{q}\int_0^T \left(\int_M \|b^\varepsilon(t,x)\, dt\|^q \, dx\right) dt \leq \frac{1}{q}\int_0^T \left(\int_M \|b(t,x)\, dt\|^q \, dx\right) dt + 1$$

for ε sufficiently small we proceed similarly to the proof of Theorem 3.4 to establish that possibly by extracting a subsequence, there exists a generalized flow Θ_t such that for all $\varphi, \psi \in C^\infty(M)$ $\Theta^n(\varphi,\psi)$ converges in law to $\Theta(\varphi,\psi)$ and $\int_0^\cdot D\tilde{\Theta}^n_s(\varphi,\psi)\, ds$ converges in law to the drift of $\tilde{\Theta}(\varphi,\psi)$. We have $D\tilde{\Theta}^n(\varphi,\psi) = \Theta^n(\varphi, \operatorname{div}(\psi\, b^{\varepsilon_n}))$ which is defined as $\int_M \varphi(x) \operatorname{div}(\psi\, b^{\varepsilon_n} g(\cdot)(x))\, dx$. Since b is time-dependent and not smooth we have to extend this definition. We let $(\tilde{\alpha}^i)_{i\geq 1}$ be

a family of smooth functions $[0,T] \to \mathbb{R}$ such that, possibly by extending the family Ψ^k defined in the proof of Theorem 3.4, linear combinations of functions $(t,x) \mapsto \tilde{\alpha}^i(t)\Psi^k(x)$ with rational coefficients are dense in $L^q([0,T]\times M)$. Now for all $\beta(t,x) = \sum_{\ell=1}^L a_\ell \tilde{\alpha}^{i_\ell}(t)\Psi^{k_\ell}(x)$ with rational a_ℓ we can assume that the processes

$$t \mapsto \int_0^t \Theta_s^n(\Phi^j, \beta(s,\cdot))\, ds = \sum_{\ell=1}^L a_\ell \int_0^t \Theta_s^n(\Phi^j, \Psi^{k_\ell})\tilde{\alpha}^{i_\ell}(s)\, ds$$

all together converge in law as $n \to \infty$ to processes $t \mapsto \int_0^t \Theta_s(\Phi^j, \beta(s,\cdot))\, ds$ which satisfy

$$\mathbb{E}\left[\int_0^T \Theta_s(\Phi^j, \beta(s,\cdot))^q\, ds\right] \leq \|\Phi^j\|_q^p \|\beta\|^q_{L^q([0,T]\times M)}, \tag{4.3}$$

from the fact that for all $n \geq 1$

$$\mathbb{E}\left[\int_0^T \Theta_s^n(\Phi^j, \beta(s,\cdot))^q\, ds\right] \leq \|\Phi^j\|_p^q \|\beta\|^q_{L^q([0,T]\times M)}. \tag{4.4}$$

Here p is the exponent conjugate to q: $p = \dfrac{q}{q-1}$.

This bound allows to define $t \mapsto \int_0^t \Theta_s(\varphi, \beta(s,\cdot))\, ds$ for all smooth φ and $\beta \in L^q([0,T]\times M)$. So $t \mapsto \int_0^t \Theta_s(\varphi, \mathrm{div}(\psi\, b))\, ds$ is well defined (recall that $\mathrm{div}(\psi b) = \langle d\psi, b\rangle$), and by an argument similar to the proof of Theorem 3.4 we see that $t \mapsto \int_0^t \Theta_s^n(\varphi, \mathrm{div}(\psi\, b^{\varepsilon_n}))\, ds$ converges in law to $t \mapsto \int_0^t \Theta_s(\varphi, \mathrm{div}(\psi\, b))\, ds$. So we can make the identification $\int_0^t D\tilde{\Theta}_s(\varphi,\psi)\, ds = \int_0^t \Theta_s(\varphi, \mathrm{div}(\psi\, b))\, ds$.

We are left to prove the bound for the kinetic energy. But this is exactly similar to the proof of Theorem 2.13. □

5. Constructing generalized flows from solutions of finite variation transport equations

In this section we aim to give an alternative construction of generalized flow with prescribed drift, using Ocone Pardoux method and weak solution of transport equations (in the sense of DiPerna and Lions).

To start with, let us consider a semimartingale flow $g(t)(x)$ satisfying $g(0)(x) \equiv x$ and

$$dg(t)(x) = \sigma(g(t)(x)) \circ dW_t + b(t, g(t)(x), \omega)dt, \tag{5.1}$$

with the same assumptions as in the beginning of section 2. In particular the vector fields σ_i are divergence free. Assume that σ and b are C^1 in the space variable. Let $\tilde{g}(t)(x)$ be the martingale flow satisfying

$$d\tilde{g}(t)(x) = \sigma(\tilde{g}(t)(x)) \circ dW_t, \qquad \tilde{g}(0)(x) \equiv x. \tag{5.2}$$

Notice that $\tilde{g}(t)$ is measure preserving. The method of Ocone and Pardoux ([20]) consists in writing

$$g(t)(x) = \tilde{g}(t)\left(\psi(t)(x)\right) \tag{5.3}$$

with $\psi(t)(x)$ a bounded variation flow to be determined. From (5.3) we get

$$dg(t)(x) = (d\tilde{g}(t))\left(\psi(t)(x)\right) + T_{\psi(t)}\tilde{g}(t)\left(d\psi(t)(x)\right) \tag{5.4}$$

where $T_{\psi(t)}\tilde{g}(t)$ is the tangent map at $\psi(t)$ to $x \mapsto \tilde{g}(t)(x)$ (we omit the dependence in ω). This together with (5.1) and (5.2) yields

$$d\psi(t)(x) = \tilde{b}(t, \psi(t)(x), \omega)\, dt \tag{5.5}$$

with

$$\tilde{b}(t, y, \omega) = \left(T_y\tilde{g}(t)(\cdot)\right)^{-1} \left(b(t, \tilde{g}(t)(y), \omega)\right). \tag{5.6}$$

For $\varphi \in C^\infty(M)$ define $\theta_t^{g,\varphi}$, $\theta_t^{\tilde{g},\varphi}$, $\theta_t^{\psi,\varphi}$ as: for $\phi \in C^\infty(M)$

(5.7)

$$\Theta^g(\varphi, \phi) = (\theta_t^{g,\varphi}, \phi)_{L^2(M)}\,,\ \Theta^{\tilde{g}}(\varphi, \phi) = \left(\theta_t^{\tilde{g},\varphi}, \phi\right)_{L^2(M)}, \Theta^{\psi}(\varphi, \phi) = \left(\theta_t^{\psi,\varphi}, \phi\right)_{L^2(M)}.$$

From (2.16) and (5.3) we get

$$\begin{aligned} \theta_t^{g,\varphi} &= \varphi \circ g(t)^{-1} \\ &= \varphi \circ \psi(t)^{-1} \circ \tilde{g}(t)^{-1}. \end{aligned} \tag{5.8}$$

and this yields

$$\Theta_t^g(\varphi, \phi) = \Theta_t^{\tilde{g}(t)} \left(\theta_t^{\psi,\varphi}, \phi\right) \tag{5.9}$$

which implies

$$\Theta_t^g(\varphi, \phi) = \Theta_t^{\psi}(\varphi, \phi \circ \tilde{g}(t)) \tag{5.10}$$

where we used the fact that $\tilde{g}(t)$ is measure preserving.

Lemma 5.1. *We have for all $\phi \in C^\infty(M)$, $t \in [0, T]$, a.s.*

$$\int_M \left(\operatorname{div} \tilde{b}(t, \cdot, \omega)\right)(x)\phi(x)\, dx = \int_M \operatorname{div} b(t, \cdot, \omega)(y) \left(\phi \circ (\tilde{g}(t))^{-1}\right)(y)\, dy. \tag{5.11}$$

In particular, if $\operatorname{div} b(t, \cdot, \omega) \equiv 0$ *then* $\operatorname{div} \tilde{b}(t, \cdot, \omega) \equiv 0$.

Proof. We will write $b(y) = b(t, y, \omega)$, $\tilde{b}(x) = \tilde{b}(t, x, \omega)$ For $\phi \in C^\infty(M)$,

$$\begin{aligned} \int_M (\operatorname{div} \tilde{b})\phi &= \int_M \langle d\phi, \tilde{b}\rangle \\ &= \int_M \left\langle d\phi, (T\tilde{g})^{-1} \circ b \circ \tilde{g}\right\rangle \\ &= \int_M \left\langle d\left(\phi \circ (\tilde{g})^{-1}\right)(\tilde{g}(x)), b(\tilde{g}(x))\right\rangle\, dx \\ &= \int_M \left\langle d\left(\phi \circ (\tilde{g})^{-1}\right)(y), b(y)\right\rangle\, dy \\ &= \int_M \left(\phi \circ (\tilde{g})^{-1}\right)(y)(\operatorname{div} b)(y)\, dy \end{aligned}$$

where we used in the fourth equality the fact that $\tilde{g}$ is measure preserving. □

Now consider a deterministic drift $b(t, x)$ such that $b \in L^1([0, T], L^q(TM))$ for some $q > 1$, and $\operatorname{div} b \equiv 0$ in the weak sense. It is easily seen that Lemma 5.1 is still valid for b, so we have a.s. $\operatorname{div} \tilde{b} \equiv 0$. Moreover a.s. $\tilde{b} \in L^1([0, T], L^q(TM))$. Under this condition we can apply Proposition II.1 in [9] and we deduce that a.s. the transport equation which is the weak version of (5.5), namely

$$\frac{\partial \theta_t}{\partial t} = -(\tilde{b} \cdot \nabla)\theta_t, \qquad \theta_0 = \varphi, \qquad \varphi \in C^\infty(M), \tag{5.12}$$

has a solution $\theta_t^{\tilde b,\varphi}$ in $\bigcap_{p\geq 1} L^\infty(0,T;L^p(M))$. Moreover since $\tilde b$ is adapted the process $\theta_t^{\tilde b,\varphi}$ can also be chosen so that all $\theta_t^{\tilde b,\varphi},\varphi\in C^\infty(M)$ and also $\tilde g, W$ are jointly adapted to the same filtration for which W is still a cylindrical Brownian motion and $\tilde g$ satisfies (5.2): instead of considering $\theta_t^{\tilde b,\varphi}$ as the limit in law of some regularized processes $\theta_t^{\tilde b^\varepsilon,\varphi}$, consider $((\theta_t^{\tilde b,\tilde\varphi^j},\tilde g,W)$ as limit in law of $((\theta_t^{\tilde b^\varepsilon,\tilde\varphi^j},\tilde g,W)$ for $(\tilde\varphi^j)_{j\geq 1}$ a dense subsequence in $C^\infty(M)$.

So by analogy to (5.9) we define

$$\Theta_t^{\sigma,b}(\varphi,\phi)=\Theta_t^{\tilde g}\left(\theta_t^{\tilde b,\varphi},\phi\right),\qquad \varphi,\phi\in C^\infty(M) \tag{5.13}$$

Proposition 5.2. *Take a deterministic drift $b(t,x)$ such that $b\in L^1([0,T],L^q(TM))$ for some $q>1$, and $\operatorname{div} b\equiv 0$ in the weak sense. Then the generalized flow $\Theta^{\sigma,b}$ defined in equation (5.13) is a generalized flow with kinetic energy*

$$\mathscr{E}_q'\left(\Theta^{\sigma,b}\right)\leq \mathscr{E}_q(b) \tag{5.14}$$

where

$$\mathscr{E}_q(b)=\frac{1}{2}\int_0^T dt\int_M dx\|b(t,x)\|^q \tag{5.15}$$

Proof. We have

$$\begin{aligned}\Theta^{\sigma,b}(\varphi,\phi)=\Theta_t^{\tilde g}\left(\theta_t^{\tilde b,\varphi},\phi\right)&=\int_M \theta_t^{\tilde b,\varphi}\left((\tilde g(t))^{-1}(x)\right)\phi(x)\,dx\\&=\int_M \theta_t^{\tilde b,\varphi}(x)\,\phi\left(\tilde g(t)(x)\right)\,dx\end{aligned} \tag{5.16}$$

and this implies

(5.17)

$$\begin{aligned}&d\tilde\Theta^{\sigma,b}(\varphi,\phi)\\&=\int_M \theta_t^{\tilde b,\varphi}(x)\left\langle d\phi,\ d^{\mathrm{It\hat o}}\tilde g(t)(x)\right\rangle\,dx+\int_M \theta_t^{\tilde b,\varphi}(x)\operatorname{div}\left(\tilde b(\phi\circ\tilde g(t))\right)(x)\,dx\,dt\\&=\int_M \theta_t^{\tilde b,\varphi}(x)\left\langle d\phi,\sigma\left(\tilde g(t)(x)\right)dW_t\right\rangle\,dx+\int_M \theta_t^{\tilde b,\varphi}(x)\left\langle d(\phi\circ\tilde g(t)),\ \tilde b\right\rangle(x)\,dx\,dt\\&=\sum_{i\geq 1}\int_M \theta_t^{\tilde b,\varphi}(x)\left\langle d\phi,\sigma_i\right\rangle\left(\tilde g(t)(x)\right)\,dx\,dW_t^i+\int_M \theta_t^{\tilde b,\varphi}(x)\left\langle d\phi,b\right\rangle\left(\tilde g(t)(x)\right)dx\,dt\\&=\sum_{i\geq 1}\int_M \theta_t^{\tilde b,\varphi}\left((\tilde g(t))^{-1}(y)\right)\left\langle d\phi,\sigma_i\right\rangle(y)\,dydW_t^i+\int_M \theta_t^{\tilde b,\varphi}\left((\tilde g(t))^{-1}(y)\right)\left\langle d\phi,b\right\rangle(y)\,dy\,dt\\&=\sum_{i\geq 1}\Theta^{\sigma,b}(\varphi,\left\langle d\phi,\sigma_i\right\rangle)\,dW_t^i+\Theta^{\sigma,b}(\varphi,\left\langle d\phi,b\right\rangle)\,dt\end{aligned}$$

where the first term in the right is the martingale part and the second term is the finite variation part. We prefer to write the last equality as

$$d\tilde\Theta^{\sigma,b}(\varphi,\phi)=\sum_{i\geq 1}\Theta^{\tilde g}\left(\theta_t^{\tilde b,\varphi},\left\langle d\phi,\sigma_i\right\rangle\right)\,dW_t^i+\Theta^{\tilde g}\left(\theta_t^{\tilde b,\varphi},\left\langle d\phi,b\right\rangle\right)\,dt. \tag{5.18}$$

From this equation the properties of a generalized flow are easily checked.

We are left to prove that $\mathscr{E}'_q(\Theta^{\sigma,b}) \leq \mathscr{E}_q(b)$. Again this can be done via a regularization procedure of $\tilde{b}$ of the form $\tilde{b}^\varepsilon = \left(e^{\varepsilon\Box}(\tilde{b}^\flat)\right)^\sharp$, an extraction of subsequence, and similar estimates as before. We leave the details to the reader.

□

Acknowledgements

The research of A.Antoniouk was supported by the grant no. 01-01-12 of National Academy of Sciences of Ukraine (under the joint Ukrainian-Russian project of NAS of Ukraine and Russian Foundation of Basic Research)."

This research was also supported by the French ANR grant ANR-09-BLAN-0364-01 ProbaGeo and by the Portuguese FCT project PTDC/MAT/104173/2008.

References

[1] A. Antoniouk and M. Arnaudon, *Variational principle for weighted porous media equation*, preprint

[2] M. Arnaudon and A. B. Cruzeiro, *Lagrangian Navier-Stokes diffusions on manifolds: variational principle and stability*, Bull. Sci. Math. 136 (2012) no. 8, 857-881

[3] Arnaudon M., Cruzeiro A.B., *Generalized Navier-Stokes flows*, arxiv:1205.6631v1 [math.PR]

[4] V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits*, Ann. Inst. Fourier 16 (1966), 316–361.

[5] V. I. Arnold and B. A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, 125, Springer-Verlag, Berlin, 1998.

[6] Y. Brenier, *The least action principle and the related concept of generalized flows for incompressible perfect fluids*, J. Amer. Math. Soc. Vol 2, no. 2 (1989), 225–255.

[7] F. Cipriano and A.B. Cruzeiro, *Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus*, Comm. Math. Phys. 275 (2007), no. 1, 255–269.

[8] P. Constantin and G. Iyer, *A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations*, Comm. Pure Appl. Math. 61 (2008), no. 3, 330-345.

[9] R.J. DiPerna and P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Inventiones Mat. 98 (1989), 511–547.

[10] Dolbeault J., Nazaret B., Savaré G., *On the Bakry-Emery criterion for linear diffusions and weighted porous media equations* Commun. Math. Sci., (2008), 6, No. 2, p. 477–494.

[11] Dolbeault J., Gentil I., Guillin A., Wang F.-Yu. *L_q-Functional inequalities and weighted porous media equations*, Potential Anal. (2008) 28, p. 35–59.

[12] D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the notion of an incompressible fluid*, Ann. of. Math. 2 92 (1970), 102–163.

[13] G.L. Eyink, *Stochastic least action principle for the incompressible Navier-Stokes equations*, Phys. D239 (2010), no. 14, 1236-1240.

[14] A. Figalli, *Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients*, J. Funct. Anal. 254 (2008), no. 1, 109-153.

[15] F. Flandoli, M. Gubinelli and E. Priola, *Well-posedness of the transport equation by stochastic perturbation*, Invent. Math. 180 (2010), no. 1, 1-53

[16] Gliklich Yu., *Global Analysis in Mathematical Physics: Geometric and Stochastic Methods.*, Springer Verlag, New York (1997), 213 p.

[17] H. Lee, *Quasi-invariant flows associated with irregular vector fields*, PhD Dissertation, Beijing Normal University and Dijon University (2011).

[18] P.A. Meyer and W.A. Zheng, *Tightness criteria for laws of semimartingales*, Annales de l'I.H.P., section B, tome 20, n. 4 (1984), 353–372.

[19] T. Nakagomi, K. Yasue and J.-C. Zambrini, *Stochastic variational derivations of the Navier-Stokes equation.* Lett. Math. Phys.,160 (1981), 337–365.

[20] D. Ocone and E. Pardoux, *A generalized Itô Ventzell formula. Application to a class of anticipating stochastic differential equations*, Ann. Inst. Poincar 25 (1989), 39–71.

[21] A. I. Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal imcompressible fluid.* Mat. Sb.,128 (1709 (1985), 82–109.
[22] Y. Watanabe, *Differential geometry on diffeomorphism groups and Lagrangian stability.* Physica D, 225 (2007), 197–203.
[23] K. Yasue, *A variational principle for the Navier-Stokes equation.* J. Funct. Anal., 51 (2) (1983), 133–141.
[24] W. A. Zheng, *Tightness results for laws of diffusion processes.* Ann. de l' I.H.P., s. B 21 n2 (1985), 103–124.

Dep. Nonlinear Analysis
Institute of Mathematics NAS Ukraine
Tereschchenkivska str, 3
Kyiv, 01 601 UKRAINE
E-mail address: antoniouk.a@gmail.com

Institut de Mathématiques de Bordeaux
CNRS: UMR 5251
Université Bordeaux 1
F33405 TALENCE Cedex, France
E-mail address: marc.arnaudon@math.u-bordeaux1.fr

GFMUL and Dep. de Matemática IST(TUL).
Av. Rovisco Pais
1049-001 Lisboa, Portugal
E-mail address: abcruz@math.ist.utl.pt