Expected volume and Euler characteristic of random submanifolds

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Abstract

In a closed manifold of positive dimension n, we estimate the expected volume and Euler characteristic for random submanifolds of codimension $r \in \{1, \ldots, n\}$ in two different settings. On one hand, we consider a closed Riemannian manifold and some positive λ. Then we take r independent random functions in the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues less than λ^2 and consider the random submanifold defined as the common zero set of these r functions. We compute asymptotics for the mean volume and Euler characteristic of this random submanifold as λ goes to infinity. On the other hand, we consider a complex projective manifold defined over the reals, equipped with an ample line bundle $\mathcal{L}$ and a rank r holomorphic vector bundle $\mathcal{E}$ that are also defined over the reals. Then we get asymptotics for the expected volume and Euler characteristic of the real vanishing locus of a random real holomorphic section of $\mathcal{E} \otimes \mathcal{L}^d$ as d goes to infinity. The same techniques apply to both settings.

Keywords: Euler characteristic, Riemannian random wave, spectral function, random polynomial, real projective manifold, ample line bundle, Bergman kernel, Gaussian field.

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1 Introduction

Zeros of random polynomials were first studied by Bloch and Pòlya [7] in the early 30's. About ten years later, Kac [22] obtained a sharp asymptotic for the expected number of real zeros of a polynomial of degree d with independent standard Gaussian coefficients, as d goes to infinity. This was later generalized to other distributions by Kostlan in [23]. In particular, he introduced a normal distribution on the space of homogeneous polynomials of degree d – known as the Kostlan distribution – which is more geometric, in the sense that it is invariant under isometries of $\mathbb{C}P^1$. Bogomolny, Bohigas and Leboeuf [8] showed that this distribution corresponds to the choice of d independent roots, uniformly distributed in the Riemann sphere.

In higher dimension, the question of the number of zeros can be generalized in at least two ways. What is the expected volume of the zero set? And what is its expected Euler characteristic? More generally, one can ask what are the expected volume and Euler characteristic of a random submanifold obtained as the zero set of some Gaussian field on a

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Riemannian manifold. In this paper, we provide an asymptotic answer to these questions in the case of Riemannian random waves and in the case of real algebraic manifolds.

Let us describe our frameworks and state the main results of this paper. See section 2 for more details. Let (M, g) be a closed (that is compact without boundary) smooth Riemannian manifold of positive dimension n, equipped with the Riemannian measure $|\mathrm{d}V_M|$ associated to g (defined below (2.1)). This induces a L^2-inner product on $\mathcal{C}^\infty(M)$ defined by:

$$\forall \phi, \psi \in \mathcal{C}^\infty(M), \qquad \langle \phi \, , \psi \rangle = \int_{x \in M} \phi(x)\psi(x) \, |\mathrm{d}V_M| \, . \tag{1.1}$$

It is well-known that the subspace $V_\lambda \subset \mathcal{C}^\infty(M)$ spanned by the eigenfunctions of the Laplacian associated to eigenvalues smaller than λ^2 has finite dimension. Let $1 \leqslant r \leqslant n$ and let $f^{(1)}, \dots, f^{(r)} \in V_\lambda$ be independent standard Gaussian vectors, we denote by Z_f the zero set of $f = (f^{(1)}, \dots, f^{(r)})$. Then, for λ large enough, Z_f is almost surely a submanifold of M of codimension r (see section 2 below) and we denote by $\mathrm{Vol}\,(Z_f)$ its Riemannian volume for the restriction of g to Z_f. We also denote by $\chi(Z_f)$ its Euler characteristic.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension n. Let V_λ be the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues smaller than λ^2. Let $f^{(1)}, \dots, f^{(r)}$ be r independent standard Gaussian vectors in V_λ, with $1 \leqslant r \leqslant n$. Then the following holds as λ goes to infinity:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_f)] = \left(\frac{\lambda}{\sqrt{n+2}}\right)^r \mathrm{Vol}\,(M) \, \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O(\lambda^{r-1}).$$

Here and throughout this paper, $\mathbb{E}[\,\cdot\,]$ denotes the mathematical expectation of the quantity between the brackets and $\mathbb{S}^n$ is, as usual, the unit Euclidean sphere in $\mathbb{R}^{n+1}$.

If $n - r$ is odd, Z_f is almost surely a smooth manifold of odd dimension. In this case, $\chi(Z_f) = 0$ almost surely. If $n - r$ is even, we get the following result.

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold of dimension n. Let V_λ be the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues smaller than λ^2. Let $f^{(1)}, \dots, f^{(r)}$ be r independent standard Gaussian vectors in V_λ, with $1 \leqslant r \leqslant n$. Then, if $n - r$ is even, the following holds as λ goes to infinity:*

$$\mathbb{E}[\chi\,(Z_f)] = (-1)^{\frac{n-r}{2}} \left(\frac{\lambda}{\sqrt{n+2}}\right)^n \mathrm{Vol}\,(M) \, \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right) \mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \, \mathrm{Vol}\,(\mathbb{S}^n) \, \mathrm{Vol}\,(\mathbb{S}^{n-1})} + O(\lambda^{n-1}).$$

We also consider the framework of the papers [15, 16] by Gayet and Welschinger, see section 2.6 for more details. Let $\mathcal{X}$ be a smooth complex projective manifold of complex dimension n. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$ and $\mathcal{E}$ be a holomorphic vector bundle over $\mathcal{X}$ of rank r. We assume that $\mathcal{X}$, $\mathcal{L}$ and $\mathcal{E}$ are equipped with compatible real structures and that the real locus $\mathbb{R}\mathcal{X}$ of $\mathcal{X}$ is non-empty.

Let $h_\mathcal{L}$ denote a Hermitian metric on $\mathcal{L}$ with positive curvature ω and $h_\mathcal{E}$ denote a Hermitian metric on $\mathcal{E}$. Both metrics are assumed to be compatible with the real structures. Then ω is a Kähler form and it induces a Riemannian metric g and a volume form $\mathrm{d}V_\mathcal{X} = \frac{\omega^n}{n!}$ on $\mathcal{X}$. For any $d \in \mathbb{N}$, the space of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ is equipped with a L^2-inner product similar to (1.1) (see section 2.6).

Let $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of real global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This is a Euclidean space for the above inner product. Let s be a standard Gaussian section in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$, we denote by Z_s the real part of its zero set. Once again, for d large enough, Z_s is almost surely a smooth submanifold of $\mathbb{R}\mathcal{X}$ of codimension r. Let $\mathrm{Vol}\,(Z_s)$ denote the Riemannian volume of Z_s and $\chi(Z_s)$ denote its Euler characteristic. We get the analogues of theorems 1.1 and 1.2 in this setting.

Theorem 1.3. *Let $\mathcal{X}$ be a complex projective manifold of dimension n defined over the reals and $r \in \{1, \dots, n\}$. Let $\mathcal{L}$ be an ample holomorphic Hermitian line bundle over $\mathcal{X}$ and $\mathcal{E}$ be a rank r holomorphic Hermitian vector bundle over $\mathcal{X}$, both equipped with real structures compatible with the one on $\mathcal{X}$. Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then the following holds as d goes to infinity:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_s)] = \left(\sqrt{d}\right)^r \mathrm{Vol}\,(\mathbb{R}\mathcal{X})\, \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O\left(\sqrt{d}^{\,r-2}\right).$$

Theorem 1.4. *Let $\mathcal{X}$ be a complex projective manifold of dimension n defined over the reals and $r \in \{1, \dots, n\}$. Let $\mathcal{L}$ be an ample holomorphic Hermitian line bundle over $\mathcal{X}$ and $\mathcal{E}$ be a rank r holomorphic Hermitian vector bundle over $\mathcal{X}$, both equipped with real structures compatible with the one on $\mathcal{X}$. Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, if $n - r$ is even, the following holds as d goes to infinity:*

$$\mathbb{E}[\chi\,(Z_s)] = (-1)^{\frac{n-r}{2}} \left(\sqrt{d}\right)^n \mathrm{Vol}\,(\mathbb{R}\mathcal{X})\, \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right) \mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi\, \mathrm{Vol}\,(\mathbb{S}^n)\, \mathrm{Vol}\,(\mathbb{S}^{n-1})} + O\left(\sqrt{d}^{\,n-2}\right).$$

In the case of random eigenfunctions of the Laplacian, theorem 1.1 was already known to Bérard [9] for hypersurfaces. See also [33, thm 1] where Zelditch shows that, in the case of hypersurfaces, $\frac{1}{\lambda}\mathbb{E}[Z_f] \longrightarrow |\mathrm{d}V_M|$ as $\lambda \to +\infty$, in the sense of currents. He also proves a similar result in the case of band limited eigenfunctions.

Let us discuss theorems 1.3 and 1.4 when $\mathcal{X}$ is $\mathbb{C}P^n$ with the standard real structure induced by the conjugation in $\mathbb{C}^{n+1}$, $\mathcal{E}$ is the trivial bundle $\mathcal{X} \times \mathbb{C}^r$ and $\mathcal{L} = \mathcal{O}(1)$ is the hyperplane bundle with its usual metric. Then $\mathbb{R}\mathcal{X} = \mathbb{R}P^n$ and ω is the Fubini-Study metric on $\mathbb{C}P^n$, normalized so that it is the quotient of the Euclidean metric on the sphere $\mathbb{S}^{2n+1}$. Besides, $\mathbb{R}H^0(\mathcal{X}, \mathcal{L}^d)$ is the space of real homogeneous polynomials of degree d in $n+1$ variables, and Z_s is the common real zero set of r independent such polynomials.

In this setting, Kostlan [23] proved that for any $d \geqslant 1$, $\mathbb{E}[\mathrm{Vol}\,(Z_s)] = \left(\sqrt{d}\right)^r \mathrm{Vol}\,(\mathbb{R}P^{n-r})$. See also the paper [30] by Shub and Smale, where they compute the expected number of common real roots for a system of n polynomials in n variables. The expected Euler characteristic of a random algebraic hypersurface of degree d in $\mathbb{R}P^n$ was computed by Podkorytov [28]. Both Kostlan's and Podkorytov's results were generalized by Bürgisser. In [10], he computed the expected volume and Euler characteristic of a submanifold of $\mathbb{R}P^n$ defined as the common zero set of r standard Gaussian polynomials $P_1, \dots, P_r$ of degree $d_1, \dots, d_r$ respectively. In particular, when these polynomials have the same degree d and $n - r$ is even, he showed that:

$$\mathbb{E}\left[\chi\left(Z_{(P_1,\dots,P_r)}\right)\right] = \left(\sqrt{d}\right)^r \sum_{p=1}^{\frac{n-r}{2}} (1-d)^p\, \frac{\Gamma\left(p + \frac{r}{2}\right)}{p!\Gamma\left(\frac{r}{2}\right)}, \tag{1.2}$$

where Γ denotes Euler's gamma function. Theorems 1.3 and 1.4 agree with these previous results.

Recently, Gayet and Welschinger computed upper and lower bounds for the asymptotics of the expected Betti numbers of random real algebraic submanifolds of a projective manifold, see [15, 16]. This relies on sharp estimates for the expected number of critical points of index $i \in \{0, \dots, n-r\}$ of a fixed Morse function $p : \mathbb{R}\mathcal{X} \to \mathbb{R}$ restricted to the random Z_s. More precisely, let $N_i(Z_s)$ denote the number of critical points of index i of $p_{/Z_s}$, let $\mathrm{Sym}(i, n-r-i)$ denote the open cone of symmetric matrices of size $n-r$ and signature $(i, n-r-i)$ and let $\mathrm{d}\mu$ denote the standard Gaussian measure on the space of symmetric matrices. Gayet and Welschinger show [16, thm 3.1.2] that:

$$\mathbb{E}[N_i(Z_s)] \underset{d\to+\infty}{\sim} \frac{\sqrt{d}^n}{\sqrt{\pi}^n}\, \mathrm{Vol}\,(\mathbb{R}\mathcal{X})\, \frac{(n-1)!}{(n-r)!2^{r-1}\Gamma\left(\frac{r}{2}\right)} e_{\mathbb{R}}(i, n-r-i), \tag{1.3}$$

where $e_{\mathbb{R}}(i, n-r-i) = \int_{\mathrm{Sym}(i,n-r-i)} |\det(A)| \, \mathrm{d}\mu(A)$.

One can indirectly deduce theorem 1.4 from this result and from [10] in the following way. By Morse theory:

$$\mathbb{E}[\chi(Z_s)] = \sum_{i=0}^{n-r} (-1)^i \mathbb{E}[N_i(Z_s)] \underset{d\to+\infty}{\sim} C_{n,r} \sqrt{d}^n \, \mathrm{Vol}\,(\mathbb{R}\mathcal{X})$$

where $C_{n,r}$ is a universal constant depending only on n and r. Specifying to the case of $\mathbb{R}P^n$, equation (1.2) gives the value of $C_{n,r}$. Gayet and Welschinger also proved a result similar to (1.3) for hypersurfaces in the case of Riemannian random waves, see [14]. It gives the order of growth of $\mathbb{E}[\chi(Z_f)]$ in theorem 1.2, for $r = 1$.

In their book [1], Taylor and Adler compute the expected Euler characteristic of the excursion sets of a centered, unit-variance Gaussian field f on a smooth manifold M. This expectation is given in terms of the Lipshitz-Killing curvatures of M for the Riemannian metric g_f induced by f, see [1, thm 12.4.1]. One can deduce from this result the expected Euler characteristic of the random hypersurface $f^{-1}(0)$, always in terms of the Lipschitz-Killing curvatures of (M, g_f). It might be possible to deduce theorems 1.2 and 1.4 from this result, in the case of hypersurfaces, when the Gaussian field $(f(x))_{x\in M}$ (resp. $(s(x))_{x\in\mathbb{R}\mathcal{X}}$) has a constant covariance function (see section 2.3 for the definitions of these objects), but one would need to estimate the Lipschitz-Killing curvatures of (M, g_f) (resp. (M, g_s)) as λ (resp. d) goes to infinity.

In a related setting, Bleher Shiffman and Zelditch [5] computed the scaling limit of the k-points correlation function for a random complex submanifold of a complex projective manifold. See also [6] in a symplectic framework. Our proofs of theorems 1.2 and 1.4 use the same formalism as these papers, adapted to our frameworks.

We now sketch the proofs of our main results in the harmonic setting. The real algebraic case is similar. The first step is to express $\mathrm{Vol}\,(Z_f)$ (resp. $\chi(Z_f)$) as the integral of some function on Z_f. In the case of the volume this is trivial, and the answer is given by the Chern-Gauss-Bonnet theorem (see section 4.2 below) in the case of the Euler characteristic. Then we use the Kac-Rice formula (see theorem 5.3) which allows us to express $\mathbb{E}[\mathrm{Vol}\,(Z_f)]$ (resp. $\mathbb{E}[\chi(Z_f)]$) as the integral on M of some explicit function that only depends on the geometry of M and on the covariance function of the smooth Gaussian field defined by f.

It turns out that the covariance function of the field associated to the r independent standard Gaussian functions $f^{(1)}, \dots, f^{(r)}$ in V_λ is given by the spectral function of the Laplacian. In the algebraic case, the covariance function is given by the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. This was already used in [5, 26].

Then, our results follow from estimates on the spectral function of the Laplacian (resp. the Bergman kernel) and their derivatives (see section 3). In the case of random waves, the estimates we need for the spectral function were proved by Bin [4], generalizing results of Hörmander [20]. In the algebraic case, much is known about the Bergman kernel [3, 5, 25, 32] but we could not find the estimates we needed in codimension higher than 1 in the litterature. These estimates are established in section 3.3 using Hörmander-Tian peak sections. Peak sections were already used in this context in [15, 16], see also [31]. The author was told by Steve Zelditch, after this paper was written, that one can deduce estimates for the Bergman kernel in higher codimension from the paper [3] by Berman, Berndtsson and Sjöstrand.

This paper is organised as follows. In section 2 we describe how our random submanifold are generated and the setting of the main theorems. Section 3 is dedicated to the estimates we need for the spectral function of the Laplacian and the Bergman kernel. In section 4 we derive an integral formula for the Euler characteristic of a submanifold. The main theorems are proved in 5, and we deal with two special cases in 6: the flat torus and the real projective space. For these examples, it is possible to compute expectations for fixed λ (resp. d) and

we recover the results of Kostlan and Bürgisser. Three appendices deal respectively with: some standard results about Gaussian vectors, a rather technical proof we postponed until the end, and a derivation of the Kac-Rice formula using Federer's coarea formula.

Contents

2 Random submanifolds

This section is concerned with the precise definition of the random submanifolds we consider. The first two subsections explain how we produce them in a quite general setting. The third one introduces the covariance kernel, which characterizes their distribution. We also describe the distribution induced on the bundle of 2-jets in terms of this kernel. Then we describe what we called the harmonic setting, before explaining how to adapt all this in the real algebraic case. This kind of random submanifolds has already been considered by Bérard [9], Zelditch [33] and Nicolaescu [26] in the harmonic case, and by Gayet and Welschinger in the real algebraic case, see [15, 16, 17]. See also [10, 11, 24] in special cases.

2.1 General setting

Let (M, g) be a smooth closed manifold of positive dimension n. We denote by $|\mathrm{d}V_M|$ the Riemannian measure on M induced by g. That is, if $x = (x_1, \dots, x_n)$ are local coordinates in an open set $U \subset M$ and ϕ is a smooth function with compact support in U,

$$\int_M \phi \, |\mathrm{d}V_M| = \int_{x \in \mathbb{R}^n} \phi(x) \sqrt{\det(g(x))} \, \mathrm{d}x_1 \dots \mathrm{d}x_n. \tag{2.1}$$

From now on we fix some $r \in \{1, \dots, n\}$, that we think of as the codimension of our random submanifolds. Let $\langle \cdot \, , \cdot \rangle$ denote the L^2-scalar product on $\mathcal{C}^\infty(M, \mathbb{R}^r)$ induced by $|\mathrm{d}V_M|$: for any f_1 and $f_2 \in \mathcal{C}^\infty(M, \mathbb{R}^r)$,

$$\langle f_1 \, , f_2 \rangle = \int_{x \in M} \langle f_1(x) \, , f_2(x) \rangle \, |\mathrm{d}V_M| \, , \tag{2.2}$$

where the inner product on the right-hand side is the standard one on $\mathbb{R}^r$.

Notation 2.1. Here and throughout this paper $\langle \cdot \, , \cdot \rangle$ will always denote the inner product on the concerned Euclidean or Hermitian space.

Let V be a subspace of $\mathcal{C}^\infty(M, \mathbb{R}^r)$ of finite dimension N. For any $f \in V$, we denote by Z_f the zero set of f. Let D denote the *discriminant locus* of V, that is the set of $f \in V$ that do not vanish transversally.

If f vanishes transversally, Z_f is a (possibly empty) smooth submanifold of M of codimension r and we denote by $|\mathrm{d}V_f|$ the Riemannian measure induced by the restriction of g to Z_f. We also denote by $\mathrm{Vol}\,(Z_f)$ the volume of Z_f and by $\chi(Z_f)$ its Euler characteristic. In the case $r = n$, when $f \notin D$, Z_f is a finite set and $|\mathrm{d}V_f|$ is the sum of the Dirac measures centered on points of Z_f.

We consider a random vector $f \in V$ with standard Gaussian distribution. That is the distribution of f admits the density function:

$$x \mapsto \frac{1}{\sqrt{2\pi}^N} \exp\left(-\frac{1}{2} \|x\|^2\right) \tag{2.3}$$

with respect to the Lebesgue measure of V. Under some further technical assumptions on V (see 2.2 below), f vanishes transversally almost surely. Hence, the random variables $\mathrm{Vol}\,(Z_f)$ and $\chi(Z_f)$ are well-defined almost everywhere, and it makes sense to compute their expectation.

For the convenience of the reader, we gathered the few results we need about Gaussian vectors in appendix A. We introduce some notations here and refer to appendix A for futher details. In the sequel, we will denote by $X \sim \mathcal{N}(m, \Lambda)$ the fact that the random vector X is distributed according to a Gaussian with mean m and variance Λ. A standard Gaussian vector is $X \sim \mathcal{N}(0, \mathrm{Id})$. We will denote by $\mathrm{d}\mu_N$ the standard Gaussian measure on a Euclidean space of dimension N, that is the measure with density (2.3) with respect to the Lebesgue measure.

2.2 The incidence manifold

Following [26], we say that V is 0-*ample* if the map $j^0_x : f \mapsto f(x)$ is onto for every $x \in M$. From now on, we assume that this is the case and we introduce an *incidence manifold* as Shub and Smale in [30] (see also [15, 16]).

Let $F : (f, x) \in V \times M \mapsto f(x) \in \mathbb{R}^r$ and let $\partial_1 F$ and $\partial_2 F$ denote the partial differentials of F with respect to the first and second variable respectively. For any $(f, x) \in V \times M$,

$$\partial_1 F(f, x) = j^0_x \qquad \text{and} \qquad \partial_2 F(f, x) = d_x f. \tag{2.4}$$

We assumed j^0_x to be surjective for every $x \in M$, thus F is a submersion. Then $\Sigma = F^{-1}(0)$ is a smooth submanifold of codimension r of $V \times M$, called the incidence manifold, and for any $(f_0, x) \in V \times M$:

$$T_{(f_0,x)}\Sigma = \{(f, v) \in V \times T_x M \mid f(x) + d_x f_0 \cdot v = 0\}.$$

We set $\pi_1 : \Sigma \to V$ and $\pi_2 : \Sigma \to M$ the projections from Σ to each factor. A vector $f \in V$ is in the range of $d_{(f_0,x)}\pi_1$ if and only if there exists some $v \in T_x M$ such that $(f, v) \in T_{(f_0,x)}\Sigma$, that is $f(x)$ is in the range of $d_x f_0$. Since V is 0-ample, the map j^0_x is onto, and $d_x f_0$ is surjective if and only if $d_{(f_0,x)}\pi_1$ is. Thus, the discriminant locus D is exactly the set of critical values of π_1. By Sard's theorem, D has measure 0 in V, both for the Lebesgue measure and for $\mathrm{d}\mu_N$, and f vanishes transversally almost surely.

We equip Σ with the restriction of the product metric on $V \times M$. Then, whenever $f \notin D$, $(\pi_1)^{-1}(f) = \{f\} \times Z_f$ is isometric to Z_f, hence we will identify these sets. Similarly, we will identify $(\pi_2)^{-1}(x) = \ker(j^0_x) \times \{x\}$ with the subspace $\ker(j^0_x)$ of V.

2.3 The covariance kernel

In this subsection we introduce the Schwarz kernel and covariance function associated to our space of random functions. It turns out (see proposition 2.3 below) that these objects are equal. The first to use this fact were Bleher, Shiffman and Zelditch in the case of complex projective manifolds [5] and in the case of symplectic manifolds [6]. In the harmonic setting this was used by Zelditch [33] and Nicolaescu [26].

In $\mathcal{C}^\infty(M, \mathbb{R}^r)$ equipped with the L^2-inner product (2.2), the orthogonal projection onto V can be represented by its *Schwartz kernel*, denoted by E. That is there exists a unique $E : M \times M \to \mathbb{R}^r \otimes \mathbb{R}^r$ such that for any smooth $f : M \to \mathbb{R}^r$, the projection of f onto V is given by:

$$x \mapsto \langle E(x, \cdot)\,, f\rangle = \int_{y \in M} \langle E(x, y)\,, f(y)\rangle\, |\mathrm{d}V_M|\,. \tag{2.5}$$

In the previous formula, the inner product on the right-hand side is the usual one on $\mathbb{R}^r$, acting on the second factor of $\mathbb{R}^r \otimes \mathbb{R}^r$. The kernel E has the following reproducing kernel property:

$$\forall f \in V,\ \forall x \in M, \quad f(x) = \langle E(x, \cdot)\,, f\rangle\,. \tag{2.6}$$

If $(f_1, \ldots, f_N)$ is any orthonormal basis of V, one can check that E is defined by:

$$E : (x, y) \mapsto \sum_{i=1}^{N} f_i(x) \otimes f_i(y). \tag{2.7}$$

This proves that E is smooth. Besides, for all $x \in M$, $E(x, x)$ is in the span of $\{\zeta \otimes \zeta \mid \zeta \in \mathbb{R}^r\}$, and for all $\zeta \in \mathbb{R}^r$:

$$\langle E(x, x)\,, \zeta \otimes \zeta\rangle = \sum_{i=1}^{N} \langle f_i(x) \otimes f_i(x)\,, \zeta \otimes \zeta\rangle = \sum_{i=1}^{N} (\langle f_i(x)\,, \zeta\rangle)^2 \geqslant 0. \tag{2.8}$$

This last equation shows that we can check the 0-amplitude condition for V on its kernel.

Lemma 2.2. *V is 0-ample if and only if, for all $x \in M$ and all $\zeta \in \mathbb{R}^r \setminus \{0\}$, we have: $\langle E(x,x)\,, \zeta \otimes \zeta \rangle > 0$. That is if and only if $E(x,x)$ is a positive-definite bilinear form on $(\mathbb{R}^r)^*$ for any $x \in M$.*

On the other hand, the standard Gaussian vector $f \in V$ defines a smooth centered Gaussian field $(f(x))_{x\in M}$ with values in $\mathbb{R}^r$. Its distribution is totally determined by its covariance function: $(x,y) \mapsto \mathrm{Cov}(f(x), f(y))$ from $M \times M$ to $\mathbb{R}^r \otimes \mathbb{R}^r$, where $\mathrm{Cov}(f(x), f(y))$ stands for the covariance form of the random vectors $f(x)$ and $f(y)$ (cf. appendix A).

Proposition 2.3. *Let V be a finite-dimensional subspace of $\mathcal{C}^\infty(M, \mathbb{R}^r)$ and E its Schwartz kernel. Let $f \sim \mathcal{N}(0, \mathrm{Id})$ in V, we have:*

$$\forall x, y \in M, \qquad \mathrm{Cov}(f(x), f(y)) = \mathbb{E}[f(x) \otimes f(y)] = E(x,y).$$

Proof. Let x and $y \in M$, the first equality is given by lemma A.7. We will now show that $\mathbb{E}[f(x) \otimes f(y)]$ satisfies condition (2.5) to prove the second equality. Let $f_0 : M \to \mathbb{R}^r$ be a smooth function and $x \in M$,

$$\begin{aligned}\int_{y\in M} \langle \mathbb{E}[f(x) \otimes f(y)]\ , f_0(y)\rangle \, |\mathrm{d}V_M| &= \int_{y\in M} \mathbb{E}[\langle f(x) \otimes f(y)\,, f_0(y)\rangle] \, |\mathrm{d}V_M| \\ &= \int_{y\in M} \mathbb{E}[f(x)\, \langle f(y)\,, f_0(y)\rangle] \, |\mathrm{d}V_M| = \mathbb{E}[f(x)\, \langle f\,, f_0\rangle]\,.\end{aligned}$$

If $f_0 \in V^\perp$, this equals 0. If $f_0 \in V$, we have:

$$\mathbb{E}[f(x)\, \langle f\,, f_0\rangle] = \mathbb{E}[\langle E(x,\cdot)\,, f\rangle\, \langle f_0\,, f\rangle] = \langle E(x,\cdot)\,, f_0\rangle = f_0(x),$$

where we used the reproducing kernel property (2.6) both for f and f_0 and applied lemma A.8 to $f \sim \mathcal{N}(0, \mathrm{Id})$. In both cases, $x \mapsto \mathbb{E}[f(x)\, \langle f\,, f_0\rangle]$ is the projection of f_0 onto V, which shows the second equality in proposition 2.3. □

This tells us that the distribution of our Gaussian field is totally determined by the Schwartz kernel E. In our cases of interest, asymptotics are known for E and its derivatives, see section 3 below. This is what allows us to derive asymptotics for the expectation of the volume and Euler characteristic of Z_f.

2.4 Random jets

Let ∇^M be the Levi-Civita connection on (M, g). For any smooth $f : M \to \mathbb{R}^r$, we denote by $\nabla^2 f = \nabla^M df$ the *Hessian* of f.

Let ∂_x (resp. ∂_y) denote the partial derivative with respect to the first (resp. second) variable for maps from $M \times M$ to $\mathbb{R}^r \otimes \mathbb{R}^r$. Likewise, we denote by $\partial_{x,x}$ (resp. $\partial_{y,y}$) the second partial derivative with respect to the first (resp. second) variable twice. As for the Hessian above, all the higher order derivatives are induced by ∇^M.

Now, let $f \sim \mathcal{N}(0, \mathrm{Id})$ in V. We will describe the distribution induced by f on the 2-jets bundle of M. Let $x \in M$, we denote by $\mathcal{J}^k_x(\mathbb{R}^r)$ the space of k-jets of smooth functions from M to $\mathbb{R}^r$ at the point x (we will only use $k \in \{0,1,2\}$). We already defined

$$\begin{aligned} j^0_x : \mathcal{C}^\infty(M, \mathbb{R}^r) &\longrightarrow \mathbb{R}^r. \\ f &\mapsto f(x)\end{aligned}$$

We define similarly,

$$\begin{aligned} j^1_x : \mathcal{C}^\infty(M,\mathbb{R}^r) &\to \mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M) \\ f &\mapsto (f(x), d_x f) \\ \text{and} \qquad j^2_x : \mathcal{C}^\infty(M,\mathbb{R}^r) &\to \mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M \oplus \mathrm{Sym}(T^*_x M)), \\ f &\mapsto (f(x), d_x f, \nabla^2_x f) \end{aligned}$$

where $\mathrm{Sym}(T^*_x M)$ denotes the space of symmetric bilinear forms on $T^*_x M$.

The map j^2_x induces an isomorphism between $\mathcal{J}^2_x(\mathbb{R}^r)$ and $\mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M \oplus \mathrm{Sym}(T^*_x M))$. In the sequel we will identify these spaces through j^2_x. Likewise, $\mathcal{J}^1_x(\mathbb{R}^r)$ and $\mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M)$ will be identified through j^1_x.

Lemma 2.4. *Let V be a finite-dimensional subspace of $\mathcal{C}^\infty(M,\mathbb{R}^r)$ and E its Schwartz kernel. Let $f \sim \mathcal{N}(0, \mathrm{Id})$ in V and $x \in M$. Then $j^2_x(f) = (f(x), d_x f, \nabla^2_x f)$ is a centered Gaussian vector, and its variance form $\mathrm{Var}(j^2_x(f))$ is characterized by:*

$$\mathrm{Var}(f(x)) = \mathbb{E}[f(x) \otimes f(x)] = E(x,x), \tag{2.9}$$

$$\mathrm{Var}(d_x f) = \mathbb{E}[\nabla_x f \otimes \nabla_x f] = (\partial_x \partial_y E)(x,x), \tag{2.10}$$

$$\mathrm{Var}(\nabla^2_x f) = \mathbb{E}\big[\nabla^2_x f \otimes \nabla^2_x f\big] = (\partial_{x,x} \partial_{y,y} E)(x,x), \tag{2.11}$$

$$\mathrm{Cov}(f(x), d_x f) = \mathbb{E}[f(x) \otimes \nabla_x f] = (\partial_y E)(x,x), \tag{2.12}$$

$$\mathrm{Cov}(f(x), \nabla^2_x f) = \mathbb{E}\big[f(x) \otimes \nabla^2_x f\big] = (\partial_{y,y} E)(x,x), \tag{2.13}$$

$$\mathrm{Cov}(d_x f, \nabla^2_x f) = \mathbb{E}\big[\nabla_x f \otimes \nabla^2_x f\big] = (\partial_x \partial_{y,y} E)(x,x). \tag{2.14}$$

Proof. The first equality on each line is given by lemmas A.4 and A.7. Then proposition 2.3 gives the second equality in (2.9). The other equalities are obtained by taking partial derivatives of (2.9) □

With this lemma, we have described the distribution of $j^2_x(f)$ only in terms of E. Since $j^0_x(f)$ and $j^1_x(f)$ are the projections of $j^2_x(f)$ onto $\mathbb{R}^r$ and $\mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M)$ respectively, their distributions are also characterized by lemma 2.4.

2.5 The harmonic setting

In this section, we describe what we called the harmonic setting, that is random eigenfunctions of the Laplacian.

Let Δ denote the Laplace-Beltrami operator on the closed Riemannian manifold (M, g). Recall the following classical facts from the theory of elliptical operators, see [13, thm 4.43].

Theorem 2.5. *1. The eigenvalues of $\Delta : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ can be arranged into a strictly increasing sequence of non-negative numbers $(\lambda^2_k)_{k\in\mathbb{N}}$ such that $\lambda^2_k \xrightarrow[k\to+\infty]{} +\infty$.*

2. The associated eigenspaces are finite-dimensional, and they are pairwise orthogonal for the L^2-inner product (1.1) on $\mathcal{C}^\infty(M)$ induced by g.

Let $\lambda \geqslant 0$, we denote by V_λ the subspace of $\mathcal{C}^\infty(M)$ spanned by the eigenfunctions of Δ associated to eigenvalues that are less or equal to λ^2. Each $f = (f^{(1)}, \dots, f^{(r)}) \in (V_\lambda)^r$ defines naturally a map from M to $\mathbb{R}^r$ so that we can see $(V_\lambda)^r$ as a subspace of $\mathcal{C}^\infty(M,\mathbb{R}^r)$. By theorem 2.5, V_λ is finite-dimensional so we can apply the construction of sections 2.1 to 2.4 to $(V_\lambda)^r$.

Since we consider a product situation, it is possible to make several simplifications. First, the scalar product (2.2) on $(V_\lambda)^r$ is induced by the one on V_λ. Thus $f = (f^{(1)}, \dots, f^{(r)})$ is a

standard Gaussian in $(V_\lambda)^r$ if and only if $f^{(1)}, \dots, f^{(r)}$ are r independent standard Gaussian in V_λ. For a $f \in (V_\lambda)^r$ satisfying this condition, $f^{(1)}, \dots, f^{(r)}$ are independent, and so are their derivatives of any order. This means that, for every $x \in M$, the matrices of $\mathrm{Var}(f(x))$, $\mathrm{Var}(d_x f)$ and $\mathrm{Var}(\nabla_x^2 f)$ in the canonical basis of $\mathbb{R}^r$ are block diagonal.

Another way to say this is that the kernel of $(V_\lambda)^r$ is a product in the following sense. We denote by $e_\lambda : M \times M \to \mathbb{R}$ the Schwartz kernel of the orthogonal projection onto V_λ in $\mathcal{C}^\infty(M)$ and by E_λ the Schwartz kernel of $(V_\lambda)^r$. The kernel e_λ is the *spectral function of the Laplacian* and precise asymptotics are known for e_λ and its derivatives, see 3.

Let $(\varphi_1, \dots, \varphi_N)$ be an orthonormal basis of V_λ. By (2.7),

$$e_\lambda : (x, y) \mapsto \sum_{i=1}^{N} \varphi_i(x)\varphi_i(y). \tag{2.15}$$

Let $(\zeta_1, \dots, \zeta_r)$ denote the canonical basis of $\mathbb{R}^r$. The maps $\varphi_i \zeta_q : M \to \mathbb{R}^r$ with $1 \leqslant i \leqslant N$ and $1 \leqslant q \leqslant r$ give an orthonormal basis of $(V_\lambda)^r$ and, for all x and $y \in M$,

$$E_\lambda(x, y) = \sum_{q=1}^{r} \sum_{i=1}^{N} (\varphi_i(x)\zeta_q) \otimes (\varphi_i(y)\zeta_q) = e_\lambda(x, y) \sum_{q=1}^{r} \zeta_q \otimes \zeta_q.$$

Lemma 2.6. *Let $(\zeta_1, \dots, \zeta_r)$ be any orthonormal basis of $\mathbb{R}^r$, for all x, $y \in M$,*

$$E_\lambda(x, y) = e_\lambda(x, y) \left(\sum_{q=1}^{r} \zeta_q \otimes \zeta_q \right).$$

An immediate consequence of this and lemma 2.2 is that $(V_\lambda)^r$ is 0-ample if and only if $e_\lambda(x, x) > 0$ for all $x \in M$. By (2.15), this is equivalent to V_λ being base-point-free, that is for every $x \in M$, there exists $f \in V_\lambda$ such that $f(x) \neq 0$.

Lemma 2.7. *For all $\lambda \geqslant 0$, $(V_\lambda)^r$ is 0-ample.*

Proof. The constant functions on M are eigenfunctions of Δ associated to the eigenvalue 0. Thus for every $\lambda \geqslant 0$, V_λ contains all constant functions on M, hence is base-point-free. By the above remark, $(V_\lambda)^r$ is then 0-ample. □

2.6 The real algebraic setting

Let us now describe more precisely the real algebraic framework. The main difference with what we did previously is that we consider sections of a rank r vector bundle instead of maps to $\mathbb{R}^r$. The local picture is the same as in sections 2.1 to 2.5, so that we can adapt everything to this setting. But the formalism is a bit heavier.

Let $\mathcal{X}$ be a smooth complex projective manifold of complex dimension n. We equip $\mathcal{X}$ with a real structure, that is with an antiholomorphic involution $c_\mathcal{X}$. We assume that its real locus, the set of fixed points of $c_\mathcal{X}$, is not empty and we denote it by $\mathbb{R}\mathcal{X}$. Let $\mathcal{L}$ be an ample holomorphic line bundle over $\mathcal{X}$ equipped with a real structure $c_\mathcal{L}$ compatible with the one on $\mathcal{X}$. By this we mean that $c_\mathcal{X} \circ \pi = \pi \circ c_\mathcal{L}$, where $\pi : \mathcal{L} \to \mathcal{X}$ stands for the projection map onto the base space. Similarly, let $\mathcal{E}$ be a holomorphic vector bundle of rank r over $\mathcal{X}$, with a compatible real structure $c_\mathcal{E}$.

Let $h_\mathcal{L}$ and $h_\mathcal{E}$ be real Hermitian metrics on $\mathcal{L}$ and $\mathcal{E}$ respectively, that is $c_\mathcal{L}^*(h_\mathcal{L}) = \overline{h_\mathcal{L}}$ and $c_\mathcal{E}^*(h_\mathcal{E}) = \overline{h_\mathcal{E}}$. We assume that $h_\mathcal{L}$ is positive in the sense that its *curvature form* ω is Kähler. Locally we have:

$$\omega_{/\Omega} = \frac{1}{2i} \partial\bar{\partial} \ln\left(h_\mathcal{L}(\zeta, \zeta)\right)$$

where ζ is any non-vanishing local holomorphic section of $\mathcal{L}$ on the open set $\Omega \subset \mathcal{X}$. This form corresponds to a Hermitian metric $g_{\mathbb{C}} = \omega(\cdot, i\cdot)$ on $\mathcal{X}$ whose real part is a Riemannian metric g. We denote by $\mathrm{d}V_{\mathcal{X}}$ the volume form $\frac{\omega^n}{n!}$ on $\mathcal{X}$.

Remark 2.8. The normalization of ω is the one of [3, 5], but differs from our references concerning peak sections [16, 31]. This will cause some discrepancies with the latter two in section 3.2. With our convention, the Fubini-Study metric on $\mathbb{R}P^n$ induced by the standard metric on the hyperplane line bundle $\mathcal{O}(1)$ is the quotient of the Euclidean metric on $\mathbb{S}^n$.

Let $d \in \mathbb{N}$, the vector bundle $\mathcal{E} \otimes \mathcal{L}^d$ comes with a real structure $c_d = c_{\mathcal{E}} \otimes c_{\mathcal{L}}^d$ compatible with $c_{\mathcal{X}}$ and a real Hermitian metric $h_d = h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d$. We equip the space $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ with the L^2 Hermitian product defined by:

$$\forall s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \qquad \langle s_1 \, , s_2 \rangle = \int_{\mathcal{X}} h_d(s_1, s_2) \, \mathrm{d}V_{\mathcal{X}}. \tag{2.16}$$

We know from the vanishing theorem of Kodaira and Serre that the space $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$ has finite dimension N_d and that N_d grows as a polynomial of degree n in d, when d goes to infinity. We denote by:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \left\{ s \in H^0\left(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d\right) \;\middle|\; c_d \circ s = s \circ c_{\mathcal{X}} \right\}$$

the space of real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$, which has real dimension N_d. The Hermitian product (2.16) induces a Euclidean inner product on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Notice that we integrate on the whole of $\mathcal{X}$, not only on the real locus, even when we consider real sections.

If $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is such that its restriction to $\mathbb{R}\mathcal{X}$ vanishes transversally, then its real zero set $Z_s = s^{-1}(0) \cap \mathbb{R}\mathcal{X}$ is a (possibly empty) submanifold of $\mathbb{R}\mathcal{X}$ of codimension r. We denote by $|\mathrm{d}V_s|$ the Riemannian measure induced on this submanifold by the metric g.

As in 2.2, we consider the incidence manifold:

$$\Sigma_d = \left\{ (s, x) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \mathbb{R}\mathcal{X} \;\middle|\; s(x) = 0 \right\}. \tag{2.17}$$

In this setting, Σ_d is the zero set of the bundle map $F_d : \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \mathbb{R}\mathcal{X} \to \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)$ over $\mathbb{R}\mathcal{X}$ defined by $F_d : (s, x) \mapsto s(x)$. In a trivialization, the situation is similar to the one in 2.2. Thus, if $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample, Σ_d is a smooth manifold equipped with two projection maps, π_1 and π_2, onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and $\mathbb{R}\mathcal{X}$ respectively. By Sard's theorem, the discriminant locus of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ then has measure 0 for any non-singular Gaussian measure, since it is the set of critical values of π_1.

Remark 2.9. Here, by $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample we mean that, for every $x \in \mathbb{R}\mathcal{X}$, the evaluation map $j_x^{0,d} : s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \mapsto s(x) \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ is onto.

Let ∇^d denote any real connection on $\mathcal{E} \otimes \mathcal{L}^d$, that is such that for every smooth section s, $\nabla^d \left(c_d \circ s \circ c_{\mathcal{X}}\right) = c_d \circ \left(\nabla^d s\right) \circ dc_{\mathcal{X}}$. For example one could choose the Chern connection. We consider the vertical component $\nabla^d F$ of the differential of F, whose kernel is the tangent space of Σ_d. For any $(s_0, x) \in \Sigma_d$ the partial derivatives of F_d are given by:

$$\partial_1^d F(s_0, x) = j_x^{0,d} \qquad \text{and} \qquad \partial_2^d F(s_0, x) = \nabla_x^d s_0. \tag{2.18}$$

Note that we only consider points of the zero section of $\mathcal{E} \otimes \mathcal{L}^d$, hence all this does not depend on the choice of ∇^d.

Let P_1 (resp. P_2) denote the projection from $\mathcal{X} \times \mathcal{X}$ onto the first (resp. second) factor. Recall that $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$ stands for the bundle $P_1^*(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \otimes P_2^*(\mathcal{E} \otimes \mathcal{L}^d)$ over $\mathcal{X} \times \mathcal{X}$. Let E_d denote the Schwartz kernel of the orthogonal projection from the space of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. It is the unique section of $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$ such

that, for every real smooth section s of $\mathcal{E}\otimes\mathcal{L}^d$, the projection of s on $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is given by:

$$x\mapsto\langle E_d(x,\cdot)\,,s\rangle=\int_{y\in\mathcal{X}}h_d(E_d(x,y),s(y))\,\mathrm{d}V_{\mathcal{X}}.$$

Here, h_d acts on the second factor of $(\overline{\mathcal{E}\otimes\mathcal{L}^d})_x\otimes(\mathcal{E}\otimes\mathcal{L}^d)_y$.

The kernel E_d satisfies a reproducing kernel property similar to (2.6):

$$\forall s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d),\ \forall x\in\mathbb{R}\mathcal{X},\quad s(x)=\langle E(x,\cdot)\,,f\rangle\in\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x. \tag{2.19}$$

If $(s_1,\dots,s_{N_d})$ is an orthonormal basis of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ then for all x and $y\in\mathcal{X}$,

$$E_d(x,y)=\sum_{i=1}^{N_d}s_i(x)\otimes s_i(y). \tag{2.20}$$

For any $x\in\mathbb{R}\mathcal{X}$, this shows $E_d(x,x)$ is in the span $\{\zeta\otimes\zeta\mid\zeta\in\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\}$. We also get the analogue of lemma 2.7.

Lemma 2.10. *$\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-ample if and only if for any $x\in\mathbb{R}\mathcal{X}$, $E_d(x,x)$ is a positive-definite bilinear form on $(\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x)^*$.*

Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, $(s(x))_{x\in\mathbb{R}\mathcal{X}}$ defines a Gaussian field with values in $\mathcal{E}\otimes\mathcal{L}^d$ and its covariance function is a section of $(\overline{\mathcal{E}\otimes\mathcal{L}^d})\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)$. The same proof as for proposition 2.3 gives the following.

Proposition 2.11. *Let E_d be the Schwartz kernel of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Let $s\sim\mathcal{N}(0,\mathrm{Id})$ in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$, we have:*

$$\forall x,y\in\mathcal{X},\qquad\mathrm{Cov}(s(x),s(y))=\mathbb{E}[s(x)\otimes s(y)]=E_d(x,y).$$

Remark 2.12. The kernel E_d is also the kernel of the orthogonal projection onto $H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ in $\Gamma\left(\mathcal{E}\otimes\mathcal{L}^d\right)$ for the Hermitian inner product (2.16), that is the *Bergman kernel* of $\mathcal{E}\otimes\mathcal{L}^d$.

As in 2.4, let $\nabla^{\mathbb{R}\mathcal{X}}$ denote the Levi-Civita connection on $(\mathbb{R}\mathcal{X},g)$. This connection and ∇^d induce the connection $\nabla^{\mathbb{R}\mathcal{X}}\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^d$ on $T^*(\mathbb{R}\mathcal{X})\otimes(\mathcal{E}\otimes\mathcal{L}^d)$. We denote by $\nabla^{2,d}$ the second covariant derivative $(\nabla^{\mathbb{R}\mathcal{X}}\otimes\mathrm{Id}+\mathrm{Id}\otimes\nabla^d)\circ\nabla^d$.

Let $x\in\mathbb{R}\mathcal{X}$ and let $\mathcal{J}^k_x(\mathcal{E}\otimes\mathcal{L}^d)$ denote the space of real k-jets of real smooth sections of $\mathcal{E}\otimes\mathcal{L}^d$ at x. On the space of smooth real sections of $\mathcal{E}\otimes\mathcal{L}^d$ we define

$$j^{1,d}_x:s\mapsto(s(x),\nabla^d_x s)\qquad\text{and}\qquad j^{2,d}_x:s\mapsto(s(x),\nabla^d_x s,\nabla^{2,d}_x s),$$

where $\nabla^d_x s$ and $\nabla^{2,d}_x s$ are implicitly restricted to $T_x\mathbb{R}\mathcal{X}$. These maps induce isomorphisms:

$$\begin{aligned}&\mathcal{J}^1_x(\mathcal{E}\otimes\mathcal{L}^d)\simeq\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes(\mathbb{R}\oplus T^*_x\mathbb{R}\mathcal{X}),&&\text{and}\\&\mathcal{J}^2_x(\mathcal{E}\otimes\mathcal{L}^d)\simeq\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes(\mathbb{R}\oplus T^*_x\mathbb{R}\mathcal{X}\oplus\mathrm{Sym}(T^*_x\mathbb{R}\mathcal{X})),\end{aligned}$$

Note that the above isomorphisms are not canonical since they depend on the choice of ∇^d.

We have the following equivalent of lemma 2.4, with the same proof, and similar notations for the partial covariant derivatives.

Lemma 2.13. *Let E_d denote the Bergman kernel of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ and let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. Let $x\in\mathbb{R}\mathcal{X}$, then $j^{2,d}_x(s)$ is a centered Gaussian vector,*

and its variance form is characterized by:

$$\mathrm{Var}(t) = \mathbb{E}[s(x) \otimes s(x)] = E_d(x,x), \tag{2.21}$$

$$\mathrm{Var}(L) = \mathbb{E}\big[\nabla^d_x s \otimes \nabla^d_x s\big] = (\partial_x \partial_y E_d)(x,x), \tag{2.22}$$

$$\mathrm{Var}(S) = \mathbb{E}\big[\nabla^{2,d}_x s \otimes \nabla^{2,d}_x s\big] = (\partial_{x,x} \partial_{y,y} E_d)(x,x), \tag{2.23}$$

$$\mathrm{Cov}(t,L) = \mathbb{E}\big[s(x) \otimes \nabla^d_x s\big] = (\partial_y E_d)(x,x), \tag{2.24}$$

$$\mathrm{Cov}(t,S) = \mathbb{E}\big[s(x) \otimes \nabla^{2,d}_x s\big] = (\partial_{y,y} E_d)(x,x), \tag{2.25}$$

$$\mathrm{Cov}(L,S) = \mathbb{E}\big[\nabla^d_x s \otimes \nabla^{2,d}_x s\big] = (\partial_x \partial_{y,y} E_d)(x,x). \tag{2.26}$$

3 Estimates for the covariance kernels

We state in this section the estimates for the kernels described above and their first and second derivatives. These estimates will allow us to compute the limit distribution for the random 2-jets induced by the Gaussian field $(f(x))_{x\in M}$ (resp. $(s(x))_{x\in\mathbb{R}\mathcal{X}}$).

In the case of the spectral function of the Laplacian e_λ, the asymptotics of 3.1 were established by Bin [4], extending results of Hörmander [20]. In the algebraic case, Bleher Shiffman and Zelditch used estimates for the related Szegö kernel, see [5, thm 3.1]. In terms of the Bergman kernel, a similmar result was established in [3]. Both these results concern line bundles. Here, we establish the estimates we need for the Bergman kernel in the case of a higher rank bundle using Hörmander-Tian peak sections (see 3.2 and 3.3 below).

3.1 The spectral function of the Laplacian

We consider the harmonic setting of section 2.5. Let $x \in M$ and let $(x_1, \dots, x_n)$ be normal coordinates centered at x. Let $(y_1, \dots, y_n)$ denote the same coordinates in a second copy of M, so that $(x_1, \dots, x_n, y_1, \dots, y_n)$ are normal coordinates around $(x,x) \in M \times M$. We denote by ∂_{x_i} (resp. ∂_{y_i}) the partial derivative with respect to x_i (resp. y_i), and similarly ∂_{x_i,x_j} (resp. ∂_{y_i,y_j}) denotes the second derivative with respect to x_i and x_j (resp. y_i and y_j). Let

$$\gamma_0 = \frac{1}{\sqrt{4\pi}^n \Gamma\left(1+\frac{n}{2}\right)}, \qquad \gamma_1 = \frac{1}{2\sqrt{4\pi}^n \Gamma\left(2+\frac{n}{2}\right)} \qquad \text{and} \qquad \gamma_2 = \frac{1}{4\sqrt{4\pi}^n \Gamma\left(3+\frac{n}{2}\right)},$$

where Γ is Euler's gamma function. Let us recall the main theorem of [4].

Theorem 3.1 (Bin)**.** *Let V_λ be as in section 2.5 and let e_λ denote its Schwartz kernel. The following asymptotics hold, uniformly in $x \in M$, as $\lambda \to +\infty$:*

$$e_\lambda(x,x) = \gamma_0 \lambda^n + O(\lambda^{n-1}), \tag{3.1}$$

$$\partial_{x_i} e_\lambda(x,x) = O(\lambda^n), \tag{3.2}$$

$$\partial_{x_i,x_k} e_\lambda(x,x) = \begin{cases} -\gamma_1 \lambda^{n+2} + O(\lambda^{n+1}) & \text{if } i = k, \\ O(\lambda^{n+1}) & \text{if } i \neq k, \end{cases} \tag{3.3}$$

$$\partial_{x_i} \partial_{y_j} e_\lambda(x,x) = \begin{cases} \gamma_1 \lambda^{n+2} + O(\lambda^{n+1}) & \text{if } i = j, \\ O(\lambda^{n+1}) & \text{if } i \neq j, \end{cases} \tag{3.4}$$

$$\partial_{x_i,x_k} \partial_{y_j} e_\lambda(x,x) = O(\lambda^{n+2}), \tag{3.5}$$

$$\partial_{x_i,x_k} \partial_{y_j,y_l} e_\lambda(x,x) = \begin{cases} 3\gamma_2 \lambda^{n+4} + O(\lambda^{n+3}) & \text{if } i = j = k = l, \\ \gamma_2 \lambda^{n+4} + O(\lambda^{n+3}) & \text{if } i = j \neq k = l \text{ or } i = k \neq j = l, \\ O(\lambda^{n+3}) & \text{otherwise.} \end{cases} \tag{3.6}$$

Since e_λ is symmetric, this also gives the asymptotics for $\partial_{y_j} e_\lambda$, $\partial_{y_j,y_l} e_\lambda$ and $\partial_{x_i}\partial_{y_j,y_l} e_\lambda$ along the diagonal. This theorem, together with lemma 2.6, gives the estimates we need for the kernel E_λ of $(V_\lambda)^r$.

We will need the following relations:

$$\frac{\gamma_0}{\gamma_1} = n+2 \qquad \text{and} \qquad \frac{\gamma_1}{\gamma_2} = n+4. \tag{3.7}$$

3.2 Hörmander-Tian peak sections

We now recall the construction of Hörmander-Tian peak sections in the framework of 2.6. Let $\mathcal{X}$ be a complex projective manifold. Let $\mathcal{E}$ be a rank r holomorphic Hermitian vector bundle and $\mathcal{L}$ be an ample holomorphic Hermitian line bundle, both defined over $\mathcal{X}$. We assume that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures, and that the Kähler metric $g_{\mathbb{C}}$ on $\mathcal{X}$ is induced by the curvature ω of $\mathcal{L}$.

The goal of this subsection is to build, for every d large enough, a family of real sections of $\mathcal{E}\otimes\mathcal{L}^d$ with prescribed 2-jets at some fixed point $x \in \mathbb{R}\mathcal{X}$. Moreover, we want this family to be orthonormal, up to an error that goes to 0 as d goes to infinity. Using these sections, we will compute (in 3.3 below) the asymptotics we need for the Bergman kernel.

Let $x \in \mathbb{R}\mathcal{X}$ and $(x_1,\dots,x_n)$ be real holomorphic coordinates centered at x and such that $(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ is orthonormal at x. The next lemma is established in [17, lemma 3.3], up to a factor π coming from different normalisations of the metric.

Lemma 3.2. *There exists a real holomorphic frame ζ_0 for $\mathcal{L}$, defined over some neighborhood of x, whose potential $-\ln(h_{\mathcal{L}}(\zeta_0,\zeta_0))$ vanishes at x, where it reaches a local minimum with Hessian $g_{\mathbb{C}}$.*

We choose such a frame ζ_0. Let $(\zeta_1,\dots,\zeta_r)$ be a real holomorphic frame for $\mathcal{E}$ over a neighborhood of x, which is orthonormal at x. Since $\mathcal{X}$ is compact, we can find $\rho > 0$, not depending on x, such that local coordinates and frames as above are defined at least on the geodesic ball of radius ρ centered at x. The following results are proved in [16, section 2.3]. See also [15, section 2.2] and the paper by Tian [31, lemmas 1.2 and 2.3], without the higher rank bundle $\mathcal{E}$ but with more details.

Proposition 3.3. *Let $p=(p_1,\dots,p_n)\in\mathbb{N}^n$, $p' > p_1+\dots+p_n$ and $q\in\{1,\dots,r\}$. There exists $d_0\in\mathbb{N}$ such that, for any $d\geqslant d_0$, there exist $C_{d,p}>0$ and $s\in\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ such that $\|s\|=1$ and*

$$s(x_1,\dots,x_n) = C_{d,p}\left(x_1^{p_1}\cdots x_n^{p_n} + O\Big(\|(x_1,\dots,x_n)\|^{2p'}\Big)\right)\left(1+O\Big(d^{-2p'}\Big)\right)\zeta_q\otimes\zeta_0^d$$

in some neighborhood of x, where the estimate $O(d^{-2p'})$ is uniform in $x\in\mathbb{R}\mathcal{X}$.

Moreover, d_0 depends on p' but does not depend on x, p, q or our choices of local coordinates and frames. Finally, $C_{d,p}$ is given by:

$$(C_{d,p})^{-2} = \int_{\left\{\|(x_1,\dots,x_n)\|\leqslant\frac{\ln(d)}{\sqrt{d}}\right\}} |x_1^{p_1}\cdots x_n^{p_n}|^2\, h_{\mathcal{L}}^d\left(\zeta_0^d,\zeta_0^d\right)\mathrm{d}V_{\mathcal{X}}.$$

The following definitions use proposition 3.3 with $p'=3$ and the corresponding d_0.

Definitions 3.4. For any $d\geqslant d_0$ and $q\in\{1,\dots,r\}$ we denote the sections of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ given by 3.3 by:

- $s_0^{d,q}$ for $p_1=\dots=p_n=0$,
- $s_i^{d,q}$ for $p_i=1$ and $\forall\, k\neq i$, $p_k=0$,

- $s^{d,q}_{i,i}$ for $p_i = 2$ and $\forall\, k \neq i$, $p_k = 0$,
- $s^{d,q}_{i,j}$ for $p_i = p_j = 1$ and $\forall\, k \notin \{i,j\}$, $p_k = 0$, when $i < j$.

Computing the values of the corresponding $C_{d,p}$ (see [15, lemma 2.5]), we get the following asymptotics as d goes to infinity. Once again, $O(d^{-1})$ is uniform in x.

Lemma 3.5. *For every $q \in \{1,\dots,r\}$, we have:*

$$s^{d,q}_0 = \sqrt{\frac{d^n}{\pi^n}}\left(1 + O\Big(\|(x_1,\dots,x_n)\|^6\Big)\right)\big(1 + O(d^{-1})\big)\,\zeta_q \otimes \zeta_0^d, \tag{3.8}$$

$$\forall i \in \{1,\dots,n\},\quad s^{d,q}_i = \sqrt{\frac{d^{n+1}}{\pi^n}}\left(x_i + O\Big(\|(x_1,\dots,x_n)\|^6\Big)\right)\big(1 + O(d^{-1})\big)\,\zeta_q \otimes \zeta_0^d, \tag{3.9}$$

$$\forall i \in \{1,\dots,n\},\quad s^{d,q}_{i,i} = \sqrt{\frac{d^{n+2}}{\pi^n}}\left(\frac{x_i^2}{\sqrt{2}} + O\Big(\|(x_1,\dots,x_n)\|^6\Big)\right)\big(1 + O(d^{-1})\big)\,\zeta_q \otimes \zeta_0^d, \tag{3.10}$$

and finally, $\forall i,j \in \{1,\dots,n\}$ such that $i < j$,

$$s^{d,q}_{i,j} = \sqrt{\frac{d^{n+2}}{\pi^n}}\left(x_i x_j + O\Big(\|(x_1,\dots,x_n)\|^6\Big)\right)\big(1 + O(d^{-1})\big)\,\zeta_q \otimes \zeta_0^d. \tag{3.11}$$

Remark 3.6. These values differ from the one given in [16, lemma 2.3.5] by a factor $\left(\int_{\mathcal{X}} \mathrm{d}V_{\mathcal{X}}\right)^{\frac{1}{2}}$ and some power of $\sqrt{\pi}$ because we do not use the same normalization for the volume form. For the same reason they also differ from [31, lemma 2.3] by a factor $\sqrt{\pi}^n$.

The sections defined in 3.4 are linearly independent, at least for d large enough. In fact, they are asymptotically orthonormal in the following sense. Let $H^d_{2,x} \subset \mathbb{R}H^0(\mathcal{X}, \mathcal{E}\otimes\mathcal{L}^d)$ denote the subspace of sections that vanish up to order 2 at x.

Lemma 3.7. *The sections $(s^{d,q}_i)_{\substack{1\leqslant q\leqslant r\\ 0\leqslant i\leqslant n}}$ and $(s^{d,q}_{i,j})_{\substack{1\leqslant q\leqslant r\\ 1\leqslant i\leqslant j\leqslant n}}$ defined in 3.4 have L^2-norm equal to 1 and their pairwise scalar product are dominated by a $O(d^{-1})$ independent of x. Moreover, their scalar product with any unit element of $H^d_{2,x}$ is dominated by some $O(d^{-1})$ not depending on x.*

3.3 The Bergman kernel

In this section we compute some asymptotics for the Bergman kernel and its derivatives. Let $x \in \mathbb{R}\mathcal{X}$ and let $(x_1,\dots,x_n)$ be real holomorphic coordinates around x such that $(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ is orthonormal at x. We denote by $(y_1,\dots,y_n)$ the same coordinates as $(x_1,\dots,x_n)$ in a second copy of $\mathcal{X}$. Let ζ_0 be a real holomorphic frame for $\mathcal{L}$ given by lemma 3.2 and $(\zeta_1,\dots,\zeta_r)$ be a real holomorphic frame for $\mathcal{E}$ that is orthonormal at x. For simplicity, we set $\zeta^d_p = \zeta_p(x)\otimes\zeta_0^d(x)$ for every $p \in \{1,\dots,r\}$ and $d \in \mathbb{N}$, so that $(\zeta^d_1,\dots,\zeta^d_r)$ is an orthonormal basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$.

Let ∇^d be any real connection on $\mathcal{E}\otimes\mathcal{L}^d$ such that, for every $p \in \{1,\dots,r\}$, $\nabla^d(\zeta_p\otimes\zeta_0^d)$ vanishes in a neighborhood of x. In this neighborhood, we have for every function f:

$$\nabla^d(f\zeta_p\otimes\zeta_0^d) = df\otimes\zeta_p\otimes\zeta_0^d \qquad\text{and}\qquad \nabla^{2,d}(f\zeta_p\otimes\zeta_0^d) = \nabla^2 f\otimes\zeta_p\otimes\zeta_0^d, \tag{3.12}$$

where $\nabla^{2,d}$ stands for the associated second covariant derivative, as in 2.6. This choice of connection may seem restrictive but the quantity we want to compute do not depend on a choice of connection so that we can choose one that suits us.

As usual, ∇^d induces a connection on $(\overline{\mathcal{E}\otimes\mathcal{L}^d})\boxtimes(\mathcal{E}\otimes\mathcal{L}^d)$. We denote by $\partial^d_{x_i}$ and $\partial^d_{y_i}$ the partial covariant derivatives with respect to x_i and y_i respectively. We also denote by $\partial^d_{x_i,x_j}$ (resp. $\partial^d_{y_i,y_j}$) the second derivative with respect to x_i and x_j (resp. y_i and y_j).

Proposition 3.8. *The following asymptotics hold as $d\to+\infty$. They are independent of $x\in\mathbb{R}\mathcal{X}$ and of the choice of the holomorphic frame $(\zeta_1,\dots,\zeta_r)$.*

$$\left\langle E_d(x,x)\,,\zeta_p^d\otimes\zeta_{p'}^d\right\rangle=\begin{cases}\dfrac{d^n}{\pi^n}+O(d^{n-1}) & \text{if } p=p',\\ O(d^{n-1}) & \text{otherwise,}\end{cases}\tag{3.13}$$

$$\left\langle \partial^d_{x_i}E_d(x,x)\,,\zeta_p^d\otimes\zeta_{p'}^d\right\rangle=O(d^{n-\frac{1}{2}}),\tag{3.14}$$

$$\left\langle \partial^d_{x_i,x_k}E_d(x,x)\,,\zeta_p^d\otimes\zeta_{p'}^d\right\rangle=O(d^{n}),\tag{3.15}$$

$$\left\langle \partial^d_{x_i}\partial^d_{y_j}E_d(x,x)\,,\zeta_p^d\otimes\zeta_{p'}^d\right\rangle=\begin{cases}\dfrac{d^{n+1}}{\pi^n}+O(d^{n}) & \text{if } p=p' \text{ and } i=j,\\ O(d^{n}) & \text{otherwise,}\end{cases}\tag{3.16}$$

$$\left\langle \partial^d_{x_i,x_k}\partial^d_{y_j}E_d(x,x)\,,\zeta_p^d\otimes\zeta_{p'}^d\right\rangle=O(d^{n+\frac{1}{2}}),\tag{3.17}$$

$$\left\langle \partial^d_{x_i,x_k}\partial^d_{y_j,y_l}E_d(x,x)\,,\zeta_p^d\otimes\zeta_{p'}^d\right\rangle=\begin{cases}2\dfrac{d^{n+2}}{\pi^n}+O(d^{n+1}) & \text{if } p=p' \text{ and } i=j=k=l,\\ \dfrac{d^{n+2}}{\pi^n}+O(d^{n+1}) & \text{if } p=p' \text{ and } i=j\neq k=l,\\ O(d^{n+1}) & \text{otherwise.}\end{cases}\tag{3.18}$$

Remark 3.9. Since $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is not a product space, the terms with $p\neq p'$ are usually not zero. However they are zero when $\mathcal{E}$ is trivial, for example.

Corollary 3.10. *For every d large enough, $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-ample.*

Proof of corollary 3.10. Let $x\in\mathbb{R}\mathcal{X}$. By (3.13), the matrix of $E_d(x,x)$ in any orthonormal basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L})_x$ is $\frac{d^n}{\pi^n}I_r(1+O(d^{-1}))$. Then $E_d(x,x)$ is positive-definite for d larger than some d_0 independent of x. By lemma 2.10, $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-ample for $d\geqslant d_0$. □

Proof of proposition 3.8. First we build an orthonormal basis of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ by applying the Gram-Schmidt process to the family of peak sections. Then we use formula (2.20) and the asymptotics of lemma 3.5 to prove the proposition.

We order the sections of definition 3.4 as follows:

$$\begin{aligned}&s_0^{d,1},\dots,s_0^{d,r},s_1^{d,1},\dots,s_1^{d,r},\dots,s_n^{d,1},\dots,s_n^{d,r},s_{1,1}^{d,1},\dots,s_{1,1}^{d,r},s_{2,2}^{d,1},\dots,s_{2,2}^{d,r},\dots,s_{n,n}^{d,1},\dots,s_{n,n}^{d,r},\\ &\quad s_{1,2}^{d,1},\dots,s_{1,2}^{d,r},s_{1,3}^{d,1},\dots,s_{1,3}^{d,r},\dots,s_{1,n}^{d,1},\dots,s_{1,n}^{d,r},s_{2,3}^{d,1},\dots,s_{2,3}^{d,r},\dots,s_{n-1,n}^{d,1},\dots,s_{n-1,n}^{d,r}.\end{aligned}\tag{3.19}$$

This family is linearly independent for d large enough and spans a space whose direct sum with $H^d_{2,x}$ is $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$. We complete it into a basis B of $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ by adding an orthonormal basis of $H^d_{2,x}$ at the end of the previous list.

We apply the Gram-Schmidt process to B, starting by the last elements and going backwards. Let $\tilde{B}$ denote the resulting orthonormal basis, and $\tilde{s}_0^{d,1},\dots,\tilde{s}_n^{d,r},\tilde{s}_{1,1}^{d,1},\dots,\tilde{s}_{n-1,n}^{d,r}$ denote its first elements. This way, $\tilde{s}_{n-1,n}^{d,r}$ is a linear combination of $s_{n-1,n}^{d,r}$ and elements of $H^d_{2,x}$, and $\tilde{s}_0^{d,1}$ is a linear combination of (possibly) all elements of B. We denote by

$(b_i)_{1\leqslant i\leqslant \frac{r(n+1)(n+2)}{2}}$ the first elements of B listed above (3.19) and by $(\tilde{b}_i)$ the corresponding elements of $\tilde{B}$.

Let $i \in \left\{1, \dots, \frac{r(n+1)(n+2)}{2}\right\}$ and assume that for any $k \leqslant i$ and any $j > i$ we have $\left\langle b_k \,, \tilde{b}_j \right\rangle = O(d^{-1})$. Note that this is the case for $i = \frac{r}{2}(n+1)(n+2)$. Then,

$$\tilde{b}_i = \frac{b_i - \sum_{j>i} \left\langle b_i \,, \tilde{b}_j \right\rangle \tilde{b}_j - \pi_i}{\left\| b_i - \sum_{j>i} \left\langle b_i \,, \tilde{b}_j \right\rangle \tilde{b}_j - \pi_i \right\|} \tag{3.20}$$

where π_i stands for the projection of b_i onto $H^d_{2,x}$. By lemma 3.7, $\|\pi_i\|^2 = \langle b_i \,, \pi_i \rangle = O(d^{-1})$. Then by our hypothesis $\left\| b_i - \sum_{j>i} \left\langle b_i \,, \tilde{b}_j \right\rangle \tilde{b}_j - \pi_i \right\|^2 = 1 + O(d^{-1})$. Using lemma 3.7 and the above hypothesis once again, we get:

$$\left\langle b_k \,, \tilde{b}_i \right\rangle = \left(1 + O(d^{-1})\right) \left(\langle b_k \,, b_i \rangle - \sum_{j>i} \left\langle b_k \,, \tilde{b}_j \right\rangle \left\langle b_i \,, \tilde{b}_j \right\rangle - \langle b_k \,, \pi_i \rangle \right) = O(d^{-1}),$$

for any $k < i$. By induction, for any $1 \leqslant i < j \leqslant \frac{r(n+1)(n+2)}{2}$, $\left\langle b_i \,, \tilde{b}_j \right\rangle = O(d^{-1})$.

Using (3.20), for any $i \in \left\{1, \dots, \frac{r(n+1)(n+2)}{2}\right\}$, $\tilde{b}_i = (b_i + O(d^{-1}) \sum_{j>i} \tilde{b}_j - \pi_i)(1 + O(d^{-1}))$. Another induction gives:

$$\tilde{b}_i = \left(b_i + O(d^{-1}) \sum_{j>i} b_j + \tilde{\pi}_i \right) \left(1 + O(d^{-1})\right), \tag{3.21}$$

for any i, where $\tilde{\pi}_i \in H^d_{2,x}$ is such that $\|\tilde{\pi}_i\|^2 = O(d^{-1})$. Moreover, all the estimates are independent of x and $i \in \left\{1, \dots, \frac{r(n+1)(n+2)}{2}\right\}$.

Among the elements of $\tilde{B}$, only $\tilde{s}^{d,1}_0, \dots, \tilde{s}^{d,r}_0$ do not vanish at x. Using formula (2.20), we get $E_d(x,x) = \sum_{1\leqslant q\leqslant r} \tilde{s}^{d,q}_0(x) \otimes \tilde{s}^{d,q}_0(x)$. Then,

$$\left\langle E_d(x,x) \,, \zeta^d_p \otimes \zeta^d_{p'} \right\rangle = \sum_{q=1}^{r} \left\langle \tilde{s}^{d,q}_0(x) \,, \zeta^d_p \right\rangle \left\langle \tilde{s}^{d,q}_0(x) \,, \zeta^d_{p'} \right\rangle.$$

Recall that $\tilde{b}_0 = \tilde{s}^{d,1}_0, \dots, \tilde{b}_r = \tilde{s}^{d,r}_0$. Because of (3.21), for all $q \in \{1, \dots, r\}$,

$$\begin{aligned} \left\langle \tilde{s}^{d,q}_0(x) \,, \zeta^d_p \right\rangle &= \left\langle s^{d,q}_0(x) \,, \zeta^d_p \right\rangle + O(d^{-1}) \left(\sum_{q'=1}^{r} \left\langle s^{d,q'}_0(x) \,, \zeta^d_p \right\rangle \right) \\ &= \begin{cases} \dfrac{1}{\sqrt{\pi}^n} \sqrt{d}^n (1 + O(d^{-1})) & \text{if } p = q, \\ O(d^{\frac{n}{2}-1}) & \text{otherwise,} \end{cases} \end{aligned} \tag{3.22}$$

where the last equality comes from equation (3.8). This etablishes (3.13).

Likewise,

$$\begin{aligned} \left\langle \partial^d_{x_i} E_d(x,x) \,, \zeta^d_p \otimes \zeta^d_{p'} \right\rangle &= \left\langle \sum_{1\leqslant q\leqslant r} \partial^d_{x_i} \tilde{s}^{d,q}_0(x) \otimes \tilde{s}^{d,q}_0(x) \,, \zeta^d_p \otimes \zeta^d_{p'} \right\rangle \\ &= \sum_{1\leqslant q\leqslant r} \left\langle \partial^d_{x_i} \tilde{s}^{d,q}_0(x) \,, \zeta^d_p \right\rangle \left\langle \tilde{s}^{d,q}_0(x) \,, \zeta^d_{p'} \right\rangle. \end{aligned}$$

The description (3.21) shows that $\partial^d_{x_i} \tilde{s}^{d,q}_0(x)$ does not necessarily vanish, but it equals:

$$O(d^{-1}) \sum_{\substack{1 \leqslant q' \leqslant r \\ 1 \leqslant j \leqslant n}} \partial^d_{x_i} s^{d,q'}_j(x).$$

By (3.9), one gets that $\left\langle \partial^d_{x_i} \tilde{s}^{d,q}_0(x), \zeta^d_p \right\rangle = O(d^{\frac{n-1}{2}})$, for all p and q. Besides by (3.22), $\left\langle \tilde{s}^{d,q}_0(x), \zeta^d_p \right\rangle = O(d^{\frac{n}{2}})$ for all p and q. This proves (3.14).

The remaining estimates can be proved in the same way, using lemma 3.5 and the fact that the estimates for corresponding elements of $\tilde{B}$ and B are the same. □

4 An integral formula for the Euler characteristic of a submanifold

The goal of this section is to derive an integral formula for the Euler characteristic of a submanifold defined as the zero set of some $f : M \to \mathbb{R}^r$, in terms of f and its derivatives. This section is independent of the previous ones and the results it contains are only useful for computing expected Euler characteristics (theorems 1.2 and 1.4).

We start by recalling the formalism of double forms, which was already used in this context by Taylor and Adler, see [1, section 7.2]. The Riemann curvature tensor and the second fundamental form of a submanifold being naturally double forms, this provides a useful way to formulate the Chern-Gauss-Bonnet theorem and the Gauss equation. This is done in subsections 4.2 and 4.3 respectively. Finally, we express the second fundamental form of a submanifold in terms of the derivatives of a defining function and prove the desired integral formula in 4.4.

4.1 The algebra of double forms

We follow the exposition of [1, pp. 157–158]. Let V be a real vector space of dimension n. For p and $q \in \{0, \ldots, n\}$ we denote by $\bigwedge^{p,q}(V^*)$ the space $\bigwedge^p(V^*) \otimes \bigwedge^q(V^*)$ of $(p+q)$-linear forms on V that are skew-symmetric in the first p and in the last q variables. The *space of double forms* on V is:

$$\bigwedge^\bullet(V^*) \otimes \bigwedge^\bullet(V^*) = \bigoplus_{0 \leqslant p,q \leqslant n} \bigwedge^{p,q}(V^*). \tag{4.1}$$

Elements of $\bigwedge^{p,q}(V^*)$ are called (p,q)-*double forms*, or *double forms of type* (p,q). We set:

$$\bigwedge^{\bullet,\bullet}(V^*) = \bigoplus_{p=0}^{n} \bigwedge^{p,p}(V^*). \tag{4.2}$$

Note that $\bigwedge^{1,1} V^*$ is the space of bilinear forms on V.

On $\bigwedge^\bullet(V^*) \otimes \bigwedge^\bullet(V^*)$ we can define a double wedge product. It extends the usual wedge product on $\bigwedge^\bullet(V^*) \simeq \bigoplus_{p=0}^n \bigwedge^{p,0}(V^*)$, so we simply denote it by $\wedge$. For pure tensors $\alpha \otimes \beta$ and $\alpha' \otimes \beta' \in \bigwedge^\bullet(V^*) \otimes \bigwedge^\bullet(V^*)$, we set:

$$(\alpha \otimes \beta) \wedge (\alpha' \otimes \beta') = (\alpha \wedge \alpha') \otimes (\beta \wedge \beta') \tag{4.3}$$

and we extend $\wedge$ to all double forms by bilinearity. This makes $\bigwedge^\bullet(V^*) \otimes \bigwedge^\bullet(V^*)$ into an algebra, of which $\bigwedge^{\bullet,\bullet}(V^*)$ is a commutative subalgebra. We denote by $\gamma^{\wedge k}$ the double wedge product of a double form $\gamma \in \bigwedge^{\bullet,\bullet}(V^*)$ with itself k times.

Lemma 4.1. *Let α be a symmetric $(1,1)$-double form on V, then for every x, y, z and $w \in V$,*

$$\alpha^{\wedge 2}((x,y),(z,w)) = 2\left(\alpha(x,z)\alpha(y,w) - \alpha(x,w)\alpha(y,z)\right).$$

Proof. Let $(e_1, \dots, e_n)$ be a basis of V and $(e_1^*, \dots, e_n^*)$ its dual basis. We have:

$$\alpha = \sum_{1 \leqslant i,k \leqslant n} \alpha_{ik} e_i^* \otimes e_k^* \qquad \text{and then} \qquad \alpha^{\wedge 2} = \sum_{1 \leqslant i,j,k,l \leqslant n} \alpha_{ik}\alpha_{jl}(e_i^* \wedge e_j^*) \otimes (e_k^* \wedge e_l^*).$$

Note that we do not restrict ourselves to indices satisfying $i < j$ and $k < l$ as is usually the case with skew-symmetric forms. By multilinearity, it is sufficient to check the result on elements of the basis. Let i, j, k and $l \in \{1, \dots, n\}$, then,

$$\begin{aligned}
\alpha^{\wedge 2}((e_i,e_j),(e_k,e_l)) &= \alpha_{ik}\alpha_{jl} - \alpha_{jk}\alpha_{il} - \alpha_{il}\alpha_{jk} + \alpha_{jl}\alpha_{ik} \\
&= 2(\alpha_{ik}\alpha_{jl} - \alpha_{il}\alpha_{jk}) \qquad \text{(since } \alpha \text{ is symmmetric)} \\
&= 2(\alpha(e_i,e_k)\alpha(e_j,e_l) - \alpha(e_i,e_l)\alpha(e_j,e_k)).
\end{aligned}$$

□

We can consider random vectors in spaces of double forms. The following technical result will be useful in the proofs of theorems 1.2 and 1.4. See [1, lemma 12.3.1] for a proof.

Lemma 4.2. *Let V be a vector space of finite dimension n. Let α be a Gaussian vector in $\bigwedge^{1,1} V^*$. If α is centered, then for any $p \leqslant \frac{n}{2}$,*

$$\mathbb{E}\left[\alpha^{\wedge 2p}\right] = \frac{(2p)!}{2^p p!}\left(\mathbb{E}\left[\alpha^{\wedge 2}\right]\right)^{\wedge p}.$$

Assume now that V is endowed with an inner product. It induces a natural inner product on $\bigwedge^\bullet(V^*)$ such that, if $(e_1, \dots, e_n)$ is an orthonormal basis of V,

$$\left\{ e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \;\middle|\; 1 \leqslant p \leqslant n \text{ and } 1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n \right\}$$

is an orthonormal basis of $\bigwedge^\bullet(V^*)$. We define the *trace operator* Tr on $\bigwedge^{\bullet,\bullet}(V^*)$ in the following way. If $\alpha \otimes \beta \in \bigwedge^{\bullet,\bullet}(V^*)$ is a pure tensor, then:

$$\operatorname{Tr}(\alpha \otimes \beta) = \langle \alpha, \beta \rangle \tag{4.4}$$

and we extend Tr to $\bigwedge^{\bullet,\bullet}(V^*)$ by linearity.

Let M be a smooth manifold of dimension n. Applying the previous construction pointwise to T_xM, we define the vector bundle $\bigwedge^\bullet(T^*M) \otimes \bigwedge^\bullet(T^*M)$ on M. Sections of this bundle are called *differential double forms* on M, and we can take the double wedge product of two such sections. Finally, if M is equipped with a Riemannian metric, we have a trace operator Tr which is defined pointwise by (4.4). This operator is $\mathcal{C}^\infty(M)$-linear and takes sections of the subbundle $\bigwedge^{\bullet,\bullet}(T^*M) = \bigoplus_{p=0}^n \bigwedge^{p,p}(T^*M)$ to smooth functions.

4.2 The Chern-Gauss-Bonnet theorem

Let (M, g) be a closed smooth Riemannian manifold of dimension n. We denote by ∇^M the Levi-Civita connection of M, and by κ its *curvature operator*. That is κ is the 2-form on M with values in the bundle $\operatorname{End}(TM) = TM \otimes T^*M$ defined by:

$$\kappa(X,Y)Z = \nabla_X^M \nabla_Y^M Z - \nabla_Y^M \nabla_X^M Z - \nabla_{[X,Y]}^M Z$$

for any vector fields X, Y and Z. Here $[X, Y]$ is the Lie bracket of X and Y.

We denote by R the *Riemann curvature tensor* of M, defined by:

$$R(X,Y,Z,W) = g(\kappa(X,Y)W,Z),$$

for any vector fields X, Y, Z and W on M. This defines a four times covariant tensor on M which is skew-symmetric in the first two and in the last two variables, hence R can naturally be seen as a $(2,2)$-double form. All this is standard material, except for the very last point, see for example [21, section 3.3].

We now state the Chern-Gauss-Bonnet theorem in terms of double forms. Recall that $|\mathrm{d}V_M|$ denotes the Riemannian measure on M (see (2.1)).

Theorem 4.3 (Chern-Gauss-Bonnet)**.** *Let M be a closed Riemannian manifold of even dimension n. Let R denote its Riemann curvature tensor and $\chi(M)$ denote its Euler characteristic. We have:*

$$\chi(M) = \frac{1}{(2\pi)^{\frac{n}{2}}\left(\frac{n}{2}\right)!}\int_M \mathrm{Tr}\left(R^{\wedge\frac{n}{2}}\right)|\mathrm{d}V_M|.$$

If M is orientable, this can be deduced from Atiyah-Singer's index theorem. The general case is treated in [27]. The above formula in terms of double forms can be found in [1, thm 12.6.18], up to a sign coming from different sign conventions in the definition of R.

Remark 4.4. If M is a closed manifold of odd dimension then $\chi(M) = 0$, see [18, cor 3.37].

4.3 The Gauss equation

Let (M,g) be a smooth Riemannian manifold of dimension n and $\tilde{M}$ be a smooth submanifold of M of codimension $r \in \{1,\dots,n-1\}$. We denote by ∇^M and $\tilde{\nabla}$ the Levi-Civita connections on M and $\tilde{M}$ respectively. Likewise, we denote by R and $\tilde{R}$ their Riemann curvature tensor. We wish to relate R and $\tilde{R}$. This is done by the Gauss equation, see proposition 4.5 below.

We denote by II the *second fundamental form* of $\tilde{M} \subset M$ which is defined as the section of $T^\perp\tilde{M} \otimes T^*\tilde{M} \otimes T^*\tilde{M}$ satisfying:

$$\mathrm{II}(X,Y) = -\left(\nabla^M_X Y - \tilde{\nabla}_X Y\right) = -\left(\nabla^M_X Y\right)^\perp \tag{4.5}$$

for any vector fields X and Y on $\tilde{M}$. Here, $\left(\nabla^M_X Y\right)^\perp$ stands for the orthogonal projection of $\nabla^M_X Y$ on $T^\perp\tilde{M}$. It is well-known that II is symmetric in X and Y, see [21, lemma 3.6.2].

The second fundamental form encodes the difference between $\tilde{R}$ and R in the following sense, see [21, thm 3.6.2].

Proposition 4.5 (Gauss equation)**.** *Let X, Y, Z and W be vector fields on $\tilde{M}$, then:*

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z)\rangle.$$

We want to write this Gauss equation in terms of double forms. Let $x \in \tilde{M}$ and X, Y, Z and $W \in T_x\tilde{M}$. Let $U \sim \mathcal{N}(0,\mathrm{Id})$ in T_xM, by lemma A.8:

$$\begin{aligned}\langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z)\rangle \\ = \mathbb{E}[\langle \mathrm{II}(X,Z), U\rangle \langle \mathrm{II}(Y,W), U\rangle - \langle \mathrm{II}(X,W), U\rangle \langle \mathrm{II}(Y,Z), U\rangle].\end{aligned}$$

Then we apply lemma 4.1 to the symmetric $(1,1)$-double form $\langle \mathrm{II}, U\rangle$ for fixed U. This gives:

$$\langle \mathrm{II}(X,Z), \mathrm{II}(Y,W)\rangle - \langle \mathrm{II}(X,W), \mathrm{II}(Y,Z)\rangle = \frac{1}{2}\,\mathbb{E}\Big[\langle \mathrm{II}, U\rangle^{\wedge 2}((X,Y),(Z,W))\Big].$$

We proved the following version of the Gauss equation.

Proposition 4.6 (Gauss equation)**.** *Let (M, g) be a Riemannian manifold and let $\tilde{M}$ be a smooth submanifold of M, such that $\dim(M) > \dim(\tilde{M}) \geqslant 1$. Let R and $\tilde{R}$ denote the Riemann curvature of M and $\tilde{M}$ respectively, and let* II *be the second fundamental form of $\tilde{M} \subset M$. Then, in the sense of double forms:*

$$\forall x \in \tilde{M}, \qquad \tilde{R}(x) = R(x) + \frac{1}{2}\,\mathbb{E}\Big[\langle \mathrm{II}(x)\,, U\rangle^{\wedge 2}\Big],$$

where $U \sim \mathcal{N}(0, \mathrm{Id})$ with values in T_xM, and $R(x)$ is implicitly restricted to $T_x\tilde{M}$.

4.4 An expression for the second fundamental form

Let us now express the second fundamental form II of a submanifold $\tilde{M}$ of M defined as the zero set of a smooth map $f : M \to \mathbb{R}^r$. For this we need some further definitions.

Let V and V' be two Euclidean spaces of dimension n and r respectively. Let $L : V \to V'$ be a linear surjection, then L^* is injective and its image is $\ker(L)^\perp$, so that LL^* is invertible.

Definition 4.7. Let $L : V \to V'$ be a surjection, the *pseudo-inverse* (or *Moore-Penrose inverse*) of L is defined as $L^\dagger = L^*(LL^*)^{-1}$ from V' to V.

The map $L^\dagger$ is the inverse of the restriction of L to $\ker(L)^\perp$. It is characterized by the fact that $LL^\dagger$ is the identity map of V' and $L^\dagger L$ is the orthogonal projection onto $\ker(L)^\perp$.

Let $f : M \to \mathbb{R}^r$ be a smooth submersion and assume that $\tilde{M} = f^{-1}(0)$. Recall that $\nabla^2 f = \nabla^M df$.

Lemma 4.8. *Let M be a Riemannian manifold and let $\tilde{M} \subset M$ be a submanifold of M defined as the zero set of the smooth submersion $f : M \to \mathbb{R}^r$. Let* II *denote the second fundamental form of $\tilde{M} \subset M$. Then,*

$$\forall x \in \tilde{M}, \quad \mathrm{II}(x) = (d_x f)^\dagger \circ \nabla^2_x f,$$

where $\nabla^2_x f$ is implicitly restricted to $T_x\tilde{M}$.

Proof. Let $x \in \tilde{M}$, since $\mathrm{II}(x)$ and $(d_x f)^\dagger$ take values in $T_x\tilde{M}^\perp = \ker(d_x f)^\perp$, we only need to prove that $d_x f \circ \mathrm{II}(x) = \nabla^2_x f$. Let X and Y be two vector fields on $\tilde{M}$. The map $df \cdot Y$ vanishes uniformly on $\tilde{M}$, hence:

$$d_x(df \cdot Y) \cdot X = (\nabla^M_X df)_x \cdot Y + d_x f \cdot (\nabla^M_X Y) = 0.$$

Then, using equation (4.5) and $\ker(d_x f) = T_x\tilde{M}$,

$$(d_x f \circ \mathrm{II}(x))(X, Y) = -d_x f \cdot (\nabla^M_X Y)^\perp = -d_x f \cdot (\nabla^M_X Y) = (\nabla^M_X df)_x \cdot Y = \nabla^2_x f(X, Y). \qquad \square$$

Proposition 4.9. *Let (M, g) be a closed Riemannian manifold of dimension n and R its Riemann curvature. Let $f : M \to \mathbb{R}^r$ be a smooth submersion and $Z_f = f^{-1}(0)$. We denote by R_f its Riemann curvature and by $|\mathrm{d}V_f|$ the Riemannian measure on Z_f.*

If $n - r$ is even, the Euler characteristic of Z_f is:

$$\chi(Z_f) = \frac{1}{(2\pi)^m m!} \int_{x \in Z_f} \mathrm{Tr}\left(R_f(x)^{\wedge m}\right) |\mathrm{d}V_f|\,, \tag{4.6}$$

where $m = \frac{n-r}{2}$. Furthermore, for all $x \in Z_f$,

$$R_f(x) = R(x) + \frac{1}{2}\mathbb{E}\Big[\big\langle \nabla^2_x f\,, (d_x f)^{\dagger *}(U)\big\rangle^{\wedge 2}\Big], \tag{4.7}$$

where U is a standard Gaussian vector in T_xM.

Proof. First, we apply the Chern-Gauss-Bonnet theorem 4.3 to Z_f, which gives (4.6). Then let $x \in Z_f$ and $U \sim \mathcal{N}(0, \mathrm{Id})$ in T_xM, by proposition 4.6,

$$R_f(x) = R(x) + \frac{1}{2}\mathbb{E}\left[\langle \mathrm{II}_f(x)\,, U\rangle^{\wedge 2}\right],$$

where II_f is the second fundamental form of $Z_f \subset M$. We conclude by lemma 4.8. □

Proposition 4.9 is also true for zero sets of sections. Let s be a section of some rank r vector bundle over M that vanishes transversally and Z_s be its zero set. Let $|\mathrm{d}V_s|$ denote the Riemannian measure on Z_s and R_s denote its Riemann tensor. As above, we can apply theorem 4.3, so that:

$$\chi(Z_s) = \frac{1}{(2\pi)^m m!}\int_{x\in Z_s} \mathrm{Tr}\left(R_s(x)^{\wedge m}\right)|\mathrm{d}V_s|\,. \tag{4.8}$$

The result of proposition 4.6 is still valid for Z_s. Besides, the same proof as in 4.8 shows that, for any connection ∇^d, the second fundamental form II_s of Z_s satisfies:

$$\forall x \in Z_s, \quad \mathrm{II}_s(x) = (\nabla^d_x s)^\dagger \circ \nabla^{2,d}_x s.$$

Remark 4.10. This is not surprising since the terms of this equality do not depend on a choice of connection and the result in a trivialisation is given by lemma 4.8.

Finally, for every connection ∇^d and every $x \in Z_s$, we get:

$$R_s(x) = R(x) + \frac{1}{2}\mathbb{E}\left[\left\langle \nabla^{2,d}_x s\,, (\nabla^d_x s)^{\dagger *}(U)\right\rangle^{\wedge 2}\right], \tag{4.9}$$

where U is a standard Gaussian vector in T_xM, as in (4.7).

5 Proofs of the main theorems

We now set to prove the main theorems. The proofs will be detailed in the harmonic case but only sketched in the real algebraic one, since they are essentially the same.

5.1 The Kac-Rice formula

First, we state the celebrated Kac-Rice formula, which is one of the key ingredients in our proofs. This formula is proved in [6, thm 4.2], see also [2, chap 6]. For the reader's convenience, we include a proof in appendix C.

Definition 5.1. Let $L : V \to V'$ be a linear map between Euclidean vector spaces. We denote by $\left|\det^\perp(L)\right|$ the *orthogonal determinant* of L:

$$\left|\det^\perp(L)\right| = \sqrt{\det(LL^*)},$$

where $L^* : V' \to V$ is the adjoint operator of L.

Remark 5.2. If L is not onto then $\left|\det^\perp(L)\right| = 0$. Else, let A be the matrix of the restriction of L to $\ker(L)^\perp$ in any orthonormal basis of $\ker(L)^\perp$ and V', we have $\left|\det^\perp(L)\right| = |\det(A)|$.

As is section 2, we consider a closed Riemannian manifold M of dimension n and a subspace $V \subset \mathcal{C}^\infty(M, \mathbb{R}^r)$ of dimension N (recall that $1 \leqslant r \leqslant n$). We assume that V is 0-ample, in the sense of section 2.2, so that

$$\Sigma = \{(f, x) \in V \times M \mid f(x) = 0\}$$

is a submanifold of codimension r of $V \times M$. Let f be a standard Gaussian vector in V. Then Z_f is almost surely a smooth submanifold of codimension r of M (see 2.2). Recall that E denotes both the Schwartz kernel of V and the covariance function of $(f(x))_{x\in M}$.

Theorem 5.3 (Kac-Rice formula)**.** *Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function,*

$$\mathbb{E}\left[\int_{x\in Z_f} \phi(f,x)\,|\mathrm{d}V_f|\right] = \frac{1}{\sqrt{2\pi}^r}\int_{x\in M} \frac{1}{\sqrt{\det E(x,x)}}\mathbb{E}\Big[\phi(f,x)\left|\det{}^{\perp}(d_x f)\right| \;\Big|\; f(x)=0\Big]\,|\mathrm{d}V_M|\,,$$

whenever one of these integrals is well-defined.

The expectation on the right-hand side is to be understood as the conditional expectation of $\phi(f,x)\left|\det{}^{\perp}(d_x f)\right|$ given $f(x)=0$. By $\det(E(x,x))$, we mean the determinant of the matrix of the bilinear form $E(x,x)$ in any orthonormal basis of $\mathbb{R}^r$.

5.2 Proof of theorem 1.1

We start with the expectation of the volume, which is the toy-model for this kind of computations. In this case, the proof is closely related to [5, 6], in a slightly different setting. The first step is to apply Kac-Rice formula above with $\phi : (f,x) \mapsto 1$. We get:

$$\mathbb{E}[\mathrm{Vol}\,(Z_f)] = \frac{1}{\sqrt{2\pi}^r}\int_{x\in M} \frac{1}{\sqrt{\det(E(x,x))}}\mathbb{E}\Big[\left|\det{}^{\perp}(d_x f)\right| \;\Big|\; f(x)=0\Big]\,|\mathrm{d}V_M|\,. \tag{5.1}$$

Let $x \in M$, $j^1_x(f) = (f(x), d_x f)$ is a Gaussian vector in $\mathbb{R}^r \otimes (\mathbb{R}\oplus T^*_x M)$ whose distribution only depends on the values of E and its derivatives at (x,x), see lemma 2.4. Thus $\mathbb{E}[\mathrm{Vol}\,(Z_f)]$ will only depend on the values of E and its derivatives along the diagonal as was expected from [5, thm 2.2].

The next step is to compute pointwise asymptotics for the integrand on the right-hand side of (5.1). We will use lemma 2.4, which describes the distribution of $j^1_x(f)$, and the estimates of section 3. Both in the harmonic and the algebraic settings, the pointwise asymptotic turns out to be universal: it does not depend on x or even on the ambient manifold. This is because the distribution of $j^1_x(f)$ is determined by the asymptotics of section 3 which are universal.

We now specify to the harmonic setting of section 2.5, that is $V = (V_\lambda)^r$ for some non-negative λ. Recall that $(V_\lambda)^r$ is 0-ample (lemma 2.7) so that equation (5.1) is valid in this case. Let $x \in M$, by lemma 2.6 and (3.1) we have:

$$\det(E_\lambda(x,x)) = (e_\lambda(x,x))^r = (\gamma_0\lambda^n)^r\left(1+O(\lambda^{-1})\right). \tag{5.2}$$

Then we want to estimate the conditional expectation in (5.1). Before going further, the asymptotics of section 3.1 suggest to consider the scaled variables:

$$(t_\lambda, L_\lambda) = \left(\frac{1}{\sqrt{\gamma_0\lambda^n}}f(x), \frac{1}{\sqrt{\gamma_1\lambda^{n+2}}}d_x f\right) \tag{5.3}$$

instead of $j^1_x(f)$. This is a centered Gaussian vector whose variance is determined by (5.3). Besides, by definition 5.1, the orthogonal determinant is homogeneous of degree r for linear maps taking values in $\mathbb{R}^r$, so that:

$$\mathbb{E}\Big[\left|\det{}^{\perp}(d_x f)\right| \;\Big|\; f(x)=0\Big] = \sqrt{\gamma_1\lambda^{n+2}}^r\,\mathbb{E}\Big[\left|\det{}^{\perp}(L_\lambda)\right| \;\Big|\; t_\lambda = 0\Big]\,. \tag{5.4}$$

Lemma 5.4. *For every $x \in M$, we have:*

$$\mathbb{E}\Big[\left|\det{}^{\perp}(L_\lambda)\right| \;\Big|\; t_\lambda = 0\Big] = \sqrt{2\pi}^r\,\frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)}(1+O(\lambda^{-1})),$$

where the error term does not depend on the point $x \in M$.

We postpone the proof of this lemma for now and conclude the proof of theorem 1.1. By (5.2), (5.4) and lemma 5.4, for every $x \in M$,

$$\frac{1}{\sqrt{2\pi}^r \sqrt{\det(E(x,x))}} \mathbb{E}\Big[\left|\det{}^{\perp}(d_x f)\right| \;\Big|\; f(x) = 0 \Big] = \left(\sqrt{\frac{\gamma_1 \lambda^{n+2}}{\gamma_0 \lambda^n}} \right)^r \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} \left(1 + O(\lambda^{-1})\right),$$

where the error term does not depend on the point $x \in M$. By (3.7) this equals:

$$\left(\frac{\lambda}{\sqrt{n+2}} \right)^r \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} \left(1 + O(\lambda^{-1})\right).$$

Plugging this into equation (5.1) gives theorem 1.1.

Remark 5.5. The same proof shows that, for any Borel measurable function $\phi : M \to \mathbb{R}$, we have:

$$\mathbb{E}\left[\int_{Z_f} \phi \, |\mathrm{d}V_f| \right] = \left(\frac{\lambda}{n+2} \right)^r \left(\int_M \phi \, |\mathrm{d}V_M| \right) \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)} + O(\lambda^{r-1}).$$

We still have to prove lemma 5.4. For this, we need to compute the variance of L_λ given $t_\lambda = 0$. Let $(x_1, \dots, x_n)$ be normal coordinates centered at x, and $(\zeta_1, \dots, \zeta_r)$ denote the canonical basis of $\mathbb{R}^r$. We equip $\mathbb{R}^r \otimes (\mathbb{R} \oplus T^*_x M)$ with the orthonormal basis:

$$(\zeta_1, \dots, \zeta_r, \zeta_1 \otimes dx_1, \dots, \zeta_1 \otimes dx_n, \zeta_2 \otimes dx_1, \dots, \zeta_2 \otimes dx_n, \dots, \zeta_r \otimes dx_1, \dots, \zeta_r \otimes dx_n). \tag{5.5}$$

Let $\Lambda(\lambda)$ denote the matrix of $\mathrm{Var}(t_\lambda, L_\lambda)$ in this basis. This matrix splits as

$$\Lambda(\lambda) = \begin{pmatrix} \Lambda_{00}(\lambda) & \Lambda_{01}(\lambda) \\ \Lambda_{10}(\lambda) & \Lambda_{11}(\lambda) \end{pmatrix}, \tag{5.6}$$

where $\Lambda_{00}(\lambda)$ and $\Lambda_{11}(\lambda)$ are the matrices of $\mathrm{Var}(t_\lambda)$ and $\mathrm{Var}(L_\lambda)$ and $\Lambda_{01}(\lambda) = {}^{\mathrm{t}}\Lambda_{10}(\lambda)$ is the matrix of $\mathrm{Cov}(t_\lambda, L_\lambda)$. We can decompose further Λ_{10} and Λ_{11} into blocks of size $r \times r$:

$$\Lambda_{10}(\lambda) = \left(\Lambda^i_{10}(\lambda)\right)_{1 \leqslant i \leqslant n}, \qquad \Lambda_{11}(\lambda) = \left(\Lambda^{i,j}_{11}(\lambda)\right)_{1 \leqslant i,j \leqslant n}. \tag{5.7}$$

By the definition (5.3) of (t_λ, L_λ), these blocks are obtained by scaling the corresponding blocks in the matrix of $\mathrm{Var}(j^1_x(f))$. Lemmas 2.4 and 2.6 tell us that each one of these blocks is a scalar matrix. Then, using the estimates of section 3.1:

$$\Lambda_{00}(\lambda) = \frac{e_\lambda(x,x)}{\gamma_0 \lambda^n} \, I_r = I_r + O(\lambda^{-1}), \tag{5.8}$$

$$\forall i \in \{1, \dots, n\}, \qquad \Lambda^i_{10}(\lambda) = \frac{\partial_{x_i} e_\lambda(x,x)}{\sqrt{\gamma_0 \gamma_1} \lambda^{n+1}} \, I_r = O(\lambda^{-1}), \tag{5.9}$$

(5.10)

$$\text{and, } \forall i,j \in \{1, \dots, n\}, \qquad \Lambda^{i,j}_{11}(\lambda) = \frac{\partial_{x_i} \partial_{y_j} e_\lambda(x,x)}{\gamma_1 \lambda^{n+2}} \, I_r = \begin{cases} I_r + O(\lambda^{-1}) & \text{if } i = j, \\ O(\lambda^{-1}) & \text{otherwise,} \end{cases}$$

where I_r stands for the identity matrix of size r. Thus $\Lambda(\lambda) = I_{r(n+1)} + O(\lambda^{-1})$ and, by corollary A.11, the distribution of L_λ conditionned on $t_\lambda = 0$ is a centered Gaussian with variance operator $\tilde{\Lambda}(\lambda) = \mathrm{Id} + O(\lambda^{-1})$. Note that these estimates do not depend on x or our choices of coordinates.

Proof of lemma 5.4. Let $\tilde{L}_\lambda \sim \mathcal{N}(0, \tilde{\Lambda}(\lambda))$ in $\mathbb{R}^r \otimes T^*_x M$. For λ large enough, $\tilde{\Lambda}(\lambda)$ is non-singular, and we have:

$$\begin{aligned}(5.11)\quad \mathbb{E}\Big[\big|\det{}^{\perp}\left(L_\lambda\right)\big| \;\Big|\; t_\lambda = 0\Big] &= \mathbb{E}\Big[\Big|\det{}^{\perp}\left(\tilde{L}_\lambda\right)\Big|\Big] \\ &= \frac{1}{\sqrt{2\pi}^{nr}\sqrt{\det(\tilde{\Lambda}(\lambda))}} \int \big|\det{}^{\perp}\left(L\right)\big| \exp\left(-\frac{1}{2}\left\langle \tilde{\Lambda}(\lambda)^{-1} L\,, L\right\rangle\right) \mathrm{d}L,\end{aligned}$$

where $\mathrm{d}L$ stands for the Lebesgue measure on $\mathbb{R}^r \otimes T^*_x M$. Beware that we see L as a linear map in the term $\big|\det{}^{\perp}\left(L\right)\big|$ but as a vector in $\tilde{\Lambda}(\lambda)^{-1}L$. The latter is not a composition.

Then $\tilde{\Lambda}(\lambda) = \mathrm{Id} + O(\lambda^{-1})$, so that $\left\|\tilde{\Lambda}(\lambda)^{-1} - \mathrm{Id}\right\|$ is bounded by $\frac{C}{\lambda}$ for some positive C. Hence, for all $L \in \mathbb{R}^r \otimes T^*_x M$,

$$\left|\left\langle \left(\tilde{\Lambda}(\lambda)^{-1} - \mathrm{Id}\right) L\,, L\right\rangle\right| \leqslant \frac{C}{\lambda}\left\|L\right\|^2,$$

and by the mean value theorem,

$$\left|\exp\left(-\frac{1}{2}\left\langle \left(\tilde{\Lambda}(\lambda)^{-1} - \mathrm{Id}\right) L\,, L\right\rangle\right) - 1\right| \leqslant \frac{C}{2\lambda}\left\|L\right\|^2 \exp\left(\frac{C}{2\lambda}\left\|L\right\|^2\right).$$

Then,

$$\begin{aligned}(5.12)\quad &\left|\int \big|\det{}^{\perp}\left(L\right)\big| \left(\exp\left(-\frac{1}{2}\left\langle \tilde{\Lambda}(\lambda)^{-1} L\,, L\right\rangle\right) - \exp\left(-\frac{\left\|L\right\|^2}{2}\right)\right) \mathrm{d}L\right| \\ &\qquad\qquad \leqslant \frac{C}{2\lambda}\int \big|\det{}^{\perp}\left(L\right)\big| \left\|L\right\|^2 \exp\left(-\frac{\left\|L\right\|^2}{2}\left(1 - \frac{C}{\lambda}\right)\right) \mathrm{d}L.\end{aligned}$$

The integral on the right-hand side of (5.12) converges to some finite limit as $\lambda \to +\infty$ by Lebesgue's dominated convergence theorem, so that:

$$(5.13)\quad \int \big|\det{}^{\perp}\left(L\right)\big| \exp\left(-\frac{1}{2}\left\langle \tilde{\Lambda}(\lambda)^{-1} L\,, L\right\rangle\right) \mathrm{d}L = \int \big|\det{}^{\perp}\left(L\right)\big| \, e^{-\frac{1}{2}\left\|L\right\|^2} \mathrm{d}L + O(\lambda^{-1}).$$

Since $\det\left(\tilde{\Lambda}(\lambda)\right) = 1 + O(\lambda^{-1})$, by (5.11), (5.13) we have:

$$\mathbb{E}\Big[\big|\det{}^{\perp}\left(L_\lambda\right)\big| \;\Big|\; t_\lambda = 0\Big] = \mathbb{E}\big[\big|\det{}^{\perp}\left(L\right)\big|\big] + O(\lambda^{-1}),$$

where L is a standard Gaussian vector in $\mathbb{R}^r \otimes T^*_x M$. The result of the lemma is given by lemma A.14. □

5.3 Proof of theorem 1.3

We now consider the real algebraic setting described in section 2.6. The proof goes along the same lines as above. Recall that $\mathcal{X}$ is a complex projective manifold of dimension n, equipped with a rank r holomorphic vector bundle $\mathcal{E}$ and an ample holomorphic line bundle $\mathcal{L}$, and that $\mathcal{X}$, $\mathcal{E}$ and $\mathcal{L}$ are endowed with compatible real structures. We are interested in the volume of the real zero set Z_s of a standard Gaussian section s in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$.

By corollary 3.10, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample for d large enough, so that we can apply Kac-Rice formula (theorem 5.3) with $\phi : (s, x) \mapsto 1$, as in the harmonic case. Note that we have to use the incidence manifold Σ_d defined by (2.17) here. As in (5.1), we get:

$$(5.14)\quad \mathbb{E}[\mathrm{Vol}\left(Z_s\right)] = \frac{1}{\sqrt{2\pi}^r}\int_{x \in \mathbb{R}\mathcal{X}} \frac{1}{\sqrt{\det(E_d(x,x))}} \mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s\right)\big| \;\Big|\; s(x) = 0\Big] \left|\mathrm{d}V_{\mathbb{R}\mathcal{X}}\right|,$$

where ∇^d is any real connection on $\mathcal{E}\otimes\mathcal{L}^d$.

Let $x\in\mathbb{R}\mathcal{X}$ and $(x_1,\dots,x_n)$ be real holomorphic coordinates around x such that $(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ is orthonormal at x. Let $(\varsigma_1^d,\dots,\varsigma_r^d)$ be an orthonormal basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$. This yields an orthonormal basis of $\mathcal{J}^2_x(\mathcal{E}\otimes\mathcal{L}^d)$ similar to (5.5).

The value of $\mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla^d_x s\right)\right| \;\middle|\; s(x)=0\right]$ does not depend on the choice of ∇^d, since $\nabla^d_x s$ does not depend on ∇^d when $s(x)=0$. We choose a connection that satisfies the conditions of section 3.3, in order to compute the pointwise asymptotic of this quantity.

The estimates of proposition 3.8 suggest to consider the scaled variables:

$$(t_d, L_d) = \left(\sqrt{\frac{\pi^n}{d^n}}s(x), \sqrt{\frac{\pi^n}{d^{n+1}}}\nabla^d_x s\right). \tag{5.15}$$

Then, (t_d, L_d) is a centered Gaussian vector in $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$, and the matrix of $\mathrm{Var}(t_d, L_d)$ in the basis described above is $I_{r(n+1)}+O(d^{-1})$. This is proved by the same kind of computation as in the harmonic case, using the estimates of proposition 3.8. The distribution of L_d given $t_d=0$ is then a centered Gaussian with variance operator $\tilde{\Lambda}(d)=\mathrm{Id}+O(d^{-1})$.

As in the previous section (cf. lemma 5.4), for every $x\in\mathbb{R}\mathcal{X}$,

$$\begin{aligned}\mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla^d_x s\right)\right| \;\middle|\; s(x)=0\right] &= \sqrt{\frac{d^{n+1}}{\pi^n}}^r\,\mathbb{E}\left[\left|\det{}^{\perp}\left(L_d\right)\right| \;\middle|\; t_d=0\right]\\ &= \sqrt{\frac{d^{n+1}}{\pi^n}}^r\,\sqrt{2\pi}^r\,\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}(1+O(d^{-1})).\end{aligned} \tag{5.16}$$

Besides, the estimate (3.13) shows that:

$$\det\left(E_d(x,x)\right) = \frac{d^{rn}}{\pi^{rn}}(1+O(d^{-1})), \tag{5.17}$$

Finally, by (5.14), (5.16) and (5.17), we have proved theorem 1.3.

As in the harmonic case, the same proof shows that for any Borel measurable function $\phi:\mathbb{R}\mathcal{X}\to\mathbb{R}$, we have:

$$\mathbb{E}\left[\int_{Z_s}\phi\,|\mathrm{d}V_s|\right] = \left(\sqrt{d}\right)^r\left(\int_{\mathbb{R}\mathcal{X}}\phi\,|\mathrm{d}V_{\mathbb{R}\mathcal{X}}|\right)\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^n\right)}+O\left(\sqrt{d}^{r-2}\right).$$

5.4 Proof of theorem 1.2

In this section we compute the expected Euler characteristic of our random submanifolds. The proof is basically the same as in the volume case: apply Kac-Rice formula then compute a pointwise asymptotic for the conditional expectation that appears in 5.3. Only, this time, we apply Kac-Rice formula to a quantity $\phi(f,x)$ that really depends on the couple $(f,x)\in\Sigma$. This makes the computations a bit more complex. Luckily, $\phi(f,x)$ only depends on the 2-jet of f at x, so we can still make pointwise computations.

Consider first the general setting of sections 2.1 to 2.4: f is a standard Gaussian vector in the finite-dimensional subspace $V\subset\mathcal{C}^\infty(M,\mathbb{R}^r)$. We assume that V is 0-ample, so that for almost every $f\in V$, Z_f is a closed submanifold of dimension $n-r$.

If $n-r$ is odd, then $\chi(Z_f)=0$ almost surely (see remark 4.4). From now on, we assume that $n-r$ is even and set $m=\frac{n-r}{2}$. If $n=r$, then Z_f is almost surely a finite set, and $\chi(Z_f)=\mathrm{Vol}\left(Z_f\right)$ is just the cardinal of Z_f. In this case theorems 1.2 and 1.1 coincide, so we need only consider the case $r<n$ in the sequel.

We denote by R the Riemann curvature of the ambient manifold M and, for any $f \in V$, we denote by R_f the Riemann curvature of Z_f. By proposition 4.9 and Kac-Rice formula 5.3,

$$\begin{aligned}(5.18)\quad \mathbb{E}[\chi(Z_f)] &= \mathbb{E}\left[\frac{1}{(2\pi)^m m!}\int_{Z_f} \mathrm{Tr}\left((R_f)^{\wedge m}\right)|\mathrm{d}V_f|\right] \\ &= \frac{1}{m!\sqrt{2\pi}^n}\int_{x\in M}\frac{1}{\sqrt{\det(E(x,x))}}\mathbb{E}\Big[\left|\det{}^{\perp}\left(d_x f\right)\right|\mathrm{Tr}(R_f(x)^{\wedge m})\;\Big|\; f(x)=0\Big]\,|\mathrm{d}V_M|\,.\end{aligned}$$

Moreover, for all $x \in M$, let $U \sim \mathcal{N}(0, \mathrm{Id})$ in T_xM be independent of f. Then, proposition 4.9 gives:

$$(5.19)\qquad R_f(x) = R(x) + \frac{1}{2}\mathbb{E}_U\Big[\left\langle \nabla_x^2 f\,, d_x f^{\dagger *}(U)\right\rangle^{\wedge 2}\Big]\,,$$

where, the notation $\mathbb{E}_U[\,\cdot\,]$ means that we only take the expectation with respect to the variable U. Here and in everything that follows, $R(x)$ and $\nabla_x^2 f$ are implicitly restricted to $\ker(d_x f) = T_x Z_f$.

As in the volume case, the next step is to compute the pointwise asymptotic for the integrand in the last term of (5.18). By (5.19), it only depends on $R(x)$ and the distribution of $j_x^2(f)$ which is characterized by lemma 2.4. This shows that the expected Euler characteristic only depends on R and the values of E and its derivatives (up to order 2 in each variable) along the diagonal in $M \times M$. It turns out that, both in the harmonic and the algebraic cases, the pointwise asymptotic is universal and no longer depends on R.

Focusing on the harmonic case $V = (V_\lambda)^r$, we already know from lemma 2.7 that $(V_\lambda)^r$ is 0-ample. Let $x \in M$, recall that $\det(E_\lambda(x,x)) = (\gamma_0\lambda^n)^r\left(1 + O(\lambda^{-1})\right)$ (see (5.2)). Then, the main task is to estimate the conditional expectation in (5.18).

We consider the scaled variables:

$$(5.20)\qquad (t_\lambda, L_\lambda, S_\lambda) = \left(\frac{1}{\sqrt{\gamma_0\lambda^n}}f(x), \frac{1}{\sqrt{\gamma_1\lambda^{n+2}}}d_x f, \frac{1}{\sqrt{\gamma_2\lambda^{n+4}}}\nabla_x^2 f\right)$$

in $\mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^*M \oplus \mathrm{Sym}(T_x^*M))$. By lemma 2.4, $(t_\lambda, L_\lambda, S_\lambda)$ is a centered Gaussian vector. We denote by $(\tilde{L}_\lambda, \tilde{S}_\lambda)$ a random variable in $\mathbb{R}^r \otimes (T_x^*M \oplus \mathrm{Sym}(T_x^*M))$ distributed as (L_λ, S_λ) given $t_\lambda = 0$. Again, $(\tilde{L}_\lambda, \tilde{S}_\lambda)$ is a centered Gaussian, see A.11.

Let $U \sim \mathcal{N}(0, \mathrm{Id})$ in T_xM be independent of f (hence of $j_x^2(f)$) and $(\tilde{L}_\lambda, \tilde{S}_\lambda)$. Then, by (5.19), (5.20) and (3.7):

$$\begin{aligned}R_f(x) &= R(x) + \frac{1}{2}\mathbb{E}_U\Big[\left\langle \nabla_x^2 f\,, d_x f^{\dagger *}(U)\right\rangle^{\wedge 2}\Big] = R(x) + \frac{1}{2}\mathbb{E}_U\left[\left\langle \sqrt{\gamma_2\lambda^{n+4}}S_\lambda\,, \frac{L_\lambda^{\dagger *}(U)}{\sqrt{\gamma_1\lambda^{n+2}}}\right\rangle^{\wedge 2}\right] \\ &= R(x) + \frac{\lambda^2}{2(n+4)}\mathbb{E}_U\left[\left\langle S_\lambda\,, L_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right].\end{aligned}$$

Besides, $\left|\det{}^{\perp}\left(d_x f\right)\right| = \sqrt{\gamma_1\lambda^{n+2}}^{\,r}\left|\det{}^{\perp}\left(L_\lambda\right)\right|$, so that,

$$\begin{aligned}(5.21)\quad &\mathbb{E}\Big[\left|\det{}^{\perp}\left(d_x f\right)\right|\mathrm{Tr}(R_f(x)^{\wedge m})\;\Big|\; f(x)=0\Big] \\ &= \sqrt{\gamma_1\lambda^{n+2}}^{\,r}\,\mathbb{E}\left[\left|\det{}^{\perp}\left(L_\lambda\right)\right|\mathrm{Tr}\left(\left(R(x) + \frac{\lambda^2}{2(n+4)}\mathbb{E}_U\left[\left\langle S_\lambda\,, L_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\;\middle|\; t_\lambda = 0\right] \\ &= \sqrt{\gamma_1\lambda^{n+2}}^{\,r}\,\mathbb{E}\left[\left|\det{}^{\perp}\left(\tilde{L}_\lambda\right)\right|\mathrm{Tr}\left(\left(R(x) + \frac{\lambda^2}{2(n+4)}\mathbb{E}_U\left[\left\langle \tilde{S}_\lambda\,, \tilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right].\end{aligned}$$

To conclude the proof, we will use the following lemmas.

Lemma 5.6. *The random vectors* $\left(\tilde{L}_\lambda, \tilde{S}_\lambda\right)$ *converge in distribution to a Gaussian vector* (L,S) *as* $\lambda \to +\infty$. *Let* $U \sim \mathcal{N}(0, \mathrm{Id})$ *in* T_xM *be independent of* $\left(\tilde{L}_\lambda, \tilde{S}_\lambda\right)$ *and* (L,S), *then:*

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(\tilde{L}_\lambda\right)\right| \operatorname{Tr}\left(\left(R(x) + \frac{\lambda^2}{2(n+4)}\mathbb{E}_U\left[\left\langle \tilde{S}_\lambda\,, \tilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right]$$
$$= \left(\frac{\lambda^2}{(n+4)}\right)^m \frac{m!}{(2m)!}\mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(\left\langle S\,, L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right](1+O(\lambda^{-1})),$$

where the error term is uniform in $x \in M$.

Lemma 5.7. *Let* (L,S) *be distributed as the limit of* $\left(\tilde{L}_\lambda, \tilde{S}_\lambda\right)$ *and* $U \sim \mathcal{N}(0, \mathrm{Id})$ *in* T_xM *be independent of* (L,S). *We have:*

$$\mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(\left\langle S\,, L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right] = \left(-\frac{n+4}{n+2}\right)^m \sqrt{2\pi}^n (2m)! \frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \operatorname{Vol}\left(\mathbb{S}^n\right)\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}.$$

Assuming these lemmas and recalling (5.2) and (5.21), the integrand in the last term of (5.18) equals:

$$(-1)^m m! \sqrt{2\pi}^n \lambda^{r+2m} \sqrt{\frac{\gamma_1}{\gamma_0}}^r \left(\frac{1}{n+2}\right)^m \frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \operatorname{Vol}\left(\mathbb{S}^n\right)\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)}(1+O(\lambda^{-1})),$$

where the error term is uniform in $x \in M$. Since $r+2m = n$ and $\frac{\gamma_0}{\gamma_1} = n+2$, we finally get:

$$\mathbb{E}[\chi(Z_f)] = (-1)^m \left(\frac{\lambda}{\sqrt{n+2}}\right)^n \frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi \operatorname{Vol}\left(\mathbb{S}^n\right)\operatorname{Vol}\left(\mathbb{S}^{n-1}\right)} \int_{x\in M} (1+O(\lambda^{-1}))\left|\mathrm{d}V_M\right|,$$

and this is theorem 1.2.

We now have to prove lemmas 5.6 and 5.7. For this we will need the following technical result which is a reformulation of [10, prop 3.12]. The proof of 5.8 is mostly tedious computations and we postpone it until appendix B.

Proposition 5.8. *Let* V *and* V' *be two Euclidean spaces of dimension* n *and* r *respectively, with* $1 \leqslant r \leqslant n$. *Let* $L \in V' \otimes V^*$ *and* $U \in V$ *be independent standard Gaussian vectors. Then,* $L^\dagger$ *is well-defined almost surely and* $\left(\left|\det{}^{\perp}(L)\right|, (L^\dagger)^* U\right)$ *has the same distribution as*

$$\left(\|X_n\|\,\|X_{n-1}\|\cdots\|X_{n-r+1}\|, \frac{U'}{\|X_{n-r+1}\|}\right),$$

where $U' \in V'$, $X_p \in \mathbb{R}^p$ *for all* $p \in \{n-r+1, \ldots, n\}$ *and* $U', X_n, \ldots, X_{n-r+1}$ *are globally independent standard Gaussian vectors.*

We start by computing the variance of $(t_\lambda, L_\lambda, S_\lambda)$. As in 5.2, we choose $(x_1, \ldots, x_n)$ normal coordinates centered at x and denote by $(\zeta_1, \ldots, \zeta_r)$ the canonical basis of $\mathbb{R}^r$. For any i and j such that $1 \leqslant i < j \leqslant n$, we set $dx_{ii} = dx_i \otimes dx_i$ and $dx_{ij} = (dx_i \otimes dx_j + dx_j \otimes dx_i)$. We complete the basis of $\mathcal{J}^1_x(\mathbb{R}^r)$ given in (5.5) into an orthonormal basis of $\mathcal{J}^2_x(\mathbb{R}^r)$ by adding the following elements (in this order) at the end of the list:

$$\begin{aligned}(5.22)\quad &\zeta_1 \otimes dx_{11}, \ldots, \zeta_r \otimes dx_{11}, \zeta_1 \otimes dx_{22}, \ldots, \zeta_r \otimes dx_{22}, \ldots, \zeta_1 \otimes dx_{nn}, \ldots, \zeta_r \otimes dx_{nn},\\ &\zeta_1 \otimes dx_{12}, \ldots, \zeta_r \otimes dx_{12}, \zeta_1 \otimes dx_{13}, \ldots, \zeta_r \otimes dx_{13}, \ldots, \zeta_1 \otimes dx_{1n}, \ldots, \zeta_r \otimes dx_{1n},\\ &\qquad\qquad \ldots, \zeta_1 \otimes dx_{(n-1)n}, \ldots, \zeta_r \otimes dx_{(n-1)n}.\end{aligned}$$

The matrix of $\mathrm{Var}(t_\lambda, L_\lambda, S_\lambda)$ with respect to this basis is:

$$\Lambda(\lambda) = \begin{pmatrix} \Lambda_{00}(\lambda) & \Lambda_{01}(\lambda) & \Lambda_{02}(\lambda) \\ \Lambda_{10}(\lambda) & \Lambda_{11}(\lambda) & \Lambda_{12}(\lambda) \\ \Lambda_{20}(\lambda) & \Lambda_{21}(\lambda) & \Lambda_{22}(\lambda) \end{pmatrix},$$

where $\Lambda_{00}(\lambda)$, $\Lambda_{10}(\lambda)$ and $\Lambda_{11}(\lambda)$ are as in (5.6) and, similarly, $\Lambda_{22}(\lambda)$ is the matrix of $\mathrm{Var}(S_\lambda)$, $\Lambda_{02}(\lambda) = {}^{\mathrm{t}}\Lambda_{20}(\lambda)$ is the matrix of $\mathrm{Cov}(t_\lambda, S_\lambda)$ and $\Lambda_{12}(\lambda) = {}^{\mathrm{t}}\Lambda_{21}(\lambda)$ is the matrix of $\mathrm{Cov}(L_\lambda, S_\lambda)$. As in (5.7), we can decompose further each of these matrices in blocks of size $r \times r$. That is, $\Lambda_{10}(\lambda)$ and $\Lambda_{11}(\lambda)$ satisfy (5.7) and,

$$\begin{aligned} \Lambda_{20}(\lambda) = \big(\Lambda_{20}^{ik}(\lambda)\big)_{1\leqslant i\leqslant k\leqslant n}, \quad & \Lambda_{21}(\lambda) = \big(\Lambda_{21}^{ik,j}(\lambda)\big)_{\substack{1\leqslant i\leqslant k\leqslant n \\ 1\leqslant j\leqslant n}} \\ \text{and} \qquad & \Lambda_{22}(\lambda) = \big(\Lambda_{22}^{ik,jl}(\lambda)\big)_{\substack{1\leqslant i\leqslant k\leqslant n \\ 1\leqslant j\leqslant l\leqslant n}}. \end{aligned} \tag{5.23}$$

By definition of $(t_\lambda, L_\lambda, S_\lambda)$, these blocks are obtained by scaling the corresponding blocks in the matrix of $\mathrm{Var}(j_x^2(f))$. By lemmas 2.4 and 2.6 the matrices in (5.23) are scalar matrices. Recalling (3.7), we set: $\gamma = -\sqrt{\frac{\gamma_1^2}{\gamma_0\gamma_2}} = -\sqrt{\frac{n+4}{n+2}}$. Then, by (5.20) and theorem 3.1 we have:

$$\Lambda_{20}^{ik}(\lambda) = \frac{\partial_{x_i,x_k} e_\lambda(x,x)}{\sqrt{\gamma_0\gamma_2}\lambda^{n+2}} I_r = \begin{cases} \gamma I_r + O(\lambda^{-1}) & \text{if } i = k, \\ O(\lambda^{-1}) & \text{if } i \neq k, \end{cases} \tag{5.24}$$

$$\Lambda_{21}^{ik,j}(\lambda) = \frac{\partial_{x_i,x_k}\partial_{y_j} e_\lambda(x,x)}{\sqrt{\gamma_1\gamma_2}\lambda^{n+3}} I_r = O(\lambda^{-1}), \tag{5.25}$$

$$\Lambda_{22}^{ik,jl}(\lambda) = \frac{\partial_{x_i,x_k}\partial_{y_j,y_l} e_\lambda(x,x)}{\gamma_2\lambda^{n+4}} I_r = \begin{cases} 3I_r + O(\lambda^{-1}) & \text{if } i = j = k = l, \\ I_r + O(\lambda^{-1}) & \text{if } i = j \neq k = l \\ & \text{or } i = k \neq j = l, \\ O(\lambda^{-1}) & \text{otherwise,} \end{cases} \tag{5.26}$$

where I_r denotes the identity matrix of size r. Similar estimates for $\Lambda_{00}(\lambda)$, $\Lambda_{10}(\lambda)$ and $\Lambda_{11}(\lambda)$ are given by (5.8), (5.9) and (5.10) respectively. Then $\Lambda(\lambda)$ writes by blocks:

$$\Lambda(\lambda) = \left(\begin{array}{c|c|cccc|c} I_r & & \gamma I_r & \gamma I_r & \cdots & \gamma I_r & \\ \hline & I_{nr} & & & & & \\ \hline \gamma I_r & & 3I_r & I_r & \cdots & I_r & \\ \gamma I_r & & I_r & 3I_r & \ddots & \vdots & \\ \vdots & & \vdots & \ddots & \ddots & I_r & \\ \gamma I_r & & I_r & \cdots & I_r & 3I_r & \\ \hline & & & & & & I_{r\frac{n(n-1)}{2}} \end{array}\right) + O(\lambda^{-1}),$$

where the empty blocks are zeros. The distribution of $(\tilde{L}_\lambda, \tilde{S}_\lambda)$, that is the distribution of (L_λ, S_λ) given $t_\lambda = 0$, is a centered Gaussian whose variance matrix is:

$$\tilde{\Lambda}(\lambda) = \left(\begin{array}{c|cccc|c} I_{nr} & & & & & \\ \hline & \beta_0 I_r & \beta I_r & \cdots & \beta I_r & \\ & \beta I_r & \beta_0 I_r & \ddots & \vdots & \\ & \vdots & \ddots & \ddots & \beta I_r & \\ & \beta I_r & \cdots & \beta I_r & \beta_0 I_r & \\ \hline & & & & & I_{r\frac{n(n-1)}{2}} \end{array}\right) + O(\lambda^{-1}), \tag{5.27}$$

where $\beta = 1 - \gamma^2 = -\dfrac{2}{n+2}$ and $\beta_0 = 3 - \gamma^2 = \dfrac{2n+2}{n+2}$ (see corollary A.11).

Let Λ denote the leading term in (5.27). Equation (5.27) shows that the random vectors $\left(\tilde{L}_\lambda, \tilde{S}_\lambda\right)$ converge in distribution to a random vector $(L, S) \sim \mathcal{N}(0, \Lambda)$ (see lemma A.12). We have:

$$\det(\Lambda) = (\beta_0 + (n-1)\beta)^r (\beta_0 - \beta)^{r(n-1)} = \left(\frac{4(2n+4)^{n-1}}{(n+2)^n}\right)^r,$$

so that Λ is non-singular, and $\tilde{\Lambda}(\lambda)$ is non-singular for λ large enough.

Proof of lemma 5.6. We have already seen that $\left(\tilde{L}_\lambda, \tilde{S}_\lambda\right)$ converges in distribution, as λ goes to infinity, to $(L, S) \sim \mathcal{N}(0, \Lambda)$. We still have to prove the estimate in the lemma. We have:

$$\left(R(x) + \frac{\lambda^2}{2(n+4)}\mathbb{E}_U\left[\left\langle \tilde{S}_\lambda, \tilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m} = \sum_{q=0}^{m} \binom{m}{q} \left(\frac{\lambda^2}{2(n+4)}\right)^q R(x)^{\wedge(m-q)} \wedge \mathbb{E}_U\left[\left\langle \tilde{S}_\lambda, \tilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]^{\wedge q}.$$

We can apply lemma 4.2 to each term in this sum. This yields:

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(\tilde{L}_\lambda\right)\right| \operatorname{Tr}\left(\left(R(x) + \frac{\lambda^2}{2(n+4)}\mathbb{E}_U\left[\left\langle \tilde{S}_\lambda, \tilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right] = \sum_{q=0}^{m} \binom{m}{q} \frac{q!}{(2q)!} \left(\frac{\lambda^2}{2(n+4)}\right)^q \mathbb{E}\left[\left|\det{}^{\perp}\left(\tilde{L}_\lambda\right)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle \tilde{S}_\lambda, \tilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right]. \tag{5.28}$$

Then it is sufficient to show that, for all $q \in \{0, \ldots, m\}$,

$$\mathbb{E}\left[\left|\det{}^{\perp}\left(\tilde{L}_\lambda\right)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle \tilde{S}_\lambda, \tilde{L}_\lambda^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right] = \mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle S, L^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right] + O(\lambda^{-1}), \tag{5.29}$$

and that these terms are finite. Then (5.28) and (5.29) yield the estimate in the lemma.

Let $q \in \{0, \ldots, m\}$, we first show that the principal part on the right-hand side of (5.29) is finite. Let $\zeta \in \mathbb{R}^r$,

$$\mathbb{E}_S\left[\operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \langle S, \zeta\rangle^{\wedge 2q}\right)\right] \tag{5.30}$$

is finite since it is the expectation of some polynomial in the coefficients of S. Thus, (5.30) only depends on ζ, and it is an homogeneous polynomial in ζ of degree $2q$.

We assumed U to be independent of (L, S), and the expression (5.27) of Λ shows that L and S are independent. Let $U' \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}^r$ and $X_{n-r+1}, \ldots, X_n$ be standard Gaussian vectors, with $X_p \in \mathbb{R}^p$, such that $U', S, X_{n-r+1}, \ldots, X_n$ are globally independent. Applying proposition 5.8, we have:

$$\begin{aligned}\mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \left\langle S, L^{\dagger *}(U)\right\rangle^{\wedge 2q}\right)\right] &= \mathbb{E}\left[\frac{\|X_n\| \cdots \|X_{n-r+2}\|}{\|X_{n-r+1}\|^{2q-1}} \operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \langle S, U'\rangle^{\wedge 2q}\right)\right] \\ &= \mathbb{E}\left[\frac{1}{\|X_{n-r+1}\|^{2q-1}}\right] \mathbb{E}\left[\operatorname{Tr}\left(R(x)^{\wedge(m-q)} \wedge \langle S, U'\rangle^{\wedge 2q}\right)\right] \prod_{p=n-r+2}^{n} \mathbb{E}[\|X_p\|].\end{aligned} \tag{5.31}$$

Since $2q-1 \leqslant 2m-1 \leqslant n-r-1$ and X_{n-r+1} is a standard Gaussian in $\mathbb{R}^{n-r+1}$, $\mathbb{E}\left[\frac{1}{\|X_{n-r+1}\|^{2q-1}}\right] < +\infty$ (see lemma A.13). The other factors on the right-hand side of (5.31) are expectations of polynomials is some standard Gaussian variables, so they are finite.

Then, $\tilde{\Lambda}(\lambda) = \Lambda + O(\lambda^{-1})$, and the same kind of computations as in the proof of lemma 5.4 gives (5.29), which concludes the proof of lemma 5.6. Note that, since M is compact, $R(x)$ is bounded, independently of x. We need this fact to ensure that the error term in (5.29) is independent of x. □

Proof of lemma 5.7. Let $(L,S) \sim \mathcal{N}(0,\Lambda)$ in $\mathbb{R}^r \otimes (T_x^*M \oplus \mathrm{Sym}(T_x^*M))$ and $U \sim \mathcal{N}(0,\mathrm{Id})$ in T_xM be independent of (L,S). By (5.31):

(5.32)
$$\mathbb{E}\left[\left|\det{}^{\perp}(L)\right| \operatorname{Tr}\left(\left\langle S\,, L^{\dagger *}(U)\right\rangle^{\wedge 2m}\right)\right] = \mathbb{E}\left[\frac{\|X_{n-r+2}\| \dots \|X_n\|}{\|X_{n-r+1}\|^{2m-1}}\right] \mathbb{E}\left[\operatorname{Tr}\left(\left\langle S\,, U'\right\rangle^{\wedge 2m}\right)\right],$$

where $U' \sim \mathcal{N}(0,\mathrm{Id})$ in $\mathbb{R}^r$, $X_p \sim \mathcal{N}(0,\mathrm{Id})$ in $\mathbb{R}^p$ for all p, and $U', S, X_{n-r+1}, \dots, X_n$ are globally independent.

Recall that we are only interested in the restriction of S to $\ker(L) \subset T_xM$. But L and S are independent, as one can see on the expression of Λ (5.27), and the distribution of S is invariant under orthogonal transformations of T_xM. Thus, we can consider S restricted to any $2m$-dimensional subspace of T_xM in our computations. For simplicity, we restrict S to V, the span of $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}}\right)$.

We now compute the term $\mathbb{E}\left[\operatorname{Tr}\left(\left\langle S_{/V}\,, U'\right\rangle^{\wedge 2m}\right)\right]$. By lemma 4.2,

$$\mathbb{E}_S\left[\left\langle S_{/V}\,, U'\right\rangle^{\wedge 2m}\right] = \frac{(2m)!}{2^m m!} \mathbb{E}_S\left[\left\langle S_{/V}\,, U'\right\rangle^{\wedge 2}\right]^{\wedge m}.$$

Assuming that $S_{/V} = \sum_{1 \leqslant i,k \leqslant 2m} S_{ik} dx_i \otimes dx_k$, with $S_{ik} \in \mathbb{R}^r$, we have:

$$\left\langle S_{/V}\,, U'\right\rangle = \sum_{1 \leqslant i,k \leqslant 2m} \left\langle S_{ik}\,, U'\right\rangle dx_i \otimes dx_k.$$

By lemma A.8,

$$\begin{aligned}\mathbb{E}_S\left[\left\langle S_{/V}\,, U'\right\rangle^{\wedge 2}\right] &= \sum_{1 \leqslant i,j,k,l \leqslant 2m} \mathbb{E}_S[\left\langle S_{ik}\,, U'\right\rangle \left\langle S_{jl}\,, U'\right\rangle] (dx_i \wedge dx_j) \otimes (dx_k \wedge dx_l) \\ &= \sum_{1 \leqslant i,j,k,l \leqslant 2m} \left\langle U'\,, \left(\Lambda^{ik,jl} U'\right)\right\rangle (dx_i \wedge dx_j) \otimes (dx_k \wedge dx_l),\end{aligned}$$

where we denoted by $\Lambda^{ik,jl}$ the covariance operator of S_{ik} and S_{jl}. Then,

$$\begin{aligned}\mathbb{E}_S\left[\left\langle S\,, U'\right\rangle^{\wedge 2m}\right] = \frac{(2m)!}{2^m m!} \sum_{\substack{1 \leqslant i_1,\dots,i_m \leqslant 2m \\ 1 \leqslant j_1,\dots,j_m \leqslant 2m \\ 1 \leqslant k_1,\dots,k_m \leqslant 2m \\ 1 \leqslant l_1,\dots,l_m \leqslant 2m}} &\left(\prod_{p=1}^m \left\langle U'\,, \left(\Lambda^{i_p k_p, j_p l_p}\right) U'\right\rangle\right) \times \\ &(dx_{i_1} \wedge dx_{j_1} \wedge \dots \wedge dx_{i_m} \wedge dx_{j_m}) \otimes (dx_{k_1} \wedge dx_{l_1} \wedge \dots \wedge dx_{k_m} \wedge dx_{l_m}) \\ = \frac{(2m)!}{2^m m!} \sum_{\sigma,\sigma' \in \mathfrak{S}_{2m}} \varepsilon(\sigma)\varepsilon(\sigma') &\prod_{p=1}^m \left\langle U'\,, \left(\Lambda^{\sigma(2p-1)\sigma'(2p-1), \sigma(2p)\sigma'(2p)}\right) U'\right\rangle (dx \otimes dx),\end{aligned}$$

where $\mathfrak{S}_{2m}$ is the set of permutations of $\{1,\dots,2m\}$, $\varepsilon : \mathfrak{S}_{2m} \to \{-1,1\}$ denotes the signature morphism and $dx = dx_1 \wedge \cdots \wedge dx_{2m}$. We get the last line by setting $\sigma(2p-1) = i_p$, $\sigma(2p) = j_p$, $\sigma'(2p-1) = k_p$ and $\sigma'(2p) = l_p$ and reordering the wedge products.

Since our local coordinates are such that $\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{2m}}\right)$ is orthonormal at x, we have $\mathrm{Tr}(dx \otimes dx) = \|dx\|^2 = 1$. Thus,

(5.33)
$$\mathrm{Tr}\left(\mathbb{E}_S\left[\langle S\,, U'\rangle^{\wedge 2m}\right]\right) = \frac{(2m)!}{2^m m!}\sum_{\sigma,\sigma'\in\mathfrak{S}_{2m}} \varepsilon(\sigma\sigma')\prod_{p=1}^{m}\left\langle U'\,,\left(\Lambda^{\sigma(2p-1)\sigma'(2p-1),\sigma(2p)\sigma'(2p)}\right)U'\right\rangle.$$

By equation (5.27), for any σ, $\sigma' \in \mathfrak{S}_{2m}$ and for any $p \in \{1,\dots,m\}$,

$$\Lambda^{\sigma(2p-1)\sigma'(2p-1),\sigma(2p)\sigma'(2p)} = K(p,\sigma,\sigma')\,\mathrm{Id},$$

where,

$$K(p,\sigma,\sigma') = \begin{cases} \beta & \text{if } \sigma(2p-1) = \sigma'(2p-1) \text{ and } \sigma(2p) = \sigma'(2p),\\ 1 & \text{if } \sigma(2p-1) = \sigma'(2p) \text{ and } \sigma(2p) = \sigma'(2p-1),\\ 0 & \text{otherwise.}\end{cases} \tag{5.34}$$

Note that $K(p,\sigma,\sigma') = K(p,\mathrm{id},\sigma^{-1}\circ\sigma')$, where id stands for the identity permutation. Then, setting $\tau = \sigma^{-1}\circ\sigma'$,

$$\begin{aligned}\sum_{\sigma,\sigma'\in\mathfrak{S}_{2m}} \varepsilon(\sigma\sigma')\prod_{p=1}^{m}\left\langle U'\,,\left(\tilde{\Lambda}^{\sigma(2p-1)\sigma'(2p-1),\sigma(2p)\sigma'(2p)}\right)U'\right\rangle &\\ = \sum_{\sigma,\sigma'\in\mathfrak{S}_{2m}} \varepsilon(\sigma\sigma')\,\|U'\|^{2m}\prod_{p=1}^{m}K(p,\sigma,\sigma') &\\ = (2m)!\,\|U'\|^{2m}\sum_{\tau\in\mathfrak{S}_{2m}}\varepsilon(\tau)\prod_{p=1}^{m}K(p,\mathrm{id},\tau). &\end{aligned} \tag{5.35}$$

From the definition (5.34) of $K(p,\mathrm{id},\tau)$, we get that $\prod_{p=1}^{m}K(p,\mathrm{id},\tau) \neq 0$ if and only if τ is a product of transpositions of the type $((2p-1)\ (2p))$. Now, if $I \subset \{1,\dots,m\}$ and $\tau = \prod_{p\in I}((2p-1)\ (2p))$, we have:

$$\prod_{p=1}^{m}K(p,\mathrm{id},\tau) = \beta^{m-|I|} \qquad \text{and} \qquad \varepsilon(\tau) = (-1)^{|I|},$$

where $|I|$ stands for the cardinal of I. Thus,

$$\begin{aligned}\sum_{\tau\in\mathfrak{S}_{2m}}\varepsilon(\tau)\prod_{p=1}^{m}K(p,\mathrm{id},\tau) = \sum_{I\subset\{1,\dots,m\}}(-1)^{|I|}\beta^{m-|I|} = \sum_{p=1}^{m}\binom{m}{p}(-1)^p\beta^{m-p} &\\ = (\beta-1)^m = (-1)^m\left(\frac{n+4}{n+2}\right)^m. &\end{aligned} \tag{5.36}$$

Finally, by equations (5.33), (5.35) and (5.36),

$$\mathbb{E}\left[\mathrm{Tr}\left(\langle S\,, U'\rangle^{\wedge 2m}\right)\right] = (-1)^m\left(\frac{n+4}{n+2}\right)^m\frac{((2m)!)^2}{2^m m!}\mathbb{E}\left[\|U'\|^{2m}\right]. \tag{5.37}$$

Then, by (5.32), (5.37) and lemma A.13,

$$\mathbb{E}\Big[\big|\det{}^{\perp}(L)\big|\operatorname{Tr}\Big(\big\langle S\,,L^{\dagger *}(U)\big\rangle^{\wedge 2m}\Big)\Big] = (-1)^m\left(\frac{n+4}{n+2}\right)^m\frac{((2m)!)^2}{2^m m!}\sqrt{2\pi}^r\frac{\operatorname{Vol}(\mathbb{S}^{n-r})}{\operatorname{Vol}(\mathbb{S}^1)}\frac{\operatorname{Vol}\left(\mathbb{S}^{n-r+1}\right)}{\operatorname{Vol}(\mathbb{S}^n)}\frac{\operatorname{Vol}\left(\mathbb{S}^{r-1}\right)}{\operatorname{Vol}(\mathbb{S}^{n-1})}.$$

We conclude the proof of lemma 5.7 by computing:

$$\sqrt{2\pi}^r\frac{(2m)!}{2^m m!}\frac{\operatorname{Vol}\mathbb{S}^{n-r}}{2} = \sqrt{2\pi}^r\frac{2^m\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{\pi}}\frac{\sqrt{\pi}^{2m+1}}{\Gamma\left(m+\frac{1}{2}\right)} = \sqrt{2\pi}^n. \qquad \square$$

5.5 Proof of theorem 1.4

Let us now adapt the proof of the previous section to the case of real algebraic submanifolds. Once again, we need only consider the case where $r < n$ and $n-r$ is even. The framework is the same as in sections 2.6 and 5.3.

We already know that $\mathbb{R}H^0(\mathcal{X},\mathcal{E}\otimes\mathcal{L}^d)$ is 0-ample by corollary 3.10. We also have the estimate (5.17) for $\det(E_d)$ along the diagonal, where E_d is the Bergman kernel of $\mathcal{E}\otimes\mathcal{L}^d$. As in the harmonic case, we use (4.8) and Kac-Rice formula:

$$\mathbb{E}[\chi(Z_s)] = \frac{1}{m!\sqrt{2\pi}^n}\int_{x\in\mathbb{R}\mathcal{X}}\frac{\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s\right)\big|\operatorname{Tr}\left(R_s(x)^{\wedge m}\right)\;\Big|\;s(x)=0\Big]}{\sqrt{\det(E_d(x,x))}}\left|\mathrm{d}V_{\mathbb{R}\mathcal{X}}\right|, \tag{5.38}$$

where ∇^d is any real connection on $(\mathcal{E}\otimes\mathcal{L}^d)$ and R_s denotes the Riemann tensor of Z_s.

We need to compute the conditional expectation in (5.38) for some fixed $x\in\mathbb{R}\mathcal{X}$. Since it does not depend on our choice of connection, we will use one that is adapted to x as in sections 3.3 and 5.3. Let $x\in\mathbb{R}\mathcal{X}$, we consider the scaled variables:

$$(t_d, L_d, S_d) = \left(\sqrt{\frac{\pi^n}{d^n}}s(x), \sqrt{\frac{\pi^n}{d^{n+1}}}\nabla^d_x s, \sqrt{\frac{\pi^n}{d^{n+2}}}\nabla^{2,d}_x s\right), \tag{5.39}$$

and $(\tilde{L}_d,\tilde{S}_d)\in\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes(T^*_x\mathbb{R}\mathcal{X}\oplus\operatorname{Sym}(T^*_x\mathbb{R}\mathcal{X}))$ distributed as (L_d,S_d) given $t_d=0$.

As in 5.3, let $(x_1,\dots,x_n)$ be real holomorphic coordinates centered at x and such that $(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n})$ is orthonormal at x. Let $(\zeta^d_1,\dots,\zeta^d_r)$ be an orthonormal basis of $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x$. We get an orthonormal basis of $\mathcal{J}^2_x(\mathcal{E}\otimes\mathcal{L}^d)$ similar to the one defined by (5.5) and (5.22).

We compute the matrix $\Lambda(d)$ of $\operatorname{Var}(t_d,L_d,S_d)$ in this basis. For this we use the estimates for the blocks of $\Lambda(d)$ given by proposition 3.8 and (5.39). Namely, for all i, j, k and $l\in\{1,\dots,n\}$, with $i\leqslant k$ and $j\leqslant l$,

$$\Lambda^{ik}_{20}(d) = O(d^{-1}), \qquad \Lambda^{ik,j}_{21}(d) = O(d^{-1}), \tag{5.40}$$

$$\Lambda^{ik,jl}_{22}(d) = \begin{cases} 2I_r+O(d^{-1}) & \text{if } i=j=k=l,\\ I_r+O(d^{-1}) & \text{if } i=j\neq k=l,\\ O(d^{-1}) & \text{otherwise.}\end{cases} \tag{5.41}$$

Recall from section 5.3 that the matrix of $\operatorname{Var}(t_d,L_d)$ in this basis is $I_{r(n+1)}+O(d^{-1})$. By corollary A.11, the distribution of (L_d,S_d) given $t_d=0$ is then a centered Gaussian whose variance matrix is, by blocks,

$$\tilde{\Lambda}(d) = \left(\begin{array}{c|c|c} I_{nr} & & \\ \hline & 2I_{nr} & \\ \hline & & I_{r\frac{n(n-1)}{2}}\end{array}\right)+O(d^{-1}), \tag{5.42}$$

where the empty blocks are zeros. Let Λ denote the leading term in equation (5.42) and let $(L,S)\sim\mathcal{N}(0,\Lambda)$ in $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes(T^*_x\mathbb{R}\mathcal{X}\oplus\mathrm{Sym}(T^*_x\mathbb{R}\mathcal{X}))$. By lemma A.12, $(\tilde{L}_d,\tilde{S}_d)$ converges in distribution to (L,S).

Let $U\sim\mathcal{N}(0,\mathrm{Id})$ in $T_x\mathbb{R}\mathcal{X}$ be independent of all the other variables. By (4.9),

$$R_s(x)=R(x)+\frac{1}{2}\mathbb{E}\Big[\big\langle\nabla^{2,d}_x s\,,(\nabla^d_x s)^{\dagger *}(U)\big\rangle^{\wedge 2}\Big]. \tag{5.43}$$

As in the harmonic case (5.21), we have:

$$\begin{aligned}\mathbb{E}\Big[\big|\det{}^{\perp}\left(\nabla^d_x s\right)\big|\,\mathrm{Tr}(R_s(x)^{\wedge m})\;\Big|\;s(x)=0\Big]&\\ =\sqrt{\frac{d^{n+1}}{\pi^n}}^{r}\;\mathbb{E}\left[\left|\det{}^{\perp}\left(\tilde{L}_d\right)\right|\mathrm{Tr}\left(\left(R(x)+d\;\mathbb{E}_U\left[\left\langle\tilde{S}_d\,,\tilde{L}^{\dagger *}_d(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right].&\end{aligned} \tag{5.44}$$

The proof of lemma 5.6 adapts immediately to this setting, so that:

$$\begin{aligned}\mathbb{E}\left[\left|\det{}^{\perp}\left(\tilde{L}_d\right)\right|\mathrm{Tr}\left(\left(R(x)+d\;\mathbb{E}_U\left[\left\langle\tilde{S}_d\,,\tilde{L}^{\dagger *}_d(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right]&\\ =d^m\frac{m!}{(2m)!}\mathbb{E}\Big[\big|\det{}^{\perp}(L)\big|\,\mathrm{Tr}\left(\big\langle S\,,L^{\dagger *}(U)\big\rangle^{\wedge 2m}\right)\Big]\,(1+O(d^{-1})),&\end{aligned} \tag{5.45}$$

where the error term is uniform in $x\in M$.

Lemma 5.9. *Let $(L,S)\sim\mathcal{N}(0,\Lambda)$ in $\mathbb{R}(\mathcal{E}\otimes\mathcal{L}^d)_x\otimes(T^*_x\mathbb{R}\mathcal{X}\oplus\mathrm{Sym}(T^*_x\mathbb{R}\mathcal{X}))$ and $U\sim\mathcal{N}(0,\mathrm{Id})$ in T_xM be independent of (L,S). We have:*

$$\mathbb{E}\Big[\big|\det{}^{\perp}(L)\big|\,\mathrm{Tr}\left(\big\langle S\,,L^{\dagger *}(U)\big\rangle^{\wedge 2m}\right)\Big]=(-1)^m\sqrt{2\pi}^n(2m)!\frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right)\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi\,\mathrm{Vol}\left(\mathbb{S}^n\right)\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}.$$

Once this lemma is proved, we get theorem 1.4 immediatly by (5.17), (5.38), (5.44), (5.45) and lemma 5.9. We sketch the proof of lemma 5.9 which is, unsurprisingly, adapted from the proof of lemma 5.7.

Proof. The only difference between lemmas 5.7 and 5.9 comes from the definition of Λ which is not the same in the algebraic case. The proof is exactly the same as the proof of 5.7 until the definition of K (5.34). The $\Lambda^{ik,jl}$ are now given by (5.42), hence we have to change the definition of K. In this setting,

$$K(p,\sigma,\sigma')=\begin{cases}1 & \text{if } \sigma(2p-1)=\sigma'(2p) \text{ and } \sigma(2p)=\sigma'(2p-1),\\ 0 & \text{otherwise,}\end{cases}$$

so that $\prod_{p=1}^m K(p,\mathrm{id},\tau)$ is 0, unless $\tau=\tau_0=\prod_{p=1}^m((2p-1)(2p))$. Then (5.36) becomes:

$$\sum_{\tau\in\mathfrak{S}_{2m}}\varepsilon(\tau)\prod_{p=1}^m K(p,\mathrm{id},\tau)=\varepsilon(\tau_0)\prod_{p=1}^m K(p,\mathrm{id},\tau_0)=(-1)^m.$$

This explains why the factor $\left(-\frac{n+4}{n+2}\right)^m$ becomes $(-1)^m$ in the algebraic case. What remains of the proof is as in 5.7. ∎

6 Two special cases

In some special cases, the covariance kernel is known explicitly. It is then possible to prove more precise results. In this section, we sketch what happens on the flat torus and in the real projective space. In these cases, we get the expectation of the volume and the Euler characteristic of our random submanifolds for fixed λ (resp. fixed d).

6.1 The flat torus

Let $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$ denote the torus of dimension n that we equip with the quotient of the Euclidean metric on $\mathbb{R}^n$. We have $\mathrm{Vol}\,(\mathbb{T}^n) = (2\pi)^n$. We identify functions on $\mathbb{T}^n$ and $(2\pi\mathbb{Z})^n$-periodic functions on $\mathbb{R}^n$. Then, the Laplacian is $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, and it is known that its eigenvalues are the integers of the form $\|p\|^2$, with $p = (p_1, \dots, p_n) \in \mathbb{N}^n$. The eigenspace associated to 0 is spanned by the constant function $x \mapsto \frac{1}{\sqrt{2\pi}^n}$. For $\lambda > 0$, the eigenspace associated to λ^2 is spanned by the normalized functions of the form:

$$x \mapsto \frac{2}{\sqrt{2\pi}^n} \sin(\langle p\,, x\rangle) \qquad \text{and} \qquad x \mapsto \frac{2}{\sqrt{2\pi}^n} \cos(\langle p\,, x\rangle),$$

where $p \in \mathbb{Z}^n$ is such that $\|p\| = \lambda$, and $\langle \cdot\,, \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^n$.

We set $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\| \leqslant \lambda\}$. After some computations, we get e_λ, the spectral function of the Laplacian on $\mathbb{T}^n$:

$$\forall \lambda \geqslant 0,\ \forall x, y \in \mathbb{T}^n, \qquad e_\lambda(x,y) = \frac{1}{(2\pi)^n} \sum_{p \in \mathbb{B}_\lambda} \cos(\langle p\,, x - y\rangle). \tag{6.1}$$

Let $\lambda \geqslant 0$ and $r \in \{1, \dots, n\}$, let V_λ be spanned by the eigenfunctions of Δ associated to eigenvalues smaller than λ^2, and let $f \sim \mathcal{N}(0, \mathrm{Id})$ in $(V_\lambda)^r$. For all $x \in \mathbb{T}^n$, $j_x^2(f)$ is a centered Gaussian variable whose variance is determined by lemma 2.4 , lemma 2.6 and the above formula (6.1). Note that e_λ and its derivatives are constant along the diagonal.

We can compute explicitly the variance of $j_x^2(f)$ (which is independent of x) and follow the same steps as in sections 5.2 and 5.4. Only, this time, we can make exact computations with λ fixed, instead of deriving asymptotics. These computations are not difficult, and similar to what we already did, so we simply state the final results. Note that the scaling of the variables has to be adapted.

Proposition 6.1. *On the flat torus $\mathbb{T}^n$, let $\lambda \geqslant 0$ and let $f_1, \dots, f_r$ be independent standard Gaussian functions in V_λ, with $1 \leqslant r \leqslant n$. We have:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_f)] = \left(\frac{1}{|\mathbb{B}_\lambda|} \sum_{(p_1,\dots,p_n) \in \mathbb{B}_\lambda} (p_1)^2 \right)^{\frac{r}{2}} (2\pi)^n \frac{\mathrm{Vol}\,(\mathbb{S}^{n-r})}{\mathrm{Vol}\,(\mathbb{S}^n)},$$

where $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\| \leqslant \lambda\}$ and $|\mathbb{B}_\lambda|$ denotes the cardinal of $\mathbb{B}_\lambda$.

This result was already known, see [29], where Rudnick and Wigman compute the variance of $\mathrm{Vol}\,(Z_f)$ when $r = 1$, and the references therein.

Proposition 6.2. *On the flat torus $\mathbb{T}^n$, let $\lambda \geqslant 0$ and let $f_1, \dots, f_r$ be independent standard Gaussian functions in V_λ, with $1 \leqslant r \leqslant n$. If $n - r$ is even, we have:*

$$\mathbb{E}[\chi(Z_f)] = (-1)^{\frac{n-r}{2}} \left(\frac{1}{|\mathbb{B}_\lambda|} \sum_{(p_1,\dots,p_n) \in \mathbb{B}_\lambda} (p_1)^2 \right)^{\frac{n}{2}} (2\pi)^n \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r+1}\right) \mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\pi\, \mathrm{Vol}\,(\mathbb{S}^n)\, \mathrm{Vol}\,(\mathbb{S}^{n-1})},$$

where $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\| \leqslant \lambda\}$ and $|\mathbb{B}_\lambda|$ denotes the cardinal of $\mathbb{B}_\lambda$.

Remark 6.3. For this last result, one of the points that make the computations tractable is that the Riemann tensor of the ambient manifold is zero.

6.2 The projective space

We consider the algebraic case with $\mathcal{X} = \mathbb{C}P^n$, $\mathcal{E} = \mathbb{C}^r \times \mathbb{C}P^n$ with the standard Hermitian metric on each fiber, and $\mathcal{L} = \mathcal{O}(1)$ the hyperplane bundle with its usual metric. Then, ω is the standard Fubini-Study form. We consider the real structures induced by the standard conjugation in $\mathbb{C}$.

Let e_d denote the Bergman kernel of $\mathcal{L}^d$ and E_d denote the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Since $\mathcal{E}$ is trivial we have a product situation, as in section 2.5, and E_d and e_d are related as in lemma 2.6. Let $(\zeta_1, \ldots, \zeta_r)$ be any orthonormal basis of $\mathbb{C}^r$, for all x, $y \in \mathbb{C}P^n$,

$$E_d(x,y) = e_d(x,y) \left(\sum_{q=1}^{r} \zeta_q \otimes \zeta_q \right). \tag{6.2}$$

In this case, $\mathbb{R}\mathcal{X} = \mathbb{R}P^n$ and the elements of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ are r-tuples of real homogeneous polynomials of degree d in $n+1$ variables. We denote by $\mathbb{R}_d^{hom}[X_0, \ldots, X_n]$ the space of real homogeneous polynomials of degree d in $X_0, \ldots, X_n$. Let $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, we set $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$, $|\alpha| = \alpha_0 + \cdots + \alpha_n$ and, if $|\alpha| = d$, we also set $\binom{d}{\alpha} = \frac{d!}{\alpha_0! \cdots \alpha_n!}$.

It is well-known that an orthonormal basis of $\mathbb{R}_d^{hom}[X_0, \ldots, X_n]$ for the inner product (2.16) is given by the sections:

$$s_\alpha = \sqrt{\frac{(n+d)!}{\pi^n d!} \binom{d}{\alpha}} X^\alpha, \qquad \text{with } |\alpha| = d, \tag{6.3}$$

see [5, 8, 10, 23]. Then, formally,

$$e_d = \frac{(n+d)!}{\pi^n d!} \sum_{|\alpha|=d} \binom{d}{\alpha} X^\alpha Y^\alpha = \frac{(n+d)!}{\pi^n d!} \langle X, Y \rangle^d. \tag{6.4}$$

More precisely, we consider the chart $(x_1, \ldots, x_n) \mapsto [1 : x_1 : \cdots : x_n]$ around the point of homogeneous coordinates $[1 : 0 : \cdots : 0]$, and the real holomorphic frame $s_{(d,0,\ldots,0)}$ for $\mathcal{O}(d)$ around this point. In these coordinates,

$$e_d(x,y) = \frac{(n+d)!}{\pi^n d!} (1 + \langle x, y \rangle)^d \left(s_{(d,0,\ldots,0)}(x) \otimes s_{(d,0,\ldots,0)}(y) \right), \tag{6.5}$$

with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Since everything is invariant under unitary transformations of $\mathbb{C}P^n$, this totally describes e_d.

Let $d \in \mathbb{N}$, $r \in \{1, \ldots, n\}$ and $s \sim \mathcal{N}(0, \mathrm{Id})$ in $(\mathbb{R}_d^{hom}[X_0, \ldots, X_n])^r$. For every $x \in \mathbb{R}P^n$, $j_x^{2,d}(s)$ is a centered Gaussian variable whose variance is determined by lemma 2.13, (6.2) and (6.5). Once again, we can compute the variance of $j_x^{2,d}(s)$ explicitly, and it does not depend on $x \in \mathbb{R}P^n$. Then, we can follow the steps of the proof of theorem 1.3, and make exact computations for fixed degree d. This yields Kostlan's result [23]. Note the we have to adapt the scaling of the variables.

Theorem 6.4 (Kostlan)**.** *In the real projective space $\mathbb{R}P^n$, let $d \in \mathbb{N}$ and let $P_1, \ldots, P_r$ be independent standard Gaussian polynomials in $\mathbb{R}_d^{hom}[X_0, \ldots, X_n]$, with $1 \leqslant r \leqslant n$. Let Z_P denote the common zero set of $P_1, \ldots, P_r$. We have:*

$$\mathbb{E}[\mathrm{Vol}\,(Z_s)] = \left(\sqrt{d}\right)^r \mathrm{Vol}\left(\mathbb{R}P^{n-r}\right).$$

Assuming that $n-r$ is even and equals $2m$, we can also adapt the proof of theorem 1.4 to get the expected Euler characteristic for fixed d. This computation is a bit more complex than on the torus since the Riemann curvature of $\mathbb{R}P^n$ is not zero.

By Kac-Rice formula and equations (6.2) and (4.9),

$$\mathbb{E}[\chi(Z_s)] = \frac{1}{m!\sqrt{2\pi}^n}\int_{x\in\mathbb{R}P^n}\frac{1}{\sqrt{e_d(x,x)}^n}\times \mathbb{E}\left[\left|\det{}^{\perp}\left(\nabla^d_x s\right)\right|\operatorname{Tr}\left(\left(R(x)+\frac{1}{2}\mathbb{E}\left[\left\langle\nabla^{2,d}_x s\,,(\nabla^d_x s)^{\dagger *}(U)\right\rangle^{\wedge 2}\right]\right)^{\wedge m}\right)\right]\left|\mathrm{d}V_{\mathbb{R}P^n}\right|, \tag{6.6}$$

where R is the Riemann curvature of $\mathbb{R}P^n$ and $U\sim\mathcal{N}(0,\mathrm{Id})$ in $T_x\mathbb{R}P^n$. We used the fact that $s(x)$, $\nabla^d_x s$ and $\nabla^{2,d}_x s$ are independent in this particular case. This is why the expectation is not conditionned on $s(x)=0$.

Since everything is invariant under orthogonal transformations of $\mathbb{R}P^n$, we only need to compute the expectation in the integrand at the point $x=[1:0:\cdots:0]$. We do this in the same chart as for (6.5) above. We get:

$$R(x)=\frac{1}{2}\sum_{1\leqslant i,j\leqslant n} dx_i\wedge dx_j\otimes dx_i\wedge dx_j.$$

Following the computations of section 5.4, we get (after a suitable rescaling) that the expectation in the integrand of (6.6) equals:

$$\sqrt{d}^r\sum_{q=0}^{m}\frac{(d-1)^q}{2^q}\binom{m}{q}\mathbb{E}\left[\frac{\|X_{n-r+2}\|\dots\|X_n\|}{\|X_{n-r+1}\|^{2q-1}}\right]\mathbb{E}\left[\operatorname{Tr}\left(R(x)^{\wedge(m-q)}\wedge\langle S\,,U'\rangle^{\wedge 2q}\right)\right], \tag{6.7}$$

where $U'\in\mathbb{R}^r$, $S\in(\mathbb{R}\mathcal{O}(d)_x)^r\otimes\mathrm{Sym}(T^*_x\mathbb{R}P^n)$ and $X_p\in\mathbb{R}^p$ (for $n-r+1\leqslant p\leqslant n$) are globally independent standard Gaussian vectors. By the same kind of computations as in the proof of lemma 5.7, we get:

$$\mathbb{E}_S\left[\langle S\,,U'\rangle^{\wedge 2}\right]=-\|U'\|^2\sum_{1\leqslant i,j\leqslant n} dx_i\wedge dx_j\otimes dx_i\wedge dx_j. \tag{6.8}$$

Restricting R and S to the span of $\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_{2m}}\right)$, we have:

$$\forall q\in\{0,\dots,m\},\qquad \operatorname{Tr}\left(\mathbb{R}(x)^{\wedge m-q}\wedge\mathbb{E}_S\left[\langle S\,,U'\rangle^{\wedge 2}\right]^{\wedge q}\right)=(-1)^q\frac{1}{2^{m-q}}(2m)!\,\|U'\|^{2q}.$$

As in the proof of 5.7, we compute the expectation of this term with respect to the variable U' by lemma A.13. The same lemma allows us to compute the value of (6.7). Finally, we recover Bürgisser's result [10].

Theorem 6.5 (Bürgisser)**.** *In the real projective space $\mathbb{R}P^n$, let $d\in\mathbb{N}$ and let $P_1,\dots,P_r$ be independent standard Gaussian polynomials in $\mathbb{R}^{hom}_d[X_0,\dots,X_n]$, with $1\leqslant r\leqslant n$. Let Z_P denote the common zero set of $P_1,\dots,P_r$. If $n-r$ is even, we have:*

$$\mathbb{E}[\chi(Z_P)]=\left(\sqrt{d}\right)^r\sum_{p=1}^{\frac{n-r}{2}}(1-d)^p\,\frac{\Gamma\left(p+\frac{r}{2}\right)}{p!\,\Gamma\left(\frac{r}{2}\right)}.$$

A Concerning Gaussian vectors

In this appendix we survey the few facts we need concerning random vectors, especially Gaussian ones. It is essentially borrowed from [26, appendix A]. We include it here for the reader's convenience.

A.1 Variance and covariance as tensors

Let V be a real vector space of finite dimension and X a random vector with values in V. For any $\xi \in V^*$, $\xi(X)$ is a real random variable. From now on, we assume that these variables are square integrable.

Definition A.1. The *expectation* (or *mean*) of X is the linear form on V^* defined by:

$$m_X : \xi \mapsto \mathbb{E}[\xi(X)]\,. \tag{A.1}$$

If $m_X = 0$, we say that X (resp. its distribution $\mathrm{d}P_X$) is *centered*.

Under the canonical isomorphism $V^{**} \simeq V$, we have $m_X = \int_V x \,\mathrm{d}P_X$.

Definition A.2. The *variance* of X is the non-negative symmetric bilinear form on V^* defined by:

$$\mathrm{Var}(X) : (\xi, \eta) \mapsto \mathbb{E}[\xi(X - m_X)\eta(X - m_X)]\,. \tag{A.2}$$

Remark A.3. Traditionally, the term "variance" is only used when V has dimension 1 and one speaks of "covariance" when $\dim(V) \geqslant 2$. We chose to use the term "covariance" for couples of distinct random vectors (see below) and "variance" otherwise. This is the convention of [2], for example.

As a bilinear form on $V^* \times V^*$, $\mathrm{Var}(X)$ is naturally an element of $V \otimes V$ and we have the following lemma.

Lemma A.4. *Let X be a random vector in V, we have:*

$$\mathrm{Var}(X) = \mathbb{E}[(X - m_X) \otimes (X - m_X)]\,. \tag{A.3}$$

Proof. For any ξ and $\eta \in V^*$, we have:

$$\begin{aligned}\mathrm{Var}(X)(\xi, \eta) &= \mathbb{E}[\xi(X - m_X)\eta(X - m_X)] = \mathbb{E}[(\xi \otimes \eta)((X - m_X) \otimes (X - m_X))] \\ &= (\xi \otimes \eta)\mathbb{E}[(X - m_X) \otimes (X - m_X)]\,.\end{aligned}$$

□

Definition A.5. The *variance operator* of X is the linear map $\Lambda_X : V^* \to V$ such that, for any ξ and $\eta \in V^*$,

$$\xi\,(\Lambda_X \eta) = \mathrm{Var}(X)(\xi, \eta). \tag{A.4}$$

By lemma A.4 we have:

$$\Lambda_X : \eta \mapsto \mathbb{E}[(X - m_X) \otimes \eta(X - m_X)]\,. \tag{A.5}$$

If $V = V_1 \oplus V_2$ and $X = (X_1, X_2)$, with X_i a random vector in V_i, then $m_X = m_{X_1} + m_{X_2}$ and the variance form $\mathrm{Var}(X)$ splits accordingly into four parts:

$$\begin{array}{lcl} \mathrm{Var}(X_1) : V_1^* \times V_1^* \to \mathbb{R}, & & \mathrm{Cov}(X_1, X_2) : V_1^* \times V_2^* \to \mathbb{R}, \\ \mathrm{Var}(X_2) : V_2^* \times V_2^* \to \mathbb{R} & \text{and} & \mathrm{Cov}(X_2, X_1) : V_2^* \times V_1^* \to \mathbb{R}. \end{array}$$

These bilinear forms are associated, as above, to the following operators:

$$\begin{array}{lcl} \Lambda_{11} : V_1^* \to V_1, & & \Lambda_{12} : V_2^* \to V_1, \\ \Lambda_{22} : V_2^* \to V_2 & \text{and} & \Lambda_{21} : V_1^* \to V_2. \end{array}$$

Since $\mathrm{Var}(X)$ is symmetric, $\mathrm{Cov}(X_1, X_2)(\xi, \eta) = \mathrm{Cov}(X_2, X_1)(\eta, \xi)$ for any ξ and η.

Definition A.6. We say that $\mathrm{Cov}(X_1, X_2)$ is the *covariance* of X_1 and X_2, and that Λ_{12} is their *covariance operator*.

As above, $\mathrm{Cov}(X_1, X_2)$ is naturally an element of $V_1 \otimes V_2$.

Lemma A.7. *Let X_1 and X_2 be random vectors in V_1 and V_2 respectively, we have:*

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[(X_1 - m_{X_1}) \otimes (X_2 - m_{X_2})]\,.$$

Moreover, for any $\eta \in V_2^*$, $\qquad \Lambda_{12}(\eta) = \mathbb{E}[(X_1 - m_{X_1}) \otimes \eta(X_2 - m_{X_2})]$.

Let $L : V \to V'$ be a linear map between finite-dimensional vector spaces and X be a random vector in V. Then $L(X)$ is a random vector in V' with $\mathrm{d}P_{L(X)} = L_*(\,\mathrm{d}P_X)$. An immediate consequence of (A.1), (A.2) and (A.4) is that:

$$m_{L(X)} = m_X \circ L^*, \tag{A.6}$$

$$\mathrm{Var}(L(X)) = \mathrm{Var}(X)(L^*\cdot, L^*\cdot), \tag{A.7}$$

and

$$\Lambda_{L(X)} = L\Lambda_X L^*, \tag{A.8}$$

where $L^* : (V')^* \to V^*$ is defined by $L^* : \xi \mapsto (\xi \circ L)$.

If X is a random vector in a Euclidean space $(V, \langle\cdot\,,\cdot\rangle)$, we can see $\mathrm{Var}(X)$ as a bilinear symmetric form on V, and Λ_X as a self-adjoint operator on V. Then, by (A.3) and (A.5) :

$$\mathrm{Var}(X) = \mathbb{E}[(X - m_X)^* \otimes (X - m_X)^*]\,,$$
$$\Lambda_X = \mathbb{E}[(X - m_X) \otimes (X - m_X)^*]\,,$$

where for any $v \in V$, we set $v^* = \langle v\,, \cdot\rangle \in V^*$.

If $V = V_1 \oplus V_2$, we can see Λ_{12} as a linear operator from V_2 to V_1 and by lemma A.7:

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[(X_1 - m_{X_1})^* \otimes (X_2 - m_{X_2})^*]\,,$$
$$\Lambda_{12} = \mathbb{E}[(X_1 - m_{X_1}) \otimes (X_2 - m_{X_2})^*]\,,$$

Lemma A.8. *Let X be a random vector in a Euclidean space V, we have:*

$$\forall v \in V, \forall w \in V, \quad \mathbb{E}[\langle v\,, X - m_X\rangle \langle w\,, X - m_X\rangle] = \langle v\,, \Lambda_X w\rangle\,. \tag{A.9}$$

Proof. Let v and $w \in V$, we have:

$$\begin{aligned}\mathbb{E}[\langle v\,, X - m_X\rangle \langle w\,, X - m_X\rangle] &= \mathrm{Var}(X)(v, w) && \text{as a bilinear form on } V,\\ &= \langle v\,, \Lambda_X w\rangle && \text{where } \Lambda_X : V \to V.\end{aligned}$$

□

A.2 Gaussian vectors

The following material can be found either in [2, section 1.2] or [1, section 1.2]. We present it in a coordinate-free fashion, in the spirit of [26].

Let $m \in \mathbb{R}$ and $\sigma \geqslant 0$, the *Gaussian* (or *normal*) distribution on $\mathbb{R}$ with expectation m and variance σ^2 is the distribution whose characteristic function is $\xi \mapsto \exp\left(im\xi - \frac{1}{2}\sigma^2\xi^2\right)$. If $\sigma = 0$, this is the Dirac measure centered at m, otherwise it has a density with respect to the Lebesgue measure, given by $x \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$.

Let V be a real vector space of dimension n, a random vector X in V is said to be *Gaussian*, or *normally distributed*, if for any $\xi \in V^*$, $\xi(X)$ is a Gaussian variable in $\mathbb{R}$. Recall that a Gaussian vector has finite moments of all orders and that its distribution is totally determined by its expectation and variance. We denote by $\mathcal{N}(m, \Lambda)$ the Gaussian distribution with expectation m and variance operator Λ and by $X \sim \mathcal{N}(m, \Lambda)$ the fact that X is distributed according to $\mathcal{N}(m, \Lambda)$. From (A.6) and (A.8) we deduce the following.

Lemma A.9. *Let $L : V \to V'$ be a linear map between finite-dimensional vector spaces and $X \sim \mathcal{N}(m, \Lambda)$ in V. Then $L(X) \sim \mathcal{N}(Lm, L\Lambda L^*)$ in V'.*

If $V = V_1 \oplus V_2$ and $X = (X_1, X_2) \sim \mathcal{N}(m, \Lambda)$ then, with the notations of section A.1, lemma A.9 shows that $X_1 \sim \mathcal{N}(m_{X_1}, \Lambda_{11})$ and $X_2 \sim \mathcal{N}(m_{X_2}, \Lambda_{22})$. Besides, X_1 and X_2 are independent if and only if $\mathrm{Cov}(X_1, X_2) = 0$, or equivalently $\Lambda_{12} = 0$.

Proposition A.10 (Regression formula)**.** *Let $X = (X_1, X_2)$ be a Gaussian vector in $V_1 \oplus V_2$. If $\mathrm{Var}(X_1)$ is non-degenerate then X_2 has the same distribution as*

$$m_{X_2} + \Lambda_{21}(\Lambda_{11})^{-1}\left(X_1 - m_{X_1}\right) + Y$$

where Y is a centered Gaussian vector in V_2 with variance operator $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$, independent of X_1.

This is shown in [2, prop 1.2]. From this, we deduce that the distribution of X_2 given $X_1 = x_1$ is Gaussian in V_2 with expectation $m_{X_2} + \Lambda_{21}(\Lambda_{11})^{-1}(x_1 - m_{X_1})$ and variance operator $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$. We use this in the case where X is centered and $x_1 = 0$.

Corollary A.11. *Let $X = (X_1, X_2)$ be a centered Gaussian vector in $V_1 \oplus V_2$ and assume that $\mathrm{Var}(X_1)$ is non-degenerate. Then the distribution of X_2 given $X_1 = 0$ is a centered Gaussian in V_2 with variance operator $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$.*

In what follows, we assume that V is a Euclidean space. Recall that in this case, we can see the variance operator of a random vector as an endomorphism of V. We will say that $\mathcal{N}(0, \mathrm{Id})$ is the *standard normal distribution* on V, where Id denotes the identity map on V.

If $\mathrm{Var}(X)$ is non-degenerate, then $\mathrm{d}P_X$ has the following density with respect to the Lebesgue measure on V:

$$x \mapsto \frac{1}{\sqrt{2\pi}^n \sqrt{\det(\Lambda_X)}} \exp\left(-\frac{1}{2}\left\langle (\Lambda_X)^{-1}(x-m), x-m\right\rangle\right).$$

If $\mathrm{Var}(X)$ is singular, $\mathrm{d}P_X$ is supported on $\ker(\Lambda_X)^\perp$.

Lemma A.12. *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of random vectors in V such that $X_k \sim \mathcal{N}(m_k, \Lambda_k)$ for all $k \in \mathbb{N}$. We assume that $m_k \xrightarrow[k\to+\infty]{} m$ and $\Lambda_k \xrightarrow[k\to+\infty]{} \Lambda$. Then X_k converges in distribution to $\mathcal{N}(m, \Lambda)$.*

Proof. Under the hypothesis of the lemma, the characteristic function of X_k converges pointwise to the characeristic function of a Gaussian vector $X \sim \mathcal{N}(m, \Lambda)$. Then, Lévy's continuity theorem gives the result. □

We conclude this appendix by computing two Gaussian expectations.

Lemma A.13. *Let $X \sim \mathcal{N}(0, \mathrm{Id})$ with values in a Euclidean space of dimension n and let $k \in \mathbb{Z}$ such that $k > -n$. Then,*

$$\mathbb{E}\left[\|X\|^k\right] = \sqrt{2\pi}^k \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n+k-1}\right)}.$$

Proof.

$$\begin{aligned}\mathbb{E}\left[\|X\|^k\right] &= \frac{1}{\sqrt{2\pi}^n}\int_V \|x\|^k e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{\sqrt{2\pi}^n}\int_0^{+\infty} r^{k+n-1}e^{-\frac{1}{2}r^2}\,\mathrm{d}r \\ &= \frac{\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{\sqrt{2\pi}^n}\int_0^{+\infty} \sqrt{2t}^{k+n-2}e^{-t}\,\mathrm{d}t = \frac{\sqrt{2\pi}^k\,\mathrm{Vol}\left(\mathbb{S}^{n-1}\right)}{2\sqrt{\pi}^{k+n}}\,\Gamma\left(\frac{k+n}{2}\right).\end{aligned}$$

□

Assuming we proved proposition 5.8 (more precisely its corollary B.3), we have the following.

Lemma A.14. *Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leqslant r \leqslant n$. Let $L \sim \mathcal{N}(0, \mathrm{Id})$ in $V' \otimes V^*$. Then:*

$$\mathbb{E}\left[\left|\det{}^{\perp}(L)\right|\right] = \sqrt{2\pi}^{r} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}.$$

Proof. By corollary B.3, $\left|\det^{\perp}(L)\right|$ is distributed as $\|X_{n-r+1}\| \cdots \|X_n\|$, with X_p a standard Gaussian in $\mathbb{R}^p$ for all p and $X_{n-r+1}, \ldots, X_n$ independent. Then, using lemma A.13,

$$\mathbb{E}\left[\left|\det^{\perp}(L)\right|\right] = \mathbb{E}[\|X_{n-r+1}\| \cdots \|X_n\|] = \prod_{p=n-r+1}^{n} \mathbb{E}[\|X_p\|] = \sqrt{2\pi}^{r} \frac{\mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}{\mathrm{Vol}\left(\mathbb{S}^{n}\right)}. \qquad \square$$

B Proof of proposition 5.8

This section is devoted to the proof of proposition 5.8 which is a reformulation of [10, prop 3.12]. Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leqslant r \leqslant n$. The space $V' \otimes V^*$ of linear maps from V to V' comes with a natural scalar product induced by those on V and V'. The set of linear maps of rank less than r is an algebraic submanifold of $V' \otimes V^*$ of codimension at least 1, hence it has measure 0 for any non-singular Gaussian measure. Let L be a standard Gaussian vector in $V' \otimes V^*$, the rank of L is r almost surely. Hence $L^\dagger$ is well-defined almost surely (recall definition 4.7).

We introduce some further notations. Let $\mathcal{B} \subset V' \otimes V^*$ denote the set of maps of rank r. We set $\mathcal{F} = \{(L, U) \in \mathcal{B} \times V \mid U \in \ker(L)^\perp\}$ and $\mathcal{S} = \{(L, U) \in \mathcal{B} \times V \mid U \in \mathbb{S}(\ker(L)^\perp)\}$. Here and in the sequel, $\mathbb{S}(\cdot)$ stands for the unit sphere of the concerned space. Given $L \in \mathcal{B}$ and $U \in V$, we denote by $\tilde{U}$ the orthogonal projection of U onto $\ker(L)^\perp$. Then we set:

$$\rho = \|\tilde{U}\|, \qquad \theta = \frac{\tilde{U}}{\|\tilde{U}\|} \qquad \text{and} \qquad \theta' = \frac{(L^\dagger)^* \theta}{\|(L^\dagger)^* \theta\|}. \tag{B.1}$$

Note that $L^*\theta' = \frac{(L^\dagger L)^* \theta}{\|(L^\dagger)^* \theta\|} = \frac{\theta}{\|(L^\dagger)^* \theta\|}$, hence $\|L^*\theta'\| = \frac{1}{\|(L^\dagger)^* \theta\|}$ and finally:

$$\theta = \frac{L^*\theta'}{\|L^*\theta'\|} \qquad \text{and} \qquad L^{\dagger *} U = L^{\dagger *} \tilde{U} = \rho L^{\dagger *} \theta = \frac{\rho}{\|L^*\theta'\|} \theta'. \tag{B.2}$$

We choose orthonormal basis $(e_1, \ldots, e_n)$ and $(e'_1, \ldots, e'_r)$, of V and V' respectively, such that $e_r = \theta$, $e'_r = \theta'$ and $(e_1, \ldots, e_r)$ is a basis of $\ker(L)^\perp$. Then,

$$\forall i \in \{1, \ldots, n\}, \quad \langle L e_i \,, \theta' \rangle = \langle e_i \,, L^*\theta' \rangle = \|L^*\theta'\| \, \langle e_i \,, \theta \rangle \,.$$

Thus the matrix of L in these basis has the form:

$$\left(\begin{array}{c|c|c} A & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 \\ \hline 0 \cdots 0 & \|L^*\theta'\| & 0 \cdots 0 \end{array}\right), \tag{B.3}$$

and $\left|\det^{\perp}(L)\right| = |\det(A)| \, \|L^*\theta'\|$.

Let π_θ and $\pi_{\theta'}$ denote the orthogonal projections along $\mathbb{R} \cdot \theta$ in V and along $\mathbb{R} \cdot \theta'$ in V' respectively. We define $L' : V \to (\mathbb{R} \cdot \theta')^\perp$ by $L' = \pi_{\theta'} \circ L \circ \pi_\theta$. Then $|\det(A)| = \left|\det^{\perp}(L')\right|$, and L' does not depend on our choice of basis. Finally, we have:

$$\left(\left|\det^{\perp}(L)\right|, L^{\dagger *} U\right) = \left(\left|\det^{\perp}(L')\right| \|L^*\theta'\|, \frac{\rho\theta'}{\|L^*\theta'\|}\right). \tag{B.4}$$

To prove proposition 5.8, we will show that $\left|\det{}^{\perp}(L')\right|$, $\|L^*\theta'\|$, ρ and θ' are independent and identify their distributions.

If L and U are independant standard Gaussians, then almost surely $L \in \mathcal{B}$ and we can consider (L, U) as a random element of $\mathcal{B} \times V$. Then $(L, \tilde{U})$ is a random element of $\mathcal{F}$ and its distribution is characterized by:

$$\begin{aligned}\mathbb{E}\left[\phi(L,\tilde{U})\right] &= \int_{L\in\mathcal{B}} \left(\int_{U\in V} \phi(L,\tilde{U})\,\mathrm{d}\mu_n(U)\right)\mathrm{d}\mu_{nr}(L)\\ &= \int_{L\in\mathcal{B}} \left(\int_{\tilde{U}\in\ker(L)^{\perp}} \phi(L,\tilde{U})\,\mathrm{d}\mu_r(\tilde{U})\right)\mathrm{d}\mu_{nr}(L),\end{aligned}$$

for any bounded continuous function $\phi : \mathcal{F} \to \mathbb{R}$. Recall that $\mathrm{d}\mu_N$ stands for the standard Gaussian measure in dimension N. We get the distribution of $(L, \theta, \rho) \in \mathcal{S} \times \mathbb{R}_+$ by a polar change of variables in the innermost integral: for any bounded continuous $\phi : \mathcal{S} \times \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}[\phi(L,\theta,\rho)] = \int_{L\in\mathcal{B}} \int_{\theta\in\mathbb{S}(\ker(L)^{\perp})} \int_{\rho=0}^{+\infty} \phi(L,\theta,\rho)\rho^{r-1}e^{-\frac{\rho^2}{2}} \frac{\mathrm{d}\rho}{\sqrt{2\pi}^r}\,\mathrm{d}\theta\,\mathrm{d}\mu_{nr}(L),$$

where $\mathrm{d}\rho$ is the Lebesgue measure on $\mathbb{R}$ and $\mathrm{d}\theta$ is the Euclidean measure on the sphere $\mathbb{S}(\ker(L)^{\perp})$.

This distribution is a product measure on $\mathcal{S} \times \mathbb{R}_+$, thus (L, θ) and ρ are independent variables. Since $(\left|\det{}^{\perp}(L')\right|, \theta', \|L^*\theta'\|)$ only depends on (L, θ), this triple is independent of ρ. Besides, ρ is distributed as the norm of a standard Gaussian vector in $\mathbb{R}^r$ since its density with respect to the Lebesgue measure is $\rho \mapsto \frac{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}{\sqrt{2\pi}^r}\rho^{r-1}e^{-\frac{\rho^2}{2}}$ on $\mathbb{R}_+$ and vanishes elsewhere. Finally, the distribution of (L, θ) satisfies:

$$\mathbb{E}[\phi(L,\theta)] = \int_{L\in\mathcal{B}} \int_{\theta\in\mathbb{S}(\ker(L)^{\perp})} \phi(L,\theta)\frac{\mathrm{d}\theta}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}\,\mathrm{d}\mu_{nr}(L), \tag{B.5}$$

for any bounded and continuous $\phi : \mathcal{S} \to \mathbb{R}$.

We will now compute the distribution of (L, θ') in $\mathcal{B} \times \mathbb{S}(V')$.

Lemma B.1. *For any bounded and continuous $\phi : \mathcal{B} \times \mathbb{S}(V') \to \mathbb{R}$,*

$$\mathbb{E}[\phi(L,\theta')] = \int_{\theta'\in\mathbb{S}(V')} \int_{L\in\mathcal{B}} \phi(L,\theta')\frac{\left|\det{}^{\perp}(L')\right|}{\|L^*\theta'\|^{r-1}}e^{-\frac{\|L\|^2}{2}}\frac{\mathrm{d}\theta'}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}\frac{\mathrm{d}L}{\sqrt{2\pi}^{nr}}, \tag{B.6}$$

where $\mathrm{d}\theta'$ is the Euclidean measure on $\mathbb{S}(V')$, $\mathrm{d}L$ is the Lebesgue measure on $V' \otimes V^*$ and L' is defined as in (B.4).

Proof. Fixing some ϕ, we see from (B.5) and (B.1) that:

$$\mathbb{E}[\phi(L,\theta')] = \int_{L\in\mathcal{B}} \int_{\theta\in\mathbb{S}(\ker(L)^{\perp})} \phi\left(L, \frac{(L^{\dagger})^*\theta}{\|(L^{\dagger})^*\theta\|}\right)e^{-\frac{\|L\|^2}{2}}\frac{\mathrm{d}\theta}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}\frac{\mathrm{d}L}{\sqrt{2\pi}^{nr}}.$$

Then we make the change of variables $\theta' = \psi(\theta) = \frac{(L^{\dagger})^*\theta}{\|(L^{\dagger})^*\theta\|}$ in the innermost integral, with L fixed. Recalling (B.2), we have $\psi^{-1} : \theta' \mapsto \frac{L^*\theta'}{\|L^*\theta'\|}$ from $\mathbb{S}(V')$ to $\mathbb{S}(\ker(L)^{\perp})$. Now, the differential of ψ^{-1} at $\theta' \in \mathbb{S}(V')$ satisfies:

$$\forall v \in (\mathbb{R}\cdot\theta')^{\perp}, \quad d_{\theta'}(\psi^{-1})\cdot v = \frac{1}{\|L^*\theta'\|}\left(L^*v - \left\langle L^*v\,,\frac{L^*\theta'}{\|L^*\theta'\|}\right\rangle\frac{L^*\theta'}{\|L^*\theta'\|}\right) = \frac{\pi_{\theta}(L^*v)}{\|L^*\theta'\|}.$$

As above, we choose an orthonormal basis $(e'_1, \dots, e'_{r-1})$ of $(\mathbb{R} \cdot \theta')^\perp$ and an orthonormal basis $(e_1, \dots, e_{r-1})$ of $(\mathbb{R} \cdot \theta \oplus \ker(L))^\perp$. In these coordinates we have:

$$\left|\det(d_{\theta'}(\psi^{-1}))\right| = \frac{\left|\det\left(\pi_\theta \circ L^*_{/(\theta')^\perp}\right)\right|}{\|L^*\theta'\|^{r-1}} = \frac{|\det(A^*)|}{\|L^*\theta'\|^{r-1}} = \frac{|\det(A)|}{\|L^*\theta'\|^{r-1}} = \frac{\left|\det{}^\perp(L')\right|}{\|L^*\theta'\|^{r-1}},$$

where A is as in (B.3) and L' as in (B.4). This proves (B.6). □

We can now compute the joint distribution of $\left(\left|\det{}^\perp(L')\right|, \theta', \|L^*\theta'\|\right)$ from the one of (L, θ'). We fix $\theta' \in \mathbb{S}(V')$ and an orthonormal basis $(e'_1, \dots e'_r)$ of V' such that $e'_r = \theta'$. The choice of $L \in V' \otimes V^*$ is equivalent to the choice of the r independent standard Gaussian vectors $L^*e'_1, \dots, L^*e'_r$ in V. For simplicity, we set $L_i = L^*e'_i$. Note that if we choose a basis for V as well, these are the rows of the matrix of L. We can rewrite (B.6) as:

$$\mathbb{E}[\phi(L, \theta')] = \int_{\theta' \in \mathbb{S}(V')} \int_{L_1, \dots, L_{r-1} \in V} \int_{L_r \in V} \phi(L, \theta') \frac{\left|\det{}^\perp(L')\right|}{\|L_r\|^{r-1}} \frac{e^{-\frac{1}{2}\sum \|L_i\|^2}\, \mathrm{d}\theta'}{\mathrm{Vol}\,(\mathbb{S}^{r-1})} \frac{\mathrm{d}L_1 \cdots \mathrm{d}L_r}{\sqrt{2\pi}^{nr}},$$

where $\mathrm{d}L_i$ denotes the Lebesgue measure in the i-th copy of V. We set $\alpha_r = \frac{L_r}{\|L_r\|}$ and $\rho_r = \|L_r\|$. Here, $L' = \pi_{\theta'} \circ L \circ \pi_{\alpha_r}$ depends on α_r and θ' but not on ρ_r. Making a polar change of variables, the above integral equals:

$$\int_{\substack{\theta' \in \mathbb{S}(V') \\ \alpha_r \in \mathbb{S}(V) \\ L_1, \dots, L_{r-1} \in V}} \int_{\rho_r=0}^{+\infty} \phi(L, \theta') \frac{\rho_r^{n-r} e^{-\frac{\rho_r^2}{2}}\, \mathrm{d}\rho_r}{\sqrt{2\pi}^{n-r+1}} \frac{e^{-\frac{1}{2}\sum_{i=1}^{r-1}\|L_i\|^2} \left|\det{}^\perp(L')\right|}{\mathrm{Vol}\,(\mathbb{S}^{r-1})} \frac{\mathrm{d}\theta'\, \mathrm{d}\alpha_r\, \mathrm{d}L_1 \cdots \mathrm{d}L_{r-1}}{\sqrt{2\pi}^{(n+1)(r-1)}}.$$

Then, $\rho_r = \|L^*\theta'\|$ is independent of $(\theta', \alpha_r, L_1, \dots, L_{r-1})$, hence of $\left(\theta', \left|\det{}^\perp(L')\right|\right)$. Moreover ρ_r is distributed as the norm of a standard Gaussian in $\mathbb{R}^{n-r+1}$, since it has the same density. Finally, $(\theta', \alpha_r, L_1, \dots, L_{r-1})$ has the density:

$$(\theta', \alpha_r, L_1, \dots, L_{r-1}) \mapsto \frac{e^{-\frac{1}{2}\sum_{i=1}^{r-1}\|L_i\|^2}}{\sqrt{2\pi}^{(n+1)(r-1)}} \frac{\left|\det{}^\perp(L')\right|}{\mathrm{Vol}\,(\mathbb{S}^{r-1})\,\mathrm{Vol}\,(\mathbb{S}^{n-r})}$$

with respect to $\mathrm{d}\theta' \otimes \mathrm{d}\alpha_r \otimes \mathrm{d}L_1 \otimes \cdots \otimes \mathrm{d}L_{r-1}$.

For $i \in \{1, \dots, r-1\}$ we denote by $L_i^\perp$ the orthogonal projection of L_i onto the orthogonal of the subspace spanned by $(\alpha_r, L_1, \dots, L_{i-1})$.

Lemma B.2. *For any $L \in \mathcal{B}$, $\left|\det{}^\perp(L')\right| = \left\|L_1^\perp\right\| \cdots \left\|L_{r-1}^\perp\right\|$.*

Proof. If one of the $L_i^\perp$ is zero, then the vectors $\alpha_r, L_1, \dots, L_{r-1}$ are linearly dependent and L is singular. Since we assumed $L \in \mathcal{B}$ this is not the case and $\left(\frac{L_1^\perp}{\|L_1^\perp\|}, \dots, \frac{L_{r-1}^\perp}{\|L_{r-1}^\perp\|}\right)$ is an orthonormal basis of $\ker(L')^\perp$. Writing the matrix of the restriction of L' to $\ker(L')^\perp$ in this basis and $(e'_1, \dots, e'_{r-1})$, we see that it is lower triangular with diagonal coefficients $\left\|L_1^\perp\right\|, \dots, \left\|L_{r-1}^\perp\right\|$. This proves the lemma. □

Let ϕ be a continuous bounded function from $\mathbb{S}(V') \times \mathbb{R}_+$ to $\mathbb{R}$. We have:

$$\mathbb{E}\left[\phi\left(\theta', \left|\det{}^\perp(L')\right|\right)\right] = \int \phi\left(\theta', \left|\det{}^\perp(L')\right|\right) \frac{e^{-\frac{1}{2}\sum\|L_i\|^2} \left|\det{}^\perp(L')\right|}{\sqrt{2\pi}^{(n+1)(r-1)}} \frac{\mathrm{d}\theta'\, \mathrm{d}\alpha_r\, \mathrm{d}L_1 \dots \mathrm{d}L_{r-1}}{\mathrm{Vol}\,(\mathbb{S}^{r-1})\,\mathrm{Vol}\,(\mathbb{S}^{n-r})}$$

$$= \int_{\alpha_r, \theta'} \int_{L_1^\perp} \cdots \int_{L_{r-1}^\perp} \phi\left(\theta', \prod \|L_i^\perp\|\right) \frac{e^{-\frac{1}{2}\sum\left\|L_i^\perp\right\|^2} \prod \left\|L_i^\perp\right\|}{\sqrt{2\pi}^{(n+1-\frac{r}{2})(r-1)}} \frac{\mathrm{d}L_{r-1}^\perp \dots \mathrm{d}L_1^\perp\, \mathrm{d}\theta'\, \mathrm{d}\alpha_r}{\mathrm{Vol}\,(\mathbb{S}^{r-1})\,\mathrm{Vol}\,(\mathbb{S}^{n-r})}.$$

Then we make polar changes of variables: for each i we set $\rho_i = \left\|L_i^\perp\right\|$ and $\alpha_i = \frac{L_i^\perp}{\left\|L_i^\perp\right\|}$. Note that, when $L_1, \dots, L_{i-1}$ are fixed, $L_i^\perp$ is a vector in a space of dimension $n-i$. We have:

$$\mathbb{E}\left[\phi\left(\theta', \left|\det{}^\perp(L')\right|\right)\right] = \int \phi\left(\theta', \prod \rho_i\right) \frac{e^{-\frac{1}{2}\sum \rho_i^2} \prod \rho_i^{n-i}}{\sqrt{2\pi}^{(n+1-\frac{r}{2})(r-1)}} \frac{\mathrm{d}\rho_1 \dots \mathrm{d}\rho_{r-1}\, \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_r\, \mathrm{d}\theta'}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right) \mathrm{Vol}\left(\mathbb{S}^{n-r}\right)}$$

$$= \int_{\rho_1,\dots,\rho_{r-1},\theta'} \phi\left(\theta', \prod \rho_i\right) \prod_{i=1}^{r-1} \left(\mathrm{Vol}\left(\mathbb{S}^{n-i}\right) \frac{e^{-\frac{\rho_i^2}{2}} \rho_i^{n-i}}{\sqrt{2\pi}^{n+1-i}} \right) \mathrm{d}\rho_1 \dots \mathrm{d}\rho_{r-1} \frac{\mathrm{d}\theta'}{\mathrm{Vol}\left(\mathbb{S}^{r-1}\right)}.$$

This shows that $\theta', \rho_1, \dots, \rho_{r-1}$ are independent variables, that θ' is uniformly distributed in $\mathbb{S}(V')$ and that, for all $i \in \{1, \dots, r-1\}$, ρ_i is distributed as the norm of a standard Gaussian vector in $\mathbb{R}^{n+1-i}$. Finally, this shows that $\left|\det{}^\perp(L')\right|$ is distributed as $\prod_{i=1}^{r-1} \rho_i$.

Putting all we have done so far together, we see that $\left(\left|\det{}^\perp(L)\right|, L^{\dagger *}U\right)$ is distributed as $\left(\|X_n\| \cdots \|X_{n-r+1}\|, \frac{\rho\theta'}{\|X_{n-r+1}\|}\right)$, where X_p is a standard Gaussian vector in $\mathbb{R}^p$ for all p. Moreover, θ' is uniformly distributed in $\mathbb{S}(V')$, ρ is distributed as the norm of a standard Gaussian vector in $\mathbb{R}^r$, and all these variables are globally independent. Finally $U' = \rho\theta'$ is a standard Gaussian in V', independent of $X_n, \dots, X_{n-r+1}$ so we have proved proposition 5.8. An immediate corollary of this is the following.

Corollary B.3. *Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leqslant r \leqslant n$. Let $L \sim \mathcal{N}(0, \mathrm{Id})$ in $V' \otimes V^*$. Then $\left|\det{}^\perp(L)\right|$ is distributed as $\|X_{n-r+1}\| \cdots \|X_n\|$, where for all $p \in \{n-r+1, \dots, n\}$, $X_p \sim \mathcal{N}(0, \mathrm{Id})$ in $\mathbb{R}^p$ and these vectors are globally independent.*

C Proof of the Kac-Rice formula

In this section, we give a proof of the Kac-Rice formula using Federer's coarea formula. This was already done by Bleher, Shiffman and Zelditch in [6, thm 4.2]. See also [2, chap 6].

C.1 The coarea formula

We start by stating the coarea formula in the case of a smooth map between smooth Riemannian manifolds. A proof in this special case can be found in [19, Appendix] (see [12, thm 3.2.12] for the general case).

Let $\pi : \tilde{M} \to M$ be a smooth map between smooth Riemannian manifolds of respective dimensions m and n. We assume that $m \geqslant n$. Let $|\mathrm{d}V_{\tilde{M}}|$ (resp. $|\mathrm{d}V_M|$) denote the Riemannian measure on $\tilde{M}$ (resp. M) induced by its metric. By Sard's theorem, for almost every $y \in M$, $\pi^{-1}(y)$ is a smooth submanifold of dimension $(m-n)$ of $\tilde{M}$. For such $y \in M$, we denote by $|\mathrm{d}V_y|$ the Riemannian measure on $\pi^{-1}(y)$ induced by the metric of $\tilde{M}$. When $m = n$, the dimension $\pi^{-1}(y)$ is 0 and $|\mathrm{d}V_y|$ is just $\sum_{x \in \pi^{-1}(y)} \delta_x$, where δ_x is the Dirac measure at x.

Theorem C.1 (Coarea formula, Federer)**.** *Let $\pi : \tilde{M} \to M$ be a smooth map between smooth Riemannian manifolds of dimension m and n respectively, with $m \geqslant n$. Let $\phi : \tilde{M} \to \mathbb{R}$ be a Borel measurable function. Then:*

$$\int_{x \in \tilde{M}} \phi(x) \left|\det{}^\perp\left(d_x\pi\right)\right| |\mathrm{d}V_{\tilde{M}}| = \int_{y \in M} \left(\int_{x \in \pi^{-1}(y)} \phi(x) |\mathrm{d}V_y| \right) |\mathrm{d}V_M|,$$

whenever one of these integrals is well-defined.

Note that the innermost integral on the right-hand side is only defined almost everywhere.

C.2 The double-fibration trick

We now describe the double-fibration trick, which consists in applying the coarea formula twice, for different fibrations. Let M_1 and M_2 be two smooth Riemannian manifolds of dimension n_1 and n_2 respectively. Let $F : M_1 \times M_2 \to \mathbb{R}^r$ be a smooth submersion, and let $\Sigma = F^{-1}(0)$. We equip Σ with the restriction of the product metric on $M_1 \times M_2$ and denote by $|\mathrm{d}V_{M_1}|$, $|\mathrm{d}V_{M_2}|$ and $|\mathrm{d}V_\Sigma|$ the Riemannian measures on the corresponding manifolds. Finally, let $\pi_1 : \Sigma \to M_1$ and $\pi_2 : \Sigma \to M_2$ be the projections from Σ to each factor. Assuming that $r \leqslant \min(n_1, n_2)$, we have $\dim(\Sigma) = n_1 + n_2 - r \geqslant \max(n_1, n_2)$. Thus we can apply the coarea formula both to π_1 and π_2.

Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function, then:

$$\begin{aligned}\int_{y_1 \in M_1} \left(\int_{x \in \pi_1^{-1}(y_1)} \phi(x) \, |\mathrm{d}V_{y_1}| \right) |\mathrm{d}V_{M_1}| &= \int_{x \in \Sigma} \phi(x) \left|\det{}^{\perp}(d_x \pi_1)\right| |\mathrm{d}V_\Sigma| \\ &= \int_{y_2 \in M_2} \left(\int_{x \in \pi_2^{-1}(y_2)} \phi(x) \frac{\left|\det{}^{\perp}(d_x \pi_1)\right|}{\left|\det{}^{\perp}(d_x \pi_2)\right|} |\mathrm{d}V_{y_2}| \right) |\mathrm{d}V_{M_2}| \,,\end{aligned} \tag{C.1}$$

whenever one of these integrals is well-defined. Note that if $\left|\det{}^{\perp}(d_x \pi_2)\right|$ vanishes then $\pi_2(x)$ is a critical value of π_2, and the set of such critical values has measure 0 in M_2.

We would like the integrand on the right-hand side to depend on F rather than on π_1 and π_2. Let $\partial_1 F$ and $\partial_2 F$ denote the partial differentials of F with respect to the first and second variable respectively. For any $x = (x_1, x_2) \in \Sigma$,

$$T_x \Sigma = \{(v_1, v_2) \in T_{x_1} M_1 \times T_{x_2} M_2 \mid \partial_1 F(x) \cdot v_1 + \partial_2 F(x) \cdot v_2 = 0\}.$$

Lemma C.2. *Let $x \in \Sigma$, then $\left|\det{}^{\perp}(d_x \pi_2)\right| = 0$ if and only if $\left|\det{}^{\perp}(\partial_1 F(x))\right| = 0$. Moreover,*

$$\left|\det{}^{\perp}(d_x \pi_1)\right| \left|\det{}^{\perp}(\partial_1 F(x))\right| = \left|\det{}^{\perp}(d_x \pi_2)\right| \left|\det{}^{\perp}(\partial_2 F(x))\right|. \tag{C.2}$$

Proof. First note that $d_x F = \partial_1 F(x) \circ d_x \pi_1 + \partial_2 F(x) \circ d_x \pi_2 = 0$ on $T_x \Sigma$. This shows that:

$$\ker(d_x \pi_1) = \{0\} \times \ker(\partial_2 F(x)) \qquad \text{and} \qquad \ker(d_x \pi_2) = \ker(\partial_1 F(x)) \times \{0\}. \tag{C.3}$$

The space $T_x \Sigma$ splits as the following orthogonal direct sum:

$$T_x \Sigma = \ker(d_x \pi_1) \oplus \ker(d_x \pi_2) \oplus G, \tag{C.4}$$

where G is the orthogonal complement of $\ker(d_x \pi_1) \oplus \ker(d_x \pi_2)$ in $T_x \Sigma$.

Then, $\left|\det{}^{\perp}(d_x \pi_2)\right| = 0$ if and only if $d_x \pi_2$ is not onto. Recalling that $\dim(M_2) = n_2$ and $\dim(\Sigma) = n_1 + n_2 - r$, this is equivalent to $\dim(\ker(d_x \pi_2)) > n_1 - r$. In the same way, $\left|\det{}^{\perp}(\partial_1 F(x))\right| = 0$ if and only if $\dim(\ker(\partial_1 F(x))) > n_1 - r$. But the kernels of $d_x \pi_2$ and $\partial_1 F(x)$ have the same dimension by (C.3), so that $\left|\det{}^{\perp}(d_x \pi_2)\right| = 0$ if and only if $\left|\det{}^{\perp}(\partial_1 F(x))\right| = 0$. A similar argument shows that $\left|\det{}^{\perp}(d_x \pi_1)\right| = 0$ if and only if $\left|\det{}^{\perp}(\partial_2 F(x))\right| = 0$. Thus the lemma is true if any of the four maps in (C.2) is singular.

From now on, we assume that these maps are all surjective. In this case, we have:

$$\begin{aligned} \dim(G) = r, \qquad & \dim(\ker(\partial_1 F(x))) = \dim(\ker(d_x \pi_2)) = n_1 - r, \\ \text{and} \qquad & \dim(\ker(\partial_2 F(x))) = \dim(\ker(d_x \pi_1)) = n_2 - r. \end{aligned}$$

We choose an orthonormal basis of $T_x M_1$ adapted to $\ker(\partial_1 F(x)) \oplus \ker(\partial_1 F(x))^\perp$ and an orthonormal basis of $T_x M_2$ adapted to $\ker(\partial_2 F(x)) \oplus \ker(\partial_2 F(x))^\perp$. From these, we deduce

orthonormal basis of $\ker(d_x\pi_1) = \{0\} \times \ker(\partial_2 F(x))$ and $\ker(d_x\pi_2) = \ker(\partial_1 F(x)) \times \{0\}$. Finally we complete the resulting basis of $\ker(d_x\pi_1) \oplus \ker(d_x\pi_2)$ in an orthonormal basis of $T_x\Sigma$ adapted to the splitting (C.4). In these basis the matrix of $d_x\pi_1$ has the form $\begin{pmatrix} 0 & I_{n_1-r} & A_1 \\ 0 & 0 & B_1 \end{pmatrix}$ where I_{n_1-r} stands for the identity matrix of size $n_1 - r$. Similarly, the matrix of $d_x\pi_2$ has the form $\begin{pmatrix} I_{n_2-r} & 0 & A_2 \\ 0 & 0 & B_2 \end{pmatrix}$, and the matrices of $\partial_1 F(x)$ and $\partial_2 F(x)$ have the form $\begin{pmatrix} 0 & C_1 \end{pmatrix}$ and $\begin{pmatrix} 0 & C_2 \end{pmatrix}$ respectively. Thus B_1, B_2, C_1 and C_2 are square matrices satisfying the following relations:

$$\left|\det{}^{\perp}(d_x\pi_1)\right| = |\det(B_1)|, \qquad \left|\det{}^{\perp}(\partial_1 F(x))\right| = |\det(C_1)|,$$
$$\left|\det{}^{\perp}(d_x\pi_2)\right| = |\det(B_2)|, \qquad \left|\det{}^{\perp}(\partial_2 F(x))\right| = |\det(C_2)|.$$

Besides the relation $\partial_1 F(x) \circ d_x\pi_1 + \partial_2 F(x) \circ d_x\pi_2 = 0$ means that $C_1 B_1 = -C_2 B_2$, hence $|\det(C_1)|\,|\det(B_1)| = |\det(C_2)|\,|\det(B_2)|$. This proves (C.2). □

An immediate consequence of (C.1) and (C.2) is the following.

Proposition C.3. *Let M_1 and M_2 be two smooth Riemannian manifolds of dimension n_1 and n_2 respectively. Let $F : M_1 \times M_2 \to \mathbb{R}^r$ be a smooth submersion, and let $\Sigma = F^{-1}(0)$. Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function. Then:*

$$\int_{y_1 \in M_1} \left(\int_{\pi_1^{-1}(y_1)} \phi(x)\, |\mathrm{d}V_{y_1}| \right) |\mathrm{d}V_{M_1}| = \int_{y_2 \in M_2} \left(\int_{\pi_2^{-1}(y_2)} \phi(x) \frac{\left|\det{}^{\perp}(\partial_2 F(x))\right|}{\left|\det{}^{\perp}(\partial_1 F(x))\right|} |\mathrm{d}V_{y_2}| \right) |\mathrm{d}V_{M_2}|,$$

whenever one of these integrals is well-defined.

C.3 Proof of theorem 5.3

Finally, we prove theorem 5.3. Let M be a closed Riemannian manifold of dimension n and V be a subspace of $\mathcal{C}^\infty(M, \mathbb{R}^r)$ of dimension N (recall that $1 \leqslant r \leqslant n$). We assume that V is 0-ample, so that $F : (f, x) \mapsto f(x)$ is a smooth submersion from $V \times M$ to $\mathbb{R}^r$ and

$$\Sigma = F^{-1}(0) = \{(f, x) \in V \times M \mid f(x) = 0\}$$

is a submanifold of codimension r of $V \times M$. Let $\mathrm{d}f$ denote the Lebesgue measure on V or on a subspace of V. Let $\phi : \Sigma \to \mathbb{R}$ be a Borel measurable function, by proposition C.3,

$$\begin{aligned}\mathbb{E}\left[\int_{x \in Z_f} \phi(f, x)\, |\mathrm{d}V_f| \right] &= \frac{1}{\sqrt{2\pi}^N} \int_{f \in V} \left(\int_{x \in Z_f} \phi(f, x) e^{-\frac{\|f\|^2}{2}} |\mathrm{d}V_f| \right) \mathrm{d}f \\ &= \frac{1}{\sqrt{2\pi}^N} \int_{x \in M} \left(\int_{f \in \ker(j_x^0)} \phi(f, x) e^{-\frac{\|f\|^2}{2}} \frac{\left|\det{}^{\perp}(\partial_2 F(x))\right|}{\left|\det{}^{\perp}(\partial_1 F(x))\right|} \mathrm{d}f \right) |\mathrm{d}V_M|.\end{aligned}$$

Recall that for all $x \in M$, $j_x^0 : f \mapsto f(x)$ is onto, since V is 0-ample. In particular, $\ker(j_x^0)$ has codimension r and V splits as $\ker(j_x^0) \oplus \ker(j_x^0)^\perp$. We recognize the innermost integral to be a conditional expectation given $f(x) = 0$ (see corollary A.11). Thus,

$$\mathbb{E}\left[\int_{x \in Z_f} \phi(f, x)\, |\mathrm{d}V_f| \right] = \frac{1}{\sqrt{2\pi}^r} \int_{x \in M} \mathbb{E}\left[\phi(f, x) \frac{\left|\det{}^{\perp}(\partial_2 F(x))\right|}{\left|\det{}^{\perp}(\partial_1 F(x))\right|} \;\middle|\; f(x) = 0 \right] |\mathrm{d}V_M|. \tag{C.5}$$

By equation (2.4), $\left|\det{}^{\perp}(\partial_2 F(x))\right| = \left|\det{}^{\perp}(d_x f)\right|$ and

$$\left|\det{}^{\perp}(\partial_1 F(x))\right| = \left|\det{}^{\perp}\left(j_x^0\right)\right| = \sqrt{\det\left(j_x^0 (j_x^0)^*\right)}. \tag{C.6}$$

Since $f \sim \mathcal{N}(0, \mathrm{Id})$, equation (A.8) shows that $j^0_x j^{0*}_x$ is the variance operator of $f(x) = j^0_x(f)$. Then, by (2.9), $\det\left(j^0_x(j^0_x)^*\right) = \det(\mathrm{Var}(f(x))) = \det(E(x,x))$. In particular, this quantity does not depend on $f \in V$. Equations (C.5), (C.6) and this last equality prove theorem 5.3.

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