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# Cascade high gain observers for nonlinear systems with delayed output measurement

Tarek Ahmed-Ali, Estelle Cherrier and Mohammed M'Saad

**Abstract**—This work presents a set of cascade high gain observers for triangular nonlinear systems with delayed output measurement. A sufficient condition ensuring the exponential convergence of the observation error towards zero is given. This result is illustrated by some simulations.

**Index Terms**—Time-delay systems, high gain observer, cascade systems

## I. INTRODUCTION

This paper deals with the design of nonlinear observers in the presence of delayed output measurement. This problem is particularly challenging since time delays affecting output measurements arise in a variety of applications. One can cite for example systems which are controlled by a remote controller. In these systems, the output data are transmitted to the controller throughout a communication system which introduces a time-delay between the process and the controller. The design of controllers for such systems can be viewed as an output feedback design based on state prediction system. In the linear case, this problem has been solved by the well-known *Smith predictor* [1] and several predictive control algorithms [2], [3]. Recently, for the nonlinear case, a new kind of chained observers which reconstruct the state at different delayed time instants for *drift observable* systems has been presented in [4]. The authors showed, by using *Gronwall* lemma, that under some conditions on the delay, exponential convergence of the chained observers is ensured. These conditions have been relaxed in [5] by using an approach based on a first-order singular partial differential equation. On the other hand, in [6] a novel predictor for linear and nonlinear systems with time delay measurement has been designed. This predictor is a set of cascade observers. Sufficient conditions based on *linear matrix inequalities* are derived to guarantee the asymptotic convergence of this predictor. In the present work, we design a set of cascade high gain observers for nonlinear triangular systems by considering a time delay in the output measurement. We will show that the general high gain observer design framework developed in [7], [8], [9], to mention a few, for delay-free output measurements can be extended to systems with delayed output. More precisely, we propose to use a suitable *Lyapunov-Krasovskii functional* and a sufficient number of high gain observers, in order to guarantee the exponential convergence of the estimated state at time  $t$  towards the

true state at time  $t$ , even if the output is affected by any constant and known delay. We will also give an explicit relation between the number of observers and the delay.

The present paper is organized as follows : In section II, we present the class of considered systems and the different assumptions. In the third one, we present the proposed observers and prove their convergence. In the last one, we illustrate our results throughout simulations on an academic example.

## II. PRELIMINARIES AND NOTATIONS

First some mathematical notations which will be used throughout the paper are introduced.

- The euclidian norm on  $\mathbb{R}^n$  will be denoted by  $\|\cdot\|$ .
- The matrix  $X^T$  represents the transposed matrix of  $X$ .
- $e_s(i) = (\underbrace{0, \dots, 0}_{s \text{ components}}, \overbrace{1}^{i^{th}}, 0, \dots, 0) \in \mathbb{R}^s, s \geq 1$  is the  $i^{th}$  vector of canonical basis of  $\mathbb{R}^s$ .
- The convex hull of  $\{x, y\}$  is denoted as  $\text{Co}(x, y) = \{\lambda x + (1 - \lambda)y, 0 \leq \lambda \leq 1\}$ .
- $\lambda_{\min}(S)$  and  $\lambda_{\max}(S)$  are the minimum and maximum eigenvalues of the square matrix  $S$ .

In this paper, we consider the following class of nonlinear systems:

$$\begin{aligned} \dot{x} &= Ax + \phi(x, u) \\ y &= Cx(t - \tau) \end{aligned} \quad (1)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \dots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \quad (2)$$

$$C = (1 \quad 0 \quad \dots \quad 0) \quad (3)$$

and

$$\phi(x, u) = \begin{pmatrix} \phi_1(x, u) \\ \vdots \\ \phi_n(x, u) \end{pmatrix} \quad (4)$$

The term  $\tau$  represents the measurement time delay,  $x(t) \in \mathbb{R}^n$  is the vector state which is supposed unavailable. The output  $y(t) \in \mathbb{R}$  is a linear function of the state  $x$  at time

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$t - \tau$ . The input  $u \in U$  where  $U$  is a compact set in  $\mathbb{R}$ . The functions  $\phi_i$ ,  $i = 1, \dots, n$  are supposed smooth. This class represents the class of uniformly observable systems. It has been shown [7], [8] that these models concern a wide variety of systems, such as bioreactors...

Throughout the paper, we assume that the following hypotheses are satisfied:

$\mathcal{H}1$ . The functions  $\phi_i(x, u)$  are triangular in  $x$ , i.e:

$$\frac{\partial \phi_i(x, u)}{\partial x_{k+1}} = 0 \quad k = i, \dots, n-1 \quad (5)$$

$\mathcal{H}2$ . The functions  $\phi_i(x, u)$  are globally Lipschitz, uniformly in  $u$ , i.e:

$$\exists \mathcal{L} > 0 \quad \text{such that} \quad \forall (x, y) \in \mathbf{R}^n \times \mathbf{R}^n, \quad \forall u \in U$$

$$\|\phi_i(x, u) - \phi_i(y, u)\| \leq \mathcal{L} \|x - y\| \quad (6)$$

$\mathcal{H}3$ . The time delay  $\tau$  is supposed constant and known.

### III. MAIN RESULTS

In this work, we consider an arbitrary long time delay  $\tau$  affecting the output measurement of system (1). The proposed nonlinear observer for system (1) is a set of  $m$  cascade high gain observers. Each one of them estimates a delayed state vector with sufficiently small delay  $\frac{\tau}{m}$ .

In order to present the proposed observer, we use the following convenient notations adopted from [4]:

$$x_j(t) = x(t - \tau + j \frac{\tau}{m})$$

, where  $j = 1, \dots, m$

Then the proposed observer can be written in the following form, for  $j = 1, \dots, m$ :

$$\begin{aligned} \dot{\hat{x}}_1 &= A\hat{x}_1 + \phi(\hat{x}_1) - \theta \Delta^{-1} S^{-1} C' C (\hat{x}_1(t - \frac{\tau}{m}) - x(t - \tau)) \\ \hat{y}_1 &= C\hat{x}_1(t - \frac{\tau}{m}) \\ &\vdots \\ \dot{\hat{x}}_j &= A\hat{x}_j + \phi(\hat{x}_j) - \theta \Delta^{-1} S^{-1} C' C (\hat{x}_j(t - \frac{\tau}{m}) - \hat{x}_{j-1}(t)) \\ \hat{y}_j &= C\hat{x}_j(t - \frac{\tau}{m}) = C\hat{x}_{j-1}(t) \end{aligned} \quad (7)$$

where  $\theta$  is a positive constant satisfying  $\theta > 1$ .

$S$  is a symmetric positive definite matrix, solution of the following algebraic Lyapunov equation:

$$SA + A^T S - C^T C = -S \quad (8)$$

and  $\Delta$  is a diagonal matrix which has the following form :

$$\Delta = \text{Diag} \left( 1, \dots, \frac{1}{\theta^{i-1}}, \dots, \frac{1}{\theta^{n-1}} \right). \quad (9)$$

We will show that the vector  $\hat{x}_j(t)$  estimates the delayed state  $x_j(t)$ ,  $j = 1, \dots, m-1$  and  $\hat{x}_m(t)$  estimates  $x(t)$ .

Before proving the exponential convergence of the proposed chained observers, we consider the case when the delay  $\tau$  is sufficiently small. Then only one high gain observer is required to estimate the state of system (1).

*Lemma 1:* Consider the following observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \phi(\hat{x}, u) - \theta \Delta^{-1} S^{-1} C^T C (\hat{x}(t - \tau) - x(t - \tau)) \\ \hat{y} &= C\hat{x}(t - \tau) \end{aligned} \quad (10)$$

Then for sufficiently large positive  $\theta$ , there exists a sufficiently small positive constant  $\tau_1$  such that  $\forall \tau \leq \tau_1$ , observer (10) converges exponentially towards system (1).

*Proof:* First let us denote the observation error as  $\tilde{x} = \hat{x} - x$ .

Then we will have:

$$\dot{\tilde{x}} = A\tilde{x} + \phi(\hat{x}, u) - \phi(x, u) - \theta \Delta^{-1} S^{-1} C^T C \tilde{x}(t - \tau) \quad (11)$$

If we apply the relation

$$\tilde{x}(t) = \tilde{x}(t - \tau) + \int_{t-\tau}^t \dot{\tilde{x}}(s) ds \quad (12)$$

and the change of coordinates  $\bar{x} = \Delta \tilde{x}$ , system (11) can be rewritten in the following manner:

$$\begin{aligned} \dot{\bar{x}} &= \theta(A - S^{-1} C^T C) \bar{x} + \Delta(\phi(\hat{x}, u) - \phi(x, u)) \\ &+ \theta S^{-1} C' C \int_{t-\tau}^t \dot{\tilde{x}}(s) ds. \end{aligned} \quad (13)$$

In order to derive an upper bound  $\tau_1$  for the delay  $\tau$ , to ensure the exponential convergence to zero of the error  $\bar{x}$ , we use the following *Lyapunov-Krasovskii functional* [10]:

$$W = \bar{x}^T S \bar{x} + \int_{t-\tau_1}^t \int_s^t \|\dot{\bar{x}}(\xi)\|^2 d\xi ds. \quad (14)$$

This functional can be written after some manipulations as follows:

$$W = \bar{x}^T S \bar{x} + \int_{t-\tau_1}^t (s - t + \tau_1) \|\dot{\bar{x}}(s)\|^2 ds \quad (15)$$

If we compute its time derivative, we obtain

$$\begin{aligned} \dot{W} &\leq \theta \bar{x}^T (A^T S + SA - 2C^T C) \bar{x} \\ &+ 2\bar{x}^T S \Delta (\phi(\hat{x}) - \phi(x)) \\ &+ 2\theta \bar{x}^T C^T C \int_{t-\tau}^t \dot{\tilde{x}}(s) ds \\ &+ \tau_1 \|\dot{\bar{x}}(t)\|^2 - \int_{t-\tau_1}^t \|\dot{\bar{x}}(s)\|^2 ds \end{aligned} \quad (16)$$

Using (8), we have

$$\begin{aligned}\dot{W} &\leq -\theta \bar{x}^T S \bar{x} + 2\bar{x}^T S \Delta(\phi(\hat{x}) - \phi(x)) \\ &+ -\theta \bar{x}^T C^T C \bar{x} + 2\theta \bar{x}^T C^T C \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \\ &+ \tau_1 \|\dot{\hat{x}}(t)\|^2 - \int_{t-\tau_1}^t \|\dot{\hat{x}}(s)\|^2 \mathbf{d}s.\end{aligned}\quad (17)$$

Note that by using the mean value theorem [11], we can write

$$\Delta(\phi(\hat{x}) - \phi(x)) = \Delta \left( \sum_{i,j=1}^{n,n} e_n(i)^T e_n(j) \frac{\partial \phi_i}{\partial x_j}(\xi) \right) \Delta^{-1} \bar{x} \quad (18)$$

where  $\xi \in \text{Cov}(x, \hat{x})$ .  
Then we will have

$$2\bar{x}^T S \Delta(\phi(\hat{x}) - \phi(x)) = 2\bar{x}^T S \Delta \left( \sum_{i,j=1}^{n,n} e_n(i)^T e_n(j) \frac{\partial \phi_i}{\partial x_j} \right) \Delta^{-1} \bar{x} \quad (19)$$

Using the triangular structure and the Lipschitz properties of the functions  $\phi_i$ , and the fact that  $\theta > 1$ , we deduce that

$$\|2\bar{x}^T S \Delta(\phi(\hat{x}) - \phi(x))\| \leq k_1 V \quad (20)$$

where  $V = \bar{x}^T S \bar{x}$  and  $k_1$  is a positive constant which does not depend on  $\theta$ .

Using the following property :

$$\begin{aligned}&2\theta \bar{x}^T C^T C \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s - \theta \bar{x}^T C^T C \bar{x} \\ &= -\theta (C\bar{x} + C \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s)^T (C\bar{x} + C \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s) \\ &+ \theta \left( \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \right)^T C^T C \left( \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \right).\end{aligned}\quad (21)$$

This means that

$$\begin{aligned}&2\theta \bar{x}^T C^T C \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s - \theta \bar{x}^T C^T C \bar{x} \\ &\leq \theta \left( \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \right)^T C^T C \left( \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \right).\end{aligned}\quad (22)$$

From this, we will have

$$\begin{aligned}\dot{W} &\leq -\theta V + k_1 V \\ &+ \theta \left( \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \right)^T C^T C \left( \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \right) \\ &+ \tau_1 \|\dot{\hat{x}}(t)\|^2 - \int_{t-\tau_1}^t \|\dot{\hat{x}}(s)\|^2 \mathbf{d}s\end{aligned}\quad (23)$$

Now, let us remark that if we use equation (13), it comes:

$$\|\dot{\hat{x}}(t)\|^2 \leq \theta^2 k_2 [V + \left\| \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s \right\|^2] \quad (24)$$

where  $k_2$  is also a positive constant which does not depend on  $\theta$ .

Using this and equation (23), we will have:

$$\begin{aligned}\dot{W} &\leq -\theta V + k_1 V + \theta I^T C^T C I \\ &+ \tau_1 \theta^2 k_2 [V + \|I\|^2] - \int_{t-\tau_1}^t \|\dot{\hat{x}}(s)\|^2 \mathbf{d}s.\end{aligned}\quad (25)$$

where  $I = \int_{t-\tau}^t \dot{\hat{x}}(s) \mathbf{d}s$ .

To prove the above lemma (1), it is sufficient to find conditions which guarantee the following inequality

$$\dot{W} + \frac{1}{\sqrt{\theta}} W < 0 \quad (26)$$

From (25), we can write

$$\begin{aligned}\dot{W} + \frac{1}{\sqrt{\theta}} W &\leq -\theta V + k_1 V + \frac{V}{\sqrt{\theta}} + \theta I^T C^T C I \\ &+ \tau_1 \theta^2 k_2 [V + \|I\|^2] + \frac{\tau_1}{\sqrt{\theta}} \int_{t-\tau_1}^t \|\dot{\hat{x}}(s)\|^2 \mathbf{d}s \\ &- \int_{t-\tau_1}^t \|\dot{\hat{x}}(s)\|^2 \mathbf{d}s.\end{aligned}\quad (27)$$

If we use the following Jensen's inequality :

$$\int_{t-\tau_1}^t \|\dot{\hat{x}}(s)\|^2 \mathbf{d}s \geq \frac{1}{\tau_1} \|I\|^2 \quad (28)$$

and if

$$\tau_1 \leq \sqrt{\theta} \quad (29)$$

we have

$$\begin{aligned}\dot{W} + \frac{1}{\sqrt{\theta}} W &\leq -(\theta - k_1 - \tau_1 \theta^2 k_2 - \frac{1}{\sqrt{\theta}}) V \\ &- \left( \frac{1}{\tau_1} - \theta - \tau_1 \theta^2 k_2 - \frac{1}{\sqrt{\theta}} \right) \|I\|^2\end{aligned}\quad (30)$$

Then, the exponential convergence to zero of the observation error  $\bar{x}$  is guaranteed if the following inequalities hold :

$$\begin{cases} \theta - k_1 - \tau_1 \theta^2 k_2 - \frac{1}{\sqrt{\theta}} \geq 0 \\ \frac{1}{\tau_1} - \theta - \tau_1 \theta^2 k_2 - \frac{1}{\sqrt{\theta}} \geq 0. \end{cases} \quad (31)$$

We can easily see, that by choosing  $\tau_1 = \frac{1}{\theta^2}$  (since  $\theta > 1$  we remark that condition (29) holds), the above inequalities will be equivalent to:

$$\begin{cases} \theta \geq k_1 + k_2 + \frac{1}{\sqrt{\theta}} \\ \theta^2 \geq \theta + k_2 + \frac{1}{\sqrt{\theta}} \end{cases} \quad (32)$$

It is obvious that the inequalities (32) are verified for sufficiently large values of  $\theta$ . Note that it is sufficient to choose

$$\theta \geq \sup \left\{ 2, \left( k_1 + k_2 + \frac{1}{\sqrt{2}} \right) \right\}.\quad (33)$$

Then, we can say that lemma 1 is verified for

$$\begin{cases} \theta \geq \sup\{2, (k_1 + k_2 + \frac{1}{\sqrt{2}})\} \\ \tau_1 = \frac{1}{\theta^2}. \end{cases} \quad (34)$$

To summarize Lemma 1, it gives the maximum delay supported by observer (10) which enables  $\hat{x}(t) \rightarrow x(t)$ , once  $\theta$  has been fixed according to conditions (34). To cope with a larger measurement delay, we propose in next paragraph a procedure to estimate  $x(t)$ , based on a chain of high-gain observers: each observer will estimate the state at a given fraction of the output delay.

#### A. Cascade high gain observers

After proving that the convergence of the observer (10) requires a small delay, we will see that when the delay is arbitrary long, a set containing a sufficient number of cascade high gain observers (7) can reconstruct the states of system (1).

*Theorem 1:* Let us consider system (1), then for any constant and known delay  $\tau$ , there exist a sufficiently large positive constant  $\theta$  and an integer  $m$  such that the observer (7) converges exponentially towards the system (1).

*Proof:* The convergence of the cascade observer will be proved step by step :

**Step 1:** We consider the first observer in the chain:

$$\begin{aligned} \dot{\hat{x}}_1 &= A\hat{x}_1 + \phi(\hat{x}_1) - \theta\Delta^{-1}S^{-1}C'C(\hat{x}_1(t - \frac{\tau}{m}) - x(t - \tau)) \\ \hat{y}_1 &= C\hat{x}_1(t - \frac{\tau}{m}) \end{aligned} \quad (35)$$

We remark that  $x(t - \tau) = x_1(t - \frac{\tau}{m})$  and consequently, if we choose  $\theta$  sufficiently large, and by choosing the integer  $m$  such that  $m \geq \theta^2\tau$ , then  $\hat{x}_1(t)$  converges towards  $x_1(t) = x(t - \tau + \frac{\tau}{m}) = x(t - (m-1)\frac{\tau}{m})$ . Indeed, we are brought back to conditions of Lemma 1, since the delay to handle with is now  $\frac{\tau}{m}$ , which is assumed smaller than  $\frac{1}{\theta^2}$ .

#### Step j:

At each step ( $j = 2, \dots, m$ ), we estimate the delayed state  $x(t - \tau + j\frac{\tau}{m})$  by using the following observer:

$$\begin{aligned} \dot{\hat{x}}_j &= A\hat{x}_j + \phi(\hat{x}_j) - \theta\Delta^{-1}S^{-1}C'C(\hat{x}_j(t - \frac{\tau}{m}) - \hat{x}_{j-1}(t)) \\ \hat{y}_j &= C\hat{x}_j(t - \frac{\tau}{m}) = C\hat{x}_{j-1}(t) \end{aligned} \quad (36)$$

It is not difficult to see that by considering the observation error vector  $\tilde{x}_j = x_j - \hat{x}_j$ , if we add and subtract the term  $\theta\Delta^{-1}S^{-1}C'Cx_{j-1}(t)$  in the previous equation, we obtain

$$\begin{aligned} \dot{\tilde{x}}_j &= A\tilde{x}_j + \phi(\hat{x}_j) - \phi(x_j) - \theta\Delta^{-1}S^{-1}C'C\tilde{x}_j(t - \frac{\tau}{m}) \\ &\quad - \theta\Delta^{-1}S^{-1}C'C\tilde{x}_{j-1}(t). \end{aligned} \quad (37)$$

If we consider the following change of coordinates  $\bar{x}_j = \Delta\tilde{x}_j$ , we will have

$$\begin{aligned} \dot{\bar{x}}_j &= \theta(A - S^{-1}C'C)\bar{x}_j + \Delta(\phi(\hat{x}_j) - \phi(x_j)) \\ &\quad + \theta S^{-1}C'C \int_{t-\frac{\tau}{m}}^t \dot{\tilde{x}}_j(s) ds - \theta S^{-1}C'C\bar{x}_{j-1}. \end{aligned} \quad (38)$$

In order to prove by recurrence the convergence of the error  $\bar{x}_j$ , we suppose that the observation error  $\bar{x}_{j-1}(t)$  converges exponentially towards zero.

Then we consider the following *Lyapunov-Krasovskii functional*

$$W_j = \bar{x}_j^T S \bar{x}_j + \int_{t-\frac{\tau}{m}}^t (s - t + \frac{\tau}{m}) \|\dot{\bar{x}}_j(s)\|^2 ds \quad (39)$$

Then its time derivative satisfies the following inequality:

$$\begin{aligned} \dot{W}_j &\leq -\theta\bar{x}_j^T S \bar{x}_j + 2\bar{x}_j^T S \Delta(\phi(\hat{x}_j) - \phi(x_j)) \\ &\quad + -\theta\bar{x}_j^T C^T C \bar{x}_j - 2\theta\bar{x}_j^T C^T C \bar{x}_{j-1} \\ &\quad + 2\theta\bar{x}_j^T C^T C \int_{t-\tau}^t \dot{\tilde{x}}_j(s) ds + \tau_1 \|\dot{\bar{x}}_j\|^2 \\ &\quad - \int_{t-\tau_1}^t \|\dot{\tilde{x}}_j(s)\|^2 ds. \end{aligned} \quad (40)$$

As in the proof of the lemma (1), we will also have :

$$\begin{aligned} \dot{W}_j &\leq -\theta V_j + k'_1 V_j + \theta I_j^T C^T C I_j \\ &\quad - 2\theta\bar{x}_j^T C^T C \bar{x}_{j-1} + \tau_1 \|\dot{\bar{x}}_j\|^2 \\ &\quad - \int_{t-\tau_1}^t \|\dot{\tilde{x}}_j(s)\|^2 ds. \end{aligned} \quad (41)$$

where  $V_j = \bar{x}_j^T S \bar{x}_j$ ,  $I_j = \int_{t-\tau}^t \dot{\tilde{x}}_j(s) ds$  and  $k'_1$  is a positive constant which does not depend on  $\theta$  and  $k'_1 \geq k_1$ .

Now, by using Young's inequality, we derive the following inequalities

$$\|\dot{\bar{x}}_j\|^2 \leq \tau_1 k'_2 \theta^2 (V_j + \|I_j\|^2 + \|\bar{x}_{j-1}\|^2) \quad (42)$$

$$-2\theta\bar{x}_j^T C^T C \bar{x}_{j-1} \leq \frac{1}{\sqrt{\theta}} V_j + \frac{\theta^2 \sqrt{\theta}}{\lambda_{\min}(S)} \|\bar{x}_{j-1}\|^2. \quad (43)$$

where  $k'_2$  is a positive constant which does not depend on  $\theta$  and  $k'_2 \geq k_2$ .

Choosing  $\tau_1 = \frac{1}{\theta^2}$ , and using (41), (42) and (43), we derive

$$\begin{aligned} \dot{W}_j + \frac{1}{\sqrt{\theta}} W_j &\leq -(\theta - k'_1 - \tau_1 \theta^2 k'_2 - \frac{2}{\sqrt{\theta}}) V_j \\ &\quad - (\frac{1}{\tau_1} - \theta - \tau_1 \theta^2 k'_2 - \frac{2}{\sqrt{\theta}}) \|I_j\|^2 \\ &\quad + (\frac{\theta^2 \sqrt{\theta}}{\lambda_{\min}(S)} + k'_2) \|\bar{x}_{j-1}\|^2 \end{aligned} \quad (44)$$

Then, we can say that if

$$\begin{cases} \theta \geq 2 + k'_1 + k'_2 \\ \tau_1 = \frac{1}{\theta^2}. \end{cases} \quad (45)$$

we will have

$$\dot{W}_j \leq -\frac{1}{\sqrt{\theta}}W_j + \left(\frac{\theta^2\sqrt{\theta}}{\lambda_{\min}(S)} + k'_2\right)\|\bar{x}_{j-1}\|^2 \quad (46)$$

Using the comparison lemma [12], we conclude that if  $\bar{x}_{j-1}$  converges exponentially towards zero, then  $\bar{x}_j$  converges also exponentially towards zero. Note that conditions (45), also ensure the convergence of the first observer ( $j = 1$ ), then we deduce, recursively, that all observation errors converge exponentially towards zero. ■

#### IV. EXAMPLE

To illustrate the obtained results, consider the following nonlinear system, affected by delayed measurements:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -2x_1(t) + 0.5 \tanh(x_1(t) + x_2(t)) + x_1(t)u(t) \\ y(t) = x_1(t - \tau) \end{cases} \quad (47)$$

The input is  $u(t) = 0.1 \sin(0.1t)$ . System (47) belongs to the considered class of triangular systems with Lipschitz nonlinearities (1).

The initial conditions for the system and for the (chained) observer(s) have been chosen as:

$$x(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \hat{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \forall t \in [-\tau, 0] \quad (48)$$

Simulations have been performed using Matlab, and the integration routine *ode23*, which specifically handles delay differential systems. The high gain parameter is set to  $\theta = 2$ . The following figures present the simulation results. Fig. 1 and Fig. 2 show the true states (solid) and the estimated states (dashed) when the output delay is set to  $0.25s$ , i.e.  $\tau = \frac{1}{\theta^2}$ . For larger delays, one observer is not sufficient to estimate the states of system (47). For instance, if we set  $\tau = 0.5s$ , then two observers are necessary to reconstruct  $x_1(t)$  and  $x_2(t)$ . This is reported by Fig. 3 and Fig. 4. If the output delay  $\tau$  is still increasing, then four observers are required to achieve state estimation, as shown by Fig. 5 and Fig. 6.

#### V. CONCLUSION

In this paper, a novel predictor based on high gain observer has been presented. This observer can be applied to the class of nonlinear uniformly observable systems. The design of adaptive observers for nonlinear systems with delayed output is under investigation.

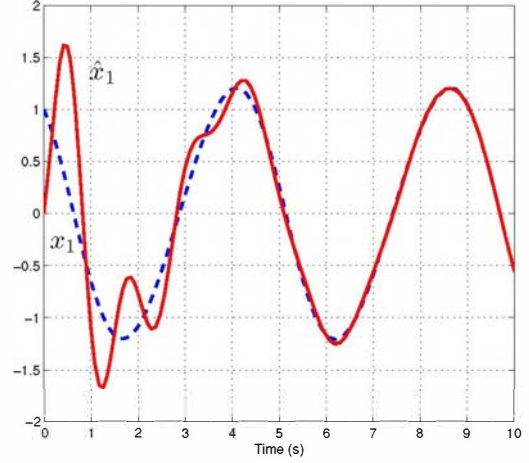


Fig. 1. Evolution of the first state for  $\tau = 0.25s$  with 1 observer

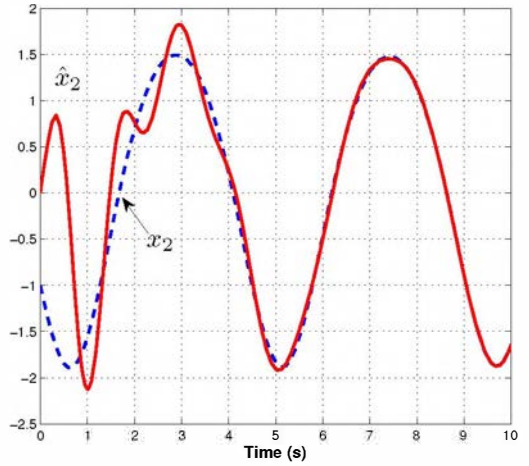


Fig. 2. Evolution of the second state for  $\tau = 0.25s$  with 1 observer

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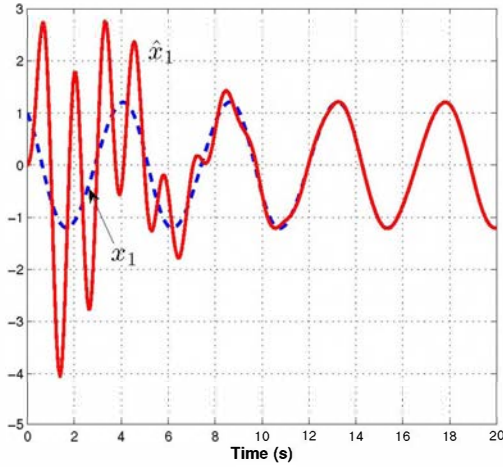


Fig. 3. Evolution of the first state for  $\tau = 0.5s$  with a cascade of 2 observers

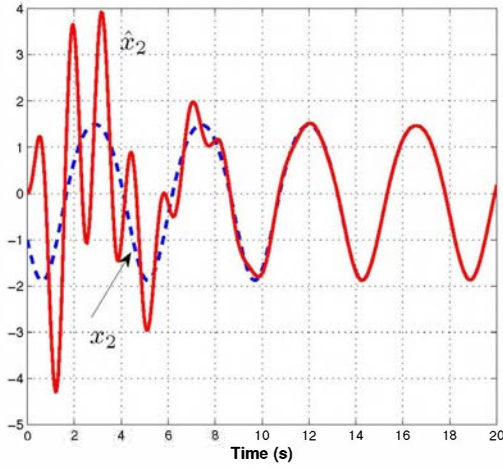


Fig. 4. Evolution of the second state for  $\tau = 0.5s$  with a cascade of 2 observers

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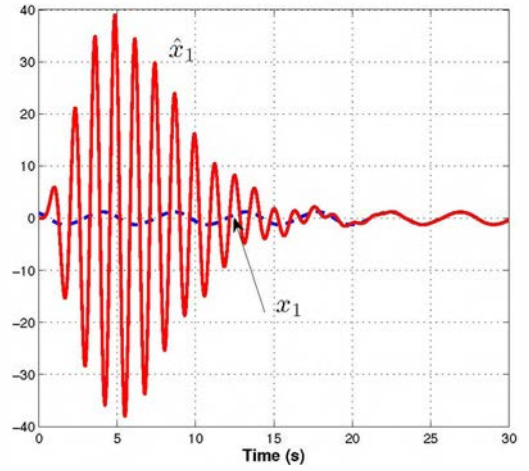


Fig. 5. Evolution of the first state for  $\tau = 1s$  with a cascade of 4 observers

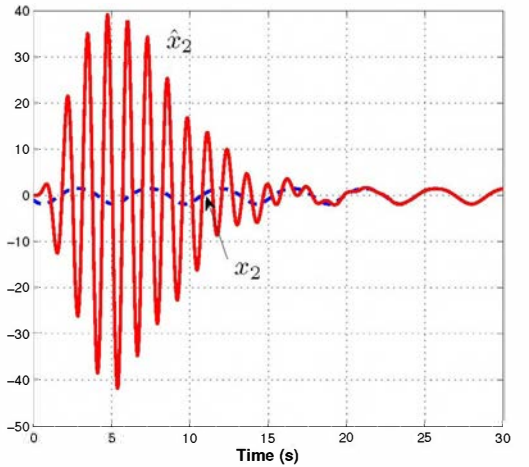


Fig. 6. Evolution of the second state for  $\tau = 1s$  with a cascade of 4 observers