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Abstract

We prove a Euler-Poincaré reduction theorem for stochastic processes taking values on a Lie group, which is a generalization of the reduction argument in [M-R] for the deterministic case. We also show examples of its application to $SO(3)$ and to the group of diffeomorphisms, which includes the Navier-Stokes equation on a bounded domain and the Camassa-Holm equation.

1 Introduction

Two approaches of stochastic perturbation of Geometric Mechanics seem to be known today. In one of them, inspired by J.M. Bismut [B] and developed by J.P. Ortega and collaborators [LC-O], the Lagrangian of the system is randomly perturbed. We shall advocate here the other approach, sometimes known as "stochastic deformation", where the Lagrangian is, essencially, the classical one but evaluated on underlying stochastic processes and their mean derivatives. This perspective was initially motivated by the quantization of classical systems ([C-Z], [Y1] , [Z]) and a probabilistic version of Feynman's path integral approach. More recently ([C-C], [A-C]), also inspired by [Y2], the Navier-Stokes equation was derived as a solution of a stochastic variational principle of this type. We showed that the Navier-Stokes equation can be viewed as the drift part of a semi-martingale which is a critical point of the functional whose Lagrangian is given by the kinetic energy expressed via a generalized time derivative. In our stochastic variational program there is no external noise perturbing Navier-Stokes equations: only the flows describing the position of the fluid particles are random.

This formulation of the Navier-Stokes equations extends naturally to the viscous case V. I. Arnold's characterization of the motion in incompressible fluid dynamics (Euler's equations) as geodesic flows on the group of volume-preserving diffeomorphisms [A].

Actually, Arnold also suggested a general framework for geodesic flows of Euler type, to be formulated on groups. Euler hydrodynamical equations turn out to be a particular case of Euler-Poincaré equations obtained in Geometric Mechanics via the celebrated Euler-Poincaré reduction principle. Formulated on general Lie groups, reduction methods (c.f. [M-R]) had an extraordinary development, from the mathematical point of view, notably in relation with geometry and dynamical systems, but also from the point of view of applications as well as in numerical analysis. Different Lie groups, both finite and infinite-dimensional, give rise to a number of equations, from the most paradigmatic ones describing the motion of a rigid body to the Euler's, Burgers, KdV or magnetohydrodynamic equations, for instance.

In this paper we establish a stochastic Euler-Poincaré reduction theorem on a general Lie group. Such theorem is the analog of the classical one, defined now for paths which are realizations of stochastic processes. Their mean velocity or drift satisfy a perturbation of the Euler-Poincaré equations with an extra term, generally dissipative, associated to the randomness of the underlying trajectories. One example is precisely Navier-Stokes equations on torus or Euclidean space, see e.g. [C-C]. But our theorem, also formulated on abstract Lie groups, allows to obtain many more equations: for example, the Navier-Stokes equations on domains and the Camassa-Holm equations.

The rest of the paper is organized as follows: in the second section, we introduce the definition and properties of semi-martingales on a general Lie group; the stochastic reduction theorem on a Lie group will be derived in section 3, and in section 4, we will consider some applications of this reduction theorem.

2 Semi-martingales in a general Lie group

The stochastic variational principle associated to the Navier-Stokes equations derived in [C-C] and [A-C] was formulated on a space of volume-preserving homeomorphisms, which is a (infinite dimensional) topological group endowed with a right-invariant metric, the same space used in the Arnold's formulation of Euler equations.

In the present work we consider a stochastic variational principle on a general Lie group G, endowed with a left-invariant (or right-invariant) metric.

For the stochastic analysis background of this work we refer to [E] or [I-W].

From now on, for simplicity, the words martingale and semi-martingale denote a time continuous L^2 integrable martingale and time continuous L^2 integrable semi-martingale respectively. The domain of our action functionals will be a set of semi-martingales. As they are not of bounded variation with respect to time, we can not use a classical derivative in time, but will replace it by a generalized mean derivative $\frac{D}{dt}$.

In a Euclidean setting the definition of the generalized time derivative corresponds to a derivative regularized by a conditional expectation with respect to the past filtration at each time. More precisely, given a semi-martingale $\xi(\cdot)$ with respect to an increasing filtration $\mathscr{F}_\cdot$ and taking values in the Euclidean space (or torus) the generalized derivative is defined by,

$$\frac{D\xi(t)}{dt} := \lim_{\varepsilon \downarrow 0} \mathbb{E}\Big[\frac{\xi(t+\varepsilon)-\xi(t)}{\varepsilon}|\mathscr{F}_t\Big] \tag{2.1}$$

In particular, as the conditional expectation of the martingale part vanishes, the generalized derivative coincides with the derivative of the bounded variation part of the semi-martingale (its drift). We refer the reader to [C-Z], [Y1] , [Z] for detailed introduction to the property of such generalized derivative on Euclidean space.

On a general manifold M a martingale can only be defined after fixing a linear connection ∇ (see [E], [I-W]). More precisely, a M-valued semi-martingale $\xi(\cdot)$ is a ∇-martingale, if for each $f \in C^\infty(M)$,

$$M_t^f := f(\xi(t)) - f(\xi(0)) - \frac{1}{2}\int_0^t \mathrm{Hess} f(\xi(s))\big(d\xi(s), d\xi(s)\big)$$

is a $\mathbb{R}^1$-valued local martingale with respect to the filtration $\mathscr{F}_\cdot$, where $\mathrm{Hess} f(x) : T_xM \times T_xM \to \mathbb{R}$ is defined by

$$\mathrm{Hess} f(x)\Big(A_1, A_2\Big) := \tilde{A}_1\tilde{A}_2 f - \nabla_{\tilde{A}_1}\tilde{A}_2 f,\ \forall A_1, A_2 \in T_xM, \tag{2.2}$$

the vector fields $\tilde{A}_j,\ j = 1, 2$ on M being smooth and such that $\tilde{A}_i(x) = A_i$.

When M is a finite dimensional manifold $\mathrm{Hess} f = \nabla df$ is the covariant derivative of the (differential) tensor field df by the connection ∇. For an infinite dimensional group the tensor field df or ∇df does not always exist due to divergence of infinite series, but the definition (2.2) is valid at least for smooth cylinder functions f. This is why we use here definition (2.2).

So for a M-valued semi-martingale $\xi(\cdot)$ it is natural to extend definition (2.1) to a ∇-generalized derivative as follows. If for each $f \in C^\infty(M)$,

$$N_t^f := f(\xi(t)) - f(\xi(0)) - \frac{1}{2}\int_0^t \mathrm{Hess} f(\xi(s))\big(d\xi(s), d\xi(s)\big) - \int_0^t A(s)f(\xi(s))ds$$

is a $\mathbb{R}^1$-valued local martingale, where the random time dependent vector $A(t)$ belongs to $T_{\xi(t)}M$ a.s., we define

$$\frac{D^\nabla \xi(t)}{dt} := A(t) \tag{2.3}$$

In fact, if M is a finite dimensional manifold with a connection ∇, there is an equivalent definition to (2.3). For simplicity, we assume that $\xi(\cdot)$ is a M-valued semi-martingale with a fixed initial point $\xi(0) = x$ and that a stochastic parallel translation $//_\cdot : T_xM \to T_{\xi(\mathrm{cot})}M$ along $\xi(\cdot)$ is associated to the connection ∇. We have $\nabla_{\circ d\xi(t)}(//_t v) = 0$ for any $v \in T_xM$, where $\circ d\xi(t)$ denotes Stratonovich differentiation. Then $\eta(t) := \int_0^t //_s^{-1} \circ d\xi(s)$ is a T_xM valued semi-martingale. As in (2.1), we take the derivative of the bounded variation part as follows,

$$\frac{D\eta(t)}{dt} := \lim_{\varepsilon\downarrow 0}\mathbb{E}\Big[\frac{\eta(t+\varepsilon) - \eta(t)}{\varepsilon}|\mathscr{F}_t\Big],$$

which is a T_xM valued process. Then we define

$$\frac{D^\nabla \xi(t)}{dt} := //_t \frac{D\eta(t)}{dt}.$$

This definition is the same as (2.3) (see, for instance, [E]).

From now on G will denote a Lie group endowed with a left invariant metric $\langle\ ,\ \rangle$ and a left invariant connection ∇. Unless explicitely stated ∇ is a general connection, not necessarily the Levi-Civita connection with respect to $\langle\ ,\ \rangle$. We let $\mathscr{G} := T_eG$; here e is the unit element of G; in particular, T_eG can be identified with the Lie algebra of G.

Taking a sequence of vectors $H_i \in \mathscr{G}$, $i = 1, 2, .., k$, and a non-random map $u(\cdot) \in C^1([0,T]; \mathscr{G})$ for some constant $T > 0$, consider the following Stratonovich SDE in the group G,

$$dg(t) = T_eL_{g(t)}\Big(\sum_i H_i \circ dW_t^i - \frac{1}{2}\sum_{i=1}^{k} \nabla_{H_i} H_i dt + u(t)dt\Big), ..., g(0) = e \tag{2.4}$$

where $T_aL_{g(t)} : T_aG \to T_{g(t)a}G$ is the differential of the left translation $L_{g(t)}(x) := g(t)x$, $\forall x \in G$ at the point $a \in G$, and where W_t is a $\mathbb{R}^k$ valued Brownian motion.

By Itô's formula and definition (2.3) we can see that

$$\frac{D^\nabla g(t)}{dt} = T_eL_{g(t)}u(t).$$

In fact the term $\frac{1}{2}\sum_i \nabla_{H_i} H_i$ corresponds to the contraction term which is the difference between the Itô and the Stratonovich stochastic integral.

In particular, if $\{H_i\}$ is an orthonormal basis of $\mathscr{G}$, ∇ is the Levi-Civita connection, $u(t) = 0$ for each t, and $\nabla_{H_i} H_i = 0$ for each i, then $g(\cdot)$ is the Brownian motion on G whose generator is the Laplace-Beltrami operator.

Note that if $H_i = 0$ for each i, then $\frac{D^\nabla g(t)}{dt}$ is the ordinary derivative with t, which does depend on the connection ∇.

Remark: By the standard theory, SDE (2.4) is, a priori, only defined on a finite dimensional Lie group. But in some special cases of infinite dimensional groups such as, for example, the group of diffeomorphism on torus, SDE (2.4) still defines a semi-martingale even when we take an infinite number of H_i, see the discussion in [A-C],[C],[C-C]. As explained below, in those references the corresponding generalized derivative for the semi-martingale is taken as (2.1) in the pointwise sense, and a connection on the group of diffeomorphisms is not used.

3 The stochastic Euler-Poincaré reduction theorem on a Lie group

3.1 The kinetic energy case (stochastic geodesics)

Let $\mathscr{S}(G)$ denote the collection of all the G-valued semi-martingales. From now on we fix a constant $T>0$ and define a stochastic action functional $J^{\nabla,\langle\ ,\ \rangle}$ on $\mathscr{S}(G)$ as follows,

(3.1)
$$J^{\nabla,\langle\ ,\ \rangle}(\xi(\cdot)) := \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big\langle T_{\xi(t)}L_{\xi(t)^{-1}}\frac{D^\nabla\xi(t)}{dt}, T_{\xi(t)}L_{\xi(t)^{-1}}\frac{D^\nabla\xi(t)}{dt}\Big\rangle dt\Big], \quad \forall \xi(\cdot)\in\mathscr{S}(G).$$

Notice that the action functional $J^{\nabla,\langle\ ,\ \rangle}$ depends on the choice of the inner product $\langle\ ,\ \rangle$ and the choice of the connection ∇, and that $T_{\xi(t)}L_{\xi(t)^{-1}}\frac{D^\nabla\xi(t)}{dt}\in\mathscr{G}$ for each t.

The Lagrangian of this action functional corresponds to the (generalized) kinetic energy. In the next subsection we consider more general Lagrangians.

For each non-random curve $v(\cdot)\in C^1([0,T];\mathscr{G})$ satisfying $v(0)=v(T)=0$, let $e_{\varepsilon,v}(\cdot)\in C^1([0,T];G)$ be the flow generated by $\varepsilon v(\cdot)$ in G, namely the solution of the following deterministic time dependent differential equation on G:

$$\begin{cases} \frac{d}{dt}e_{\varepsilon,v}(t) = \varepsilon T_e L_{e_{\varepsilon,v}(t)}\dot{v}(t), \\ e_{\varepsilon,v}(0)=e, \end{cases} \tag{3.2}$$

We say that a G-valued semi-martingale $g(\cdot)$ is a *critical point* of the action functional $J^{\nabla,\langle\ ,\ \rangle}$ if for any $v(\cdot)\in C^1([0,T];\mathscr{G})$ satisfying $v(0)=v(T)=0$,

$$\frac{dJ^{\nabla,\langle\ ,\ \rangle}(g_{\varepsilon,v}(.))}{d\varepsilon}\Big|_{\varepsilon=0}=0, \tag{3.3}$$

where $g_{\varepsilon,v}(t):=g(t)e_{\varepsilon,v}(t),\ t\in[0,1]$.

For the variation of $e_{\varepsilon,v}$ the following lemma holds,

Lemma 3.1. *We have,*

$$\frac{d}{d\varepsilon}e_{\varepsilon,v}(t)\big|_{\varepsilon=0}=v(t)$$
$$\frac{d}{d\varepsilon}e^{-1}_{\varepsilon,v}(t)\big|_{\varepsilon=0}=-v(t)$$

Proof. In the proof we omit the index v in $e_{\varepsilon,v}$ for simplicity. If $\frac{\hat{D}}{dt}$ denotes the covariant derivative on G via the Levi-Civita connection, then

$$\frac{\hat{D}}{dt}\frac{d}{d\varepsilon}e_\varepsilon(t)=\frac{\hat{D}}{d\varepsilon}\frac{d}{dt}e_\varepsilon(t)=\frac{\hat{D}}{d\varepsilon}\big(\varepsilon T_eL_{e_\varepsilon(t)}\dot{v}(t)\big)=T_eL_{e_\varepsilon(t)}\dot{v}(t)+\varepsilon\frac{\hat{D}}{d\varepsilon}\big(T_eL_{e_\varepsilon(t)}\dot{v}(t)\big)$$

Let $X(t) := \frac{d}{d\varepsilon} e_\varepsilon(t)\big|_{\varepsilon=0}$; taking $\varepsilon = 0$ above, and noting that $e_0(t) = e$ for each t, we derive,

$$\frac{d}{dt} X(t) = \dot{v}(t),$$

Then, as $v(0) = 0$, we deduce that $X(t) = v(t)$.

Since $e_\varepsilon(t) e_\varepsilon^{-1}(t) = e$ for each ε, differentiating with respect to ε we obtain $\frac{d}{d\varepsilon} e_\varepsilon^{-1}(t)$ $= -T_e R_{e_\varepsilon^{-1}(t)}\, T_{e_\varepsilon(t)} L_{e_\varepsilon^{-1}(t)} \frac{de_\varepsilon(t)}{d\varepsilon}$, where TR is the differential of right translation. Hence we have,

$$\frac{d}{d\varepsilon} e_\varepsilon^{-1}(t)\big|_{\varepsilon=0} = -v(t)$$

$\square$

From now on, for each $u, v \in \mathscr{G}$, we define $\nabla_u v \in \mathscr{G}$ by $\nabla_u v := \nabla_{U(x)} V(x)\Big|_{x=e}$, where $U(x)$, $V(x)$ are the left invariant vector fields (or right invariant if the metric and connection are right invariant) such that $U(e) = u$, $V(e) = v$.

We now present the stochastic Euler-Poincaré reduction theorem in the kinetic energy case, a sufficient and necessary condition for the critical points of $J^{\nabla, \langle\ ,\ \rangle}$.

Theorem 3.2. *Suppose that G is a Lie group with a left invariant metric $\langle\ ,\ \rangle$ and a left invariant connection ∇. The G-valued semi-martingale $g(\cdot)$ defined by (2.4) is a critical point of $J^{\nabla, \langle\ ,\ \rangle}$ if and only if the non-random curve $u(\cdot) \in C^1([0,T]; \mathscr{G})$ satisfies the following equation:*

$$\frac{d}{dt} u(t) = ad^*_{\tilde{u}(t)} u(t) + K(u(t)), \tag{3.4}$$

where

$$\tilde{u}(t) := u(t) - \frac{1}{2} \sum_i \nabla_{H_i} H_i, \tag{3.5}$$

for each $u \in \mathscr{G}$, $ad^*_u : \mathscr{G} \to \mathscr{G}$ is the adjoint of $ad_u : \mathscr{G} \to \mathscr{G}$ with respect to the metric $\langle\ ,\ \rangle$,

$$\langle ad^*_u v,\ w\rangle = \langle v,\ ad_u w\rangle \quad \forall u, v, w \in \mathscr{G}, \tag{3.6}$$

and the operator $K : \mathscr{G} \to \mathscr{G}$ is defined as follows

$$\langle K(u), v\rangle = -\Big\langle u, \frac{1}{2} \sum_i \big(\nabla_{ad_v H_i} H_i + \nabla_{H_i}(ad_v H_i)\big)\Big\rangle, \quad \forall u, v \in \mathscr{G}. \tag{3.7}$$

Proof. In the proof, we omit the index v in $e_{\varepsilon,v}(\cdot)$ and in $g_{\varepsilon,v}(\cdot)$ for simplicity. As $g_\varepsilon(t) = g(t)e_\varepsilon(t)$, by Itô formula we have,

$$
\begin{aligned}
dg_\varepsilon(t) &= \sum_i T_eL_{g_\varepsilon(t)}H_i^\varepsilon(t)\circ dW_t^i + T_eL_{g_\varepsilon(t)}\Big(Ad_{e_\varepsilon^{-1}(t)}\big(-\frac{1}{2}\nabla_{H_i}H_i + u(t)\big)\Big)dt \\
&+ T_eL_{g_\varepsilon(t)}\big(T_{e_\varepsilon(t)}L_{e_\varepsilon^{-1}(t)}\dot{e}_\varepsilon(t)\big)dt,
\end{aligned}
\tag{3.8}
$$

where $H_i^\varepsilon(t) := Ad_{e_\varepsilon^{-1}(t)}H_i$. From the definition of $e_\varepsilon(t)$, we know that $T_{e_\varepsilon(t)}L_{e_\varepsilon^{-1}(t)}\dot{e}_\varepsilon(t) = \varepsilon\dot{v}(t)$. Then for each $f\in C^\infty(G)$,

$$
\begin{aligned}
N_t^f &:= f(g_\varepsilon(t)) - f(g_\varepsilon(0)) - \frac{1}{2}\int_0^t \mathrm{Hess}f(g_\varepsilon(s))\big(dg_\varepsilon(s), dg_\varepsilon(s)\big) \\
&- \frac{1}{2}\sum_i\int_0^t T_eL_{g_\varepsilon(s)}\big(\nabla_{H_i^\varepsilon(s)}H_i^\varepsilon(s)\big)f(g_\varepsilon(s))ds \\
&- \int_0^t T_eL_{g_\varepsilon(s)}\Big(Ad_{e_\varepsilon^{-1}(s)}\big(-\frac{1}{2}(\sum_i\nabla_{H_i}H_i) + u(s)\big) + \varepsilon\dot{v}(s)\Big)f(g_\varepsilon(s))ds
\end{aligned}
$$

is a local martingale.

By the definition of generalized derivative above,

$$
\begin{aligned}
&T_{g_\varepsilon(t)}L_{g_\varepsilon^{-1}(t)}\frac{D^\nabla g_\varepsilon(t)}{dt} \\
&= \sum_i \frac{1}{2}\nabla_{H_i^\varepsilon(t)}H_i^\varepsilon(t) + Ad_{e_\varepsilon^{-1}(t)}\big(-\frac{1}{2}(\sum_i\nabla_{H_i}H_i) + u(t)\big) + \varepsilon\dot{v}(t)
\end{aligned}
$$

Using Lemma 3.1,

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\Big(Ad_{e_\varepsilon^{-1}(t)}\big(-\frac{1}{2}(\sum_i\nabla_{H_i}H_i) + u(t)\big)\Big)\Big|_{\varepsilon=0} \\
&= -ad_{v(t)}\big(-\frac{1}{2}(\sum_i\nabla_{H_i}H_i) + u(t)\big) = ad_{\big(-\frac{1}{2}(\sum_i\nabla_{H_i}H_i)+u(t)\big)}v(t)
\end{aligned}
$$

Notice that $H_i^0(t) = H_i$ for every t and that, by Lemma 3.1, $\frac{dH_i^\varepsilon(t)}{d\varepsilon}\big|_{\varepsilon=0} = -ad_{v(t)}H_i$. We obtain,

$$
\frac{d}{d\varepsilon}\nabla_{H_i^\varepsilon(t)}H_i^\varepsilon(t)\big|_{\varepsilon=0} = -\nabla_{ad_{v(t)}H_i}H_i - \nabla_{H_i}(ad_{v(t)}H_i)
$$

Recall that $T_{g(t)}L_{g(t)^{-1}}\frac{D^\nabla g(t)}{dt}=u(t)$. We derive,

(3.9)
$$\frac{dJ^{\nabla,\langle\ ,\ \rangle}(g_\varepsilon(.))}{d\varepsilon}\Big|_{\varepsilon=0}=\mathbb{E}\int_0^T\langle\frac{d}{d\varepsilon}(T_{g_\varepsilon(t)}L_{g_\varepsilon^{-1}(t)}\frac{D^\nabla g_\varepsilon(t)}{dt})\big|_{\varepsilon=0},u(t)\rangle dt$$
$$=\int_0^T\big\langle u(t),\dot v(t)+ad_{\left(-\frac{1}{2}(\sum_i\nabla_{H_i}H_i)+u(t)\right)}v(t)-\frac{1}{2}\sum_i\left(\nabla_{ad_{v(t)}H_i}H_i+\nabla_{H_i}(ad_{v(t)}H_i)\right)\big\rangle dt$$
$$=\int_0^T\big\langle-\dot u(t)+ad^*_{\tilde u(t)}u(t)+K(u(t)),v(t)\big\rangle dt$$

where in the second step we could drop the expectation $\mathbb{E}$ since $u(t),v(t)$ are non-random and in the last step we used integration by parts with respect to time and the condition $v(0)=v(T)=0$. Definitions (3.5), (3.6) and (3.7) were also used.

By definition, $g(\cdot)$ is a critical point of $J^{\nabla,\langle\ ,\ \rangle}$ if and only if $\frac{dJ^{\nabla,\langle\ ,\ \rangle}(g_{\varepsilon,v}(.))}{d\varepsilon}\big|_{\varepsilon=0}=0$ for each $v\in C^1([0,T];\mathscr{G})$. The result follows from (3.9), which implies equation (3.4) since v is arbitrary. □

Remark 1. If $H_i=0$, then $K(u)=0$ and equation (3.4) reduces to the standard Euler-Poincaré equation, see for example [A-K], [M-R].

Remark 2. As we can deduce from the computation, since $u(t)$ is assumed to be non-random, for each ε, the expression $T_{g_\varepsilon(t)}L_{g_\varepsilon^{-1}(t)}\frac{D^\nabla g_\varepsilon(t)}{dt}$ is non-random and does not depend on the initial point $g(0)$, this is why we can take the test vector curves $v(t)$ to be non-random here. For the case $u\in C^1([0,T];\mathscr{G})$ to be adapted and random, we must take test vectors $v\in C^1([0,T];\mathscr{G})$ to be adapted and random. The above proof still holds and we can obtain an equation (3.4) which holds almost surely in the underlying probability space.

Remark 3. The critical equation (3.4) depends on the metric, connection and the choice of $\{H_i\}$. The term $K(u)$ defined by (3.7) depends on the metric, the connection and the choice of $\{H_i\}$ whereas ad^* depends on the metric only.

Remark 4. If G is the group of diffeomorphisms on the torus the SDE (2.4) becomes equation (4.4) of next section. We can check that the Itô formula (3.8) holds by direct computation. Then the proof of Theorem 3.2 is still valid, and the conclusion is true in this case. See the section 4.2 for more details.

3.2 The general form of the stochastic Euler-Poncaré reduction

As mentioned above, we can also consider the critical point of an action functional J induced by some more general Lagrangian function. In fact, suppose $l:\mathscr{G}\to\mathbb{R}$ is a function whose (functional) derivative $\frac{\delta l}{\delta w}:\mathscr{G}\times\mathscr{G}^*\to\mathbb{R}$ exists in the following sense

$$\frac{d}{d\varepsilon}l\big(u+\varepsilon v\big)\Big|_{\varepsilon=0}={}_{\mathscr{G}^*}\Big(\frac{\delta l}{\delta w}(u),v\Big)_{\mathscr{G}},\ \forall\ u,v\in\mathscr{G},$$

where $\mathscr{G}^*$ is the dual space of $\mathscr{G}$ and ${}_{\mathscr{G}^*}\Big(\cdot,\cdot\Big)_{\mathscr{G}}$ denotes the pairing of $\mathscr{G}^*$ and $\mathscr{G}$.

We define the action functional $J^{\nabla,l}$ on $\mathscr{S}(G)$ as follows,

$$J^{\nabla,l}(\xi(\cdot)):=\mathbb{E}\Big[\int_0^T l\Big(T_{\xi(t)}L_{\xi(t)^{-1}}\frac{D^\nabla\xi(t)}{dt}\Big)dt\Big],\ \ \forall\ \xi(\cdot)\in\mathscr{S}(G).$$

With the same formulation as (3.3), we still say that a G-valued semi-martingale $g(\cdot)$ is a *critical point* of $J^{\nabla,l}$ if for every $v(\cdot)\in C^1([0,T];\mathscr{G})$ with $v(0)=v(T)=0$,

$$\frac{dJ^{\nabla,l}(g_{\varepsilon,v}(.))}{d\varepsilon}\Big|_{\varepsilon=0}=0,$$

where $g_{\varepsilon,v}(t):=g(t)e_{\varepsilon,v}(t),\ t\in[0,1]$, and $e_{\varepsilon,v}$ is defined by (3.2).

Then following the same computation in Theorem 3.2, we obtain

(3.10)

$$\begin{aligned}
&\frac{dJ^{\nabla,l}(g_\varepsilon(.))}{d\varepsilon}\Big|_{\varepsilon=0}=\mathbb{E}\int_0^T\frac{d}{d\varepsilon}l\Big(T_{g_\varepsilon(t)}L_{g_\varepsilon^{-1}(t)}\frac{D^\nabla g_\varepsilon(t)}{dt}\Big)\Big|_{\varepsilon=0}dt\\
&=\int_0^T {}_{\mathscr{G}^*}\Big(\frac{\delta l}{\delta w}\big(u(t)\big),\dot v(t)+ad_{\tilde u(t)}v(t)-\frac{1}{2}\sum_i\big(\nabla_{ad_{v(t)}H_i}H_i+\nabla_{H_i}(ad_{v(t)}H_i)\big)\Big)_{\mathscr{G}}dt\\
&=\int_0^T {}_{\mathscr{G}^*}\Big(-\frac{d}{dt}\big(\frac{\delta l}{\delta w}\big(u(t)\big)\big)+ad^*_{\tilde u(t)}\frac{\delta l}{\delta w}\big(u(t)\big)+K\Big(\frac{\delta l}{\delta w}\big(u(t)\big)\Big),v(t)\Big)_{\mathscr{G}}dt,
\end{aligned}$$

where $\tilde u(t)$ is defined by (3.5), $ad^*_u:\mathscr{G}^*\to\mathscr{G}^*$, $u\in\mathscr{G}$ is the dual of ad_u in the following sense

$${}_{\mathscr{G}^*}\Big(ad^*_u\mu,v\Big)_{\mathscr{G}}={}_{\mathscr{G}^*}\Big(\mu,ad_uv\Big)_{\mathscr{G}},\quad u,v\in\mathscr{G},\ \mu\in\mathscr{G}^*,$$

and $K:\mathscr{G}^*\to\mathscr{G}^*$ is defined by

$${}_{\mathscr{G}^*}\Big(K(\mu),v\Big)_{\mathscr{G}}={}_{\mathscr{G}^*}\Big(\mu,-\frac{1}{2}\sum_i\big(\nabla_{ad_{v(t)}H_i}H_i+\nabla_{H_i}(ad_{v(t)}H_i)\big)\Big)_{\mathscr{G}},\quad v\in\mathscr{G},\ \mu\in\mathscr{G}^*.$$

Note that here the definition of ad^* and K are slightly different from that in (3.6) and (3.7) since we do not fix a metric $\langle,\rangle$ on $\mathscr{G}$.

From the arguments above (notably (3.10)) we have the following result on the characterization of critical points of a action functional induced by a general Lagrangian.

Theorem 3.3. *The G-valued semi-martingale $g(\cdot)$ with the form (2.4) is a critical point of $J^{\nabla,l}$ if and only if the following equations for $u(t)$ holds*

$$-\frac{d}{dt}\Big(\frac{\delta l}{\delta w}\big(u(t)\big)\Big)+ad^*_{\tilde{u}(t)}\frac{\delta l}{\delta w}\big(u(t)\big)+K\Big(\frac{\delta l}{\delta w}\big(u(t)\big)\Big)=0. \tag{3.11}$$

In particular, if we choose $l(u)=\frac{\langle u,u\rangle}{2}$ for a metric $\langle,\rangle$ on $\mathscr{G}$, we obtain equation (3.4).

3.3 The right invariant case

For a Lie group G with a right invariant metric and right invariant connection, we can define a composition map $\diamond$ by $a\diamond b:=ba,\ \forall a,b\in G$. Then the original metric and connection are left invariant under the composition $\diamond$ and we can also define the semi-martingale $g(\cdot)$, the action functional $J(g(\cdot))$ and the perturbed semi-martingales $g_{\varepsilon,v}$ using the composition $\diamond$. For example, one can check that the semi-martingale $g(\cdot)$ in (2.4) is changed to the following one

$$dg(t)=T_eR_{g(t)}\Big(\sum_i H_i\circ dB_t^i-\frac{1}{2}\sum_{i=1}^k\nabla_{H_i}H_idt+u(t)dt\Big), \tag{3.12}$$

where $T_eR_{g(t)}$ is the differential of right translation with $g(t)$ at the point $x=e$. The action functional J in (3.1) is defined by right translation if we use the composition $\diamond$ on G. We also say that $g(\cdot)$ is a critical point if $\frac{d}{d\varepsilon}\big(J^{\nabla,\langle\ ,\ \rangle}(g_{\varepsilon,v})\big)\big|_{\varepsilon=0}=0$ for each $v\in C^1([0,T];\mathscr{G})$ with $v(0)=v(T)=0$. By the same procedure as above, we can derive the following theorem on a Lie group with right invariant metric and connection.

Theorem 3.4. *Suppose that G is a Lie group with a right invariant metric $\langle\ ,\ \rangle$ and a right invariant connection ∇. The G-valued semi-martingale $g(\cdot)$ defined in (3.12) is a critical point of $J^{\nabla,\langle\ ,\ \rangle}$ if and only if $u(\cdot)\in C^1([0,T];\mathscr{G})$ satisfies the following equation,*

$$\frac{d}{dt}u(t)=-ad^*_{\tilde{u}(t)}u(t)-K(u(t)), \tag{3.13}$$

where $\tilde{u}$ and $K:\mathscr{G}\to\mathscr{G}$ are defined in (3.5) and (3.7) respectively.

As the same procedure, for a general Lagrangian the right-invariant version of equation (3.11) will be

$$\frac{d}{dt}\Big(\frac{\delta l}{\delta w}\big(u(t)\big)\Big)+ad^*_{\tilde{u}(t)}\frac{\delta l}{\delta w}\big(u(t)\big)+K\Big(\frac{\delta l}{\delta w}\big(u(t)\big)\Big)=0.$$

Under some special conditions, the operator $K(u)$ defined in (3.7) coincides with the de Rham-Hodge operator on the Lie group. More precisely we have the following result,

Proposition 3.5. *Suppose that G is a Lie group with a right invariant metric $\langle \ , \ \rangle$, and ∇ is the (right invariant) Levi-Civita connection with respect to $\langle \ , \ \rangle$. If we assume that $\nabla_{H_i} H_i = 0$ for each i, we have,*

$$K(u) = -\frac{1}{2} \sum_i \big(\nabla_{H_i} \nabla_{H_i} u + \mathbf{R}(u, H_i) H_i\big), \quad \forall u \in \mathscr{G},$$

here $\mathbf{R}$ is the Riemannian curvature tensor with respect to ∇. In particular, if $\{H_i\}$ is an orthonormal basis of $\mathscr{G}$, then $K(u) = -\frac{1}{2}\Box u := -\frac{1}{2}\big(\Delta u + Ric(u)\big)$, where $\Delta u := \Delta U(x)|_{x=e}$ for the right invariant vector fields $U(x) := T_e R_x u, \ \forall u \in \mathscr{G}, \ x \in G$.

Proof. Notice that for each $v \in \mathscr{G}$,

$$\begin{aligned}
&\nabla_{ad_v H_i} H_i + \nabla_{H_i}(ad_v H_i) \\
&= -\nabla_{[v,H_i]} H_i - \nabla_{H_i}[v, H_i] \\
&= -\nabla_{[v,H_i]} H_i - \nabla_{H_i}(\nabla_v H_i - \nabla_{H_i} v) \\
&= -\nabla_{[v,H_i]} H_i - \nabla_v \nabla_{H_i} H_i - \nabla_{[H_i,v]} H_i - \mathbf{R}(H_i, v) H_i + \nabla_{H_i} \nabla_{H_i} v \\
&= \mathbf{R}(v, H_i) H_i + \nabla_{H_i} \nabla_{H_i} v
\end{aligned} \tag{3.14}$$

In the first step above, we used the property $ad_v u = -[v, u]$ for every $u, v \in \mathscr{G}$ if we view u, v as right invariant vector fields on G. In the second step we used the fact that ∇ is torsion free and in the third the definition of the Riemannian curvature tensor. Finally we used the assumption $\nabla_{H_i} H_i = 0$.

Then by (3.7), for each $u, v \in \mathscr{G}$,

$$\begin{aligned}
\langle K(u), v \rangle &= -\frac{1}{2}\Big\langle u, \sum_i \big(\nabla_{ad_v H_i} H_i + \nabla_{H_i}(ad_v H_i)\big)\Big\rangle \\
&= -\frac{1}{2}\Big\langle u, \sum_i \big(\mathbf{R}(v, H_i) H_i + \nabla_{H_i} \nabla_{H_i} v\big)\Big\rangle \\
&= -\frac{1}{2}\Big\langle \sum_i \big(\nabla_{H_i} \nabla_{H_i} u + \mathbf{R}(u, H_i) H_i\big), v\Big\rangle,
\end{aligned}$$

where in the last step we used the property $\langle \nabla_u v, w \rangle = -\langle v, \nabla_u w \rangle$ for $u, v, w \in \mathscr{G}$ since ∇ is Riemannian with respect the metric $\langle \ , \ \rangle$; we also used the symmetric property of the curvature tensor $\mathbf{R}$.

Since v is arbitrary, we get,

$$K(u) = -\frac{1}{2} \sum_i \big(\nabla_{H_i} \nabla_{H_i} u + \mathbf{R}(u, H_i) H_i\big),$$

If $\{H_i\}$ is an orthonormal basis of $\mathscr{G}$, define the right invariant vector fields $\tilde{H}_i(x) := T_e R_x H_i, \ U(x) := T_e R_x u, \ \forall x \in G$. Then $\Delta U(x) = \sum_i \nabla^2 U(x)(\tilde{H}_i(x), \tilde{H}_i(x)) =$

$\sum_i \big(\nabla_{\tilde H_i} \nabla_{\tilde H_i} U(x) - \nabla_{\nabla_{\tilde H_i} \tilde H_i} U(x)\big)$, hence $\Delta u = \Delta U(x)|_{x=e} = \sum_i \big(\nabla_{H_i}\nabla_{H_i} u - \nabla_{H_i} H_i u\big)$ $= \sum_i \nabla_{H_i}\nabla_{H_i} u$ since $\nabla_{H_i} H_i = 0$. Also notice that $\sum_i \mathbf{R}(u, H_i)H_i = \mathrm{Ric}(u)$, so we have $K(u) = -\frac{1}{2}\big(\Delta u + \mathrm{Ric}(u)\big)$.

□

4 Some applications

4.1 The rigid body (SO(3))

To describe the motion of a rigid body, the configuration space is the space of matrices $G = SO(3)$, see [A-K] and [M-R]. Then $T_e G = so(3)$, the space of 3×3 skew symmetric matrices. Take a basis of $so(3)$, namely

$$E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This basis satisfies the following relations,

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2. \tag{4.1}$$

For $v \in so(3)$ with the form $v = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$, $v_j \in \mathbb{R}^1,\ j = 1,2,3$, we have $v = v_1E_1 + v_2E_2 + v_3E_3$. We define $\hat v \in \mathbb{R}^3$ to be the unique element such that $v\eta = \hat v \times \eta$ for each $\eta \in \mathbb{R}^3$; in fact, it easy to check that $\hat v := (v_1, v_2, v_3)$.

Take $I = (I_1, I_2, I_3)$ such that $I_j > 0,\ j = 1,2,3$ and define an inner product in $so(3)$ as follows,

$$\langle v, v\rangle^I := \sum_{j=1}^{3} I_j v_j^2, \quad \forall v \in so(3) \text{ with } \hat v = (v_1, v_2, v_3),$$

We extend $\langle\ ,\ \rangle^I$ to $SO(3)$ by left translation, then we get a left invariant metric, which we still write as $\langle\ ,\ \rangle^I$. In particular, if $H_i = 0$ for each i in the semi-martingale (2.4), then $g(t)^{-1}\frac{dg(t)}{dt} = u(t)$, and $\widehat{u(t)}$ is the angular velocity vector. In the definition of the Lagrangian in (3.1), if we choose the metric to be $\langle\ ,\ \rangle^I$, then the Lagrangian is the kinetic energy with moment of inertia I. See the discussion in [A-K], [M-R].

Let ∇^I be the Levi-Civita connection with respect to $\langle\ ,\ \rangle^I$. By (4.1) and the formula for the Levi-Civita connection, we derive,

$$
\begin{aligned}
&\nabla^I_{E_1}E_1=0,\quad \nabla^I_{E_1}E_2=\frac{1}{2}\big(1+\frac{I_2-I_1}{I_3}\big)E_3,\quad \nabla^I_{E_2}E_1=\frac{1}{2}\big(-1+\frac{I_2-I_1}{I_3}\big)E_3\\
&\nabla^I_{E_2}E_2=0,\quad \nabla^I_{E_2}E_3=\frac{1}{2}\big(1+\frac{I_3-I_2}{I_1}\big)E_1,\quad \nabla^I_{E_3}E_2=\frac{1}{2}\big(-1+\frac{I_3-I_2}{I_1}\big)E_1\\
&\nabla^I_{E_3}E_3=0,\quad \nabla^I_{E_3}E_1=\frac{1}{2}\big(1+\frac{I_1-I_3}{I_2}\big)E_2,\quad \nabla^I_{E_1}E_3=\frac{1}{2}\big(-1+\frac{I_1-I_3}{I_2}\big)E_2
\end{aligned}
\tag{4.2}
$$

Take $H_i:=\frac{1}{\sqrt{I_i}}E_i$ for $i=1,2,3$ in SDE (2.4); $\{H_i\}_{i=1}^3$ is an orthonormal basis of $so(3)$. By (4.1) and (4.2), for each $v\in so(3)$ with $\hat{v}=(v_1,v_2,v_3)$,

$$\sum_i\big(\nabla^I_{ad_vH_i}H_i+\nabla^I_{H_i}(ad_vH_i)\big)=\frac{1}{I_1I_2I_3}\big((I_2-I_3)^2v_1E_1+(I_3-I_1)^2v_2E_2+(I_1-I_2)^2v_3E_3\big)$$

Then by (3.7), for every $u\in so(3)$ with $\hat{u}=(u_1,u_2,u_3)$,

$$K(u)=-\frac{1}{2}\frac{1}{I_1I_2I_3}\big((I_2-I_3)^2u_1E_1+(I_3-I_1)^2u_2E_2+(I_1-I_2)^2u_3E_3\big)$$

From [M-R], we know for each $u\in so(3)$ with $\hat{u}=(u_1,u_2,u_3)$, the adjoint of ad with respect to $\langle\ ,\ \rangle^I$ has the following expression,

$$ad^*_u(u)=\frac{u_2u_3(I_2-I_3)}{I_1}E_1+\frac{u_3u_1(I_3-I_1)}{I_2}E_2+\frac{u_1u_2(I_1-I_2)}{I_3}E_3.$$

Replacing in the equation (3.4), if the semi-martingale $g(\cdot)$ in (2.4) is a critical point of $J^{\nabla^I,\langle\ ,\ \rangle^I}$, and writting $\widehat{u(t)}=(u_1(t),u_2(t),u_3(t))$, the vector $\hat{u}$ satisfies the following equations,

$$\begin{cases}
I_1\dot{u}_1(t)=(I_2-I_3)u_2(t)u_3(t)-\frac{(I_2-I_3)^2}{2I_2I_3}u_1(t)\\
I_2\dot{u}_2(t)=(I_3-I_1)u_1(t)u_3(t)-\frac{(I_3-I_1)^2}{2I_1I_3}u_2(t)\\
I_3\dot{u}_3(t)=(I_1-I_2)u_1(t)u_2(t)-\frac{(I_1-I_2)^2}{2I_1I_2}u_3(t)
\end{cases}$$

Remark. These equations are perturbations of the standard Euler-Poincaré equations. In particular, as in the argument in Proposition 3.4, the extra term is the Hodge Laplacian operator applied to $u(t)$ on (the Lie algebra) $so(3)$. Hence the above equation may also be viewed as a version of the viscous Euler-Poincaré equation.

More generally, using properties (4.1) and (4.2), we can compute equation (3.4) for the critical point of functional $J^{\nabla^{I'},\langle\ ,\ \rangle^I}$ where $I,I'\in\mathbb{R}^3$ may be different. In particular, for $I'=(1,1,1)$, by (4.2), $\nabla^{I'}_{E_i}E_j+\nabla^{I'}_{E_j}E_i=0$ for each i,j, which implies that $K(u)=0$ for each $u\in so(3)$ for the metric $\langle\ ,\ \rangle^I$ and the connection $\nabla^{I'}$. Therefore in this case, equation (3.4) is the same as the standard Euler-Poincaré equation. Nevertheless we stress that, even for this classical motion, we have associated the deterministic velocity trajectories to Lagrangian paths which are random.

4.2 Equations of fluid dynamics (volume preserving diffeomorphisms on the torus)

We shall discuss the two dimensional torus $\mathbb{T}^2$ for simplicity, although the torus of any dimension or even a more general compact Riemannian manifold can be considered as well. Let

$$G_V^s := \{g := \mathbb{T}^2 \to \mathbb{T}^2 \text{ is a volume preserving bijection map, } g, g^{-1} \in H^s\},$$

where H^s is the s-th order Sobolev space. If $s > 2$ the space G_V^s is an C^∞ infinite dimensional Hilbert manifold (see [E-M]). The composition operation on G_V^s will be the composition of $\mathbb{T}^2$ maps. If $s > 2$, G_V^s is also a topological group (not quite a Lie group since left translation is not smooth), see [E-M], and

$$g_V^s := T_e G_V^s = \{X \in H^s(\mathbb{T}^2; T\mathbb{T}^2),\ \pi(X) = e,\ \mathrm{div} X = 0\}$$

is the "Lie algebra" of G_V^s, where e is the identity map in $\mathbb{T}^2$.

We consider the inner products $\langle\ ,\rangle^0$ and $\langle\ ,\rangle^1$ on g_V^s defined as follows,

$$\langle X, Y\rangle^0 := \int_{\mathbb{T}^2} \langle X(x), Y(x)\rangle_x dx, \quad \forall X, Y \in g_V^s,$$

$$\langle X, Y\rangle^1 := \int_{\mathbb{T}^2} \langle X(x), Y(x)\rangle_x dx + \int_{\mathbb{T}^2} \langle \nabla X(x), \nabla Y(x)\rangle_x dx, \quad \forall X, Y \in g_V^s,$$

where $\langle\ ,\ \rangle$, ∇ are the standard metric and corresponding Levi-Civita connection on $\mathbb{T}^2$ (∇ coincides with the ordinary derivative on $\mathbb{T}^2$). We extend $\langle\ ,\rangle^0$, $\langle\ ,\rangle^1$ to right invariant metrics on G_V^s by right translation, which we still write as $\langle\ ,\rangle^0$ and $\langle\ ,\rangle^1$.

By Theorem 9.1 and 9.6 in [E-M], there exists a right invariant Levi-Civita connection ∇^0 with respect to $\langle\ ,\ \rangle^0$. In particular,

$$\nabla_X^0 Y = P_e\big(\nabla_X Y\big), \quad \forall X, Y \in g_V^s, \tag{4.3}$$

where ∇ is the Levi-Civita connection on $\mathbb{T}^2$ and P_e is the orthogonal projection (with respect to L^2) onto $g_V^s = \{X \in H^s(T\mathbb{T}^2),\ \mathrm{div} X = 0\}$, determined by the Hodge decomposition $H^s(T\mathbb{T}^2) := g_V^s \bigoplus dH^{s+1}(\mathbb{T}^2)$. From now on, for $X \in g_V^s$ when we use ∇ we view $X \in T\mathbb{T}^2$ as a vector field on $\mathbb{T}^2$ and when we use ∇^0 we view X as an element in g_V^s.

We want to make some remarks about the SDE (3.12) and its perturbation on the infinite dimensional group G_V^s. We take $H_i \in g_V^s$, $1 \leqslant i \leqslant m$, $u \in C^1([0,1]; g_V^s)$; then, as in [C-C], we consider the SDE on G_V^s as follows,

$$dg(t,\theta) = \sum_i H_i(g(t,\theta)) \circ dW_t^i + \tilde{u}(t, g(t,\theta))dt, \quad g(0,\theta) = \theta, \tag{4.4}$$

where $\tilde{u}(t) := u(t) - \sum_i \frac{1}{2}\nabla^0_{H_i} H_i$. We assume that H_i and u are regular enough so that $g(t,\cdot) \in G^s_V$ for each t, see e.g. the standard theory of stochastic flows in [K]. Notice that $H_i(g(t,\cdot)) = T_e R_{g(t)} H_i$, and therefore (4.4) can be viewed as the SDE (3.12) on the infinite dimensional group G^s_V.

Consider a smooth cylindrical function $F(g) := f\big(g(\theta_1), g(\theta_2), ...g(\theta_l)\big)$, $\forall g \in G^s_V$, where $f \in C^\infty\big((\mathbb{T}^2)^l\big)$ and $\theta_j \in \mathbb{T}^2$, $1 \leqslant j \leqslant l$. Applying Itô's formula to equation (4.4) we get,

(4.5)

$$F(g(t)) - F(g(0)) = N^F_t + \sum_i \sum_{j,k=1}^{l} \int_0^t H_{i,j}(g(s,\theta_j))\Big(H_{i,k}(g(s,\theta_k)) f\big(g(s,\theta_1), ..., g(s,\theta_l)\big)\Big) ds$$

$$+ \sum_{k=1}^{l} \int_0^t \tilde{u}_k(s, g(s,\theta_k)) f\big(g(s,\theta_1), ..., g(s,\theta_l)\big) ds$$

$$= N^F_t + \sum_i \int_0^t \big(T_e R_{g(s)} H_i\big)\big(T_e R_{g(s)} H_i\big) F(g(s)) ds + \int_0^t \big(T_e R_{g(s)} \tilde{u}\big) F(g(s)) ds,$$

where $H_{i,k} f := \langle H_i, \nabla_k f\rangle$, ∇_k denotes the gradient with the k-th variable of $f(x_1, .., x_l) \in C^\infty\big((\mathbb{T}^2)^l\big)$, and the term $\tilde{u}_k f$ has the same meaning. The term

$$N^F_t := \sum_i \int_0^t \big(T_e R_{g(s)} H_i\big) F(g(s)) dW^i_s$$

is a martingale. If we fix the connection ∇^0, by (2.2), (4.4) and (4.5), we have,

$$\mathrm{Hess}^0 F\big(g(t)\big)\big(dg(t), dg(t)\big) = \sum_i \mathrm{Hess}^0 F\big(g(t)\big)\big(T_e R_{g(t)} H_i, T_e R_{g(t)} H_i\big) dt$$

$$= \sum_i \Big(\big(T_e R_{g(t)} H_i\big)\big(T_e R_{g(t)} H_i\big) F(g(t)) dt - \nabla^0_{T_e R_{g(t)} H_i}(T_e R_{g(t)} H_i) F(g(t)) dt\Big).$$

Hence, by definition (2.1), we have $T_{g(t)} R_{g(t)^{-1}} \frac{D^{\nabla^0} g(t)}{dt} = u(t)$, which is the same as in the finite dimensional case.

Next we consider the variations of $g(t)$; the flow $e_{\varepsilon,v}$ defined by (3.2) in G^s_V is the solution of the equation,

$$\begin{cases} \frac{de_{\varepsilon,v}(t,\theta)}{dt} = \varepsilon \dot{v}(t, e_{\varepsilon,v}(t,\theta)) \\ e_{\varepsilon,v}(0,\theta) = \theta, \end{cases}$$

where $v \in C^1([0,1]; g^s_V)$ with $v(0) = v(T) = 0$. Notice that it coincides with the perturbation taken in [C-C]. For $g_\varepsilon(t,\theta) := e_{\varepsilon,v}(t) g(t)(\theta) = e_{\varepsilon,v}(t, g(t,\theta))$, by Itô's formula we get,

$$
\begin{aligned}
dg_\varepsilon(t,\theta) &= \sum_i \big(T_{g(t,\theta)}e_\varepsilon(t,g(t,\theta))\big)H_i(g(t,\theta))\circ dW_t^i \\
&+ \big(T_{g(t,\theta)}e_\varepsilon(t,g(t,\theta))\big)\tilde{u}(t,g(t,\theta))dt + \varepsilon\dot{v}(t,g_\varepsilon(t,\theta))dt \\
&= \sum_i (Ad_{e_\varepsilon(t)}H_i)(g_\varepsilon(t,\theta))\circ dW_t^i + (Ad_{e_\varepsilon(t)}\tilde{u}(t))(g_\varepsilon(t,\theta))dt + \varepsilon\dot{v}(t,g_\varepsilon(t,\theta))dt,
\end{aligned}
\tag{4.6}
$$

where we omit the index v in $e_{\varepsilon,v}$ for simplicity, $T_x e_\varepsilon(t,x)$ denotes the differential of the map $e_\varepsilon(t,.)$ at the point $x\in\mathbb{T}^2$, and in the second step above we use the property $(Ad_{e_\varepsilon(t)}H_i)(\theta) = \big(T_{e_\varepsilon^{-1}(t,\theta)}e_\varepsilon(t,e_\varepsilon^{-1}(t,\theta))\big)H_i(e_\varepsilon^{-1}(t,\theta))$.

Notice that equation (4.6) corresponds to (3.8) on G_V^s(right invariant metric case). Hence from equation (4.6) and the same procedure of the analysis for $g(t)$ above, we derive the following equality

$$
T_{g_\varepsilon(t)}R_{g_\varepsilon(t)^{-1}}\frac{D^{\nabla^0}g_\varepsilon(t)}{dt} = \sum_i \frac{1}{2}\nabla^0_{H_i^\varepsilon(t)}H_i^\varepsilon(t) + Ad_{e_\varepsilon(t)}\tilde{u}(t) + \varepsilon\dot{v}(t),
$$

where $H_i^\varepsilon(t) = Ad_{e_\varepsilon(t)}H_i$. Then we can take the derivative with respect to ε of $T_{g_\varepsilon(t)}R_{g_\varepsilon(t)^{-1}}\frac{D^{\nabla^0}g_\varepsilon(t)}{dt}$ and the proof of Theorem 3.4 can still be applied to this case, which means that Theorem 3.4 still holds on the infinite dimensional group G_V^s.

We choose some suitable basis of g_V^s as in [C-C]. We consider such basis indexed by k in a subset of $\mathbb{Z}^2$ having an unique representative of the equivalence class defined by the relation $k\simeq k'$ if $k+k'=0$. More precisely we choose the vectors $\{A_k,B_k\}_{k=1}^\infty$ of the following form,

$$
A_k(\theta) = \lambda(|k|)(A_k^1(\theta),A_k^2(\theta)),\quad \text{with } A_k^1(\theta) = k_2\cos(k\cdot\theta),\ A_k^2(\theta) = -k_1\cos(k\cdot\theta),
$$

$$
B_k(\theta) = \lambda(|k|)(B_k^1(\theta),B_k^2(\theta)),\quad \text{with } B_k^1(\theta) = k_2\sin(k\cdot\theta),\ B_k^2(\theta) = -k_1\sin(k\cdot\theta),
$$

where $\theta=(\theta_1,\theta_2)\in\mathbb{T}^2$, $k=(k_1,k_2)\in\mathbb{Z}^2$, $k\cdot\theta = k_1\theta_1+k_2\theta_2$ and $\lambda(|k|)$ is a constant depending only on $|k|=|k_1|+|k_2|$. Since $\nabla_{A_k}A_k=0,\ \ \nabla_{B_k}B_k=0\ \ \forall k$ (see the proof of Lemma 2.1 in [C-C]), the SDE (4.4) becomes,

$$
dg(t,\theta) = \sum_k \big(A_k(g(t,\theta))\circ dW_t^{k,1} + B_k(g(t,\theta))\circ dW_t^{k,2}\big) + u(t,g(t,\theta))dt,\quad g(0,\theta)=\theta,
\tag{4.7}
$$

If we assume $u(t,\cdot)\in T\mathbb{T}^2$ to be regular enough and $\lambda(|k|)$ decaying to 0 fast enough as $|k|$ tends to infinity, then a weak solution of (4.7) exists, see [C-C]. Moreover the Stratonovich and the Itô integrals in the equation coincide.

Note that in the proof Theorem 3.2, when $\{A_k,B_k\}$ is an infinite sequence, if $\lambda(|k|)$ decays to 0 fast enough as $|k|$ tends to infinity, we can change the derivation with respect to ε and the infinite sum in indeces k and the conclusion of the Theorem is true. But for simplicity, from now on we assume that $u(\cdot,\cdot)$ is regular enough, and $\{A_k, B_k\}$ is a finite sequence, i.e., there exists an integer $m > 0$, such that $\lambda(|k|) = 0$ for each k with $|k| > m$. Furthermore, by the proof of Theorem 2.2 in [C-C], we have the following characterization,

$$\sum_{|k|\leqslant m} \big(A_k A_k f + B_k B_k f\big) = \nu \Delta f, \quad \forall f \in C^2(\mathbb{T}^2), \tag{4.8}$$

where $\nu := \frac{1}{2}\sum_{k\leqslant m} \lambda^2(|k|)k_1^2$.

So the infinite dimensional (projected) Laplacian, when computed on smooth cylindrical functions with only one variable, coincides with the usual Laplacian on the torus.

Proposition 4.1. *The semi-martingale $g(\cdot,\cdot)$ in (4.7) is a critical point of the action functional $J^{\nabla^0,\langle\ ,\ \rangle^0}$ (see (3.1) for definition), if and only if, for some function p, u satisfies the following Navier-Stokes equation on time interval $t \in [0,T]$,*

$$\begin{cases} \frac{\partial u}{\partial t} = -u\cdot\nabla u + \frac{\nu}{2}\Delta u + \nabla p(t) \\ \mathrm{div} u = 0. \end{cases} \tag{4.9}$$

The semi-martingale $g(\cdot,\cdot)$ in (4.7) is a critical point of the action functional $J^{\nabla^0,\langle\ ,\ \rangle^1}$ if and only if, for some function p, u satisfies the viscous Camassa-Holm equation on time interval $t \in [0,T]$,

$$\begin{cases} \frac{\partial v}{\partial t} = -u\cdot\nabla v - \sum_{j=1}^2 v_j \nabla u_j + \frac{\nu}{2}\Delta v + \nabla p(t) \\ v = u - \Delta u \\ \mathrm{div} u = 0 \end{cases} \tag{4.10}$$

Proof. In order to apply Theorem 3.4, we just need to give an explicit expression of $ad_u^*(u)$ and $K(u)$ in (3.6), (3.7) for the different metrics and connections.

For each $X \in H^s(T\mathbb{T}^2)$ and $Y \in g_V^s$,

$$\langle P_e X, Y\rangle^0 = \int_{\mathbb{T}^2} \langle (P_e X)(x), Y(x)\rangle dx = \int_{\mathbb{T}^2} \langle X(x), Y(x)\rangle dx.$$

Therefore, for each $u, v \in g_V^s$ regular enough,

$$\begin{aligned} &\langle u, \nabla^0_{ad_v A_k} A_k + \nabla^0_{A_k}(ad_v A_k)\rangle^0 = \int_{\mathbb{T}^2} \langle u, P_e\big(\nabla_{ad_v A_k} A_k + \nabla_{A_k}(ad_v A_k)\big)\rangle dx \\ &= -\int_{\mathbb{T}^2} \langle u, (\nabla_{[v,A_k]} A_k + \nabla_{A_k}[v, A_k])\rangle dx \end{aligned}$$

Note that ∇ is the Levi-Civita connection on $\mathbb{T}^2$, $\nabla_{A_k} A_k = 0$, and the Riemannian curvature on $\mathbb{T}^2$ is zero; by the same computation as in (3.14) we have,

$$\nabla_{[v,A_k]} A_k + \nabla_{A_k}[v, A_k] = -\nabla_{A_k}\nabla_{A_k} v$$

An analogous identity holds for B_k, so combining the computations above,

$$\sum_k \langle u, \nabla^0_{ad_v A_k} A_k + \nabla^0_{A_k}(ad_v A_k) + \nabla^0_{ad_v B_k} B_k + \nabla^0_{B_k}(ad_v B_k)\rangle^0$$
$$= \sum_k \int_{\mathbb{T}^2} \langle u, \nabla_{A_k}\nabla_{A_k} v + \nabla_{B_k}\nabla_{B_k} v\rangle dx$$
$$= \int_{\mathbb{T}^2} \langle u, \nu\Delta v\rangle dx = \int_{\mathbb{T}^2} \langle \nu\Delta u, v\rangle dx = \langle \nu\Delta u, v\rangle^0,$$

where in the second step above we used property (4.8), in the third the integration by parts formula on $\mathbb{T}^2$, and the last step is due to the fact that $\Delta u \in g_V^s$ for $u \in g_V^s$ regular enough. So by definition (3.7), we have $K(u) = -\frac{\nu}{2}\Delta u$ for the metric $\langle\ ,\ \rangle^0$ and the connection ∇^0.

Another proof of this equality was given in [C], using the characterization of K in Proposition 3.4 and a direct computation of the operator K via the computation of the Ricci tensor for the Levi-Civita connection with respect to the metric $\langle\ ,\ \rangle^0$.

From [A-K] or [M-R], for the metric $\langle\ ,\ \rangle^0$, we have $ad^*_u(u) = P_e(\nabla_u u) = P_e(u\cdot\nabla u)$.

As a result the reduced Euler-Poincaré equation (3.13) for $J^{\nabla^0,\langle\ ,\ \rangle^0}$ is the Navier-Stokes equation (4.9). The pressure term p is derived by a standard L^2 duality argument.

Now we consider the metric $\langle\ ,\ \rangle^1$. For each $X \in H^s(T\mathbb{T}^2)$ and $Y \in g_V^s$,

$$\langle P_e X, Y\rangle^1 = \int_{\mathbb{T}^2} \langle (P_e X)(x), Y(x)\rangle dx + \int_{\mathbb{T}^2} \langle \nabla(P_e X)(x), \nabla Y(x)\rangle dx$$
$$= \int_{\mathbb{T}^2} \langle X(x), Y(x)\rangle dx + \int_{\mathbb{T}^2} \langle \nabla X(x), \nabla Y(x)\rangle dx,$$

Notice also that $\langle u, \Delta v\rangle^1 = \langle \Delta u, v\rangle^1$ for $u, v \in g_V^s$, due to the integration by parts formula on $\mathbb{T}^2$. So we can follow the same steps as we did for the metric $\langle\ ,\ \rangle^0$ above, and obtain $K(u) = -\frac{\nu}{2}\Delta u$ for the metric $\langle\ ,\ \rangle^1$ and connection ∇^0 (the connection is still ∇^0 here).

From Theorem 3.2 in [S] (notice that the definition of Laplacian in [S] has a different sign from the Laplacian here), and since $P_e(1-\Delta)^{-1} = (1-\Delta)^{-1}P_e$ on $T\mathbb{T}^2$, for the metric $\langle\ ,\ \rangle^1$, we have,

$$ad^*_u(u) = (1-\Delta)^{-1}\Big(P_e\Big(u\cdot\nabla(u-\Delta u) + \sum_{j=1}^{2}(u_j - \Delta u_j)\nabla u_j\Big)\Big).$$

Combining the above together, the reduced Euler-Poincaré equation (3.13) for $J^{\nabla^0,\langle\ ,\ \rangle^1}$ is the viscous Cassama-Holm equation (4.10). $\square$

Concerning the result of the above theorem on the Navier-Sokes equation, it was first derived in [C-C] and later generalized to incompressible Brownian flows in compact manifolds (examples of such flows are known, more generally, in compact symmetric spaces), where the same formula as (4.8) is valid if we replace the Laplacian by the Laplace-Hodge operator (c.f. [A-C], Theorem 2.2.). The appearance of this operator is actually an illustration of Proposition 3.4.

For the standard Camassa-Holm equation we refer to [C-H] and [H-M-R], for viscous Camassa-Holm equation we refer to [F-H-T] and [V]. Our result is new for this equation.

Remark 1. The generalized derivative in [A-C] and [C-C] for stochastic processes is essentially taken in the pointwise sense, and the choice of a connection on the space G_V^s is not needed. Although this is adapted to the reduction for a solution of the Navier-Stokes equation, it seems not possible to be applied to the viscous Camassa Holm equation, and that it is necessary to define the generalized derivative associated with a connection on G_V^s as we do in this article.

Remark 2. For simplicity we assume here that u is regular, so that u is the classical solution of the corresponding PDE. But to check the proof of Theorem 3.2, we only need the test vectors v to be regular enough and under such cases, a less regular u is still a weak solution.

Remark 3. We can define a H^n metric as $\langle X, Y\rangle^n := \int \sum_{i=0}^n \langle \nabla^i X(x), \nabla^i Y(x)\rangle dx$ for each $X, Y \in g_V^s$, the corresponding critical equation (3.13) for $J^{\nabla^0, \langle\ ,\ \rangle^n}$ is as follows,

$$\begin{cases} \frac{\partial u}{\partial t} = -ad_u^*(u) + \frac{\nu}{2}\Delta u, \\ \mathrm{div} u = 0, \end{cases}$$

where the duality in ad^* here is defined by (3.6) for the metric $\langle\ ,\rangle^n$.

Remark 4. For the volume-preserving diffeomorphisms group on higher dimensional torus, we can also choose an suitable basis of the corresponding Lie algebra, see [C-M]. Then we can get the Navier-Stokes and viscous Camassa-Holm equation in higher dimensional torus by the stochastic reduction procedure above.

Remark 5. In fact, in [F-H-T] and [V], the following "second grade fluid equation" is studied

$$\begin{cases} \frac{\partial v}{\partial t} = -u \cdot \nabla v - \sum_{j=1}^2 v_j \nabla u_j + \frac{\nu}{2}\Delta v + \nabla p(t) \\ v = u - \alpha \Delta u \\ \mathrm{div} u = 0, \end{cases} \tag{4.11}$$

where $\alpha \geqslant 0$ is a non-negative constant. In particular, when $\alpha = 0$, it is the Navier-Stokes equation, and when $\alpha = 1$ it reduces to (4.10). Following the same procedure as in the proof of Proposition 4.1, we can verify that the semi-martingale $g(\cdot, \cdot)$ in (4.7) is a critical point of the action functional $J^{\nabla^0, \langle\ ,\ \rangle^\alpha}$, if and only if u satisfies the equation

(4.11), where $\langle\ ,\ \rangle^{\alpha}$ is a metric on g_V^s defined by

$$\langle X,Y\rangle^{\alpha} := \int_{\mathbb{T}^2} \langle X(x),Y(x)\rangle_x dx + \alpha \int_{\mathbb{T}^2} \langle \nabla X(x), \nabla Y(x)\rangle_x dx, \quad \forall X,Y \in g_V^s.$$

4.3 Navier-Stokes equation on a bounded domain

Suppose D is a smooth bounded open domain on $\mathbb{R}^2$, and denote the boundary and the closure of D by ∂D and $\bar{D}$, respectively. As in [E-M], let

$$G_V^s(D) := \{g : \bar{D} \to \bar{D} \text{ is a volume preserving map, } g(x) \in \partial D \text{ for every } x \in \partial D,\ g, g^{-1} \in H^s\}.$$

If $s > 2$, then $G_V^{s,D}$ is a C^∞ topological group with the composition operation defined as the composition of maps from $\bar{D}$ to $\bar{D}$. In particular, for $G_V^{s,0}(D) := \{g \in G_V^s(D) : g(x) = x \text{ for every } x \in \partial D\}$, then the "Lie algebra" for $G_V^{s,0}(D)$ is as follows,

$$\begin{aligned} & g_V^{s,0}(D) := T_e G_V^{s,0}(D) \\ & = \{X \in H^s(\bar{D}; T\bar{D}),\ \pi(X) = e,\ \mathrm{div} X = 0,\ X(x) = 0 \text{ for every } x \in \partial D\}. \end{aligned}$$

One can consider other subgroups of $G_V^s(D)$ with specified boundary conditions and the "Lie algebras" of such subgroups will be the vector fields on D with adequate boundary conditions.

By the same procedure, we can also consider the SDE on $G_V^{s,0}(D)$ as follows,

$$dg(t,\theta) = \sum_i H_i(g(t,\theta)) \circ dW_t^i + u(t, g(t,\theta))dt, \quad g(0,\theta) = \theta, \theta \in \bar{D},$$

where we assume that H_i, $u(t) \in g_V^{s,0}(D)$ are regular enough.

In the deterministic case, i.e. when $H_i = 0$, such framework is adopted in [E-M] to study the geodesic spray on $G_V^{s,0}(D)$ as a characterization of the Euler equation on D with specific boundary condition. But for the stochastic case, different from the case for volume preserving maps on torus introduced in Section 4.2, it seems not possible to find suitable sequences of vector fields $H_i \in g_V^{s,0}(D)$ which ensure that the generator of the above SDE is the Laplacian operator, due to the restriction on the boundary value.

So here we need to adopt an alternative way to formulate the stochastic reduction for the Navier-Stokes equation on D. By Corollary 3.2 in [K-M-P-T], given $u \in C^1([0,T]; g_V^{s,0}(D))$, for every t, there exists an extension $\bar{u}(t) \in H^s(\mathbb{R}^2)$ of $u(t)$ such that $u(t,x) = \bar{u}(t,x)$ for every $x \in D$, $\mathrm{div}\bar{u}(t) = 0$, and $\bar{u}(t)$ has compact support in $\mathbb{R}^2$. Then for a fixed $\nu > 0$, taking $H_1(x) = (\sqrt{\nu}, 0)$, $H_2(x) = (0, \sqrt{\nu})$, $u = \bar{u}$ in SDE (4.4), we consider,

$$dg(t,\theta) = \sum_{i=1}^2 H_i(g(t,\theta)) \circ dW_t^i + \bar{u}(t, g(t,\theta))dt, \quad g(0,\theta) = \theta, \theta \in \mathbb{R}^2. \tag{4.12}$$

Since, by the proof of Corollary 3.2 in [K-M-P-T], there exists a compact set $K \subseteq \mathbb{R}^2$, such that supp $\bar{u}(t) \subseteq K$ for every $t \in [0,T]$, we can view $\bar{u}(t)$ as a vector field on a torus (not necessarily with periodicity 1), and SDE (4.12) can also be viewed as a SDE on the space of diffeomorphisms on such torus. Hence, taking s sufficiently big to ensure the needed regularity for $\bar{u}(t)$, and for every $v \in C^1([0,T]; g_V^s)$, we can repeat the computation in Section 4.2 and define the perturbed stochastic Lagrangian paths $g_{\varepsilon,v}(t,\theta)$ as well as the generalized derivative $\frac{D^{\nabla^0} g_{\varepsilon,v}}{dt}$, where ∇^0 is the Levi-Civita connection on G_V^s with respect to the metric $\langle\ ,\rangle^0$ defined by (4.3). Therefore the action functional $J^{\nabla^0,\langle\ ,\ \rangle^0}$ (defined by (3.1)) is well defined for every $g_{\varepsilon,v}(\cdot)$.

Proposition 4.2. *The vector field $u \in C^1([0,T]; g_V^{s,0}(D))$ is a solution of the Navier-Stokes equation on D,*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = -u(t,x)\cdot\nabla u(t,x) + \frac{\nu}{2}\Delta u(t,x) + \nabla p(t,x),\ x \in D, \\ \mathrm{div} u(t,x) = 0,\ x \in D, \\ u(t,x) = 0, \quad x \in \partial D \end{cases} \tag{4.13}$$

if and only if for every $v \in C^1([0,T]; C_0^\infty(D))$ satisfying $\mathrm{div} v = 0$ *and* $v(0) = v(T) = 0$*, we have*

$$\frac{dJ^{\nabla^0,\langle\ ,\ \rangle^0}(g_{\varepsilon,\bar{v}}(.))}{d\varepsilon}\Big|_{\varepsilon=0} = 0, \tag{4.14}$$

where $C_0^\infty(D)$ denotes the set of smooth functions whose supports are compact sets contained in D, and $\bar{v} \in C_0^\infty(\mathbb{R}^2)$ is the extension of v such that $\bar{v}(x) = v(x)$ for every $x \in D$ and $\bar{v}(x) = 0$ for every $x \notin D$.

Proof. As in the computations in Theorem 3.2, and Section 4.2, for every $v \in C^1([0,T]; C_0^\infty(D))$ with $\mathrm{div} v = 0$, $v(0) = v(T) = 0$ and any extension $\bar{u}$ of u with compact support ,

$$\begin{aligned} &\frac{dJ^{\nabla^0,\langle\ ,\ \rangle^0}(g_{\varepsilon,\bar{v}}(.))}{d\varepsilon}\Big|_{\varepsilon=0} \\ &= \int_0^T \int_{\mathbb{R}^2} \Big\langle \bar{u}(t,x), \frac{\partial \bar{v}(t,x)}{\partial t} + [\bar{u}(t,x), \bar{v}(t,x)] + \sum_{i=1}^2 \frac{1}{2}\nabla_{H_i}\nabla_{H_i}\bar{v}(t,x)\Big\rangle dxdt \\ &= \int_0^T \int_D \Big\langle -\frac{u(t,x)}{\partial t} - u(t,x)\cdot\nabla u(t,x) + \frac{\nu}{2}\Delta u(t,x), v(t,x)\Big\rangle dxdt, \end{aligned}$$

where, in the last step, we use the integration by parts formula, the property that $\bar{v} = 0$ for every $x \notin D$, and the boundary condition $u(t,x) = 0$, $v(t,x) = 0$ for every $x \in \partial D$ and $v(0) = v(T) = 0$. Hence by Proposition 1.1, Section 1.4 in [T] we know that u is a solution to (4.13) if only if (4.14) is true.

In particular, the result does not depend on the choice of extension of u. $\square$

Remark 1. Comparing with the case of the Navier-Stokes equation on torus (see e.g. Proposition 4.1), the solution of the Navier-Stokes equation on a bounded domain with no-slip boundary condition is characterized as the drift of a semi-martingale which can be seen as a critical point under some perturbation. This perturbation is not exactly generated by the "Lie algebra" g_V^s (i.e. $g_V^{s,0}(D)$) for the action functional $J^{\nabla^0, \langle\ ,\ \rangle^0}$.

Remark 2. If we consider the action functional $J^{\nabla^0, \langle\ ,\ \rangle^1}$ associated with the connection ∇^0 and inner product $\langle\ ,\ \rangle^1$, with similar arguments to those in Proposition 4.1 and 4.2, we will obtain a characterization of the Camassa-Holm equation on a smooth bounded domain D.

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REFERENCES

[A-C] M. Arnaudon and A.B. Cruzeiro, *Lagrangian Navier-Stokes diffusions on manifolds: variational principle and stability,* Bull. Sci. Math. 136 (2012), no. 8, 857–881.

[A] V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits,*Ann. Inst. Fourier 16 (1966), 316–361.

[A-K] V. I. Arnold and B. Khesin, *Topological methods in hydrodynamics,* Applied Math. Series 125, Springer (1998).

[B] J. M. Bismut, *Mécanique aléatoire,* Lecture Notes in Mathematics, 866, Springer (1981).

[C-H] R. Camassa, D.D. Holm, *A completely integrable dispersive shallow water equation with peaked solutions,* Phys. Rev. Lett. 71 (1993), 1661–1664.

[C-Z] K. L. Chung and J. C. Zambrini, *Introduction to random time and quantum randomness,* World Scientific (2003).

[C-C] F. Cipriano and A.B. Cruzeiro, *Navier-Stokes equation and diffusions on the group of homeomorphisms of the torus,* Comm. Math. Phys. 275 (2007), no. 1, 255–269.

[C] A.B. Cruzeiro, *Hydrodynamics, probability and the geometry of the diffeomorphisms group,* Seminar on Stoch. Analysis, Random Fields and Applications IV, R. C. Dalang. M. Dozzy, F. Russo ed, Birkhauser P.P. 63 (2011).

[C-M] A.B. Cruzeiro and P. Malliavin, *Nonergodicity of Euler fluid dynamics on* *tori versus positivity of the Arnold-Ricci tensor,* J. Funct. Anal. 257 no. 1, (2008), 1903–1925.

[E] M. Emery, *Stochastic calculus in manifolds,* Springer, Universitext (1989).

[E-M] D.G. Ebin and J.E. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid,* Ann of Math. 92 (1970), 102–163.

[F-H-T] C. Foias, D. D. Holm and E.S. Titi, *The three-dimensional viscous Camassa-Holm equations and their relation to the Navier-Stokes equations and turbulence theory,* J. Dyn. Diff. Eqns. 14, (2002), 1–35.

[H-M-R] D.D. Holm, J.E. Marsden and T. Ratiu, *The Euler-Poincaré equations and semidirect products with applications to continuum theories,* Adv. Math. 137 no 1 (1998), 1–81.

[I-W] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes,* North-Holland (1981).

[K-M-P-T] T. Kato, M. Mitrea, G. Ponce and M. Taylor, *Extension and representation of divergence-free vector fields on bounded domains,* Math. Res. Lett. 7 no 5-6 (2000), 643–650.

[K] H. Kunita, *Stochastic flows and stochastic differential equations,* Cambridge University Press, 1997.

[LC-O] J.A. Lázaro-Camí and J.P. Ortega, *Stochastic Hamiltonian dynamical systems,* Reports on Math. Phys, 61 no. 1 (2008), 65–122.

[M-R] J. E. Marsden and T. S. Ratiu, *Introduction to Mechanics and Symmetry: a basic exposition of classical mechanical systems,* Springer, Texts in Applied Math. (2003).

[S] S. Shkoller, *Geometry and curvature of diffeomorphism groups with H1 metric and mean hydrodynamics,* J. Funct. Anal. 160 (1998), 337–365.

[T] R. Teman, *Navier-Stokes equations,* North-Holland, Amsterdam, (1986).

[V] J. Vukadinovic, *On the backwards behavior of the solutions of the 2D periodic viscous Camassa-Holm equations,* J. Dyn. Diff. Eqns. 14 no. 1 (2002), 37–62.

[Y1] K. Yasue, *Stochastic calculus of variations,* Lett. Math. Phys. 4 no. 4 (1980), 357–360.

[Y2] K. Yasue, *A variational principle for the Navier-Stokes equation,* J. Funct. Anal. 51 no. 2 (1983), 133–141.

[Z] J. C. Zambrini, *Variational processes and stochastic versions of mechanics,* J. Math. Phys. 27 no. 9 (1986), 2307–2330.

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