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ON THE PRODUCT OF FUNCTIONS IN BMO AND H^1 OVER SPACES OF HOMOGENEOUS TYPE

LUONG DANG KY

ABSTRACT. Let \mathcal{X} be an RD-space, which means that \mathcal{X} is a space of homogeneous type in the sense of Coifman-Weiss with the additional property that a reverse doubling property holds in \mathcal{X} . The aim of the present paper is to study the product of functions in BMO and H^1 in this setting. Our results generalize some recent results in [4] and [10].

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

A famous result of C. Fefferman state that $BMO(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$. Although, for $f \in BMO(\mathbb{R}^n)$ and $g \in H^1(\mathbb{R}^n)$, the point-wise product fg may not be an integrable function, one (see [2]) can view the product of f and g as a distribution, denoted by $f \times g$. Such a distribution can be written as the sum of an integrable function and a distribution in a new Hardy space, so-called Hardy space of Musielak-Orlicz type (see [1, 8]). A complete study about the product of functions in BMO and \mathcal{H}^1 has been firstly done by Bonami, Iwaniec, Jones and Zinsmeister [2]. Recently, Li and Peng [10] generalized this study to the setting of Hardy and BMO spaces associated with Schrödinger operators. In particular, Li and Peng showed that if $L = -\Delta + V$ is a Schrödinger operator with the potential V belongs to the reverse Hölder class RH_q for some $q \geq n/2$, then one can view the product of $b \in BMO_L(\mathbb{R}^n)$ and $f \in H_L^1(\mathbb{R}^n)$ as a distribution $b \times f$ which can be written the sum of an integrable function and a distribution in $H_L^\varphi(\mathbb{R}^n, d\mu)$. Here $H_L^\varphi(\mathbb{R}^n, d\mu)$ is the weighted Hardy-Orlicz space associated with L , related to the Orlicz function $\varphi(t) = t/\log(e+t)$ and the weight $d\nu(x) = dx/\log(e+|x|)$. More precisely, they proved the following.

Theorem A. *For each $f \in H_L^1(\mathbb{R}^n)$, there exist two bounded linear operators $\mathcal{L}_f : BMO_L(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ and $\mathcal{H}_f : BMO_L(\mathbb{R}^n) \rightarrow H_L^\varphi(\mathbb{R}^n, d\nu)$ such that for every $b \in BMO_L(\mathbb{R}^n)$,*

$$b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).$$

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Let (\mathcal{X}, d, μ) be a space of homogeneous type in the sense of Coifman-Weiss. Following Han, Müller and Yang [7], we say that (\mathcal{X}, d, μ) is an *RD-space* if μ satisfies *reverse doubling property*, i.e., there exists a constant $C > 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \geq C\mu(B(x, r)).$$

A typical example for such RD-spaces is the Carnot-Carathéodory space with doubling measure. We refer to the seminal paper of Han, Müller and Yang [7] (see also [5, 6, 12, 13]) for a systematic study of the theory of function spaces in harmonic analysis on RD-spaces.

Let (\mathcal{X}, d, μ) be an RD-space. Recently, in analogy with the classical result of Bonami-Iwaniec-Jones-Zinsmeister, Feuto proved in [4] that:

Theorem B. *For each $f \in H^1(\mathcal{X})$, there exist two bounded linear operators $\mathcal{L}_f : BMO(\mathcal{X}) \rightarrow L^1(\mathcal{X})$ and $\mathcal{H}_f : BMO(\mathcal{X}) \rightarrow H^\varphi(\mathcal{X}, d\nu)$ such that for every $b \in BMO(\mathcal{X})$,*

$$b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).$$

Here the weight $d\nu(x) = d\mu(x)/\log(e + d(x_0, x))$ with $x_0 \in \mathcal{X}$ and the Orlicz function φ is as in Theorem A. It should be pointed out that in [4], for $f = \sum_{j=1}^{\infty} \lambda_j a_j$, the author defined the distribution $b \times f$ as

$$(1.1) \quad b \times f := \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j + \sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j$$

by proving that the second series is convergent in $H^\varphi(\mathcal{X}, d\nu)$. This is made possible by the fact that $H^\varphi(\mathcal{X}, d\nu)$ is complete and is continuously imbedded into the space of distributions $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ (see Section 2), which is not established in [4]. Moreover one has to prove that Definition (1.1) does not depend on the atomic decomposition of f . In this paper, we give a definition for the distribution $b \times f$ (see Section 3) which is similar to that of Bonami-Iwaniec-Jones-Zinsmeister.

Our first main result can be read as follows.

Theorem 1.1. *For each $f \in H^1(\mathcal{X})$, there exist two bounded linear operators $\mathcal{L}_f : BMO(\mathcal{X}) \rightarrow L^1(\mathcal{X})$ and $\mathcal{H}_f : BMO(\mathcal{X}) \rightarrow H^{\log}(\mathcal{X})$ such that for every $b \in BMO(\mathcal{X})$,*

$$b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).$$

Here $H^{\log}(\mathcal{X})$ is the Musielak-Orlicz Hardy space related to the Musielak-Orlicz function $\varphi(x, t) = \frac{t}{\log(e+d(x_0, x))+\log(e+t)}$ (see Section 2). Theorem 1.1 is an improvement of Theorem B since $H^{\log}(\mathcal{X})$ is a proper subspace of $H^\varphi(\mathcal{X}, d\nu)$.

Let ρ be an *admissible function* (see Section 2). Recently, Yang and Zhou [12, 13] introduced and studied Hardy spaces and Morrey-Campanato spaces related to the function ρ . There, they established that $BMO_\rho(\mathcal{X})$ is the dual

space of $H_\rho^1(\mathcal{X})$. Similar to the classical case, we can define the product of functions $b \in BMO_\rho(\mathcal{X})$ and $f \in H_\rho^1(\mathcal{X})$ as distributions $b \times f \in (\mathcal{G}_0^\varepsilon(\beta, \gamma))'$.

Our next main result is as follows.

Theorem 1.2. *For each $f \in H_\rho^1(\mathcal{X})$, there exist two bounded linear operators $\mathcal{L}_{\rho,f} : BMO_\rho(\mathcal{X}) \rightarrow L^1(\mathcal{X})$ and $\mathcal{H}_{\rho,f} : BMO_\rho(\mathcal{X}) \rightarrow H^{\log}(\mathcal{X})$ such that for every $b \in BMO_\rho(\mathcal{X})$,*

$$b \times f = \mathcal{L}_{\rho,f}(b) + \mathcal{H}_{\rho,f}(b).$$

When $\mathcal{X} \equiv \mathbb{R}^n, n \geq 3$, and $\rho(x) \equiv \sup\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1\}$, where $L = -\Delta + V$ is as in Theorem A, one has $BMO_\rho(\mathcal{X}) \equiv BMO_L(\mathbb{R}^n)$ and $H_\rho^1(\mathcal{X}) \equiv H_L^1(\mathbb{R}^n)$. So, Theorem 1.2 is an improvement of Theorem A since $H^{\log}(\mathbb{R}^n)$ is a proper subspace of $H_L^\varphi(\mathbb{R}^n, d\nu)$ (see [9]).

The following conjecture is suggested by A. Bonami and F. Bernicot.

Conjecture. *There exist two bounded bilinear operators $\mathcal{L} : BMO(\mathcal{X}) \times H^1(\mathcal{X}) \rightarrow L^1(\mathcal{X})$ and $\mathcal{H} : BMO(\mathcal{X}) \times H^1(\mathcal{X}) \rightarrow H^{\log}(\mathcal{X})$ such that*

$$b \times f = \mathcal{L}(b, f) + \mathcal{H}(b, f).$$

It should be pointed out that when $\mathcal{X} = \mathbb{R}^n$ and $H^{\log}(\mathcal{X})$ is replaced by $H^\varphi(\mathbb{R}^n, d\nu)$, the above conjecture is just Conjecture 1.7 of [2], which answered recently by Bonami, Grellier and Ky [1] (see also [9]).

Throughout the whole paper, C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. We write $f \sim g$ if there exists a constant $C > 1$ such that $C^{-1}f \leq g \leq Cf$.

The paper is organized as follows. In Section 2, we present some notations and preliminaries about BMO type spaces and Hardy type spaces on RD-spaces. Section 3 is devoted to prove Theorem 1.1. Finally, we give the proof for Theorem 1.2 in Section 4.

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2. SOME PRELIMINARIES AND NOTATIONS

Let d be a quasi-metric on a set \mathcal{X} , that is, d is a nonnegative function on $\mathcal{X} \times \mathcal{X}$ satisfying

- (a) $d(x, y) = d(y, x)$,
- (b) $d(x, y) > 0$ if and only if $x \neq y$,
- (c) there exists a constant $\kappa \geq 1$ such that for all $x, y, z \in \mathcal{X}$,

$$(2.1) \quad d(x, z) \leq \kappa(d(x, y) + d(y, z)).$$

A trip (\mathcal{X}, d, μ) is called a *space of homogeneous type* in the sense of Coifman-Weiss [3] if μ is a regular Borel measure satisfying *doubling property*, i.e. there exists a constant $C > 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Following Han, Müller and Yang [7], (\mathcal{X}, d, μ) is called an *RD-space* if (\mathcal{X}, d, μ) is a space of homogeneous type and μ also satisfies *reverse doubling property*, i.e. there exists a constant $C > 1$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \geq C\mu(B(x, r)).$$

Set $\text{diam}(\mathcal{X}) := \sup_{x, y \in \mathcal{X}} d(x, y)$. It should be pointed out that (\mathcal{X}, d, μ) is an RD-space if and only if there exist constants $0 < \mathfrak{d} \leq \mathfrak{n}$ and $C > 1$ such that for all $x \in \mathcal{X}$, $0 < r < \text{diam}(\mathcal{X})/2$, and $1 \leq \lambda < \text{diam}(\mathcal{X})/(2r)$,

$$(2.2) \quad C^{-1}\lambda^{\mathfrak{d}}\mu(B(x, r)) \leq \mu(B(x, \lambda r)) \leq C\lambda^{\mathfrak{n}}\mu(B(x, r)).$$

Here and what in follows, for $x, y \in \mathcal{X}$ and $r > 0$, we denote $V_r(x) := \mu(B(x, r))$ and $V(x, y) := \mu(B(x, d(x, y)))$.

Definition 2.1. Let $x_0 \in \mathcal{X}$, $r > 0$, $0 < \beta \leq 1$ and $\gamma > 0$. A function f is said to belong to the space of test functions, $\mathcal{G}(x_0, r, \beta, \gamma)$, if there exists a positive constant C_f such that

$$(i) \quad |f(x)| \leq C_f \frac{1}{V_r(x_0) + V(x_0, x)} \left(\frac{r}{r + d(x_0, x)} \right)^\gamma \quad \text{for all } x \in \mathcal{X};$$

$$(ii) \quad |f(x) - f(y)| \leq C_f \left(\frac{d(x, y)}{r + d(x_0, x)} \right)^\beta \frac{1}{V_r(x_0) + V(x_0, x)} \left(\frac{r}{r + d(x_0, x)} \right)^\gamma \quad \text{for all } x, y \in \mathcal{X}$$

satisfying that $d(x, y) \leq \frac{r + d(x_0, x)}{2\kappa}$.

For any $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, we define

$$\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} := \inf\{C_f : (i) \text{ and } (ii) \text{ hold}\}.$$

Let ρ be a positive function on \mathcal{X} . Following Yang and Zhou [13], the function ρ is said to be *admissible* if there exist positive constants C_0 and k_0 such that for all $x, y \in \mathcal{X}$,

$$\rho(y) \leq C_0[\rho(x)]^{1/(1+k_0)}[\rho(x) + d(x, y)]^{k_0/(1+k_0)}.$$

Throughout the whole paper, we always assume that \mathcal{X} is an RD-space with $\mu(\mathcal{X}) = \infty$, and ρ is an admissible function on \mathcal{X} . Also we fix $x_0 \in \mathcal{X}$.

In Definition 2.1, it is easy to see that $\mathcal{G}(x_0, 1, \beta, \gamma)$ is a Banach space. For simplicity, we write $\mathcal{G}(\beta, \gamma)$ instead of $\mathcal{G}(x_0, 1, \beta, \gamma)$. Let $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon]$, we define the space $\mathcal{G}_0^\epsilon(\beta, \gamma)$ to be the completion of $\mathcal{G}(\epsilon, \epsilon)$ in $\mathcal{G}(\beta, \gamma)$, and denote by $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$ the space of all continuous linear functionals on $\mathcal{G}_0^\epsilon(\beta, \gamma)$. We say that f is a *distribution* if f belongs to $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$.

Remark that, for any $x \in \mathcal{X}$ and $r > 0$, one has $\mathcal{G}(x, r, \beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$ with equivalent norms, but of course the constants are depending on x and r .

Let f be a distribution in $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$. We define *the grand maximal functions* $\mathcal{M}(f)$ and $\mathcal{M}_\rho(f)$ as following

$$\mathcal{M}(f)(x) := \sup\{|\langle f, \varphi \rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r > 0\},$$

$$\mathcal{M}_\rho(f)(x) := \sup\{|\langle f, \varphi \rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x, r, \beta, \gamma)} \leq 1 \text{ for some } r \in (0, \rho(x))\}.$$

Let $L^{\log}(\mathcal{X})$ (see [1, 8] for details) be the Musielak-Orlicz type space of μ -measurable functions f such that

$$\int_{\mathcal{X}} \frac{|f(x)|}{\log(e + |f(x)|) + \log(e + d(x_0, x))} d\mu(x) < \infty.$$

For $f \in L^{\log}(\mathcal{X})$, we define the "norm" of f as

$$\|f\|_{L^{\log}} = \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \frac{\frac{|f(x)|}{\lambda}}{\log(e + \frac{|f(x)|}{\lambda}) + \log(e + d(x_0, x))} d\mu(x) \leq 1 \right\}.$$

Definition 2.2. Let $\epsilon \in (0, 1)$ and $\beta, \gamma \in (0, \epsilon)$.

(i) The Hardy space $H^1(\mathcal{X})$ is defined by

$$H^1(\mathcal{X}) = \{f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))' : \|f\|_{H^1} := \|\mathcal{M}(f)\|_{L^1} < \infty\}.$$

(ii) The Hardy space $H_\rho^1(\mathcal{X})$ is defined by

$$H_\rho^1(\mathcal{X}) = \{f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))' : \|f\|_{H_\rho^1} := \|\mathcal{M}_\rho(f)\|_{L^1} < \infty\}.$$

(iii) The Hardy space $H^{\log}(\mathcal{X})$ is defined by

$$H^{\log}(\mathcal{X}) = \{f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))' : \|f\|_{H^{\log}} := \|\mathcal{M}(f)\|_{L^{\log}} < \infty\}.$$

It is clear that $H^1(\mathcal{X}) \subset H_\rho^1(\mathcal{X})$ and $H^1(\mathcal{X}) \subset H^{\log}(\mathcal{X})$ with the inclusions are continuous. It should be pointed out that the Musielak-Orlicz Hardy space $H^{\log}(\mathcal{X})$ is a proper subspace of the weighted Hardy-Orlicz space $\mathcal{H}^\rho(\mathcal{X}, \nu)$ studied in [4]. We refer to [8] for an introduction to Musielak-Orlicz Hardy spaces on the Euclidean space \mathbb{R}^n .

Definition 2.3. Let $q \in (1, \infty]$.

(i) A measurable function \mathbf{a} is called an (H^1, q) -atom related to the ball $B(x, r)$ if

- (a) $\text{supp } \mathbf{a} \subset B(x, r)$,
- (b) $\|\mathbf{a}\|_{L^q} \leq (V_r(x))^{1/q-1}$,
- (c) $\int_{\mathcal{X}} \mathbf{a}(y) d\mu(y) = 0$.

(ii) A measurable function \mathbf{a} is called an (H_ρ^1, q) -atom related to the ball $B(x, r)$ if $r \leq 2\rho(x)$ and \mathbf{a} satisfies (a) and (b), and when $r < \rho(x)$, \mathbf{a} also satisfies (c).

The following results were established in [5, 13].

Theorem 2.1. Let $\epsilon \in (0, 1)$, $\beta, \gamma \in (0, \epsilon)$ and $q \in (1, \infty]$. Then, we have:

(i) The space $H^1(\mathcal{X})$ coincides with the Hardy space $H_{\text{at}}^{1,q}(\mathcal{X})$ of Coifman-Weiss. More precisely, $f \in H^1(\mathcal{X})$ if and only if f can be written as $f =$

$\sum_{j=1}^{\infty} \lambda_j a_j$ where the a_j 's are (H^1, q) -atoms and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$. Moreover,

$$\|f\|_{H^1} \sim \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

(ii) $f \in H^1_{\rho}(\mathcal{X})$ if and only if f can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where the a_j 's are (H^1_{ρ}, q) -atoms and $\{\lambda_j\}_{j=1}^{\infty} \in \ell^1$. Moreover,

$$\|f\|_{H^1_{\rho}} \sim \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}.$$

Here and what in follows, for any ball $B \subset \mathcal{X}$ and $g \in L^1_{\text{loc}}(\mathcal{X})$, we denote by g_B the average value of g over the ball B and denote

$$MO(g, B) := \frac{1}{\mu(B)} \int_B |g(x) - g_B| d\mu(x).$$

Recall (see [3]) that a function $f \in L^1_{\text{loc}}(\mathcal{X})$ is said to be in $BMO(\mathcal{X})$ if

$$\|f\|_{BMO} = \sup_B MO(f, B) < \infty,$$

where the supremum is taken all over balls $B \subset \mathcal{X}$.

Definition 2.4. Let ρ be an admissible function and $\mathcal{D} := \{B(x, r) \subset \mathcal{X} : r \geq \rho(x)\}$. A function $f \in L^1_{\text{loc}}(\mathcal{X})$ is said to be in $BMO_{\rho}(\mathcal{X})$ if

$$\|f\|_{BMO_{\rho}} = \|f\|_{BMO} + \sup_{B \in \mathcal{D}} \frac{1}{\mu(B)} \int_B |f(x)| d\mu(x) < \infty.$$

The following results are well-known, see [3, 5, 12].

Theorem 2.1. (i) The space $BMO(\mathcal{X})$ is the dual space of $H^1(\mathcal{X})$.

(ii) The space $BMO_{\rho}(\mathcal{X})$ is the dual space of $H^1_{\rho}(\mathcal{X})$.

3. THE PRODUCT OF FUNCTIONS IN $BMO(\mathcal{X})$ AND $H^1(\mathcal{X})$

Remark that if $g \in \mathcal{G}(\beta, \gamma)$, then

$$(3.1) \quad \|g\|_{L^{\infty}} \leq C \frac{1}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)}$$

and

$$(3.2) \quad \|g\|_{L^1} \leq (C + \sum_{j=0}^{\infty} 2^{-j\gamma}) \|g\|_{\mathcal{G}(\beta, \gamma)} \leq C \|g\|_{\mathcal{G}(\beta, \gamma)}.$$

Proposition 3.1. *Let $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. Then, g is a pointwise multiplier of $BMO(\mathcal{X})$ for all $g \in \mathcal{G}(\beta, \gamma)$. More precisely,*

$$\|gf\|_{BMO} \leq C \frac{1}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)} \|f\|_{BMO^+}$$

for all $f \in BMO(\mathcal{X})$. Here and what in follows,

$$\|f\|_{BMO^+} := \|f\|_{BMO} + \frac{1}{V_1(x_0)} \int_{B(x_0, 1)} |f(x)| d\mu(x).$$

Using Proposition 3.1, for $b \in BMO(\mathcal{X})$ and $f \in H^1(\mathcal{X})$, one can define the distribution $b \times f \in (\mathcal{G}_0^\varepsilon(\beta, \gamma))'$ by the rule

$$(3.3) \quad \langle b \times f, \phi \rangle := \langle \phi b, f \rangle$$

for all $\phi \in \mathcal{G}_0^\varepsilon(\beta, \gamma)$, where the second bracket stands for the duality bracket between $H^1(\mathcal{X})$ and its dual $BMO(\mathcal{X})$.

Proof of Proposition 3.1. By (3.1) and the pointwise multipliers characterization of $BMO(\mathcal{X})$ (see [11, Theorem 1.1]), it is sufficient to show that

$$(3.4) \quad \log(e + 1/r) MO(g, B(a, r)) \leq C \frac{1}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)}$$

and

$$(3.5) \quad \log(e + d(x_0, a) + r) MO(g, B(a, r)) \leq C \frac{1}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)}$$

hold for all balls $B(a, r) \subset \mathcal{X}$. It is easy to see that (3.4) follows from (3.1) and the Lipschitz property of g (see (ii) of Definition 2.1). Let us now establish (3.5). If $r < 1$, then by (3.5) follows from the Lipschitz property of g and the fact that $\lim_{\lambda \rightarrow \infty} \frac{\log(\lambda)}{\lambda^\beta} = 0$. Otherwise, we consider the following two cases:

(a) The case: $1 \leq r \leq \frac{1}{4\kappa^3} d(x_0, a)$. Then, for every $x, y \in B(a, r)$, one has $d(x_0, a) \leq \frac{4\kappa^3}{4\kappa^2 - 1}$ and $d(x, y) \leq \frac{d(x_0, x)}{2\kappa}$. Hence, the Lipschitz property of g yields

$$|g(x) - g(y)| \leq C \|g\|_{\mathcal{G}(\beta, \gamma)} \frac{1}{V_1(x_0)} \left(\frac{1}{d(x_0, a)} \right)^\gamma.$$

This implies that (3.5) holds since $\lim_{\lambda \rightarrow \infty} \frac{\log(\lambda)}{\lambda^\gamma} = 0$.

(b) The case: $r > \frac{1}{4\kappa^3}d(x_0, a)$. Then, one has $B(x_0, r) \subset B(a, \kappa(4\kappa^3 + 1)r)$. Hence, by (2.2), we get

$$\begin{aligned} \log(e + d(x_0, a) + r)MO(g, B(a, r)) &\leq C \frac{\log(2r)}{V_r(x_0)} \|g\|_{L^1} \\ &\leq C \frac{\log(2r)}{r^{\mathfrak{d}}} \frac{1}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)} \\ &\leq C \frac{1}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)}. \end{aligned}$$

This proves (3.5) and thus the proof of Proposition 3.1 is finished. \square

Next we define $L^\Xi(\mathcal{X})$ as the space of μ -measurable functions f such that

$$\int_{\mathcal{X}} \frac{e^{|f(x)|} - 1}{(1 + d(x_0, x))^{2n}} d\mu(x) < \infty.$$

Then, the norm on the space $L^\Xi(\mathcal{X})$ is defined by

$$\|f\|_{L^\Xi} = \inf \left\{ \lambda > 0 : \int_{\mathcal{X}} \frac{e^{|f(x)|/\lambda} - 1}{(1 + d(x_0, x))^{2n}} d\mu(x) \leq 1 \right\}.$$

Recall the following two lemmas due to Feuto [4].

Lemma 3.1. *For every $f \in BMO(\mathcal{X})$,*

$$\|f - f_{B(x_0, 1)}\|_{L^\Xi} \leq C \|f\|_{BMO}.$$

Lemma 3.2. *Let $q \in (1, \infty]$. Then,*

$$\|(\mathfrak{b} - \mathfrak{b}_B)\mathcal{M}(\mathfrak{a})\|_{L^1} \leq C \|\mathfrak{b}\|_{BMO}$$

for all $\mathfrak{b} \in BMO(\mathcal{X})$ and for all (H^1, q) -atom \mathfrak{a} related to the ball B .

The main point in the proof of Theorem 1.1 is the following.

Proposition 3.2. (i) *For any $f \in L^1(\mathcal{X})$ and $g \in L^\Xi(\mathcal{X})$, we have*

$$\|fg\|_{L^{\log}} \leq 64n^2 \|f\|_{L^1} \|g\|_{L^\Xi}.$$

(ii) *For any $f \in L^1(\mathcal{X})$ and $g \in BMO(\mathcal{X})$, we have*

$$\|fg\|_{L^{\log}} \leq C \|f\|_{L^1} \|g\|_{BMO+}.$$

Proof. (i) If $\|f\|_{L^1} = 0$ or $\|g\|_{L^\Xi} = 0$, then there is nothing to prove. Otherwise, we may assume that $\|f\|_{L^1} = \|g\|_{L^\Xi} = \frac{1}{8n}$ since homogeneity of the norms. Then, we need to prove that

$$\int_{\mathcal{X}} \frac{|f(x)g(x)|}{\log(e + |f(x)g(x)|) + \log(e + d(x_0, x))} d\mu(x) \leq 1.$$

Indeed, by using the following two inequalities

$$\log(e + ab) \leq 2(\log(e + a) + \log(e + b)), \quad a, b \geq 0,$$

and

$$\frac{ab}{\log(e + ab)} \leq a + (e^b - 1), \quad a, b \geq 0,$$

we obtain that, for every $x \in \mathcal{X}$,

$$\begin{aligned} & \frac{(1 + d(x_0, x))^{2n} |f(x)g(x)|}{4n(\log(e + |f(x)g(x)|) + \log(e + d(x_0, x)))} \\ & \leq \frac{(1 + d(x_0, x))^{2n} |f(x)g(x)|}{2(\log(e + |f(x)g(x)|) + \log(e + (1 + d(x_0, x))^{2n}))} \\ & \leq \frac{(1 + d(x_0, x))^{2n} |f(x)||g(x)|}{\log(e + (1 + d(x_0, x))^{2n} |f(x)||g(x)|)} \\ & \leq (1 + d(x_0, x))^{2n} |f(x)| + (e^{|g(x)|} - 1). \end{aligned}$$

This together with the fact $8n(e^{|g(x)|} - 1) \leq e^{8n|g(x)|} - 1$ give

$$\begin{aligned} & \int_{\mathcal{X}} \frac{|f(x)g(x)|}{\log(e + |f(x)g(x)|) + \log(e + d(x_0, x))} d\mu(x) \\ & \leq 4n\|f\|_{L^1} + \frac{1}{2} \int_{\mathcal{X}} \frac{e^{8n|g(x)|} - 1}{(1 + d(x_0, x))^{2n}} d\mu(x) \\ & \leq \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

which completes the proof of (i).

(ii) It follows directly from (i) and Lemma 3.1. □

Now we ready to give the proof for Theorem 1.1.

Proof of Theorem 1.1. By (i) of Theorem 2.1, f can be written as

$$f = \sum_{j=1}^{\infty} \lambda_j a_j$$

where the a_j 's are (H^1, ∞) -atoms related to the balls B_j 's and $\sum_{j=1}^{\infty} |\lambda_j| \leq C\|f\|_{H^1}$. Therefore, for all $b \in BMO(\mathcal{X})$, we have

$$(3.6) \quad \left\| \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j \right\|_{L^1} \leq \sum_{j=1}^{\infty} |\lambda_j| \| (b - b_{B_j}) a_j \|_{L^1} \leq C \|b\|_{BMO} \|f\|_{H^1}.$$

By this and Definition (3.3), we see that the series $\sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j$ converges to $b \times f - \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j$ in $(\mathcal{G}_0^\varepsilon(\beta, \gamma))'$. Consequently, if we define the

decomposition operators as

$$\mathcal{L}_f(b) = \sum_{j=1}^{\infty} \lambda_j (b - b_{B_j}) a_j$$

and

$$\mathcal{H}_f(b) = \sum_{j=1}^{\infty} \lambda_j b_{B_j} a_j,$$

where the sums are in $(\mathcal{G}_0^\epsilon(\beta, \gamma))'$, then it is clear that $\mathcal{L}_f : BMO(\mathcal{X}) \rightarrow L^1(\mathcal{X})$ is a bounded linear operator, since (3.6), and for every $b \in BMO(\mathcal{X})$,

$$b \times f = \mathcal{L}_f(b) + \mathcal{H}_f(b).$$

Now we only need to prove that the distribution $\mathcal{H}_f(b)$ is in $H^{\log}(\mathcal{X})$. Indeed, by Lemma 3.2 and (ii) of Proposition 3.2, we get

$$\begin{aligned} \|\mathcal{M}(\mathcal{H}_f(b))\|_{L^{\log}} &\leq \left\| \sum_{j=1}^{\infty} |\lambda_j| |b_{B_j}| \mathcal{M}(a_j) \right\|_{L^{\log}} \\ &\leq \left\| \sum_{j=1}^{\infty} |\lambda_j| |b - b_{B_j}| \mathcal{M}(a_j) \right\|_{L^1} + \left\| b \sum_{j=1}^{\infty} |\lambda_j| \mathcal{M}(a_j) \right\|_{L^{\log}} \\ &\leq C \|f\|_{H^1} \|b\|_{BMO^+}. \end{aligned}$$

This proves that \mathcal{H}_f is bounded from $BMO(\mathcal{X})$ into $H^{\log}(\mathcal{X})$, and thus ends the proof of Theorem 1.1. \square

4. THE PRODUCT OF FUNCTIONS IN $BMO_\rho(\mathcal{X})$ AND $H_\rho^1(\mathcal{X})$

For $f \in BMO_\rho(\mathcal{X})$, a standard argument gives

$$(4.1) \quad \|f\|_{BMO^+} \leq C \log(\rho(x_0) + 1/\rho(x_0)) \|f\|_{BMO_\rho}.$$

Proposition 4.1. *Let $\beta \in (0, 1]$ and $\gamma \in (0, \infty)$. Then, g is a pointwise multiplier of $BMO_\rho(\mathcal{X})$ for all $g \in \mathcal{G}(\beta, \gamma)$. More precisely, for every $f \in BMO_\rho(\mathcal{X})$,*

$$\|gf\|_{BMO_\rho} \leq C \frac{\log(\rho(x_0) + 1/\rho(x_0))}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)} \|f\|_{BMO_\rho}.$$

Proof. By Proposition 3.1, (4.1) and (3.1), we get

$$\begin{aligned} \|gf\|_{BMO_\rho} &\leq \|gf\|_{BMO} + \|g\|_{L^\infty} \sup_{B \in \mathcal{D}} \frac{1}{\mu(B)} \int_B |f(x)| d\mu(x) \\ &\leq C \frac{\log(\rho(x_0) + 1/\rho(x_0))}{V_1(x_0)} \|g\|_{\mathcal{G}(\beta, \gamma)} \|f\|_{BMO_\rho}. \end{aligned}$$

\square

Using Proposition 4.1, for $b \in BMO_\rho(\mathcal{X})$ and $f \in H_\rho^1(\mathcal{X})$, one can define the distribution $b \times f \in (\mathcal{G}_0^\epsilon(\beta, \gamma))'$ by the rule

$$(4.2) \quad \langle b \times f, \phi \rangle := \langle \phi b, f \rangle$$

for all $\phi \in \mathcal{G}_0^\epsilon(\beta, \gamma)$, where the second bracket stands for the duality bracket between $H_\rho^1(\mathcal{X})$ and its dual $BMO_\rho(\mathcal{X})$.

Proof of Theorem 1.2. By (ii) of Theorem 2.1, there exist a sequence of (H_ρ^1, ∞) -atoms $\{a_j\}_{j=1}^\infty$ related to the sequence of balls $\{B(x_j, r_j)\}_{j=1}^\infty$ and $\sum_{j=1}^\infty |\lambda_j| \leq C\|f\|_{H_\rho^1}$ such that

$$f = \sum_{j=1}^\infty \lambda_j a_j = f_1 + f_2,$$

where $f_1 = \sum_{r_j < \rho(x_j)} \lambda_j a_j \in H^1(\mathcal{X})$ and $f_2 = \sum_{r_j \geq \rho(x_j)} \lambda_j a_j$.

We define the decomposition operators as following

$$\mathcal{L}_{\rho, f}(b) = \mathcal{L}_{f_1}(b) + b f_2$$

and

$$\mathcal{H}_{\rho, f}(b) = \mathcal{H}_{f_1}(b),$$

where the operators \mathcal{L}_{f_1} and \mathcal{H}_{f_1} are as in Theorem 1.1. Then, Theorem 1.1 together with (4.1) give

$$\begin{aligned} \|\mathcal{L}_{\rho, f}(b)\|_{L^1} &\leq \|\mathcal{L}_{f_1}(b)\|_{L^1} + \sum_{r_j \geq \rho(x_j)} |\lambda_j| \|b a_j\|_{L^1} \\ &\leq C\|f_1\|_{H^1} \|b\|_{BMO} + C\|b\|_{BMO_\rho} \sum_{r_j \geq \rho(x_j)} |\lambda_j| \\ &\leq C\|f\|_{H_\rho^1} \|b\|_{BMO_\rho} \end{aligned}$$

and

$$\|\mathcal{H}_{\rho, f}(b)\|_{H^{\log}} \leq C\|f_1\|_{H^1} \|b\|_{BMO^+} \leq C\|f\|_{H_\rho^1} \|b\|_{BMO_\rho}.$$

This proves that the linear operator $\mathcal{L}_{\rho, f} : BMO_\rho(\mathcal{X}) \rightarrow L^1(\mathcal{X})$ is bounded and the linear operator $\mathcal{H}_{\rho, f} : BMO_\rho(\mathcal{X}) \rightarrow H^{\log}(\mathcal{X})$ is bounded. Moreover,

$$\begin{aligned} b \times f &= b \times f_1 + b \times f_2 \\ &= (\mathcal{L}_{f_1}(b) + \mathcal{H}_{f_1}(b)) + b f_2 \\ &= \mathcal{L}_{\rho, f}(b) + \mathcal{H}_{\rho, f}(b), \end{aligned}$$

which ends the proof of Theorem 1.2. \square

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