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Expected volume and Euler characteristic of random submanifolds

Thomas Letendre *

August 2, 2014

Abstract

In a closed manifold of positive dimension n , we estimate the expected volume and Euler characteristic for random submanifolds of codimension $r \in \{1, \dots, n\}$ in two different settings. On one hand, we consider a closed Riemannian manifold and some positive λ . Then we take r independent random functions in the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues less than λ^2 and consider the random submanifold defined as the common zero set of these r functions. We compute asymptotics for the mean volume and Euler characteristic of this random submanifold as λ goes to infinity. On the other hand, we consider a complex projective manifold defined over the reals, equipped with an ample line bundle \mathcal{L} and a rank r holomorphic vector bundle \mathcal{E} that are also defined over the reals. Then we get asymptotics for the expected volume and Euler characteristic of the vanishing locus of a random real holomorphic section of $\mathcal{E} \otimes \mathcal{L}^d$ as d goes to infinity. The same techniques apply to both settings.

Keywords: Euler characteristic, Riemannian random wave, spectral function, random polynomial, real projective manifold, ample line bundle, Bergman kernel, Gaussian field.

Mathematics Subject Classification 2010: 14P25, 32A25, 34L20, 60D05, 60G60.

1 Introduction

Zeros of random polynomials were first studied by Bloch and Pòlya [7] in the early 30's. About ten years later, Kac [22] obtained a sharp asymptotic for the expected number of real zeros of a polynomial of degree d with independent standard Gaussian coefficients, as d goes to infinity. This was later generalized to other distributions by Kostlan in [23]. In particular, he introduced a normal distribution on the space of homogeneous polynomials of degree d – known as the Kostlan distribution – which is more geometric, in the sense that it is invariant under isometries of $\mathbb{C}P^1$. Bogomolny, Bohigas and Leboeuf [8] showed that this distribution corresponds to the choice of d independent roots, uniformly distributed in the Riemann sphere.

In higher dimension, the question of the number of zeros can be generalized in at least two ways. What is the expected volume of the zero set? And what is its expected Euler characteristic? More generally, one can ask what are the expected volume and Euler characteristic of a random submanifold obtained as the zero set of some Gaussian field on a

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Riemannian manifold. In this paper, we provide an asymptotic answer to these questions in the case of Riemannian random waves and in the case of real algebraic manifolds.

Let us describe our frameworks and state the main results of this paper. See section 5 for more details. Let (M, g) be a closed smooth Riemannian manifold of positive dimension n equipped with the Riemannian measure $|dV_M|$ associated to g . This induces a L^2 -inner product on $\mathcal{C}^\infty(M)$ defined by:

$$\forall \phi, \psi \in \mathcal{C}^\infty(M), \quad \langle \phi, \psi \rangle = \int_{x \in M} \phi(x)\psi(x) |dV_M|.$$

It is well-known that the subspace $V_\lambda \subset \mathcal{C}^\infty(M)$ spanned by the eigenfunctions of the Laplacian associated to eigenvalues smaller than λ^2 has finite dimension. Let $1 \leq r \leq n$ and let $f \in (V_\lambda)^r$ be a standard Gaussian vector, we denote by Z_f the zero set of f . Then, for λ large enough, Z_f is almost surely a submanifold of M of codimension r (see section 7 below) and we denote by $\text{Vol}(Z_f)$ its Riemannian volume for the restriction of g to Z_f . We also denote by $\chi(Z_f)$ its Euler characteristic.

Theorem 1.1. *Let (M, g) be a closed Riemannian manifold of dimension n . Let V_λ be the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues smaller than λ^2 . Let f be a standard Gaussian vector in $(V_\lambda)^r$, with $1 \leq r \leq n$. Then the following holds as λ goes to infinity:*

$$\mathbb{E}[\text{Vol}(Z_f)] = \left(\frac{\lambda}{\sqrt{n+2}} \right)^r \text{Vol}(M) \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)} + O(\lambda^{r-1}).$$

Here and throughout this paper, $\mathbb{E}[\cdot]$ denotes the mathematical expectation of the quantity between the brackets.

If $n-r$ is odd, Z_f is almost surely a smooth manifold of odd dimension. In this case, $\chi(Z_f) = 0$ almost surely. If $n-r$ is even, we get the following result.

Theorem 1.2. *Let (M, g) be a closed Riemannian manifold of dimension n . Let V_λ be the direct sum of the eigenspaces of the Laplace-Beltrami operator associated to eigenvalues smaller than λ^2 . Let f be a standard Gaussian vector in $(V_\lambda)^r$, with $1 \leq r \leq n$. Then, if $n-r$ is even, the following holds as λ goes to infinity:*

$$\mathbb{E}[\chi(Z_f)] = (-1)^{\frac{n-r}{2}} \left(\frac{\lambda}{\sqrt{n+2}} \right)^n \text{Vol}(M) \frac{\text{Vol}(\mathbb{S}^{n-r+1}) \text{Vol}(\mathbb{S}^{r-1})}{\pi \text{Vol}(\mathbb{S}^n) \text{Vol}(\mathbb{S}^{n-1})} + O(\lambda^{n-1}).$$

We also consider the framework of the papers [15, 16] by Gayet and Welschinger, see section 5.6 for more details. Let \mathcal{X} be a smooth complex projective manifold of complex dimension n . Let \mathcal{L} be an ample holomorphic line bundle over \mathcal{X} and \mathcal{E} be a holomorphic vector bundle over \mathcal{X} of rank r . We assume that \mathcal{X} , \mathcal{L} and \mathcal{E} are equipped with compatible real structures and that the real locus $\mathbb{R}\mathcal{X}$ of \mathcal{X} is non-empty. Let $h_{\mathcal{L}}$ denote a Hermitian metric on \mathcal{L} with positive curvature ω and $h_{\mathcal{E}}$ denote a Hermitian metric on \mathcal{E} . Both metrics are assumed to be compatible with the real structures. Then ω is a Kähler form on \mathcal{X} and it induces a Riemannian metric g and a volume form $dV_{\mathcal{X}} = \frac{\omega^n}{n!}$ on \mathcal{X} . For any $d \in \mathbb{N}$, the space $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ is then equipped with the L^2 -inner product:

$$\forall s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \quad \langle s_1, s_2 \rangle = \int_{x \in \mathcal{X}} (h_{\mathcal{E}} \otimes h_{\mathcal{L}}^d)(s_1(x), s_2(x)) dV_{\mathcal{X}}.$$

Let $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the space of real global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$. This is a Euclidean space for the above inner product. Let s be a standard Gaussian section in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. We denote by Z_s the real part of its zero set. Once again, for d large enough, Z_s is almost surely a smooth submanifold of $\mathbb{R}\mathcal{X}$ of codimension r . Let $\text{Vol}(Z_s)$ denote the Riemannian volume of Z_s and $\chi(Z_s)$ denote its Euler characteristic. We get the analogues of theorems 1.1 and 1.2 in this setting.

Theorem 1.3. *Let \mathcal{X} be a complex projective manifold of dimension n defined over the reals and $r \in \{1, \dots, n\}$. Let \mathcal{L} be an ample holomorphic Hermitian line bundle over \mathcal{X} and \mathcal{E} be a rank r holomorphic Hermitian vector bundle over \mathcal{X} , both equipped with real structures compatible with the one on \mathcal{X} . Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then the following holds as d goes to infinity:*

$$\mathbb{E}[\text{Vol}(Z_s)] = (\sqrt{d})^r \text{Vol}(\mathbb{R}\mathcal{X}) \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)} + O(\sqrt{d}^{r-2}).$$

Theorem 1.4. *Let \mathcal{X} be a complex projective manifold of dimension n defined over the reals and $r \in \{1, \dots, n\}$. Let \mathcal{L} be an ample holomorphic Hermitian line bundle over \mathcal{X} and \mathcal{E} be a rank r holomorphic Hermitian vector bundle over \mathcal{X} , both equipped with real structures compatible with the one on \mathcal{X} . Let s be a standard Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Then, if $n - r$ is even, the following holds as d goes to infinity:*

$$\mathbb{E}[\chi(Z_s)] = (-1)^{\frac{n-r}{2}} (\sqrt{d})^n \text{Vol}(\mathbb{R}\mathcal{X}) \frac{\text{Vol}(\mathbb{S}^{n-r+1}) \text{Vol}(\mathbb{S}^{r-1})}{\pi \text{Vol}(\mathbb{S}^n) \text{Vol}(\mathbb{S}^{n-1})} + O(\sqrt{d}^{n-2}).$$

In the case of random eigenfunctions of the Laplacian, theorem 1.1 was already known to Bérard [9] for hypersurfaces. See also [33, theorem 1] where Zelditch shows that, in the case of hypersurfaces, $\frac{1}{\lambda} \mathbb{E}[Z_f] \rightarrow |dV_M|$ as $\lambda \rightarrow +\infty$, in the sense of currents. He also proves a similar result in the case of band limited eigenfunctions.

Let us discuss theorems 1.3 and 1.4 when \mathcal{X} equals $\mathbb{C}P^n$ with the standard real structure induced by the conjugation in \mathbb{C}^{n+1} , with \mathcal{E} equal to the trivial bundle $\mathcal{X} \times \mathbb{C}^r$ and $\mathcal{L} = \mathcal{O}(1)$ with its usual metric. Then $\mathbb{R}\mathcal{X} = \mathbb{R}P^n$ and ω is the Fubini-Study metric on $\mathbb{C}P^n$, normalized so that it is the quotient of the Euclidean metric on the sphere \mathbb{S}^{2n+1} . Besides, $\mathbb{R}H^0(\mathcal{X}, \mathcal{L}^d)$ is the space of real homogeneous polynomials of degree d in $n+1$ variables, and Z_s is the set of simultaneous real zeros of r independent Gaussian such polynomials. Then, theorems 1.3 and 1.4 read respectively:

$$(1.1) \quad \mathbb{E}[\text{Vol}(Z_s)] = (\sqrt{d})^r \text{Vol}(\mathbb{R}P^{n-r}) + O(\sqrt{d}^{r-2}) \quad \text{and,}$$

$$(1.2) \quad \mathbb{E}[\chi(Z_s)] = (-1)^{\frac{n-r}{2}} (\sqrt{d})^n \frac{\text{Vol}(\mathbb{R}P^{n-r+1}) \text{Vol}(\mathbb{R}P^{r-1})}{\pi \text{Vol}(\mathbb{R}P^{n-1})} + O(\sqrt{d}^{n-2}).$$

In this setting (1.1) was proved by Kostlan [23]. In fact, Kostlan proved that for any $d \geq 1$, $\mathbb{E}[\text{Vol}(Z_s)] = (\sqrt{d})^r \text{Vol}(\mathbb{R}P^{n-r})$. See also the paper [30] by Shub and Smale, where they compute the expected number of common real roots for a system of n polynomials in n variables. The expected Euler characteristic of a random algebraic hypersurface of degree d in $\mathbb{R}P^n$ was computed by Podkorytov [28]. Both Kostlan's and Podkorytov's results were generalized by Bürgisser. In [10], he computed the expected volume and Euler characteristic of a submanifold of $\mathbb{R}P^n$ defined as the common zero set of r standard Gaussian polynomials P_1, \dots, P_r of degree d_1, \dots, d_r respectively. In particular, when these polynomials have the same degree d and $n - r$ is even, he showed that:

$$(1.3) \quad \mathbb{E}[\chi(Z_{(P_1, \dots, P_r)})] = (\sqrt{d})^r \sum_{p=1}^{\frac{n-r}{2}} (1-d)^p \frac{\Gamma(p + \frac{r}{2})}{p! \Gamma(\frac{r}{2})},$$

where Γ denotes Euler's gamma function. Equation (1.2) agrees with this result.

Recently, Gayet and Welschinger computed upper and lower bounds for the asymptotics of the expected Betti numbers of random real algebraic submanifolds of a projective manifold, see [15, 16]. This relies on sharp estimates for the expected number of critical points of index

$i \in \{0, \dots, n-r\}$ of a fixed Morse function $p : \mathbb{R}\mathcal{X} \rightarrow \mathbb{R}$ restricted to the random Z_s . More precisely, let $N_i(Z_s)$ denote the number of critical points of index i of $p|_{Z_s}$, let $\text{Sym}(i, n-r-i)$ denote the open cone of symmetric matrices of size $n-r$ and signature $(i, n-r-i)$ and let $d\mu$ denote the standard Gaussian measure on the space of symmetric matrices. Gayet and Welschinger show [16, theorem 3.1.2] that:

$$(1.4) \quad \mathbb{E}[N_i(Z_s)] \underset{d \rightarrow +\infty}{\sim} \frac{\sqrt{d}^n}{\sqrt{\pi}^n} \text{Vol}(\mathbb{R}\mathcal{X}) \frac{(n-1)!}{(n-r)!2^{r-1}\Gamma\left(\frac{r}{2}\right)} e_{\mathbb{R}}(i, n-r-i),$$

$$\text{where } e_{\mathbb{R}}(i, n-r-i) = \int_{\text{Sym}(i, n-r-i)} |\det(A)| d\mu(A).$$

One can indirectly deduce theorem 1.4 from this result and from [10] in the following way. By Morse theory:

$$\mathbb{E}[\chi(Z_s)] = \sum_{i=0}^{n-r} (-1)^i \mathbb{E}[N_i(Z_s)] \underset{d \rightarrow +\infty}{\sim} C_{n,r} \sqrt{d}^n \text{Vol}(\mathbb{R}\mathcal{X})$$

where $C_{n,r}$ is a universal constant depending only on n and r . Specifying to the case of $\mathbb{R}P^n$, equation (1.3) gives the value of $C_{n,r}$. Gayet and Welschinger also proved a result similar to (1.4) for hypersurfaces in the case of Riemannian random waves, see [14]. It gives the order of growth of $\mathbb{E}[\chi(Z_f)]$ in theorem 1.2, for $r=1$.

In their book [1], Taylor and Adler compute the expected Euler characteristic of the excursion sets of a centered, unit-variance Gaussian field f on a smooth manifold M . This expectation is given in terms of the Lipschitz-Killing curvatures of M for the Riemannian metric g_f induced by f , see [1, theorem 12.4.1]. One can deduce from this result the expected Euler characteristic of the random hypersurface $f^{-1}(0)$, always in terms of the Lipschitz-Killing curvatures of (M, g_f) . It might be possible to deduce theorems 1.2 and 1.4 from this result, in the case of hypersurfaces, when the Gaussian field $(f(x))_{x \in M}$ (resp. $(s(x))_{x \in \mathbb{R}\mathcal{X}}$) has a constant covariance function (see section 5.3 for the definitions of these objects), but one would need to estimate the Lipschitz-Killing curvatures of (M, g_f) (resp. (M, g_s)) as λ (resp. d) goes to infinity.

In a related setting, Bleher Shiffman and Zelditch [5] computed the scaling limit of the “ k -points correlation function” for a random complex submanifold of a complex projective manifold. See also [6] in a symplectic framework. In particular, for $k=1$, one can deduce the expected volume of a random submanifold from their result. It is possible to adapt the formalism of those papers to our frameworks and to use it to prove our results on the expected volume (theorems 1.1 and 1.3). We give a different proof of these results which is the toy-model for the proof of theorems 1.2 and 1.4.

We now sketch the proofs of our main results in the harmonic setting. The real algebraic case is similar. The first step is to express $\text{Vol}(Z_f)$ (resp. $\chi(Z_f)$) as the integral of some function on Z_f . In the case of the volume this is trivial, and the answer is given by the Chern-Gauss-Bonnet theorem (see section 3.2 below) in the case of the Euler characteristic. Then we apply Federer’s coarea formula twice (see section 2) as in [15, 26]. This allows us to express $\mathbb{E}[\text{Vol}(Z_f)]$ (resp. $\mathbb{E}[\chi(Z_f)]$) as the integral on M of some explicit function that only depends on the geometry of M and on the covariance function of the smooth Gaussian field defined by f .

It turns out that the covariance function of the field associated to a standard Gaussian $f \in (V_\lambda)^r$ is given by the Schwartz kernel of the orthogonal projection from $\mathcal{C}^\infty(M, \mathbb{R}^r)$ onto $(V_\lambda)^r$, for the L^2 -inner product defined above. In the harmonic case, this kernel is related to the spectral function of the Laplacian. In the algebraic case, the covariance function is given by the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. This was already used in [5, 26].

Then, our results follow from estimates on the kernels and their derivatives (see section 6). In the case of random waves, the estimates we need for the spectral function of the Laplacian were proved by Bin [4], generalizing results of Hörmander [20]. In the algebraic case, much is known about the Bergman kernel [3, 5, 25, 32] but we could not find the estimates we needed in codimension higher than 1 in the literature. These estimates are established in section 6.3 using Hörmander-Tian peak sections. Peak sections were already used in this context in [15, 16], see also [31]. The author was told by Steve Zelditch, after this paper was written, that one can deduce estimates for the Bergman kernel in higher codimension from the paper [3] by Berman, Berndtsson and Sjöstrand.

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2 The double-fibration trick

In this section, we start by recalling Federer's coarea formula in the case of a smooth map between smooth Riemannian manifolds. A proof in this special case can be found in the appendix of [19]. See [12, theorem 3.2.12] for the general case. Then we describe what Nicolaescu [26] refers to as the double-fibration trick, that is apply the coarea formula twice for different fibrations, which provides an alternative to the use of Kac-Rice type formulas. This was already used in this context in [15, 26].

2.1 The coarea formula

Let $L : V \rightarrow V'$ be a linear map between two Euclidean vector spaces of dimension m and n respectively. We define the *orthogonal determinant* $|\det^\perp(L)|$ of L in the following way.

Definition 2.1. Let A be the matrix of the restriction of L to $\ker(L)^\perp$ in any orthonormal basis of $\ker(L)^\perp$ and V' . Then we set:

$$|\det^\perp(L)| = \begin{cases} |\det(A)| & \text{if } L \text{ has rank } n, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, one can set $|\det^\perp(L)| = \sqrt{\det(LL^*)}$ where L^* denotes the adjoint operator of L . In particular, $|\det^\perp(L)|$ does not depend on a choice of orthonormal basis.

Let $\pi : \tilde{M} \rightarrow M$ be a smooth map between two smooth Riemannian manifolds of respective dimensions m and n . We assume that $m \geq n$. Let $|dV_{\tilde{M}}|$ (resp. $|dV_M|$) denote the Riemannian measure on \tilde{M} (resp. M) induced by its metric. By Sard's theorem, for almost every $y \in M$, $\pi^{-1}(y)$ is a smooth submanifold of dimension $(m - n)$ of \tilde{M} . For such $y \in M$, we denote by $|dV_y|$ the Riemannian measure on $\pi^{-1}(y)$ induced by the metric of \tilde{M} . When $m = n$, the dimension $\pi^{-1}(y)$ is 0 and $|dV_y|$ is just $\sum_{x \in \pi^{-1}(y)} \delta_x$ where δ_x is the Dirac measure at the point x . The following result is proved in the appendix of [19].

Theorem 2.2 (Smooth coarea formula). *Let $\pi : \tilde{M} \rightarrow M$ be a smooth map between smooth Riemannian manifolds of dimension m and n respectively, with $m \geq n$. Let $\phi : \tilde{M} \rightarrow \mathbb{R}$ be a Borel measurable function. Then:*

$$\int_{x \in \tilde{M}} \phi(x) |\det^\perp(d_x \pi)| |dV_{\tilde{M}}| = \int_{y \in M} \left(\int_{x \in \pi^{-1}(y)} \phi(x) |dV_y| \right) |dV_M|,$$

whenever one of these integrals is well-defined.

Note that the innermost integral on the right-hand side is only defined almost everywhere.

2.2 The double-fibration trick

Let M_1 and M_2 be two smooth Riemannian manifolds of dimension n_1 and n_2 respectively. Let $F : M_1 \times M_2 \rightarrow \mathbb{R}^r$ be a smooth submersion, and let $\Sigma = F^{-1}(0)$. We equip Σ with the restriction of the product metric on $M_1 \times M_2$ and denote by $|dV_{M_1}|$, $|dV_{M_2}|$ and $|dV_\Sigma|$ the Riemannian measures on the corresponding manifolds. Finally, let $\pi_1 : \Sigma \rightarrow M_1$ and $\pi_2 : \Sigma \rightarrow M_2$ be the projections from Σ to each factor.

Assuming that $r \leq \min(n_1, n_2)$, we have $\dim(\Sigma) = n_1 + n_2 - r \geq \max(n_1, n_2)$. Thus we can apply the coarea formula twice, for π_1 and π_2 . Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a Borel measurable

function, then:

$$(2.1) \quad \int_{y_1 \in M_1} \left(\int_{x \in \pi_1^{-1}(y_1)} \phi(x) |dV_{y_1}| \right) |dV_{M_1}| = \int_{x \in \Sigma} \phi(x) |\det^\perp(d_x \pi_1)| |dV_\Sigma| \\ = \int_{y_2 \in M_2} \left(\int_{x \in \pi_2^{-1}(y_2)} \phi(x) \frac{|\det^\perp(d_x \pi_1)|}{|\det^\perp(d_x \pi_2)|} |dV_{y_2}| \right) |dV_{M_2}|,$$

whenever one of these integrals is well-defined. Note that if $|\det^\perp(d_x \pi_2)|$ vanishes then $\pi_2(x)$ is a critical value of π_2 , and the set of such critical values has measure 0 in M_2 .

Let $\partial_1 F$ and $\partial_2 F$ denote the partial differentials of F with respect to the first and second variable respectively. For any $x = (x_1, x_2) \in \Sigma$,

$$T_x \Sigma = \{(v_1, v_2) \in T_{x_1} M_1 \times T_{x_2} M_2 \mid \partial_1 F(x) \cdot v_1 + \partial_2 F(x) \cdot v_2 = 0\}.$$

Lemma 2.3. *Let $x \in \Sigma$, then $|\det^\perp(d_x \pi_2)| = 0$ if and only if $|\det^\perp(\partial_1 F(x))| = 0$. Moreover,*

$$(2.2) \quad |\det^\perp(d_x \pi_1)| |\det^\perp(\partial_1 F(x))| = |\det^\perp(d_x \pi_2)| |\det^\perp(\partial_2 F(x))|.$$

Proof. First note that $d_x F = \partial_1 F(x) \circ d_x \pi_1 + \partial_2 F(x) \circ d_x \pi_2 = 0$ on $T_x \Sigma$. This shows that:

$$(2.3) \quad \ker(d_x \pi_1) = \{0\} \times \ker(\partial_2 F(x)) \quad \text{and} \quad \ker(d_x \pi_2) = \ker(\partial_1 F(x)) \times \{0\}.$$

The space $T_x \Sigma$ splits as the following orthogonal direct sum:

$$(2.4) \quad T_x \Sigma = \ker(d_x \pi_1) \oplus \ker(d_x \pi_2) \oplus G,$$

where G is the orthogonal complement of $\ker(d_x \pi_1) \oplus \ker(d_x \pi_2)$.

Then, $|\det^\perp(d_x \pi_2)| = 0$ means that $d_x \pi_2$ is not surjective. Recalling that $\dim(M_2) = n_2$ and $\dim(\Sigma) = n_1 + n_2 - r$, this is equivalent to $\dim(\ker(d_x \pi_2)) > n_1 - r$. In the same way, $|\det^\perp(\partial_1 F(x))| = 0$ if and only if $\dim(\ker(\partial_1 F(x))) > n_1 - r$. But $\ker(d_x \pi_2)$ and $\ker(\partial_1 F(x))$ have the same dimension by (2.3), so that $|\det^\perp(d_x \pi_2)| = 0$ if and only if $|\det^\perp(\partial_1 F(x))| = 0$. A similar argument shows that $|\det^\perp(d_x \pi_1)| = 0$ if and only if $|\det^\perp(\partial_2 F(x))| = 0$. Thus the lemma is true if any of the four maps in (2.2) is singular.

From now on, we assume that these maps are all surjective. In this case, we have:

$$\dim(G) = r, \quad \dim(\ker(\partial_1 F(x))) = \dim(\ker(d_x \pi_2)) = n_1 - r, \\ \text{and} \quad \dim(\ker(\partial_2 F(x))) = \dim(\ker(d_x \pi_1)) = n_2 - r.$$

We choose an orthonormal basis of $T_x M_1$ adapted to $\ker(\partial_1 F(x)) \oplus \ker(\partial_1 F(x))^\perp$ and an orthonormal basis of $T_x M_2$ adapted to $\ker(\partial_2 F(x)) \oplus \ker(\partial_2 F(x))^\perp$. From these, we deduce orthonormal basis of $\ker(d_x \pi_1) = \{0\} \times \ker(\partial_2 F(x))$ and $\ker(d_x \pi_2) = \ker(\partial_1 F(x)) \times \{0\}$. Finally we complete the resulting basis of $\ker(d_x \pi_1) \oplus \ker(d_x \pi_2)$ in an orthonormal basis of $T_x \Sigma$ adapted to the splitting (2.4).

In these basis the matrix of $d_x \pi_1$ has the form $\begin{pmatrix} 0 & I_{n_1-r} & A_1 \\ 0 & 0 & B_1 \end{pmatrix}$ where I_{n_1-r} stands for the

identity matrix of size $n_1 - r$. Similarly, the matrix of $d_x \pi_2$ has the form $\begin{pmatrix} I_{n_2-r} & 0 & A_2 \\ 0 & 0 & B_2 \end{pmatrix}$,

and the matrices of $\partial_1 F(x)$ and $\partial_2 F(x)$ have the form $\begin{pmatrix} 0 & C_1 \end{pmatrix}$ and $\begin{pmatrix} 0 & C_2 \end{pmatrix}$ respectively. Thus B_1, B_2, C_1 and C_2 are square matrices satisfying the following relations:

$$|\det^\perp(d_x \pi_1)| = |\det(B_1)|, \quad |\det^\perp(\partial_1 F(x))| = |\det(C_1)|, \\ |\det^\perp(d_x \pi_2)| = |\det(B_2)|, \quad |\det^\perp(\partial_2 F(x))| = |\det(C_2)|.$$

Besides the relation $\partial_1 F(x) \circ d_x \pi_1 + \partial_2 F(x) \circ d_x \pi_2 = 0$ means that $C_1 B_1 = -C_2 B_2$, hence $|\det(C_1)| |\det(B_1)| = |\det(C_2)| |\det(B_2)|$. This proves (2.2). \square

An immediate consequence of (2.1) and (2.2) is the following.

Proposition 2.4. *Let M_1 and M_2 be two smooth Riemannian manifolds of dimension n_1 and n_2 respectively. Let $F : M_1 \times M_2 \rightarrow \mathbb{R}^r$ be a smooth submersion, and let $\Sigma = F^{-1}(0)$. Let $\phi : \Sigma \rightarrow \mathbb{R}$ be a Borel measurable function. Then:*

$$\int_{y_1 \in M_1} \left(\int_{\pi_1^{-1}(y_1)} \phi(x) |dV_{y_1}| \right) |dV_{M_1}| = \int_{y_2 \in M_2} \left(\int_{\pi_2^{-1}(y_2)} \phi(x) \frac{|\det^\perp(\partial_2 F(x))|}{|\det^\perp(\partial_1 F(x))|} |dV_{y_2}| \right) |dV_{M_2}|,$$

where $\pi_1 : \Sigma \rightarrow M_1$ and $\pi_2 : \Sigma \rightarrow M_2$ denote the projections from Σ to each factor.

3 Material from Riemannian geometry

This section is mostly concerned with double forms. We start by recalling the formalism of double forms, which was already used in this context by Taylor and Adler, see [1, section 7.2]. The Riemann curvature tensor and the second fundamental form of a submanifold being naturally double forms, this provides a useful way to formulate the Chern-Gauss-Bonnet theorem and the Gauss equation. Finally we express the second fundamental form of a submanifold in terms of the derivatives of a defining function.

3.1 The algebra of double forms

We follow the exposition of [1, pp. 157–158]. Let V be a real vector space of dimension n . For p and $q \in \{0, \dots, n\}$ we denote by $\Lambda^{p,q}(V^*)$ the space $\Lambda^p(V^*) \otimes \Lambda^q(V^*)$ of $(p+q)$ -linear forms on V that are skew-symmetric in the first p and in the last q variables. The *space of double forms* on V is:

$$\Lambda^\bullet(V^*) \otimes \Lambda^\bullet(V^*) = \bigoplus_{0 \leq p, q \leq n} \Lambda^{p,q}(V^*).$$

Elements of $\Lambda^{p,q}(V^*)$ are called (p, q) -double forms, or *double forms of type (p, q)* . We also set: $\Lambda^{\bullet,\bullet}(V^*) = \bigoplus_{p=0}^n \Lambda^{p,p}(V^*)$. Note that the space $\Lambda^{1,1} V^*$ of $(1, 1)$ -double forms is merely the space of bilinear forms on V .

On $\Lambda^\bullet(V^*) \otimes \Lambda^\bullet(V^*)$ we can define a double wedge product. It extends the usual wedge product on $\Lambda^\bullet(V^*) \simeq \bigoplus_{p=0}^n \Lambda^{p,0}(V^*)$, so we simply denote it by \wedge . For pure tensors $\alpha \otimes \beta$ and $\alpha' \otimes \beta' \in \Lambda^\bullet(V^*) \otimes \Lambda^\bullet(V^*)$, we define:

$$(\alpha \otimes \beta) \wedge (\alpha' \otimes \beta') = (\alpha \wedge \alpha') \otimes (\beta \wedge \beta')$$

and we extend \wedge to all double forms by bilinearity. This makes $\Lambda^\bullet(V^*) \otimes \Lambda^\bullet(V^*)$ into an algebra, of which $\Lambda^{\bullet,\bullet}(V^*)$ is a commutative subalgebra. We denote by $\gamma^{\wedge k}$ the double wedge product of a double form $\gamma \in \Lambda^{\bullet,\bullet}(V^*)$ with itself k times.

Lemma 3.1. *Let α be a symmetric $(1, 1)$ -double form on V , then for every x, y, z and $w \in V$,*

$$\alpha^{\wedge 2}((x, y), (z, w)) = 2(\alpha(x, z)\alpha(y, w) - \alpha(x, w)\alpha(y, z)).$$

Proof. Let (e_1, \dots, e_n) be a basis of V and (e_1^*, \dots, e_n^*) its dual basis. We have:

$$\alpha = \sum_{1 \leq i, k \leq n} \alpha_{ik} e_i^* \otimes e_k^* \quad \text{and then} \quad \alpha^{\wedge 2} = \sum_{1 \leq i, j, k, l \leq n} \alpha_{ik} \alpha_{jl} (e_i^* \wedge e_j^*) \otimes (e_k^* \wedge e_l^*).$$

Note that we do not restrict ourselves to indices satisfying $i < j$ and $k < l$ as is usually the case with skew-symmetric forms. By multilinearity, it is sufficient to check the result on elements of the basis. Let i, j, k and $l \in \{1, \dots, n\}$, then,

$$\begin{aligned} \alpha^{\wedge 2}((e_i, e_j), (e_k, e_l)) &= \alpha_{ik}\alpha_{jl} - \alpha_{jk}\alpha_{il} - \alpha_{il}\alpha_{jk} + \alpha_{jl}\alpha_{ik} \\ &= 2(\alpha_{ik}\alpha_{jl} - \alpha_{il}\alpha_{jk}) && \text{using the the symmetry of } \alpha, \\ &= 2(\alpha(e_i, e_k)\alpha(e_j, e_l) - \alpha(e_i, e_l)\alpha(e_j, e_k)). \end{aligned} \quad \square$$

Assume now that V is endowed with an inner product. It induces a natural inner product on $\bigwedge^{\bullet}(V^*)$ such that, if (e_1, \dots, e_n) is an orthonormal basis of V then

$$\left\{ e_{i_1}^* \wedge \dots \wedge e_{i_p}^* \mid 1 \leq p \leq n \text{ and } 1 \leq i_1 < i_2 < \dots < i_p \leq n \right\}$$

is an orthonormal basis of $\bigwedge^{\bullet}(V^*)$. We define the *trace operator* Tr on $\bigwedge^{\bullet, \bullet}(V^*)$ in the following way. If $\alpha \otimes \beta \in \bigwedge^{\bullet, \bullet}(V^*)$ is a pure tensor, then:

$$\text{Tr}(\alpha \otimes \beta) = \langle \alpha, \beta \rangle$$

and we extend Tr to $\bigwedge^{\bullet, \bullet}(V^*)$ by linearity.

Notation 3.2. Here and throughout this paper $\langle \cdot, \cdot \rangle$ will always denote the inner product on the concerned Euclidean or Hermitian space.

Let M be a smooth manifold of dimension n . Applying the previous construction pointwise to $T_x M$, we define the vector bundle $\bigwedge^{\bullet}(T^*M) \otimes \bigwedge^{\bullet}(T^*M)$ on M . Sections of this bundle are called *differential double forms* on M , and we can take the double wedge product of two such sections. Finally, if M is equipped with a Riemannian metric, we have a trace operator Tr which is $\mathcal{C}^\infty(M)$ -linear and takes sections of the subbundle $\bigwedge^{\bullet, \bullet}(T^*M) = \bigoplus_{p=0}^n \bigwedge^{p,p}(T^*M)$ to smooth functions.

3.2 The Chern-Gauss-Bonnet theorem

Let (M, g) be a smooth Riemannian manifold of dimension n . We denote by ∇^M the Levi-Civita connection of M , and by κ its *curvature operator*. That is κ is the 2-form on M with values in $\text{End}(TM)$ defined by:

$$\kappa(X, Y)Z = \nabla_X^M \nabla_Y^M Z - \nabla_Y^M \nabla_X^M Z - \nabla_{[X, Y]}^M Z$$

for any vector fields X, Y and Z . We denote by R the *Riemann curvature tensor* of M , defined by:

$$R(X, Y, Z, W) = g(\kappa(X, Y)W, Z),$$

for any vector fields X, Y, Z and W on M . This defines a four times covariant tensor on M which is skew-symmetric in the first two and in the last two variables, thus R can naturally be seen as a double form of type $(2, 2)$. All this is standard material, except for the very last point, see for example [21, section 3.3]. We now state the Chern-Gauss-Bonnet theorem in terms of double forms.

Theorem 3.3 (Chern-Gauss-Bonnet). *Let M be a closed (that is compact without boundary) Riemannian manifold of even dimension n . Let R denote its Riemann curvature tensor and $\chi(M)$ denote its Euler characteristic. We have:*

$$\chi(M) = \frac{1}{(2\pi)^{\frac{n}{2}} \left(\frac{n}{2}\right)!} \int_M \text{Tr}(R^{\wedge \frac{n}{2}}) |dV_M|,$$

where $|dV_M|$ is the Riemannian measure on M .

If M is orientable, this can be deduced from Atiyah-Singer's index theorem. The general case is treated in [27]. The above formula in terms of double forms can be found in [1, theorem 12.6.18], up to a sign coming from different sign conventions in the definition of R .

3.3 The Gauss equation

Let (M, g) be a smooth Riemannian manifold of dimension n and \tilde{M} be a smooth submanifold of M of codimension $r \in \{1, \dots, n-1\}$. We denote by ∇^M and $\tilde{\nabla}$ the Levi-Civita connections on M and \tilde{M} respectively. Likewise, we denote by R and \tilde{R} their Riemann curvature tensor. We denote by Π the *second fundamental form* of $\tilde{M} \subset M$ which is defined as the section of $T^*\tilde{M} \otimes T^*\tilde{M} \otimes T^\perp\tilde{M}$ satisfying:

$$(3.1) \quad \Pi(X, Y) = - \left(\nabla_X^M Y - \tilde{\nabla}_X Y \right) = - \left(\nabla_X^M Y \right)^\perp$$

for any vector fields X and Y on \tilde{M} . Here, $\left(\nabla_X^M Y \right)^\perp$ stands for the component of $\nabla_X^M Y$ in $T^\perp\tilde{M}$. It is well known that Π is symmetric in X and Y , see [21, lemma 3.6.2]. The second fundamental form encodes the difference between \tilde{R} and R in the following sense, see [21, theorem 3.6.2].

Proposition 3.4 (Gauss equation). *Let X, Y, Z and W be vector fields on \tilde{M} , then*

$$(3.2) \quad R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(\Pi(X, W), \Pi(Y, Z)) - g(\Pi(X, Z), \Pi(Y, W)).$$

Let Ξ denote the four times covariant tensor on \tilde{M} defined by:

$$(3.3) \quad \Xi(X, Y, Z, W) = g(\Pi(X, Z), \Pi(Y, W)) - g(\Pi(X, W), \Pi(Y, Z))$$

for any vector fields X, Y, Z and W on \tilde{M} . Then (3.2) can be rewritten as: $\tilde{R} = R + \Xi$.

Remark 3.5. If \tilde{M} is an hypersurface, we can think of $\Pi(x)$ as a $(1, 1)$ -double form on $T_x\tilde{M}$ by contracting it with a unit normal vector to $T_x\tilde{M}$. Then $\Xi = \frac{1}{2} \Pi^{\wedge 2}$ whatever our choice of normal vector. If \tilde{M} is of higher codimension this does not even make sense, but we will find a good substitute using Gaussian vectors, see lemma 4.16 below.

3.4 An expression for the second fundamental form

Let us now express the second fundamental form Π of a submanifold \tilde{M} of M defined as the zero set of a smooth map. For this we need further definitions. Let V and V' be two Euclidean spaces of dimension n and r respectively. Let $L : V \rightarrow V'$ be a linear map of rank r , then L^* is injective and its image is $\ker(L)^\perp$, so that LL^* is invertible in $\text{End}(V')$.

Definition 3.6. Let $L : V \rightarrow V'$ be a linear map of rank r , the *pseudo-inverse* (or *Moore-Penrose inverse*) of L is defined as $L^\dagger = L^*(LL^*)^{-1}$ from V' to V .

The map L^\dagger is the inverse of the restriction of L to $\ker(L)^\perp$. It is characterized by the fact that LL^\dagger is the identity map on V' and $L^\dagger L$ is the orthogonal projection onto $\ker(L)^\perp$. Let $f : M \rightarrow \mathbb{R}^r$ be a smooth submersion and assume that $\tilde{M} = f^{-1}(0)$. Let $\nabla^2 f = \nabla^M df$ denote the *Hessian* of f .

Lemma 3.7. *Let M be a Riemannian manifold and let $\tilde{M} \subset M$ be a submanifold of M defined as the zero set of the smooth submersion $f : M \rightarrow \mathbb{R}^r$. Let Π denote the second fundamental form of $\tilde{M} \subset M$. Then,*

$$\forall x \in \tilde{M}, \quad \Pi(x) = (d_x f)^\dagger \circ \nabla_x^2 f.$$

Proof. Since both $\Pi(x)$ and $(d_x f)^\dagger$ take values in $T_x\tilde{M}^\perp = \ker(d_x f)^\perp$, we only need to prove that, for any $x \in \tilde{M}$, $d_x f \circ \Pi(x) = \nabla_x^2 f$. Let X and Y be two vector fields on \tilde{M} . The map $df \cdot Y$ vanishes uniformly on \tilde{M} . Hence for all $x \in \tilde{M}$:

$$d_x(df \cdot Y) \cdot X = (\nabla_X^M df)_x \cdot Y + d_x f \cdot (\nabla_X^M Y) = 0.$$

Then, using that $\ker(d_x f) = T_x\tilde{M}$,

$$(d_x f \circ \Pi(x))(X, Y) = -d_x f \cdot (\nabla_X^M Y)^\perp = -d_x f \cdot (\nabla_X^M Y) = (\nabla_X^M df)_x \cdot Y = \nabla_x^2 f(X, Y). \quad \square$$

4 Some results about Gaussian vectors

We start this section by a quick reminder of some basic facts about random vectors, especially Gaussian ones. We then state and prove one of the key ingredients in the proof of theorems 1.2 and 1.4. The last subsection is devoted to a probabilistic version of the Gauss equation and some useful technical result on Gaussian double forms. The first two subsections are essentially borrowed from [26, appendix A]. We include them here for the reader's convenience and to introduce some notations.

4.1 Basics of random vectors

Let V be a real vector space of finite dimension and X be a random vector with values in V . For any $\xi \in V^*$, $\xi(X)$ is a real random variable, and from now on, we assume that these variables have finite first and second moments (we will shortly restrict ourselves to Gaussian variables anyway). The *expectation* (or *mean*) of X is the linear form on V^* defined by:

$$(4.1) \quad m_X : \xi \mapsto \mathbb{E}[\xi(X)].$$

Under the canonical isomorphism $V^{**} \simeq V$, $m_X = \int_V x \, dP_X$ where dP_X stands for the distribution of X . If $m_X = 0$, we say that X (resp. dP_X) is *centered*.

The *variance* of X is the non-negative symmetric bilinear form on V^* defined by:

$$(4.2) \quad \text{Var}(X) : (\xi, \eta) \mapsto \mathbb{E}[\xi(X - m_X)\eta(X - m_X)].$$

As a bilinear map from $V^* \times V^*$ to \mathbb{R} , $\text{Var}(X)$ is naturally an element of $V \otimes V$.

Lemma 4.1. *Let X be a random vector in V with finite second moment. We have:*

$$\text{Var}(X) = \mathbb{E}[(X - m_X) \otimes (X - m_X)].$$

Proof. For any ξ and $\eta \in V^*$, we have:

$$\begin{aligned} \text{Var}(X)(\xi, \eta) &= \mathbb{E}[\xi(X - m_X)\eta(X - m_X)] = \mathbb{E}[(\xi \otimes \eta)((X - m_X) \otimes (X - m_X))] \\ &= (\xi \otimes \eta)\mathbb{E}[(X - m_X) \otimes (X - m_X)]. \quad \square \end{aligned}$$

The *variance operator* of X is the linear map $\Lambda_X : V^* \rightarrow V$ such that, for any ξ and $\eta \in V^*$,

$$(4.3) \quad \xi(\Lambda_X \eta) = \text{Var}(X)(\xi, \eta).$$

By lemma 4.1 we have:

$$\Lambda_X : \eta \mapsto \mathbb{E}[(X - m_X) \otimes \eta(X - m_X)].$$

If $V = V_1 \oplus V_2$ and $X = (X_1, X_2)$ with X_i a random vector in V_i , then $m_X = m_{X_1} + m_{X_2}$ and the variance form $\text{Var}(X)$ splits accordingly into four parts:

$$\begin{aligned} \text{Var}(X_1) : V_1^* \times V_1^* &\rightarrow \mathbb{R}, & \text{Cov}(X_1, X_2) : V_1^* \times V_2^* &\rightarrow \mathbb{R}, \\ \text{Var}(X_2) : V_2^* \times V_2^* &\rightarrow \mathbb{R} & \text{and} & \text{Cov}(X_2, X_1) : V_2^* \times V_1^* &\rightarrow \mathbb{R}. \end{aligned}$$

These bilinear maps are associated, as above, to the following operators:

$$\begin{aligned} \Lambda_{11} : V_1^* &\rightarrow V_1, & \Lambda_{12} : V_2^* &\rightarrow V_1, \\ \Lambda_{22} : V_2^* &\rightarrow V_2, & \text{and} & \Lambda_{21} : V_1^* &\rightarrow V_2. \end{aligned}$$

Since $\text{Var}(X)$ is symmetric, $\text{Cov}(X_1, X_2)(\xi, \eta) = \text{Cov}(X_2, X_1)(\eta, \xi)$ for any ξ and η . Equivalently $(\Lambda_{12})^* = \Lambda_{21}$. We say that $\text{Cov}(X_1, X_2)$ is the *covariance form* of X_1 and X_2 , and that Λ_{12} is their *covariance operator*. As above, $\text{Cov}(X_1, X_2)$ is naturally an element of $V_1 \otimes V_2$.

Lemma 4.2. *Let X_1 and X_2 be random vectors in V_1 and V_2 respectively, with finite second moments. We have:*

$$\text{Cov}(X_1, X_2) = \mathbb{E}[(X_1 - m_{X_1}) \otimes (X_2 - m_{X_2})].$$

Moreover, for any $\eta \in V_2^*$, $\Lambda_{12}(\eta) = \mathbb{E}[(X_1 - m_{X_1}) \otimes \eta(X_2 - m_{X_2})]$.

Assuming that V is endowed with an inner product, we denote by v^* the linear form $\langle v, \cdot \rangle$, for any $v \in V$. The Euclidean structure allows us to see $\text{Var}(X)$ as a bilinear symmetric form on V , and Λ_X as a self-adjoint operator on V . Then, lemma 4.1 gives:

$$\Lambda_X = \mathbb{E}[(X - m_X) \otimes (X - m_X)^*].$$

If $X = (X_1, X_2)$, one can see Λ_{12} as a linear operator from V_2 to V_1 and by lemma 4.2:

$$\Lambda_{12} = \mathbb{E}[(X_1 - m_{X_1}) \otimes (X_2 - m_{X_2})^*].$$

Lemma 4.3. *Let V be a Euclidean vector space and let X be a random vector in V with finite second moment. We have:*

$$(4.4) \quad \forall v \in V, \forall w \in V, \quad \mathbb{E}[\langle v, X - m_X \rangle \langle w, X - m_X \rangle] = \langle v, \Lambda_X w \rangle.$$

Moreover, if $V = V_1 \oplus V_2$ and $X = (X_1, X_2)$ with $X_i \in V_i$, then:

$$(4.5) \quad \forall v \in V_1, \forall w \in V_2, \quad \mathbb{E}[\langle v, X_1 - m_{X_1} \rangle \langle w, X_2 - m_{X_2} \rangle] = \langle v, \Lambda_{12} w \rangle.$$

Proof. Let v and $w \in V$, we have:

$$\begin{aligned} \mathbb{E}[\langle v, X - m_X \rangle \langle w, X - m_X \rangle] &= \text{Var}(X)(v, w) && \text{as a bilinear form on } V, \\ &= \langle v, \Lambda_X w \rangle && \text{where } \Lambda_X : V \rightarrow V. \end{aligned}$$

This proves (4.4). The proof of (4.5) is similar. \square

Let $L : V \rightarrow V'$ be a linear map between finite-dimensional vector spaces and X be a random vector in V . Then $L(X)$ is a random vector in V' whose distribution is given by $dP_{L(X)} = L_*(dP_X)$. An immediate consequence of definitions (4.1), (4.2) and (4.3) is that $m_{L(X)} = L_*(m_X)$ and $\text{Var}(L(X)) = L_*(\text{Var}(X))$, that is:

$$(4.6) \quad \forall \xi \in V^*, \quad m_{L(X)}(\xi) = m_X(L^*\xi),$$

$$(4.7) \quad \forall \xi \in V^*, \forall \eta \in V^*, \quad \text{Var}(L(X))(\xi, \eta) = \text{Var}(X)(L^*\xi, L^*\eta), \quad \text{and}$$

$$(4.8) \quad \Lambda_{L(X)} = L\Lambda_X L^*.$$

4.2 Gaussian vectors

The following material can be found either in [2, section 1.2] or in [1, section 1.2]. In both cases, it is written in a coordinate-dependent fashion in \mathbb{R}^n . We present it in a more intrinsic way, in the spirit of [26].

Let $m \in \mathbb{R}$ and $\sigma \geq 0$, the *Gaussian* (or *normal*) distribution on \mathbb{R} with expectation m and variance σ^2 is the distribution whose Fourier transform is $y \mapsto \exp(imy - \frac{1}{2}\sigma^2 y^2)$. If $\sigma = 0$, this is the Dirac measure at m , otherwise it has a density with respect to the Lebesgue measure, given by $x \mapsto \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$.

Let V be a real vector space of dimension n , a random vector X in V is said to be *Gaussian*, or *normally distributed*, if for any $\xi \in V^*$, $\xi(X)$ is a Gaussian variable in \mathbb{R} . Recall that such a distribution has finite moments of all orders and that the distribution of

a Gaussian vector in V is totally determined by its expectation and variance. We denote by $\mathcal{N}(m, \Lambda)$ the Gaussian distribution with expectation equal to m and variance operator equal to Λ . We also denote by $X \sim \mathcal{N}(m, \Lambda)$ the fact that X is distributed according to $\mathcal{N}(m, \Lambda)$. The following lemma is a consequence of (4.6), (4.8) and of the above definition of a Gaussian vector.

Lemma 4.4. *Let $L : V \rightarrow V'$ be a linear map between finite-dimensional vector spaces and $X \sim \mathcal{N}(m, \Lambda)$ in V . Then $L(X) \sim \mathcal{N}(Lm, L\Lambda L^*)$ in V' .*

If $V = V_1 \oplus V_2$ and $X = (X_1, X_2) \sim \mathcal{N}(m, \Lambda)$ then, with the notations of section 4.1, lemma 4.4 shows that $X_1 \sim \mathcal{N}(m_{X_1}, \Lambda_{11})$ and $X_2 \sim \mathcal{N}(m_{X_2}, \Lambda_{22})$. Besides, X_1 and X_2 are independent if and only if $\text{Cov}(X_1, X_2) = 0$, or equivalently $\Lambda_{12} = 0$.

Proposition 4.5 (Regression formula). *Let $X = (X_1, X_2)$ be a Gaussian vector in $V_1 \oplus V_2$. If $\text{Var}(X_1)$ is non-degenerate then X_2 has the same distribution as*

$$m_{X_2} + \Lambda_{21}(\Lambda_{11})^{-1}(X_1 - m_{X_1}) + Y$$

where Y is a centered Gaussian vector in V_2 , independent of X_1 , with variance operator equal to $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$.

This is shown in [2, proposition 1.2]. See also [26, lemma A.2]. From this, we deduce that the distribution of X_2 given that $X_1 = x_1$ is a Gaussian vector in V_2 with expectation equal to $m_{X_2} + \Lambda_{21}(\Lambda_{11})^{-1}(x_1 - m_{X_1})$ and variance operator equal to $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$. We will make use of this in the special case where X is centered and $x_1 = 0$.

Corollary 4.6. *Let $X = (X_1, X_2)$ be a centered Gaussian vector in $V_1 \oplus V_2$ and assume that $\text{Var}(X_1)$ is non-degenerate. Then the distribution of X_2 given that $X_1 = 0$ is a centered Gaussian in V_2 with variance operator $\Lambda_{22} - \Lambda_{21}(\Lambda_{11})^{-1}\Lambda_{12}$.*

In what follows, we assume that V is a Euclidean space. Recall that in this case, we can see the variance operator of a random vector as an endomorphism of V . We will say that $\mathcal{N}(0, \text{Id})$ is the *standard normal distribution* on V , where Id denotes the identity map on V . If $X \sim \mathcal{N}(0, \text{Id})$, then, as a linear map from V^* to V , $\Lambda_X : v^* \mapsto v$ is the isomorphism induced by $\langle \cdot, \cdot \rangle$.

If $\text{Var}(X)$ is non-degenerate, then dP_X has the following density with respect to the Lebesgue measure on V :

$$x \mapsto \frac{1}{\sqrt{2\pi}^n \sqrt{\det(\Lambda_X)}} \exp\left(-\frac{1}{2} \langle (\Lambda_X)^{-1}(x - m), x - m \rangle\right).$$

Notation 4.7. We will denote by $d\mu_n$ the standard Gaussian measure on any Euclidean space of dimension n .

Lemma 4.8. *Let $X \sim \mathcal{N}(0, \text{Id})$ with values in a Euclidean space of dimension n and let $k \in \mathbb{R}$ such that $k > -n$. Then,*

$$\mathbb{E}[\|X\|^k] = \sqrt{2\pi}^k \frac{\text{Vol}(\mathbb{S}^{n-1})}{\text{Vol}(\mathbb{S}^{n+k-1})}.$$

Proof.

$$\begin{aligned} \mathbb{E}[\|X\|^k] &= \frac{1}{\sqrt{2\pi}^n} \int_V \|x\|^k e^{-\frac{1}{2}\|x\|^2} dx = \frac{\text{Vol}(\mathbb{S}^{n-1})}{\sqrt{2\pi}^n} \int_0^{+\infty} r^{k+n-1} e^{-\frac{1}{2}r^2} dr \\ &= \frac{\text{Vol}(\mathbb{S}^{n-1})}{\sqrt{2\pi}^n} \int_0^{+\infty} \sqrt{2t}^{k+n-2} e^{-t} dt = \frac{\sqrt{2\pi}^k \text{Vol}(\mathbb{S}^{n-1})}{2\sqrt{\pi}^{k+n}} \Gamma\left(\frac{k+n}{2}\right). \quad \square \end{aligned}$$

4.3 A useful result

Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leq r \leq n$. The space $V' \otimes V^*$ of linear maps from V to V' comes with a natural scalar product induced by those on V and V' . Let L be a standard Gaussian vector with values in $V' \otimes V^*$. This means that L is a random linear map from V to V' such that, in any orthonormal basis, the coefficients of the matrix of L are independent real standard Gaussian variables. The set of linear maps of rank less than r is an algebraic submanifold of $V' \otimes V^*$ of codimension at least 1, hence it has measure 0 for the Lebesgue measure and for any non-singular Gaussian measure. Thus the rank of L is almost surely r . Let U be a standard Gaussian vector in V independent of L , the random variable $(|\det^\perp(L)|, (L^\dagger)^*U)$ is defined almost surely (recall the definition 3.6 of L^\dagger).

The remainder of this subsection is devoted to the proof of the following technical result. This is a reformulation of proposition 3.12 in [10]. It will be one of the key arguments in the proof of theorems 1.2 and 1.4. The proof is mostly tedious computations.

Proposition 4.9. *Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leq r \leq n$. Let $L \sim \mathcal{N}(0, \text{Id})$ in $V' \otimes V^*$, let $U \sim \mathcal{N}(0, \text{Id})$ in V and let $U' \sim \mathcal{N}(0, \text{Id})$ in V' . For all $p \in \{n - r + 1, \dots, n\}$, let $X_p \sim \mathcal{N}(0, \text{Id})$ in \mathbb{R}^p such that $U', X_n, \dots, X_{n-r+1}$ are globally independent. Then $(|\det^\perp(L)|, (L^\dagger)^*U)$ has the same distribution as $(\|X_n\| \|X_{n-1}\| \cdots \|X_{n-r+1}\|, \frac{U'}{\|X_{n-r+1}\|})$.*

Let us introduce some further notations. Let $\mathcal{B} \subset V' \otimes V^*$ denote the set of maps of rank r . We set $\mathcal{F} = \{(L, U) \in \mathcal{B} \times V \mid U \in \ker(L)^\perp\}$ and $\mathcal{S} = \{(L, U) \in \mathcal{B} \times V \mid U \in \mathbb{S}(\ker(L)^\perp)\}$. Here and in the sequel, $\mathbb{S}(\cdot)$ stands for the unit sphere of the concerned space. Given $L \in \mathcal{B}$ and $U \in V$, we denote by \tilde{U} the orthogonal projection of U onto $\ker(L)^\perp$. Then we set:

$$\rho = \|\tilde{U}\|, \quad \theta = \frac{\tilde{U}}{\|\tilde{U}\|} \quad \text{and} \quad \theta' = \frac{(L^\dagger)^*\theta}{\|(L^\dagger)^*\theta\|}.$$

Note that $L^*\theta' = \frac{(L^\dagger L)^*\theta}{\|(L^\dagger)^*\theta\|} = \frac{\theta}{\|(L^\dagger)^*\theta\|}$, hence $\|L^*\theta'\| = \frac{1}{\|(L^\dagger)^*\theta\|}$ and finally:

$$(4.9) \quad \theta = \frac{L^*\theta'}{\|L^*\theta'\|} \quad \text{and} \quad L^\dagger U = L^\dagger \tilde{U} = \rho L^\dagger \theta = \frac{\rho}{\|L^*\theta'\|} \theta'.$$

We choose orthonormal basis (e_1, \dots, e_n) and (e'_1, \dots, e'_r) , of V and V' respectively, such that $e_r = \theta$, $e'_r = \theta'$ and (e_1, \dots, e_r) is a basis of $\ker(L)^\perp$. Then,

$$\forall i \in \{1, \dots, n\}, \quad \langle L e_i, \theta' \rangle = \langle e_i, L^* \theta' \rangle = \|L^* \theta'\| \langle e_i, \theta \rangle.$$

Thus the matrix of L in these basis has the form:

$$(4.10) \quad \left(\begin{array}{c|c|c} A & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 \\ \hline 0 \cdots 0 & \|L^* \theta'\| & 0 \cdots 0 \end{array} \right),$$

and $|\det^\perp(L)| = |\det(A)| \|L^* \theta'\|$.

Let π_θ and $\pi_{\theta'}$ denote the orthogonal projections along $\mathbb{R} \cdot \theta$ in V and along $\mathbb{R} \cdot \theta'$ in V' respectively. We define $L' : V \rightarrow (\mathbb{R} \cdot \theta')^\perp$ by $L' = \pi_{\theta'} \circ L \circ \pi_\theta$. Then $|\det(A)| = |\det^\perp(L')|$, and L' does not depend on our choice of basis. Finally, we have:

$$(|\det^\perp(L)|, L^\dagger U) = \left(|\det^\perp(L')| \|L^* \theta'\|, \frac{\rho \theta'}{\|L^* \theta'\|} \right).$$

To prove proposition 4.9, we will show that $|\det^\perp(L')|$, $\|L^*\theta'\|$, ρ and θ' are independent and identify their distributions.

If L and U are independent standard Gaussians, then almost surely $L \in \mathcal{B}$ and we can consider (L, U) as a random element of $\mathcal{B} \times V$. Then (L, \tilde{U}) is a random element of \mathcal{F} and its distribution is characterized by:

$$\mathbb{E}[\phi(L, \tilde{U})] = \int_{L \in \mathcal{B}} \left(\int_{\tilde{U} \in \ker(L)^\perp} \phi(L, \tilde{U}) e^{-\frac{\|\tilde{U}\|^2}{2}} \frac{d\tilde{U}}{\sqrt{2\pi}^r} \right) d\mu_{nr}(L),$$

for any bounded continuous function $\phi : \mathcal{F} \rightarrow \mathbb{R}$, where $d\tilde{U}$ stands for the Lebesgue measure on $\ker(L)^\perp$. From this, we compute the distribution of $(L, \theta, \rho) \in \mathcal{S} \times \mathbb{R}_+$ by a polar change of variables in the innermost integral: for any bounded continuous $\phi : \mathcal{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}$,

$$\mathbb{E}[\phi(L, \theta, \rho)] = \int_{L \in \mathcal{B}} \int_{\theta \in \mathbb{S}(\ker(L)^\perp)} \int_{\rho=0}^{+\infty} \phi(L, \theta, \rho) \rho^{r-1} e^{-\frac{\rho^2}{2}} \frac{d\rho}{\sqrt{2\pi}^r} d\theta d\mu_{nr}(L),$$

where $d\rho$ is the Lebesgue measure on \mathbb{R} and $d\theta$ is the Euclidean measure on the sphere $\mathbb{S}(\ker(L)^\perp)$. This distribution is a product measure on $\mathcal{S} \times \mathbb{R}_+$ thus (L, θ) and ρ are independent variables. Since $(|\det^\perp(L')|, \theta', \|L^*\theta'\|)$ only depends on (L, θ) , this triple is independent of ρ . Besides, ρ has the distribution of the norm of a standard Gaussian vector in \mathbb{R}^r : its density with respect to the Lebesgue measure is $\rho \mapsto \frac{\text{Vol}(\mathbb{S}^{r-1})}{\sqrt{2\pi}^r} \rho^{r-1} e^{-\frac{\rho^2}{2}}$ on \mathbb{R}_+ and vanishes elsewhere. Finally, the distribution of (L, θ) satisfies:

$$(4.11) \quad \mathbb{E}[\phi(L, \theta)] = \int_{L \in \mathcal{B}} \int_{\theta \in \mathbb{S}(\ker(L)^\perp)} \phi(L, \theta) \frac{d\theta}{\text{Vol}(\mathbb{S}^{r-1})} d\mu_{nr}(L),$$

for any bounded and continuous $\phi : \mathcal{S} \rightarrow \mathbb{R}$. We will now compute the distribution of (L, θ') in $\mathcal{B} \times \mathbb{S}(V')$.

Lemma 4.10. *For any bounded and continuous $\phi : \mathcal{B} \times \mathbb{S}(V') \rightarrow \mathbb{R}$,*

$$(4.12) \quad \mathbb{E}[\phi(L, \theta')] = \int_{\theta' \in \mathbb{S}(V')} \int_{L \in \mathcal{B}} \phi(L, \theta') \frac{|\det^\perp(L')|}{\|L^*\theta'\|^{r-1}} e^{-\frac{\|L\|^2}{2}} \frac{d\theta'}{\text{Vol}(\mathbb{S}^{r-1})} \frac{dL}{\sqrt{2\pi}^{nr}}$$

where $d\theta'$ stands for the Euclidean measure on $\mathbb{S}(V')$ and dL stands for the Lebesgue measure on $V' \otimes V^*$.

Proof. Fixing some ϕ , we see from (4.11) that:

$$\mathbb{E}[\phi(L, \theta')] = \int_{L \in \mathcal{B}} \int_{\theta \in \mathbb{S}(\ker(L)^\perp)} \phi \left(L, \frac{(L^\dagger)^*\theta}{\|(L^\dagger)^*\theta\|} \right) e^{-\frac{\|L\|^2}{2}} \frac{d\theta}{\text{Vol}(\mathbb{S}^{r-1})} \frac{dL}{\sqrt{2\pi}^{nr}}.$$

Then we make the change of variables $\theta' = \psi(\theta) = \frac{(L^\dagger)^*\theta}{\|(L^\dagger)^*\theta\|}$ in the innermost integral, with L fixed. Recalling (4.9) we have $\psi^{-1} : \theta' \mapsto \frac{L^*\theta'}{\|L^*\theta'\|}$, from $\mathbb{S}(V')$ to $\mathbb{S}(\ker(L)^\perp)$. Now the differential of ψ^{-1} at $\theta' \in \mathbb{S}(V')$ satisfies:

$$\forall v \in (\mathbb{R} \cdot \theta')^\perp, \quad d_{\theta'} \psi^{-1} \cdot v = \frac{1}{\|L^*\theta'\|} \left(L^*v - \left\langle L^*v, \frac{L^*\theta'}{\|L^*\theta'\|} \right\rangle \frac{L^*\theta'}{\|L^*\theta'\|} \right) = \frac{\pi_\theta(L^*v)}{\|L^*\theta'\|}.$$

As above, we choose an orthonormal basis (e'_1, \dots, e'_{r-1}) of $(\mathbb{R} \cdot \theta')^\perp$ and an orthonormal basis (e_1, \dots, e_{r-1}) of $(\mathbb{R} \cdot \theta \oplus \ker(L))^\perp$. In these coordinates we have:

$$|\det(d_{\theta'} \psi^{-1})| = \frac{|\det(\pi_\theta \circ L^*_{/(\theta')^\perp})|}{\|L^*\theta'\|^{r-1}} = \frac{|\det(A^*)|}{\|L^*\theta'\|^{r-1}} = \frac{|\det(A)|}{\|L^*\theta'\|^{r-1}} = \frac{|\det^\perp(L')|}{\|L^*\theta'\|^{r-1}},$$

where A is the same as in (4.10). This proves (4.12). \square

We can now compute the joint distribution of $(|\det^\perp(L')|, \theta', \|L^*\theta'\|)$ from the one of (L, θ') . If we fix $\theta' \in \mathbb{S}(V')$ and an orthonormal basis (e'_1, \dots, e'_r) of V' such that $e'_r = \theta'$, the choice of $L \in V' \otimes V^*$ is equivalent to the choice of $L^*e'_1, \dots, L^*e'_r \sim \mathcal{N}(0, \text{Id})$ independent in V . For simplicity, we set $L_i = L^*e'_i$. Note that if we choose a basis for V as well, these are the rows of the matrix of L . We can rewrite (4.12) as:

$$\mathbb{E}[\phi(L, \theta')] = \int_{\theta' \in \mathbb{S}(V')} \int_{L_1, \dots, L_{r-1} \in V} \int_{L_r \in V} \phi(L, \theta') \frac{|\det^\perp(L')|}{\|L_r\|^{r-1}} \frac{e^{-\frac{1}{2} \sum \|L_i\|^2} d\theta' dL_1 \cdots dL_r}{\text{Vol}(\mathbb{S}^{r-1}) \sqrt{2\pi}^{nr}},$$

where dL_i denotes the Lebesgue measure in the i -th copy of V . We set $\alpha_r = \frac{L_r}{\|L_r\|}$ and $\rho_r = \|L_r\|$. Here, $L' = \pi_{\theta'} \circ L \circ \pi_{\alpha_r}$ depends on α_r and θ' but not on ρ_r . Making a polar change of variables, the above integral equals:

$$\int_{\substack{\theta' \in \mathbb{S}(V') \\ \alpha_r \in \mathbb{S}(V) \\ L_1, \dots, L_{r-1} \in V}} \int_{\rho_r=0}^{+\infty} \phi(L, \theta') \frac{\rho_r^{n-r} e^{-\frac{\rho_r^2}{2}} d\rho_r}{\sqrt{2\pi}^{n-r+1}} \frac{e^{-\frac{1}{2} \sum_{i=1}^{r-1} \|L_i\|^2} |\det^\perp(L')| d\theta' d\alpha_r dL_1 \cdots dL_{r-1}}{\text{Vol}(\mathbb{S}^{r-1}) \sqrt{2\pi}^{(n+1)(r-1)}}.$$

Then $\rho_r = \|L^*\theta'\|$ is independent of $(\theta', \alpha_r, L_1, \dots, L_{r-1})$, hence of $(\theta', |\det^\perp(L')|)$. Moreover, looking at its density, we see that ρ_r has the distribution of the norm of a standard Gaussian vector in \mathbb{R}^{n-r+1} . Besides $(\theta', \alpha_r, L_1, \dots, L_{r-1})$ has the density:

$$(\theta', \alpha_r, L_1, \dots, L_{r-1}) \mapsto \frac{e^{-\frac{1}{2} \sum_{i=1}^{r-1} \|L_i\|^2} |\det^\perp(L')|}{\sqrt{2\pi}^{(n+1)(r-1)} \text{Vol}(\mathbb{S}^{r-1}) \text{Vol}(\mathbb{S}^{n-r})}$$

with respect to $d\theta' \otimes d\alpha_r \otimes dL_1 \otimes \cdots \otimes dL_{r-1}$. For $i \in \{1, \dots, r-1\}$ we denote by L_i^\perp the orthogonal projection of L_i onto the orthogonal of the subspace spanned by $(\alpha_r, L_1, \dots, L_{i-1})$.

Lemma 4.11. *For any $L \in \mathcal{B}$, $|\det^\perp(L')| = \|L_1^\perp\| \cdots \|L_{r-1}^\perp\|$.*

Proof. If one of the L_i^\perp is zero, then the vectors $\alpha_r, L_1, \dots, L_{r-1}$ are linearly dependent and L is singular. Since we assumed $L \in \mathcal{B}$ this is not the case and $\left(\frac{L_1^\perp}{\|L_1^\perp\|}, \dots, \frac{L_{r-1}^\perp}{\|L_{r-1}^\perp\|}\right)$ is an orthonormal basis of $\ker(L')^\perp$. Writing the matrix of the restriction of L' to $\ker(L')^\perp$ in this basis and (e'_1, \dots, e'_{r-1}) , we see that it is lower triangular with diagonal coefficients $\|L_1^\perp\|, \dots, \|L_{r-1}^\perp\|$. This proves the lemma. \square

Let ϕ be a continuous bounded function from $\mathbb{S}(V') \times \mathbb{R}_+$ to \mathbb{R} . We have:

$$\begin{aligned} \mathbb{E}[\phi(\theta', |\det^\perp(L')|)] &= \int \phi(\theta', |\det^\perp(L')|) \frac{e^{-\frac{1}{2} \sum \|L_i\|^2} |\det^\perp(L')| d\theta' d\alpha_r dL_1 \cdots dL_{r-1}}{\sqrt{2\pi}^{(n+1)(r-1)} \text{Vol}(\mathbb{S}^{r-1}) \text{Vol}(\mathbb{S}^{n-r})} \\ &= \int_{\alpha_r, \theta'} \int_{L_1^\perp} \cdots \int_{L_{r-1}^\perp} \phi(\theta', \prod \|L_i^\perp\|) \frac{e^{-\frac{1}{2} \sum \|L_i^\perp\|^2} \prod \|L_i^\perp\| dL_{r-1}^\perp \cdots dL_1^\perp d\theta' d\alpha_r}{\sqrt{2\pi}^{(n+1-\frac{r}{2})(r-1)} \text{Vol}(\mathbb{S}^{r-1}) \text{Vol}(\mathbb{S}^{n-r})}. \end{aligned}$$

Then we make polar changes of variables, for each i we set $\rho_i = \|L_i^\perp\|$ and $\alpha_i = \frac{L_i^\perp}{\|L_i^\perp\|}$. Note that, when L_1, \dots, L_{i-1} are fixed, L_i^\perp is a vector in a space of dimension $n-i$. We have:

$$\begin{aligned} \mathbb{E}[\phi(\theta', |\det^\perp(L')|)] &= \int \phi(\theta', \prod \rho_i) \frac{e^{-\frac{1}{2} \sum \rho_i^2} \prod \rho_i^{n-i} d\rho_1 \cdots d\rho_{r-1} d\alpha_1 \cdots d\alpha_r d\theta'}{\sqrt{2\pi}^{(n+1-\frac{r}{2})(r-1)} \text{Vol}(\mathbb{S}^{r-1}) \text{Vol}(\mathbb{S}^{n-r})} \\ &= \int_{\rho_1, \dots, \rho_{r-1}, \theta'} \phi(\theta', \prod \rho_i) \prod_{i=1}^{r-1} \left(\text{Vol}(\mathbb{S}^{n-i}) \frac{e^{-\frac{\rho_i^2}{2}} \rho_i^{n-i}}{\sqrt{2\pi}^{n+1-i}} \right) d\rho_1 \cdots d\rho_{r-1} \frac{d\theta'}{\text{Vol}(\mathbb{S}^{r-1})}. \end{aligned}$$

This shows that $\theta', \rho_1, \dots, \rho_{r-1}$ are independent variables, that θ' is uniformly distributed in $\mathbb{S}(V')$ and that, for all $i \in \{1, \dots, r-1\}$, ρ_i is distributed as the norm of a standard Gaussian vector in \mathbb{R}^{n+1-i} . Finally, this shows that $|\det^\perp(L')|$ is distributed as $\prod_{i=1}^{r-1} \rho_i$.

Putting all we have done so far together, we see that $(|\det^\perp(L)|, L^{\dagger*}U)$ is distributed as $(\|X_n\| \cdots \|X_{n-r+1}\|, \frac{\rho\theta'}{\|X_{n-r+1}\|})$ where X_p is a standard Gaussian vector in \mathbb{R}^p for all p . Moreover, θ' is uniformly distributed in $\mathbb{S}(V')$, ρ is distributed as the norm of a standard Gaussian vector in \mathbb{R}^r , and all these variables are globally independent. Finally $U' = \rho\theta'$ is a standard Gaussian in V' , independent of X_n, \dots, X_{n-r+1} so we have proved proposition 4.9. An immediate corollary of this is the following.

Corollary 4.12. *Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leq r \leq n$. Let $L \sim \mathcal{N}(0, \text{Id})$ in $V' \otimes V^*$.*

Then $|\det^\perp(L)|$ has the same distribution as $\|X_{n-r+1}\| \cdots \|X_n\|$, where $X_p \sim \mathcal{N}(0, \text{Id})$ in \mathbb{R}^p for all $p \in \{n-r+1, \dots, n\}$, and these vectors are globally independent.

Remark 4.13. This can also be seen as follows. Fix basis for V and V' and notice that $|\det^\perp(L)|$ is the r -dimensional volume of the parallelepiped spanned by the lines L_1, \dots, L_r of the matrix of L . Using that the volume of a parallelepiped is basis times height ($r-1$) times we get: $|\det^\perp(L)| = \|L_1^\perp\| \|L_2^\perp\| \cdots \|L_r^\perp\|$. Here we denoted by L_j^\perp the component of L_j in the orthogonal complement of the space spanned by (L_1, \dots, L_{j-1}) . Since the lines of L are independent the distribution of L_j is the same as that of L_j given (L_1, \dots, L_{j-1}) , that is $d\mu_n$. Then, for (L_1, \dots, L_{j-1}) fixed, L_j^\perp is the orthogonal projection of L_j onto a space of dimension $n-j+1$, hence its distribution is $d\mu_{n-j+1}$. Moreover, $\|L_j^\perp\|$ is independent of (L_1, \dots, L_{j-1}) , even if L_j is not. One can deduce corollary 4.12 from this.

Lemma 4.14. *Let V and V' be two Euclidean spaces of dimension n and r respectively, with $1 \leq r \leq n$. Let $L \sim \mathcal{N}(0, \text{Id})$ in $V' \otimes V^*$. Then:*

$$\mathbb{E}[|\det^\perp(L)|] = \sqrt{2\pi}^r \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)}.$$

Proof. Since by corollary 4.12 $|\det^\perp(L)|$ is distributed as $\|X_{n-r+1}\| \cdots \|X_n\|$ with X_p standard Gaussian in \mathbb{R}^p and the X_p 's independent, we have:

$$\begin{aligned} \mathbb{E}[|\det^\perp(L)|] &= \mathbb{E}[\|X_{n-r+1}\| \cdots \|X_n\|] = \prod_{p=n-r+1}^n \mathbb{E}[\|X_p\|] = \prod_{p=n-r+1}^n \sqrt{2\pi} \frac{\text{Vol}(\mathbb{S}^{p-1})}{\text{Vol}(\mathbb{S}^p)} \\ &= \sqrt{2\pi}^r \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)}. \end{aligned}$$

The last but one equality comes from lemma 4.8. □

4.4 Random double forms

We will use the following result in the proof of theorems 1.2 and 1.4. See [1, lemma 12.3.1] for a proof.

Lemma 4.15. *Let V be a vector space of finite dimension n . Let $\alpha \in \bigwedge^{1,1} V^*$ be a Gaussian bilinear form on V . If α is centered, then for any $p \leq \frac{n}{2}$,*

$$\mathbb{E}[\alpha^{\wedge 2p}] = \frac{(2p)!}{2^p p!} (\mathbb{E}[\alpha^{\wedge 2}])^{\wedge p}.$$

Going back to the setting of 3.3, let \tilde{M} be a smooth submanifold of the Riemannian manifold (M, g) , with $\dim(M) > \dim(\tilde{M}) \geq 1$. Let \mathbb{I} be the second fundamental form of $\tilde{M} \subset M$, defined by equation (3.1), and let Ξ be defined by (3.3). Let $x \in \tilde{M}$ and $U \sim \mathcal{N}(0, \text{Id})$ be a random vector in $T_x M$. Recalling lemma 4.3, we get the following: for any X, Y, Z and $W \in T_x \tilde{M}$,

$$\begin{aligned} \Xi(X, Y, Z, W) &= \langle \mathbb{I}(X, Z), \mathbb{I}(Y, W) \rangle - \langle \mathbb{I}(X, W), \mathbb{I}(Y, Z) \rangle \\ &= \mathbb{E}[\langle \mathbb{I}(X, Z), U \rangle \langle \mathbb{I}(Y, W), U \rangle] - \mathbb{E}[\langle \mathbb{I}(X, W), U \rangle \langle \mathbb{I}(Y, Z), U \rangle] \\ &= \mathbb{E}[\langle \mathbb{I}(X, Z), U \rangle \langle \mathbb{I}(Y, W), U \rangle - \langle \mathbb{I}(X, W), U \rangle \langle \mathbb{I}(Y, Z), U \rangle] \\ &= \frac{1}{2} \mathbb{E} \left[\langle \mathbb{I}, U \rangle^2 ((X, Y), (Z, W)) \right], \end{aligned}$$

where we applied lemma 3.1 to the random symmetric $(1, 1)$ -double form $\langle \mathbb{I}, U \rangle$. We have proved the following version of the Gauss equation.

Lemma 4.16. *Let (M, g) be a Riemannian manifold and let \tilde{M} be a smooth submanifold of M , such that $\dim(M) > \dim(\tilde{M}) \geq 1$. Let R and \tilde{R} denote the Riemann curvature of M and \tilde{M} respectively, and let \mathbb{I} be the second fundamental form of $\tilde{M} \subset M$. Then:*

$$\forall x \in \tilde{M}, \quad \tilde{R} = R + \frac{1}{2} \mathbb{E} \left[\langle \mathbb{I}, U \rangle^2 \right],$$

where $U \sim \mathcal{N}(0, \text{Id})$ with values in $T_x M$.

5 Random submanifolds

This section is concerned with the precise definition of the random submanifolds we consider. The first two subsections explain how we produce them in a quite general setting. The third one introduces the covariance kernel, which characterizes their distribution. We also describe the distribution induced on the bundle of 2-jets in terms of this kernel. Then we describe what we called the harmonic setting, before explaining how to adapt all this in the real algebraic case. This kind of random submanifolds has already been considered by Bérard [9], Zelditch [33] and Nicolaescu [26] in the harmonic case, and by Gayet and Welschinger in the real algebraic case, see [15, 16, 17]. See also [10, 11, 24] in special cases.

5.1 General setting

Let (M, g) be a smooth closed manifold of positive dimension n . We denote by $|dV_M|$ the Riemannian measure on M induced by g . Let $\langle \cdot, \cdot \rangle$ denote the L^2 -scalar product induced by $|dV_M|$, that is for any φ_1 and $\varphi_2 \in \mathcal{C}^\infty(M)$,

$$(5.1) \quad \langle \varphi_1, \varphi_2 \rangle = \int_{x \in M} \varphi_1(x) \varphi_2(x) |dV_M|.$$

Let V be a subspace of $\mathcal{C}^\infty(M)$ of finite dimension N' .

From now on we fix some $r \in \{1, \dots, n\}$ that will be the codimension of our random submanifolds. We equip V^r with the L^2 -scalar product defined by:

$$(5.2) \quad \forall f_1, f_2 \in V^r, \quad \langle f_1, f_2 \rangle = \int_{x \in M} \langle f_1(x), f_2(x) \rangle |dV_M|,$$

where the inner product on the right-hand side is the standard one on \mathbb{R}^r . Let $N = rN'$ denote the dimension of V^r .

Let $f \in V^r$, we denote by Z_f the zero set of $f : M \rightarrow \mathbb{R}^r$. If f vanishes transversally then Z_f is a smooth submanifold of M of codimension r and we denote by $|dV_f|$ the Riemannian measure induced by the restriction of g to Z_f . We also denote by $\text{Vol}(Z_f)$ the volume of Z_f and by $\chi(Z_f)$ its Euler characteristic. In the case $r = n$, when f vanishes transversally Z_f is a finite set, so we define $|dV_f| = \sum_{x \in Z_f} \delta_x$, where δ_x stands for the Dirac measure at the point x . Let D denote the *discriminant locus* of V^r , that is the set of $f \in V^r$ that do not vanish transversally.

In the sequel, we consider a random $f \in V^r$ with standard Gaussian distribution. This means that the components of f are independent standard Gaussian vectors in V . Recall that $d\mu_{N'}$ stands for the standard Gaussian measure on V and $d\mu_N = d\mu_{N'} \otimes \cdots \otimes d\mu_{N'}$ stands for the standard Gaussian measure on V^r . Under some further technical assumptions on V (see 5.2 below), f vanishes transversally almost surely. Hence, the random variables $\text{Vol}(Z_f)$ and $\chi(Z_f)$ are well defined almost everywhere, and it makes sense to compute their expectation.

5.2 The incidence manifold

Following [26], we say that V^r is *0-ample* if the map $j_x^0 : f \mapsto f(x)$ is onto for every $x \in M$. For $r = 1$ this is just saying that V has no base point. From now on, we assume that V^r is 0-ample. Note that it is equivalent to assume that V has no base point.

We introduce an *incidence manifold*, following the idea of Shub and Smale [30], see also [15, 16]. Let $F : (f, x) \in V^r \times M \mapsto f(x) \in \mathbb{R}^r$ and $\Sigma = F^{-1}(0)$. For any $(f_0, x) \in V^r \times M$, we have:

$$d_{(f_0, x)}F : (f, v) \mapsto f(x) + d_x f_0 \cdot v.$$

Then $d_{(f_0, x)}F$ is onto, indeed $\partial_1 F(f_0, x) = j_x^0$ is already onto. Thus F is a submersion and Σ is a smooth submanifold of codimension r of $V^r \times M$. As in 2.2, we set $\pi_1 : \Sigma \rightarrow V^r$ and $\pi_2 : \Sigma \rightarrow M$ the projections from Σ to each factor. Let $(f_0, x) \in V^r \times M$, we have:

$$T_{(f_0, x)}\Sigma = \{(f, v) \in V^r \times T_x M \mid d_x f_0 \cdot v + f(x) = 0\},$$

$$\text{and} \quad d_{(f_0, x)}\pi_1 : (f, v) \in T_{(f_0, x)}\Sigma \mapsto f \in V^r.$$

Let $f \in V^r$, it is in the range of $d_{(f_0, x)}\pi_1$ if and only if there exists some $v \in T_x M$ such that $(f, v) \in T_{(f_0, x)}\Sigma$, that is $f(x)$ is in the range of $d_x f_0$. Since V^r is 0-ample, the map j_x^0 is onto, and $d_x f_0$ is surjective if and only if $d_{(f_0, x)}\pi_1$ is. Thus the discriminant locus D is exactly the set of critical values of π_1 . By Sard's theorem, D has measure 0 in V^r , both for the Lebesgue measure and $d\mu_N$, and f vanishes transversally almost surely.

Later on we will apply the double-fibration trick of section 2.2 for the fibrations induced by π_1 and π_2 . For this we will need the partial differentials of F :

$$(5.3) \quad \partial_1 F(f, x) = j_x^0 \quad \text{and} \quad \partial_2 F(f, x) = d_x f.$$

We also equip Σ with the restriction of the product metric on $V^r \times M$. Then, whenever $f \notin D$, $(\pi_1)^{-1}(f) = \{f\} \times Z_f$ is isometric to Z_f , hence we will identify those sets. Similarly, we will identify $(\pi_2)^{-1}(x) = \ker(j_x^0) \times \{x\}$ with $\ker(j_x^0)$.

5.3 The covariance kernel

In this subsection we introduce the Schwarz kernel and covariance function associated to our space of random functions. It turns out (see proposition 5.2 below) that these objects are in fact the same. The first to use this fact seem to have been Bleher, Shiffman and Zelditch in the case of complex projective manifolds [5] and in the case of symplectic manifolds [6]. In the harmonic setting this was used by Zelditch [33] and Nicolaescu [26].

We denote by $\underline{\mathbb{R}}^r$ the trivial bundle $\mathbb{R}^r \times M$ over M and by ∇ the trivial flat connection on $\underline{\mathbb{R}}^r$. This connection and ∇^M induce the connection $\nabla^M \otimes \text{Id} + \text{Id} \otimes \nabla$ on $T^*M \otimes \underline{\mathbb{R}}^r$. Let $\nabla^2 = (\nabla^M \otimes \text{Id} + \text{Id} \otimes \nabla) \circ \nabla$ denote the second covariant derivative on $\underline{\mathbb{R}}^r$. In what follows, we identify canonically smooth functions from M to \mathbb{R}^r with sections of $\underline{\mathbb{R}}^r$. Under this identification, ∇ corresponds to the exterior differential d and, ∇^2 corresponds to the usual Hessian defined in 3.3. Let P_1 (resp. P_2) denote the projection from $M \times M$ onto the first (resp. second) factor. Recall that $\underline{\mathbb{R}}^r \boxtimes \underline{\mathbb{R}}^r$ denotes the bundle $P_1^*(\underline{\mathbb{R}}^r) \otimes P_2^*(\underline{\mathbb{R}}^r)$ over $M \times M$.

In $\mathcal{C}^\infty(M)$ equipped with the L^2 -inner product (5.1), the orthogonal projection onto V can be represented by its *Schwartz kernel*, denoted by e . That is there exists a unique $e : M \times M \rightarrow \mathbb{R}$ such that for any smooth $\varphi : M \rightarrow \mathbb{R}$, the projection of φ onto V is given by $x \mapsto \langle e(x, \cdot), \varphi \rangle$. This function has the following *reproducing kernel property* :

$$\forall \varphi \in V, \forall x \in M, \quad \varphi(x) = \langle e(x, \cdot), \varphi \rangle = \int_{y \in M} e(x, y) \varphi(y) |dV_M|.$$

Likewise, there exists a kernel for the orthogonal projection onto V^r in $\mathcal{C}^\infty(M, \mathbb{R}^r)$. It is the unique section E of $\underline{\mathbb{R}}^r \boxtimes \underline{\mathbb{R}}^r$ such that, for any smooth $f : M \rightarrow \mathbb{R}^r$, the projection of f onto V^r is given by:

$$(5.4) \quad x \mapsto \langle E(x, \cdot), f \rangle = \int_{y \in M} \langle E(x, y), f(y) \rangle |dV_M|.$$

In the previous formula, the inner product on the right-hand side is the usual one on \mathbb{R}^r , acting on the second factor of $\mathbb{R}^r \otimes \mathbb{R}^r$. Note that $\langle E(x, y), f(y) \rangle \in (\underline{\mathbb{R}}^r)_x$, the fiber of $\underline{\mathbb{R}}^r$ over x . Again, E has a reproducing kernel property:

$$(5.5) \quad \forall f \in V^r, \forall x \in M, \quad f(x) = \int_{y \in M} \langle E(x, y), f(y) \rangle |dV_M|.$$

Remark 5.1. It may seem odd to consider E as a section instead of a map from $M \times M$ to $\mathbb{R}^r \otimes \mathbb{R}^r$. It turns out to be the right point of view when we consider sections of a vector bundle instead of functions, as will be the case in the real algebraic setting.

If (f_1, \dots, f_N) is any orthonormal basis of V^r , one can check that E is defined by:

$$(5.6) \quad E : (x, y) \mapsto \sum_{i=1}^N f_i(x) \otimes f_i(y).$$

Then E is smooth, for all $x \in M$, $E(x, x)$ is in the span of $\{\zeta \otimes \zeta \mid \zeta \in \mathbb{R}^r\}$, and for all $\zeta \in \mathbb{R}^r$:

$$(5.7) \quad \langle E(x, x), \zeta \otimes \zeta \rangle = \sum_{i=1}^N \langle f_i(x) \otimes f_i(x), \zeta \otimes \zeta \rangle = \sum_{i=1}^N (\langle f_i(x), \zeta \rangle)^2 \geq 0.$$

Thus V^r is 0-ample if and only if for all $x \in M$ and $\zeta \in \mathbb{R}^r \setminus \{0\}$, $\langle E(x, x), \zeta \otimes \zeta \rangle > 0$. That is if and only if $E(x, x)$ is a positive-definite bilinear form on $(\mathbb{R}^r)^*$ for any $x \in M$.

Let $(\varphi_1, \dots, \varphi_{N'})$ be an orthonormal basis of V . As above, e can be expressed as:

$$e : (x, y) \mapsto \sum_{i=1}^{N'} \varphi_i(x) \varphi_i(y).$$

Thus e is smooth and $e(x, \cdot) \in V$ for all $x \in M$. Moreover $e(x, x) = \sum_{i=1}^{N'} \varphi_i(x)^2 \geq 0$, and equality holds if and only if x is a base point of V . Then, e and E are related as follows.

Let $(\zeta_1, \dots, \zeta_r)$ denote the canonical basis of \mathbb{R}^r and let $\varphi_i^{(q)} = \varphi_i \zeta_q$ for $1 \leq i \leq N'$ and $1 \leq q \leq r$. Then $(\varphi_i^{(q)})_{(i,j)}$ is an orthonormal basis of V^r and, for all x and $y \in M$,

$$E(x, y) = \sum_{q=1}^r \sum_{i=1}^{N'} \varphi_i^{(q)}(x) \otimes \varphi_i^{(q)}(y) = \sum_{q=1}^r \left(\sum_{i=1}^{N'} \varphi_i(x) \varphi_i(y) \right) \zeta_q \otimes \zeta_q = \sum_{q=1}^r e(x, y) \zeta_q \otimes \zeta_q.$$

Thus, for any $x, y \in M$ and any $\zeta, \zeta' \in \mathbb{R}^r$,

$$(5.8) \quad \langle E(x, y), \zeta \otimes \zeta' \rangle = e(x, y) \langle \zeta, \zeta' \rangle.$$

On the other hand, the standard Gaussian vector $f \in V^r$ defines a smooth centered Gaussian field $(f(x))_{x \in M}$ indexed by M , with values in \mathbb{R}^r . Its distribution is totally determined by its covariance function, that is the section of $\underline{\mathbb{R}^r} \boxtimes \underline{\mathbb{R}^r}$ defined by $(x, y) \mapsto \text{Cov}(f(x), f(y))$. By this we mean that the covariance function determines all finite-dimensional marginal distributions (cf. [1, section 1.2], for example).

Proposition 5.2. *Let V be a finite-dimensional subspace of $C^\infty(M)$ and let E be the Schwartz kernel of V^r . Let $f \sim \mathcal{N}(0, \text{Id})$ in V^r . Then,*

$$\forall x, y \in M, \quad \text{Cov}(f(x), f(y)) = \mathbb{E}[f(x) \otimes f(y)] = E(x, y).$$

Proof. Let x and $y \in M$, the first equality is given by lemma 4.2. We will now show that $\mathbb{E}[f(x) \otimes f(y)]$ satisfies condition (5.4) to prove the second equality. Let $f_0 : M \rightarrow \mathbb{R}^r$ be a smooth function and $x \in M$,

$$\begin{aligned} \int_{y \in M} \langle \mathbb{E}[f(x) \otimes f(y)], f_0(y) \rangle |dV_M| &= \int_{y \in M} \mathbb{E}[\langle f(x) \otimes f(y), f_0(y) \rangle] |dV_M| \\ &= \int_{y \in M} \mathbb{E}[f(x) \langle f(y), f_0(y) \rangle] |dV_M| = \mathbb{E}[f(x) \langle f, f_0 \rangle]. \end{aligned}$$

If $f_0 \in (V^r)^\perp$, this equals 0. If $f_0 \in V^r$, we have:

$$\mathbb{E}[f(x) \langle f, f_0 \rangle] = \mathbb{E}[\langle E(x, \cdot), f \rangle \langle f_0, f \rangle] = \langle E(x, \cdot), f_0 \rangle = f_0(x),$$

where we applied lemma 4.3 to $f \sim \mathcal{N}(0, \text{Id})$ and made use of the reproducing kernel property (5.5) for f_0 . In both cases, $x \mapsto \langle \mathbb{E}[f(x) \otimes f], f_0 \rangle$ is the projection of f_0 onto V^r . This shows the second equality in proposition 5.2. \square

Remark 5.3. Similarly, for the field induced by $\varphi \sim \mathcal{N}(0, \text{Id})$ in V , we have:

$$\forall (x, y) \in M \times M, \quad \mathbb{E}[\varphi(x)\varphi(y)] = e(x, y).$$

This tells us that the distribution of our Gaussian field is totally determined by the Schwartz kernel E . It is the main motivation for the use of the L^2 -scalar product on V . In our cases of interest, asymptotics are known for E and its derivatives, see section 6 below. This is what allows us to derive asymptotics for the expectation of the volume and Euler characteristic of Z_f .

5.4 Random jets

We will now describe the distribution induced by our random f on the bundle of 2-jets. Let $x \in M$, we denote by $\mathcal{J}_x^k(\underline{\mathbb{R}^r})$ the vector space of k -jets of smooth sections of $\underline{\mathbb{R}^r}$ at x . Let $\text{Sym}(T_x^*M)$ denote the space of symmetric bilinear forms on T_x^*M , and let

$$\begin{aligned} j_x^2 : C^\infty(M, \mathbb{R}^r) &\rightarrow \mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^*M \oplus \text{Sym}(T_x^*M)), \\ f &\mapsto (f(x), \nabla_x f, \nabla_x^2 f) \end{aligned}$$

where ∇ and ∇^2 are as in section 5.3. This maps induces an isomorphism between $\mathcal{J}_x^2(\underline{\mathbb{R}}^r)$ and $\mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^*M \oplus \text{Sym}(T_x^*M))$, and in the sequel we will identify these spaces via j_x^2 . Note that the isomorphism depends on our choices of connection on $\underline{\mathbb{R}}^r$ and T^*M . We still denote by j_x^2 the restriction of j_x^2 to V^r , which takes $f \in V^r$ to its 2-jet. Likewise, we denote by $j_x^1 : f \mapsto (f(x), \nabla_x f)$ the map that takes a smooth function to its 1-jet in $\mathcal{J}_x^1(\underline{\mathbb{R}}^r) \simeq \mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^*M)$.

Since j_x^2 is linear and $f \sim \mathcal{N}(0, \text{Id})$, lemma 4.4 shows that $j_x^2(f)$ is a centered Gaussian random vector. Let $t \in \mathbb{R}^r$, $L \in \mathbb{R}^r \otimes T_x^*M$ and $S \in \mathbb{R}^r \otimes \text{Sym}(T_x^*M)$ be random vectors such that (t, L, S) is distributed as $j_x^2(f)$. Then t , L and S are three centered Gaussian vectors, and their variances and covariances are determined by $\text{Var}(j_x^2(f))$.

Let ∂_x (resp. ∂_y) denote the partial covariant derivative with respect to the first (resp. second) variable for sections of $\underline{\mathbb{R}}^r \boxtimes \underline{\mathbb{R}}^r$. When these sections are seen as functions from $M \times M$ to $\mathbb{R}^r \otimes \mathbb{R}^r$, ∂_x and ∂_y are just the usual partial differentials. Likewise, we denote by $\partial_{x,x}$ (resp. $\partial_{y,y}$) the second partial derivative with respect to the first (resp. second) variable twice.

Lemma 5.4. *Let V be a finite-dimensional subspace of $\mathcal{C}^\infty(M)$ and let E be the Schwartz kernel of V^r . Let $f \sim \mathcal{N}(0, \text{Id})$ in V^r . Let $x \in M$ and let $(t, L, S) \in \mathcal{J}_x^2(\underline{\mathbb{R}}^r)$ be distributed as $j_x^2(f)$, the 2-jet of f at x . Then (t, L, S) is a centered Gaussian vector, and its variance is characterized by the following relations:*

$$\begin{aligned} (5.9) \quad & \text{Var}(t) = \mathbb{E}[f(x) \otimes f(x)] = E(x, x), \\ (5.10) \quad & \text{Var}(L) = \mathbb{E}[\nabla_x f \otimes \nabla_x f] = (\partial_x \partial_y E)(x, x), \\ (5.11) \quad & \text{Var}(S) = \mathbb{E}[\nabla_x^2 f \otimes \nabla_x^2 f] = (\partial_{x,x} \partial_{y,y} E)(x, x), \\ (5.12) \quad & \text{Cov}(t, L) = \mathbb{E}[f(x) \otimes \nabla_x f] = (\partial_y E)(x, x), \\ (5.13) \quad & \text{Cov}(t, S) = \mathbb{E}[f(x) \otimes \nabla_x^2 f] = (\partial_{y,y} E)(x, x), \\ (5.14) \quad & \text{Cov}(L, S) = \mathbb{E}[\nabla_x f \otimes \nabla_x^2 f] = (\partial_x \partial_{y,y} E)(x, x). \end{aligned}$$

Proof. The vector t is the projection on the factor \mathbb{R}^r of (t, L, S) , thus it is distributed as $f(x)$. Hence $\text{Var}(t) = \text{Var}(f(x)) = \mathbb{E}[f(x) \otimes f(x)] = E(x, x)$, by proposition 5.2. This proves the first equation (5.9). Likewise, one shows that $\text{Cov}(t, L) = \text{Cov}(f(x), \nabla_x f)$. Then by lemma 4.2 this equals $\mathbb{E}[f(x) \otimes \nabla_x f]$, and the first equality in (5.12) is proved. Applying ∂_y to the relation given by proposition 5.2, one gets:

$$\mathbb{E}[f(x) \otimes \nabla_y f] = (\partial_y E)(x, y).$$

For $x = y$, this establishes the second part of (5.12). The remaining relations are proved in the same way. \square

With this lemma, we have characterized the distribution of $j_x^2(f)$ only in terms of E . Since $j_x^0(f)$ and $j_x^1(f)$ are the projections of $j_x^2(f)$ onto \mathbb{R}^r and $\mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^*M)$ respectively, their distributions are also given by lemma 5.4. Precisely $j_x^0(f)$ has the same distribution as t and $j_x^1(f)$ has the same distribution as (t, L) .

5.5 The harmonic setting

Let Δ denote the Laplace-Beltrami operator on the closed Riemannian manifold (M, g) . Recall the following classical facts from the theory of elliptical operators, see [13, theorem 4.43].

Theorem 5.5. *1. The eigenvalues of $\Delta : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$ can be arranged into a strictly increasing sequence of non-negative numbers $(\lambda_k^2)_{k \in \mathbb{N}}$ such that $\lambda_k^2 \xrightarrow[k \rightarrow +\infty]{} +\infty$.*

2. The associated eigenspaces are finite-dimensional, and they are pairwise orthogonal for the L^2 -inner product defined by equation (5.2)

Let $\lambda \geq 0$, we denote by V_λ the subspace of $\mathcal{C}^\infty(M)$ spanned by the eigenfunctions of Δ associated to eigenvalues that are less or equal to λ^2 . This space is finite-dimensional and we can apply the construction of sections 5.1 to 5.4 to V_λ . We denote by e_λ the Schwartz kernel of the projection onto V_λ . In this case, e_λ is known as the *spectral function of the Laplacian*. We also denote by E_λ the Schwartz kernel of $(V_\lambda)^r$.

Remark 5.6. Note that for every $\lambda \geq 0$, V_λ is base-point-free since the constant functions on M are eigenfunctions of Δ associated to the eigenvalue 0.

5.6 The real algebraic setting

Let us now describe more precisely the framework of the algebraic case. It is the same as in [15, 16]. Let \mathcal{X} be a smooth complex projective manifold of complex dimension n . We equip \mathcal{X} with a real structure, that is with an antiholomorphic involution $c_\mathcal{X}$. We assume that its real locus, the set of fixed points of $c_\mathcal{X}$, is not empty and we denote it by $\mathbb{R}\mathcal{X}$. Let \mathcal{L} be an ample holomorphic line bundle over \mathcal{X} equipped with a real structure $c_\mathcal{L}$ compatible with the one on \mathcal{X} . By this we mean that $c_\mathcal{X} \circ \pi = \pi \circ c_\mathcal{L}$, where $\pi : \mathcal{L} \rightarrow \mathcal{X}$ stands for the projection map onto the base space. Similarly, let \mathcal{E} be a holomorphic vector bundle of rank r over \mathcal{X} , with a compatible real structure $c_\mathcal{E}$.

Let $h_\mathcal{L}$ and $h_\mathcal{E}$ be real Hermitian metrics on \mathcal{L} and \mathcal{E} respectively, that is $c_\mathcal{L}^*(h_\mathcal{L}) = \overline{h_\mathcal{L}}$ and $c_\mathcal{E}^*(h_\mathcal{E}) = \overline{h_\mathcal{E}}$. We assume that $h_\mathcal{L}$ is positive in the sense that its *curvature form* ω is Kähler. Locally we have:

$$\omega_{/\Omega} = \frac{1}{2i} \partial \bar{\partial} \ln(h_\mathcal{L}(\zeta, \zeta))$$

where ζ is any non-vanishing local holomorphic section of \mathcal{L} on the open set $\Omega \subset \mathcal{X}$. This form corresponds to a Hermitian metric $g_\mathbb{C} = \omega(\cdot, i\cdot)$ on \mathcal{X} whose real part is a Riemannian metric g on \mathcal{X} . We denote by $dV_\mathcal{X}$ the volume form $\frac{\omega^n}{n!}$ on \mathcal{X} .

Remark 5.7. The normalization of ω is the one of [3, 5], but differs from our references concerning peak sections [16, 31]. This will cause some discrepancies with the latter two in section 6.2. With our convention, the Fubini-Study metric on $\mathbb{R}P^n$ induced by the standard metric on $\mathcal{O}(1)$ is the quotient of the Euclidean metric on \mathbb{S}^n .

Let $d \in \mathbb{N}$, the vector bundle $\mathcal{E} \otimes \mathcal{L}^d$ comes with a real structure $c_d = c_\mathcal{E} \otimes c_\mathcal{L}^d$ compatible with $c_\mathcal{X}$ and a real Hermitian metric $h_d = h_\mathcal{E} \otimes h_\mathcal{L}^d$. We equip the space $\Gamma(\mathcal{E} \otimes \mathcal{L}^d)$ of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ with the L^2 Hermitian product defined by:

$$(5.15) \quad \forall s_1, s_2 \in \Gamma(\mathcal{E} \otimes \mathcal{L}^d), \quad \langle s_1, s_2 \rangle = \int_{\mathcal{X}} h_d(s_1, s_2) dV_\mathcal{X}.$$

We know from the vanishing theorem of Kodaira and Serre that the space $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ of global holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$ has finite dimension N_d and that N_d grows as a polynomial of degree n in d , when d goes to infinity. We denote by:

$$\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) = \{s \in H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \mid c_d \circ s = s \circ c_\mathcal{X}\}$$

the space of real holomorphic sections of $\mathcal{E} \otimes \mathcal{L}^d$, which also has dimension N_d . The Hermitian product (5.15) induces a Euclidean inner product on $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Notice that we integrate on the whole of \mathcal{X} and not only on the real locus, even when we consider real sections. Finally, we equip $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ with the standard Gaussian measure $d\mu_{N_d}$.

If $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is such that its restriction to $\mathbb{R}\mathcal{X}$ vanishes transversally, then its real zero set $Z_s = s^{-1}(0) \cap \mathbb{R}\mathcal{X}$ is a (possibly empty) submanifold of $\mathbb{R}\mathcal{X}$ of codimension r ,

and we denote by $|dV_s|$ the Riemannian measure induced on this submanifold by the metric g . As in 5.2, we consider the incidence manifold:

$$\Sigma = \{(s, x) \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \mathbb{R}\mathcal{X} \mid s(x) = 0\}.$$

In this setting Σ is the zero set of the bundle map $F : \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \times \mathbb{R}\mathcal{X} \rightarrow \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)$ over $\mathbb{R}\mathcal{X}$ defined by $F(s, x) = s(x)$. In a trivialization, the situation is similar to the one in 5.2. Thus, if $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample, Σ is a smooth manifold equipped with two projection maps, π_1 and π_2 , onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and $\mathbb{R}\mathcal{X}$ respectively. By Sard's theorem, the discriminant locus of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ has measure 0, since it is the set of critical values of π_1 , as in section 5.2. Here, by 0-ample we mean that $j_x^0 : \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \rightarrow \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ is onto for every $x \in \mathbb{R}\mathcal{X}$.

Let ∇^d denote any real connection on $\mathcal{E} \otimes \mathcal{L}^d$, that is such that for every smooth section s , $\nabla^d(c_d \circ s \circ c_{\mathcal{X}}) = c_d \circ (\nabla^d s) \circ dc_{\mathcal{X}}$. For example one could choose the Chern connection. We consider the vertical component $\nabla^d F$ of the differential of F , whose kernel is the tangent space to Σ . For any (s_0, x) we have:

$$(5.16) \quad \nabla_{(s_0, x)}^d F \cdot (s, v) = s(x) + \nabla_x^d s_0 \cdot v.$$

As in 5.2, we have $\partial_1^d F(s_0, x) = j_x^0$ and $\partial_2^d F(s_0, x) = \nabla_x^d s_0$. Note that we only consider point of the zero section of $\mathcal{E} \otimes \mathcal{L}^d$, hence all this does not depend on the choice of ∇^d .

Let E_d denote the Schwartz kernel of the orthogonal projection from the space of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. It is the section of $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$ over $\mathcal{X} \times \mathcal{X}$ such that, for every real smooth section s of $\mathcal{E} \otimes \mathcal{L}^d$, the projection of s onto $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is given by:

$$x \mapsto \langle E_d(x, \cdot), s \rangle = \int_{y \in \mathcal{X}} h_d(E_d(x, y), s(y)) dV_{\mathcal{X}}.$$

Here, h_d acts on the second factor of $(\overline{\mathcal{E} \otimes \mathcal{L}^d})_x \otimes (\mathcal{E} \otimes \mathcal{L}^d)_y$. The kernel E_d satisfies a reproducing kernel property similar to (5.5). If (s_1, \dots, s_{N_d}) is an orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ then for all x and $y \in \mathcal{X}$,

$$(5.17) \quad E_d(x, y) = \sum_{i=1}^{N_d} s_i(x) \otimes s_i(y).$$

For any x and $y \in \mathbb{R}\mathcal{X}$ this shows that $E_d(x, y) \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_y$, and for $x = y$, $E_d(x, x)$ is in the span $\{\zeta \otimes \zeta \mid \zeta \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x\}$. Moreover, $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample if and only if for any $x \in \mathbb{R}\mathcal{X}$, $E_d(x, x)$ is a positive-definite bilinear form on $(\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x)^*$.

This time, $(s(x))_{x \in \mathcal{X}}$ defines a Gaussian field with values in $\mathcal{E} \otimes \mathcal{L}^d$ and its covariance function is a section of $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$ over $\mathcal{X} \times \mathcal{X}$. With the same proof as 5.2 we have:

$$\forall x, y \in \mathcal{X}, \quad \text{Cov}(s(x), s(y)) = \mathbb{E}[s(x) \otimes s(y)] = E_d(x, y).$$

Remark 5.8. The kernel E_d is also the kernel of the orthogonal projection onto $H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ in the space of smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$, for the Hermitian inner product (5.15). It is called the *Bergman kernel* of $\mathcal{E} \otimes \mathcal{L}^d$.

Let $x \in \mathbb{R}\mathcal{X}$ and, as in 5.4, let $\mathbb{R}\mathcal{J}_x^k(\mathcal{E} \otimes \mathcal{L}^d)$ denote the space of real k -jets of real smooth sections of $\mathcal{E} \otimes \mathcal{L}^d$ at x . Let $j_x^{1,d} : s \mapsto (s(x), \nabla_x^d s)$ and $j_x^{2,d} : s \mapsto (s(x), \nabla_x^d s, \nabla_x^{2,d} s)$ be defined on the space of smooth real sections of $\mathcal{E} \otimes \mathcal{L}^d$. Here $\nabla_x^d s$ and $\nabla_x^{2,d} s$ are implicitly restricted to $T_x \mathbb{R}\mathcal{X}$. These two maps induce isomorphisms:

$$\begin{aligned} \mathbb{R}\mathcal{J}_x^1(\mathcal{E} \otimes \mathcal{L}^d) &\simeq \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes (\mathbb{R} \oplus T_x^* \mathbb{R}\mathcal{X}), & \text{and} \\ \mathbb{R}\mathcal{J}_x^2(\mathcal{E} \otimes \mathcal{L}^d) &\simeq \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes (\mathbb{R} \oplus T_x^* \mathbb{R}\mathcal{X} \oplus \text{Sym}(T_x^* \mathbb{R}\mathcal{X})), \end{aligned}$$

and we still denote by $j_x^{1,d}$ and $j_x^{2,d}$ their restrictions to $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Note that the above isomorphisms are, once again, non-canonical since they depend on our choices of connections. Since $s \sim \mathcal{N}(0, \text{Id})$, the same arguments as in 5.4 show that $j_x^{2,d}(s)$ is a centered Gaussian vector in $\mathbb{R}\mathcal{J}_x^2(\mathcal{E} \otimes \mathcal{L}^d)$. Let $t \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$, $L \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes T_x^*\mathbb{R}\mathcal{X}$ and $S \in \mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x \otimes \text{Sym}(T_x^*\mathbb{R}\mathcal{X})$ be random vectors such that (t, L, S) is distributed as $j_x^{2,d}(s)$. Then t , L and S are centered Gaussian vectors and we have the following equivalent of lemma 5.4, with the same proof and similar notations.

Lemma 5.9. *Let E_d denote the kernel of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and let s be a standar Gaussian vector in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Let $x \in \mathbb{R}\mathcal{X}$ and let $(t, L, S) \in \mathbb{R}\mathcal{J}_x^2(\mathcal{E} \otimes \mathcal{L}^d)$ be distributed as $j_x^{2,d}(s)$, the 2-jet of s at x . Then (t, L, S) is a centered Gaussian vector, and its variance is characterized by:*

$$\begin{aligned}
(5.18) \quad & \text{Var}(t) = \mathbb{E}[s(x) \otimes s(x)] = E_d(x, x), \\
(5.19) \quad & \text{Var}(L) = \mathbb{E}[\nabla_x^d s \otimes \nabla_x^d s] = (\partial_x \partial_y E_d)(x, x), \\
(5.20) \quad & \text{Var}(S) = \mathbb{E}[\nabla_x^{2,d} s \otimes \nabla_x^{2,d} s] = (\partial_{x,x} \partial_{y,y} E_d)(x, x), \\
(5.21) \quad & \text{Cov}(t, L) = \mathbb{E}[s(x) \otimes \nabla_x^d s] = (\partial_y E_d)(x, x), \\
(5.22) \quad & \text{Cov}(t, S) = \mathbb{E}[s(x) \otimes \nabla_x^{2,d} s] = (\partial_{y,y} E_d)(x, x), \\
(5.23) \quad & \text{Cov}(L, S) = \mathbb{E}[\nabla_x^d s \otimes \nabla_x^{2,d} s] = (\partial_x \partial_{y,y} E_d)(x, x).
\end{aligned}$$

6 Estimates for the covariance kernels

We state in this section the estimates for the kernels described above and their first and second derivatives. These estimates will allow us to compute the limit distribution for the random 2-jets induced by the Gaussian field $(f(x))_{x \in M}$ (resp. $(s(x))_{x \in \mathcal{X}}$).

In the case of the spectral function of the Laplacian e_λ , the asymptotics of 6.1 were established by Bin [4], extending results of Hörmander [20]. In the algebraic case, Bleher Shiffman and Zelditch used estimates for the related Szegő kernel instead of the Bergman kernel, see [5, theorem 3.1]. In terms of the Bergman kernel, a similmar result was established in [3]. Both these results concern line bundles. Here, we establish the estimates we need for the Bergman kernel in the case of a higher rank bundle using Hörmander-Tian peak sections (see 6.2 and 6.3 below).

6.1 The spectral function of the Laplacian

We consider the harmonic setting of section 5.5. Let M be a Riemannian manifold of dimension n , let $x \in M$ and let (x_1, \dots, x_n) be normal coordinates centered at x . Let (y_1, \dots, y_n) denote the same coordinates in a second copy of M , so that $(x_1, \dots, x_n, y_1, \dots, y_n)$ are normal coordinates around $(x, x) \in M \times M$. We denote by ∂_{x_i} (resp. ∂_{y_i}) the partial derivative with respect to x_i (resp. y_i), and similarly we denote by ∂_{x_i, x_j} (resp. ∂_{y_i, y_j}) the second derivative with respect to x_i and x_j (resp. y_i and y_j). Let

$$\gamma_0 = \frac{1}{\sqrt{4\pi}^n \Gamma(1 + \frac{n}{2})}, \quad \gamma_1 = \frac{1}{2\sqrt{4\pi}^n \Gamma(2 + \frac{n}{2})} \quad \text{and} \quad \gamma_2 = \frac{1}{4\sqrt{4\pi}^n \Gamma(3 + \frac{n}{2})}.$$

We have:

$$(6.1) \quad \frac{\gamma_0}{\gamma_1} = n + 2 \quad \text{and} \quad \frac{\gamma_1}{\gamma_2} = n + 4.$$

Let us recall the main theorem of [4].

Theorem 6.1 (Bin). *Let V_λ be as in section 5.5 and let e_λ denote its Schwartz kernel. The following asymptotics hold uniformly in $x \in M$ as $\lambda \rightarrow +\infty$:*

$$(6.2) \quad e_\lambda(x, x) = \gamma_0 \lambda^n + O(\lambda^{n-1}),$$

$$(6.3) \quad \partial_{x_i} e_\lambda(x, x) = O(\lambda^n),$$

$$(6.4) \quad \partial_{x_i, x_k} e_\lambda(x, x) = \begin{cases} -\gamma_1 \lambda^{n+2} + O(\lambda^{n+1}) & \text{if } i = k, \\ O(\lambda^{n+1}) & \text{if } i \neq k, \end{cases}$$

$$(6.5) \quad \partial_{x_i} \partial_{y_j} e_\lambda(x, x) = \begin{cases} \gamma_1 \lambda^{n+2} + O(\lambda^{n+1}) & \text{if } i = j, \\ O(\lambda^{n+1}) & \text{if } i \neq j, \end{cases}$$

$$(6.6) \quad \partial_{x_i, x_k} \partial_{y_j} e_\lambda(x, x) = O(\lambda^{n+2}),$$

$$(6.7) \quad \partial_{x_i, x_k} \partial_{y_j, y_l} e_\lambda(x, x) = \begin{cases} 3\gamma_2 \lambda^{n+4} + O(\lambda^{n+3}) & \text{if } i = j = k = l, \\ \gamma_2 \lambda^{n+4} + O(\lambda^{n+3}) & \text{if } i = j \neq k = l \text{ or } i = k \neq j = l, \\ O(\lambda^{n+3}) & \text{otherwise.} \end{cases}$$

Since e_λ is symmetric, this also gives the asymptotics for $\partial_{y_j} e_\lambda$, $\partial_{y_j, y_l} e_\lambda$ and $\partial_{x_i} \partial_{y_j, y_l} e_\lambda$ along the diagonal. This theorem, together with equation (5.8), gives the estimates we need for the kernel E_λ of $(V_\lambda)^r$. Let $(\zeta_1, \dots, \zeta_r)$ be an orthonormal basis of \mathbb{R}^r . With obvious notations, the following estimates hold as $\lambda \rightarrow +\infty$, and they do not depend on $x \in M$ or our choice of basis.

$$(6.8) \quad \langle E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle = \begin{cases} \gamma_0 \lambda^n + O(\lambda^{n-1}) & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases}$$

$$(6.9) \quad \langle \partial_{x_i} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle = O(\lambda^n),$$

$$(6.10) \quad \langle \partial_{x_i, x_k} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle = \begin{cases} -\gamma_1 \lambda^{n+2} + O(\lambda^{n+1}) & \text{if } p = q \text{ and } i = k, \\ O(\lambda^{n+1}) & \text{otherwise,} \end{cases}$$

$$(6.11) \quad \langle \partial_{x_i} \partial_{y_j} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle = \begin{cases} \gamma_1 \lambda^{n+2} + O(\lambda^{n+1}) & \text{if } p = q \text{ and } i = j, \\ O(\lambda^{n+1}) & \text{otherwise,} \end{cases}$$

$$(6.12) \quad \langle \partial_{x_i, x_k} \partial_{y_j} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle = O(\lambda^{n+2}),$$

$$(6.13) \quad \langle \partial_{x_i, x_k} \partial_{y_j, y_l} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle = \begin{cases} 3\gamma_2 \lambda^{n+4} + O(\lambda^{n+3}) & \text{if } p = q \text{ and } i = j = k = l, \\ \gamma_2 \lambda^{n+4} + O(\lambda^{n+3}) & \text{if } p = q \text{ and} \\ & (i = j \neq k = l \text{ or } i = k \neq j = l), \\ O(\lambda^{n+3}) & \text{otherwise.} \end{cases}$$

As above, E_λ is symmetric, so this also gives the asymptotics for $\partial_{y_j} E_\lambda$, $\partial_{y_j, y_l} E_\lambda$ and $\partial_{x_i} \partial_{y_j, y_l} E_\lambda$ along the diagonal, as $\lambda \rightarrow +\infty$.

6.2 Hörmander-Tian peak sections

We now recall the construction of Hörmander-Tian peak sections in the framework of 5.6. Let \mathcal{X} be a complex projective manifold. Let \mathcal{E} be a rank r holomorphic Hermitian vector bundle and \mathcal{L} be an ample holomorphic Hermitian line bundle, both defined over \mathcal{X} . We assume that \mathcal{X} , \mathcal{E} and \mathcal{L} are endowed with compatible real structures, and that the Kähler metric $g_{\mathbb{C}}$ on \mathcal{X} is induced by the curvature ω of \mathcal{L} . Let $x \in \mathbb{R}\mathcal{X}$ and (x_1, \dots, x_n) be real holomorphic coordinates centered at x and such that $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ is orthonormal at x .

The next lemma is established in [17, lemma 3.3], up to a factor π coming from different normalisations of the metric.

Lemma 6.2. *In this setting, there exists a real holomorphic frame ζ_0 for \mathcal{L} , defined over some neighborhood of x , whose potential $-\ln(h_{\mathcal{L}}(\zeta_0, \zeta_0))$ vanishes at x , where it reaches a local minimum with Hessian $g_{\mathbb{C}}$.*

We choose such a frame ζ_0 . Let $(\zeta_1, \dots, \zeta_r)$ be a real holomorphic frame for \mathcal{E} over a neighborhood of x , which is orthonormal at x . Since \mathcal{X} is compact, we can find $\rho > 0$, not depending on x , such that local coordinates and frames as above are defined at least on the geodesic ball of radius ρ centered at x . The following results are proved in [16, section 2.3]. See also [15, section 2.2] and the paper by Tian [31, lemmas 1.2 and 2.3], without the higher rank bundle \mathcal{E} but with more details.

Proposition 6.3. *Let $p = (p_1, \dots, p_n) \in \mathbb{N}^n$, $p' > p_1 + \dots + p_n$ and $q \in \{1, \dots, r\}$. There exists $d_0 \in \mathbb{N}$ such that, for any $d \geq d_0$, there exist $C_{d,p} > 0$ and $s \in \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ such that $\|s\| = 1$ and, in some neighborhood of x ,*

$$s(x_1, \dots, x_n) = C_{d,p} \left(x_1^{p_1} \cdots x_n^{p_n} + O\left(\|(x_1, \dots, x_n)\|^{2p'}\right) \right) \left(1 + O\left(d^{-2p'}\right) \right) \zeta_q \otimes \zeta_0^d,$$

where the estimate $O(d^{-2p'})$ is uniform in $x \in \mathbb{R}\mathcal{X}$. Moreover, $C_{d,p}$ is given by:

$$(C_{d,p})^{-2} = \int_{\{\|(x_1, \dots, x_n)\| \leq \frac{\ln(d)}{\sqrt{d}}\}} |x_1^{p_1} \cdots x_n^{p_n}|^2 h_{\mathcal{L}}^d(\zeta_0^d, \zeta_0^d) dV_{\mathcal{X}},$$

and d_0 does not depend on x , p , q or our choices of local coordinates and frames.

Note that d_0 does depend on p' . The following definitions make use of proposition 6.3 with $p' = 3$ and the corresponding d_0 .

Definitions 6.4. For any $d \geq d_0$ and $q \in \{1, \dots, r\}$ we denote the sections of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ given by 6.3 by:

- $s_0^{d,q}$ for $p_1 = \dots = p_n = 0$,
- $s_i^{d,q}$ for $p_i = 1$ and $\forall k \neq i, p_k = 0$,
- $s_{i,i}^{d,q}$ for $p_i = 2$ and $\forall k \neq i, p_k = 0$,
- $s_{i,j}^{d,q}$ for $p_i = p_j = 1$ and $\forall k \notin \{i, j\}, p_k = 0$, when $i < j$.

Computing the values of the corresponding $C_{d,p}$ (see [15, lemma 2.5]), we get the following asymptotics as d goes to infinity. Once again, $O(d^{-1})$ is uniform in x .

Lemma 6.5. *For every $q \in \{1, \dots, r\}$, we have:*

$$(6.14) \quad s_0^{d,q} = \frac{1}{\sqrt{\pi}^n} \sqrt{d}^n \left(1 + O\left(\|(x_1, \dots, x_n)\|^6\right) \right) \left(1 + O(d^{-1}) \right) \zeta_q \otimes \zeta_0^d,$$

$$(6.15) \quad \forall i \in \{1, \dots, n\}, \quad s_i^{d,q} = \frac{1}{\sqrt{\pi}^n} \sqrt{d}^{n+1} \left(x_i + O\left(\|(x_1, \dots, x_n)\|^6\right) \right) \left(1 + O(d^{-1}) \right) \zeta_q \otimes \zeta_0^d,$$

$$(6.16) \quad \forall i \in \{1, \dots, n\}, \quad s_{i,i}^{d,q} = \frac{1}{\sqrt{\pi}^n} \sqrt{d}^{n+2} \left(\frac{x_i^2}{\sqrt{2}} + O\left(\|(x_1, \dots, x_n)\|^6\right) \right) \left(1 + O(d^{-1}) \right) \zeta_q \otimes \zeta_0^d,$$

and finally, $\forall i, j \in \{1, \dots, n\}$ such that $i < j$,

$$(6.17) \quad s_{i,j}^{d,q} = \frac{1}{\sqrt{\pi^n}} \sqrt{d^{n+2}} \left(x_i x_j + O\left(\|(x_1, \dots, x_n)\|^6\right) \right) (1 + O(d^{-1})) \zeta_q \otimes \zeta_0^d.$$

Remark 6.6. These values differ from the one given in [16, lemma 2.3.5] by a factor $(\int_{\mathcal{X}} dV_{\mathcal{X}})^{\frac{1}{2}}$ and some power of $\sqrt{\pi}$ because we do not use the same normalization for the volume form. For the same reason they also differ from [31, lemma 2.3] by a factor $\sqrt{\pi^n}$.

The sections defined in 6.4 are linearly independent, at least for d large enough. In fact, they are asymptotically orthonormal in the following sense. Let $H_{2,x}^d \subset \mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ denote the subspace of sections that vanish up to order 2 at $x \in \mathcal{X}$.

Lemma 6.7. *The sections $(s_{i,j}^{d,q})_{\substack{1 \leq q \leq r \\ 0 \leq i \leq n}}$ and $(s_{i,j}^{d,q})_{\substack{1 \leq q \leq r \\ 1 \leq i \leq j \leq n}}$ defined in 6.4 have L^2 -norm equal to 1 and their pairwise scalar product are dominated by a $O(d^{-1})$ independent of x . Moreover, their scalar product with any unit element of $H_{2,x}^d$ is dominated by some $O(d^{-1})$ not depending on x .*

6.3 The Bergman kernel

Let \mathcal{X} be a complex projective manifold of dimension n , let \mathcal{E} be a rank r holomorphic vector bundle over \mathcal{X} and let \mathcal{L} be an ample holomorphic line bundle over \mathcal{X} . We assume that \mathcal{X} , \mathcal{E} and \mathcal{L} equipped with compatible real structures and that \mathcal{E} and \mathcal{L} are equipped with Hermitian metrics $h_{\mathcal{E}}$ and $h_{\mathcal{L}}$. Finally, we assume that $\mathbb{R}\mathcal{X}$ is equipped with the Riemannian metric induced by the curvature of \mathcal{L} . Let $x \in \mathbb{R}\mathcal{X}$ and let (x_1, \dots, x_n) be real holomorphic coordinates around x as in 6.2. We denote by (y_1, \dots, y_n) the same coordinates as (x_1, \dots, x_n) in a second copy of \mathcal{X} . Let ζ_0 be a real holomorphic frame for \mathcal{L} given by lemma 6.2 and $(\zeta_1, \dots, \zeta_r)$ be a real holomorphic frame for \mathcal{E} that is orthonormal at x .

Let ∇^d be any real connection on $\mathcal{E} \otimes \mathcal{L}^d$ such that $\nabla^d(\zeta_p \otimes \zeta_0^d)$ vanishes in a neighborhood of x for every $p \in \{1, \dots, r\}$. We set $\nabla^{2,d} = (\nabla^{\mathcal{X}} \otimes \text{Id} + \text{Id} \otimes \nabla^d) \circ \nabla^d$ the associated second covariant derivative, where $\nabla^{\mathcal{X}}$ is the Levi-Civita connection of \mathcal{X} . In a neighborhood of x , we have for every function f :

$$(6.18) \quad \nabla^d(f \zeta_p \otimes \zeta_0^d) = df \otimes \zeta_p \otimes \zeta_0^d \quad \text{and} \quad \nabla^{2,d}(f \zeta_p \otimes \zeta_0^d) = \nabla^2 f \otimes \zeta_p \otimes \zeta_0^d.$$

As usual, ∇^d induces a connection on $(\overline{\mathcal{E} \otimes \mathcal{L}^d}) \boxtimes (\mathcal{E} \otimes \mathcal{L}^d)$. We denote by $\partial_{x_i}^d$ and $\partial_{y_i}^d$ the partial covariant derivatives with respect to x_i and y_i respectively. We also denote by ∂_{x_i, x_j}^d (resp. ∂_{y_i, y_j}^d) the second derivative with respect to x_i and x_j (resp. y_i and y_j). For simplicity, let ζ_p^d denote $\zeta_p(x) \otimes \zeta_0^d(x)$ for every $p \in \{1, \dots, r\}$ and $d \in \mathbb{N}$, so that $(\zeta_1^d, \dots, \zeta_r^d)$ is an orthonormal basis of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$.

Proposition 6.8. *In this setting, the following asymptotics hold as $d \rightarrow +\infty$, and they are*

independent of $x \in \mathbb{R}\mathcal{X}$ and of the choice of the holomorphic frame $(\zeta_1, \dots, \zeta_r)$.

$$(6.19) \quad \langle E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle = \begin{cases} \frac{d^n}{\pi^n} + O(d^{n-1}) & \text{if } p = p', \\ O(d^{n-1}) & \text{otherwise,} \end{cases}$$

$$(6.20) \quad \langle \partial_{x_i}^d E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle = O(d^{n-\frac{1}{2}}),$$

$$(6.21) \quad \langle \partial_{x_i, x_k}^d E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle = O(d^n),$$

$$(6.22) \quad \langle \partial_{x_i}^d \partial_{y_j}^d E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle = \begin{cases} \frac{d^{n+1}}{\pi^n} + O(d^n) & \text{if } p = p' \text{ and } i = j, \\ O(d^n) & \text{otherwise,} \end{cases}$$

$$(6.23) \quad \langle \partial_{x_i, x_k}^d \partial_{y_j}^d E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle = O(d^{n+\frac{1}{2}}),$$

$$(6.24) \quad \langle \partial_{x_i, x_k}^d \partial_{y_j, y_l}^d E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle = \begin{cases} 2 \frac{d^{n+2}}{\pi^n} + O(d^{n+1}) & \text{if } p = p' \text{ and } i = j = k = l, \\ \frac{d^{n+2}}{\pi^n} + O(d^{n+1}) & \text{if } p = p' \text{ and } i = j \neq k = l, \\ O(d^{n+1}) & \text{otherwise.} \end{cases}$$

Proof. First we build an orthonormal basis of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ by applying the Gram-Schmidt process to the family of peak sections. Then we use formula (5.17) and the asymptotics of lemma 6.5 to prove the proposition. We order the sections of definition 6.4 as follows:

$$\begin{aligned} & s_0^{d,1}, \dots, s_0^{d,r}, s_1^{d,1}, \dots, s_1^{d,r}, \dots, s_n^{d,1}, \dots, s_n^{d,r}, s_{1,1}^{d,1}, \dots, s_{1,1}^{d,r}, s_{2,2}^{d,1}, \dots, s_{2,2}^{d,r}, \dots, s_{n,n}^{d,1}, \dots, s_{n,n}^{d,r}, \\ & s_{1,2}^{d,1}, \dots, s_{1,2}^{d,r}, s_{1,3}^{d,1}, \dots, s_{1,3}^{d,r}, \dots, s_{1,n}^{d,1}, \dots, s_{1,n}^{d,r}, s_{2,3}^{d,1}, \dots, s_{2,3}^{d,r}, \dots, s_{n-1,n}^{d,1}, \dots, s_{n-1,n}^{d,r}. \end{aligned}$$

This family is linearly independent for d large enough and spans a space whose direct sum with $H_{2,x}^d$ is $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$. Recall that $H_{2,x}^d$ is the subspace of section whose 2-jet at x vanishes. We complete it into a basis B of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ by adding an orthonormal basis of $H_{2,x}^d$ at the end of the previous list. We apply the Gram-Schmidt process to B , starting by the last elements and going backwards. Let \tilde{B} denote the resulting orthonormal basis, and $\tilde{s}_0^{d,1}, \dots, \tilde{s}_n^{d,r}, \tilde{s}_{1,1}^{d,1}, \dots, \tilde{s}_{n-1,n}^{d,r}$ denote its first elements. This way, $\tilde{s}_{n-1,n}^{d,r}$ is a linear combination of $s_{n-1,n}^{d,r}$ and elements of $H_{2,x}^d$, and $\tilde{s}_0^{d,1}$ is a linear combination of (possibly) all elements of B .

We denote by (b_i) the first elements of B listed above (with $1 \leq i \leq \frac{r(n+1)(n+2)}{2}$), and by (\tilde{b}_i) the corresponding elements of \tilde{B} . Let $i \in \left\{1, \dots, \frac{r(n+1)(n+2)}{2}\right\}$ and assume that for any $k \leq i$ and any $j > i$ we have $\langle b_k, \tilde{b}_j \rangle = O(d^{-1})$. Note that this is the case for $i = \frac{r}{2}(n+1)(n+2)$. Then,

$$(6.25) \quad \tilde{b}_i = \frac{b_i - \sum_{j>i} \langle b_i, \tilde{b}_j \rangle \tilde{b}_j - \pi_i}{\left\| b_i - \sum_{j>i} \langle b_i, \tilde{b}_j \rangle \tilde{b}_j - \pi_i \right\|}$$

where π_i stands for the projection of b_i onto $H_{2,x}^d$. By lemma 6.7, $\|\pi_i\|^2 = \langle b_i, \pi_i \rangle = O(d^{-1})$. Then by our hypothesis $\left\| b_i - \sum_{j>i} \langle b_i, \tilde{b}_j \rangle \tilde{b}_j - \pi_i \right\|^2 = 1 + O(d^{-1})$. Using lemma 6.7 and the above hypothesis once again, we get:

$$\langle b_k, \tilde{b}_i \rangle = (1 + O(d^{-1})) \left(\langle b_k, b_i \rangle - \sum_{j>i} \langle b_k, \tilde{b}_j \rangle \langle b_i, \tilde{b}_j \rangle - \langle b_k, \pi_i \rangle \right) = O(d^{-1}),$$

for any $k < i$. By induction, for any $1 \leq i < j \leq \frac{r(n+1)(n+2)}{2}$, $\langle b_i, \tilde{b}_j \rangle = O(d^{-1})$. Using (6.25), for any $i \in \left\{1, \dots, \frac{r(n+1)(n+2)}{2}\right\}$, $\tilde{b}_i = (b_i + O(d^{-1}) \sum_{j>i} \tilde{b}_j - \tilde{\pi}_i)(1 + O(d^{-1}))$. Another induction gives:

$$(6.26) \quad \tilde{b}_i = \left(b_i + O(d^{-1}) \sum_{j>i} b_j + \tilde{\pi}_i \right) (1 + O(d^{-1})),$$

for any i , where $\tilde{\pi}_i \in H_{2,x}^d$ is such that $\|\tilde{\pi}_i\|^2 = O(d^{-1})$. Moreover, all the estimates are independent of x and $i \in \left\{1, \dots, \frac{r(n+1)(n+2)}{2}\right\}$.

Among the elements of \tilde{B} , only $\tilde{s}_0^{d,1}, \dots, \tilde{s}_0^{d,r}$ do not vanish at x . Using formula (5.17), we get $E_d(x, x) = \sum_{1 \leq q \leq r} \tilde{s}_0^{d,q}(x) \otimes \tilde{s}_0^{d,q}(x)$. Then,

$$\langle E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle = \sum_{q=1}^r \langle \tilde{s}_0^{d,q}(x), \zeta_p^d \rangle \langle \tilde{s}_0^{d,q}(x), \zeta_{p'}^d \rangle.$$

Because of the above description (6.26) of $\tilde{b}_0 = \tilde{s}_0^{d,1}, \dots, \tilde{b}_r = \tilde{s}_0^{d,r}$, for all $q \in \{1, \dots, r\}$,

$$(6.27) \quad \begin{aligned} \langle \tilde{s}_0^{d,q}(x), \zeta_p^d \rangle &= \langle s_0^{d,q}(x), \zeta_p^d \rangle + O(d^{-1}) \left(\sum_{q'=1}^r \langle s_0^{d,q'}(x), \zeta_p^d \rangle \right) \\ &= \begin{cases} \frac{1}{\sqrt{\pi^n}} \sqrt{d^n} (1 + O(d^{-1})) & \text{if } p = q, \\ O(d^{\frac{n}{2}-1}) & \text{otherwise,} \end{cases} \end{aligned}$$

where the last equality comes from lemma 6.5, equation (6.14). This establishes (6.19). Likewise,

$$\begin{aligned} \langle \partial_{x_i}^d E_d(x, x), \zeta_p^d \otimes \zeta_{p'}^d \rangle &= \left\langle \sum_{1 \leq q \leq r} \partial_{x_i}^d \tilde{s}_0^{d,q}(x) \otimes \tilde{s}_0^{d,q}(x), \zeta_p^d \otimes \zeta_{p'}^d \right\rangle \\ &= \sum_{1 \leq q \leq r} \langle \partial_{x_i}^d \tilde{s}_0^{d,q}(x), \zeta_p^d \rangle \langle \tilde{s}_0^{d,q}(x), \zeta_{p'}^d \rangle. \end{aligned}$$

The description (6.26) shows that $\partial_{x_i}^d \tilde{s}_0^{d,q}(x)$ does not necessarily vanish, but it equals:

$$O(d^{-1}) \sum_{\substack{1 \leq q' \leq r \\ 1 \leq j \leq n}} \partial_{x_i}^d s_j^{d,q'}(x).$$

By (6.15), one gets that $\langle \partial_{x_i}^d \tilde{s}_0^{d,q}(x), \zeta_p^d \rangle = O(d^{\frac{n-1}{2}})$, for all p and q . Besides by (6.27), $\langle \tilde{s}_0^{d,q}(x), \zeta_p^d \rangle = O(d^{\frac{n}{2}})$ for all p and q . This proves (6.20). The remaining estimates can be proved in exactly the same way, using lemma 6.5 and the fact that the estimates for the elements of \tilde{B} are the same as those for the elements of B . \square

7 Proofs of the main theorems

We now set to prove the main theorems. The proofs will be detailed in the harmonic case but only sketched in the real algebraic one, since they are essentially the same. We start with the expectation of the volume. In this case, the first part of the proof is closely related to [5, 6], in a slightly different setting.

7.1 Proof of theorem 1.1

For a start, we consider the general setting described in sections 5.1 to 5.4. That is M is a closed Riemannian manifold of dimension n , V is a subspace of $\mathcal{C}^\infty(M)$ of finite dimension without base point, and f is a standard Gaussian vector in V^r . We denote by N the dimension of V^r . For almost every $f \in V^r$, Z_f is a smooth submanifold of M , and its volume is $\text{Vol}(Z_f) = \int_{Z_f} |dV_f|$. We have:

$$\mathbb{E}[\text{Vol}(Z_f)] = \int_{f \in V^r} \left(\int_{Z_f} |dV_f| \right) d\mu_N = \int_{f \in V^r} \left(\int_{Z_f} \frac{e^{-\frac{1}{2}\|f\|^2}}{\sqrt{2\pi}^N} |dV_f| \right) df,$$

where df stands for the Lebesgue measure on V^r . Then, using the incidence manifold of section 5.2 we apply the double-fibration trick of proposition 2.4.

$$\begin{aligned} \mathbb{E}[\text{Vol}(Z_f)] &= \int_{x \in M} \left(\int_{\ker(j_x^0)} \frac{e^{-\frac{1}{2}\|f\|^2}}{\sqrt{2\pi}^N} \frac{|\det^\perp(\partial_2 F(f, x))|}{|\det^\perp(\partial_1 F(f, x))|} df \right) |dV_M| \\ &= \frac{1}{\sqrt{2\pi}^r} \int_{x \in M} \frac{1}{|\det^\perp(j_x^0)|} \left(\int_{f \in \ker(j_x^0)} |\det^\perp(d_x f)| d\mu_{(N-r)} \right) |dV_M|. \end{aligned}$$

Where $j_x^0 : f \mapsto f(x)$ and the last equality comes from equation (5.3) and the fact that $|\det^\perp(j_x^0)|$ does not depend on f . We also used that, for every $x \in M$, $\ker(j_x^0)$ has dimension $N - r$, since V^r being 0-ample.

Let us express this in terms of the Schwartz kernel E of V^r . Let $x \in M$ and recall from section 2.1 that $|\det^\perp(j_x^0)| = \sqrt{\det(j_x^0 j_x^{0*})}$. Since $f \sim \mathcal{N}(0, \text{Id})$, equation (4.8) shows that $j_x^0 j_x^{0*}$ is the variance operator of $f(x) = j_x^0(f)$. But $\det(\Lambda_{f(x)}) = \det(\text{Var}(f(x)))$, that is the determinant of the matrix, in any orthonormal basis, of $\text{Var}(f(x))$ seen as a bilinear form. By lemma 5.4, $\text{Var}(f(x)) = E(x, x)$ hence $|\det^\perp(j_x^0(f))| = \sqrt{\det(E(x, x))}$. Then, $V^r = \ker(j_x^0)^\perp \oplus \ker(j_x^0)$ and by corollary 4.6 applied to this splitting,

$$\int_{f \in \ker(j_x^0)} |\det^\perp(d_x f)| d\mu_{(N-r)} = \mathbb{E}[|\det^\perp(d_x f)| \mid f(x) = 0].$$

Thus,

$$(7.1) \quad \mathbb{E}[\text{Vol}(Z_f)] = \frac{1}{\sqrt{2\pi}^r} \int_{x \in M} \frac{1}{\sqrt{\det(E(x, x))}} \mathbb{E}[|\det^\perp(d_x f)| \mid f(x) = 0] |dV_M|.$$

Let $x \in M$, $\mathbb{E}[|\det^\perp(d_x f)| \mid f(x) = 0] = \mathbb{E}[|\det^\perp(L)| \mid t = 0]$ where (t, L) is a random vector in $\mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^* M)$ distributed as $j_x^1(f)$. Recall section 5.4, (t, L) is a centered Gaussian vector whose variance is given by lemma 5.4. In particular, the distribution of (t, L) only depends on the values of E and its derivatives at (x, x) . Thus $\mathbb{E}[\text{Vol}(Z_f)]$ will only depend on the values of E and its derivatives along the diagonal as was expected from [5, theorem 2.2].

Let (x_1, \dots, x_n) be normal coordinates centered at x , and $(\zeta_1, \dots, \zeta_r)$ be an orthonormal basis of \mathbb{R}^r . We equip $\mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^* M)$ with the orthonormal basis:

$$(7.2) \quad (\zeta_1, \dots, \zeta_r, \zeta_1 \otimes dx_1, \dots, \zeta_1 \otimes dx_n, \zeta_2 \otimes dx_1, \dots, \zeta_2 \otimes dx_n, \dots, \zeta_r \otimes dx_1, \dots, \zeta_r \otimes dx_n).$$

With respect to this basis, the matrix of the variance of (t, L) is:

$$(7.3) \quad \Lambda = \begin{pmatrix} \Lambda_{00} & \Lambda_{01} \\ \Lambda_{10} & \Lambda_{11} \end{pmatrix},$$

where Λ_{00} and Λ_{11} are the matrices of $\text{Var}(t)$ and $\text{Var}(L)$ respectively, and $\Lambda_{10} = {}^t\Lambda_{01}$ is the matrix of $\text{Cov}(L, t)$. We can decompose further Λ_{10} and Λ_{11} into blocks of size $r \times r$:

$$(7.4) \quad \Lambda_{10} = (\Lambda_{10}^i)_{1 \leq i \leq n}, \quad \Lambda_{11} = (\Lambda_{11}^{i,j})_{1 \leq i, j \leq n}.$$

Then by lemma 5.4 we have:

$$\begin{aligned} \Lambda_{00} &= (\langle E(x, x), \zeta_p \otimes \zeta_q \rangle)_{1 \leq p, q \leq r}, \\ \forall i \in \{1, \dots, n\}, \quad \Lambda_{10}^i &= (\langle \partial_{x_i} E(x, x), \zeta_p \otimes \zeta_q \rangle)_{1 \leq p, q \leq r}, \\ \forall i, j \in \{1, \dots, n\}, \quad \Lambda_{11}^{i,j} &= (\langle \partial_{x_i} \partial_{y_j} E(x, x), \zeta_p \otimes \zeta_q \rangle)_{1 \leq p, q \leq r}. \end{aligned}$$

We now specify to the harmonic setting of section 5.5, that is $V = V_\lambda$ is spanned by the eigenfunctions of the Laplacian associated to eigenvalues smaller than λ^2 . The estimates of section 6.1 suggest to consider the rescaled variables $t_\lambda = \frac{1}{\sqrt{\gamma_0 \lambda^n}} f(x)$ and $L_\lambda = \frac{1}{\sqrt{\gamma_1 \lambda^{n+2}}} d_x f$.

We denote by $\Lambda(\lambda) = \begin{pmatrix} \Lambda_{00}(\lambda) & \Lambda_{01}(\lambda) \\ \Lambda_{10}(\lambda) & \Lambda_{11}(\lambda) \end{pmatrix}$ the variance matrix of the *rescaled* variables (t_λ, L_λ) . The $r \times r$ matrices $\Lambda_{10}^i(\lambda)$ and $\Lambda_{11}^{i,j}(\lambda)$ are defined as above. Using the estimates of section 6.1:

$$(7.5) \quad \Lambda_{00}(\lambda) = \left(\frac{\langle E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle}{\gamma_0 \lambda^n} \right)_{1 \leq p, q \leq r} = I_r + O(\lambda^{-1}),$$

$$(7.6) \quad \forall i \in \{1, \dots, n\}, \quad \Lambda_{10}^i(\lambda) = \left(\frac{\langle \partial_{x_i} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle}{\sqrt{\gamma_0 \gamma_1} \lambda^{n+1}} \right)_{1 \leq p, q \leq r} = O(\lambda^{-1}),$$

and, $\forall i, j \in \{1, \dots, n\}$,

$$(7.7) \quad \Lambda_{11}^{i,j}(\lambda) = \left(\frac{\langle \partial_{x_i} \partial_{y_j} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle}{\gamma_1 \lambda^{n+2}} \right)_{1 \leq p, q \leq r} = \begin{cases} I_r + O(\lambda^{-1}) & \text{if } i = j, \\ O(\lambda^{-1}) & \text{otherwise,} \end{cases}$$

where I_r stands for the identity matrix of size r . Thus $\Lambda(\lambda) = I_{r(n+1)} + O(\lambda^{-1})$ and by corollary 4.6, the distribution of L_λ conditioned on $t_\lambda = 0$ is a centered Gaussian with variance operator $\tilde{\Lambda}(\lambda) = \text{Id} + O(\lambda^{-1})$.

Lemma 7.1. *Let L be a standard Gaussian vector in $\mathbb{R}^r \otimes T_x^* M$, then:*

$$\mathbb{E}[|\det^\perp(L_\lambda)| \mid t_\lambda = 0] = \mathbb{E}[|\det^\perp(L)|] + O(\lambda^{-1}).$$

Proof. By Lebesgue's theorem,

$$\int |\det^\perp(L)| \exp\left(-\frac{1}{2} \langle \tilde{\Lambda}(\lambda)^{-1} L, L \rangle\right) dL \xrightarrow{\lambda \rightarrow +\infty} \int |\det^\perp(L)| e^{-\frac{\|L\|^2}{2}} dL.$$

Note that $\tilde{\Lambda}(\lambda)$ is invertible for λ large enough and that these integrals are finite since $|\det^\perp(L)|$ is the absolute value of a polynomial in the coefficients of L . Thus we have:

$$\begin{aligned} \mathbb{E}[|\det^\perp(L_\lambda)| \mid t_\lambda = 0] &= \frac{1}{\sqrt{2\pi}^{nr} \sqrt{\det(\tilde{\Lambda}(\lambda))}} \int |\det^\perp(L)| \exp\left(-\frac{1}{2} \langle \tilde{\Lambda}(\lambda)^{-1} L, L \rangle\right) dL \\ &= \frac{1}{\sqrt{2\pi}^{nr}} \int |\det^\perp(L)| \exp\left(-\frac{1}{2} \langle \tilde{\Lambda}(\lambda)^{-1} L, L \rangle\right) dL + O(\lambda^{-1}). \end{aligned}$$

Besides,

$$\begin{aligned} & \left| \int |\det^\perp(L)| \exp\left(-\frac{1}{2} \langle \tilde{\Lambda}(\lambda)^{-1} L, L \rangle\right) dL - \int |\det^\perp(L)| e^{-\frac{\|L\|^2}{2}} dL \right| \\ & \leq \int |\det^\perp(L)| e^{-\frac{\|L\|^2}{2}} \left| \exp\left(-\frac{1}{2} \langle (\tilde{\Lambda}(\lambda)^{-1} - \text{Id})L, L \rangle\right) - 1 \right| dL, \end{aligned}$$

and $(\tilde{\Lambda}(\lambda)^{-1} - \text{Id})$ is bounded by $\frac{C}{\lambda}$ for some positive C . Then for any L ,

$$\left| \langle (\tilde{\Lambda}(\lambda)^{-1} - \text{Id})L, L \rangle \right| \leq \frac{C}{\lambda} \|L\|^2,$$

and

$$\left| \exp\left(-\frac{1}{2} \langle (\tilde{\Lambda}(\lambda)^{-1} - \text{Id})L, L \rangle\right) - 1 \right| \leq \left| e^{\frac{C}{2\lambda} \|L\|^2} - 1 \right| \leq \frac{C}{2\lambda} \|L\|^2 e^{\frac{C}{2\lambda} \|L\|^2},$$

by the mean value theorem. Then for $\lambda \geq 2C$, the above integral is less than:

$$\frac{C}{2\lambda} \int |\det^\perp(L)| \|L\|^2 e^{-\frac{1}{4} \|L\|^2} \frac{dL}{\sqrt{2\pi}^{nr}} = O(\lambda^{-1}).$$

Finally we have proved:

$$\mathbb{E}[|\det^\perp(L_\lambda)| \mid t_\lambda = 0] = \frac{1}{\sqrt{2\pi}^{nr}} \int |\det^\perp(L)| e^{-\frac{\|L\|^2}{2}} dL + O(\lambda^{-1}).$$

Note that all the above estimates do not depend on x or our choices of coordinates. \square

By remark 5.6, V_λ has no base point. Thus $(V_\lambda)^r$ is 0-ample for all $\lambda \geq 0$, and we can use formula (7.1). Besides, by equation (6.8) the matrix of $E_\lambda(x, x)$ in the orthonormal basis $(\zeta_1, \dots, \zeta_r)$ is $\gamma_0 \lambda^n (I_r + O(\lambda^{-1}))$. Hence, $\det(E_\lambda(x, x)) = \gamma_0^r \lambda^{nr} (1 + O(\lambda^{-1}))$.

$$\begin{aligned} \mathbb{E}[\text{Vol}(Z_f)] &= \frac{1}{\sqrt{2\pi}^r} \int_{x \in M} \frac{1}{\sqrt{\det(E_\lambda(x, x))}} \mathbb{E}\left[\left| \det^\perp(\sqrt{\gamma_1 \lambda^{n+2}} L_\lambda) \right| \mid t_\lambda = 0 \right] |dV_M| \\ &= \frac{1}{\sqrt{2\pi}^r} \int_M \left(\frac{\gamma_1 \lambda^{n+2}}{\gamma_0 \lambda^n} \right)^{\frac{r}{2}} \mathbb{E}[|\det^\perp(L_\lambda)| \mid t_\lambda = 0] (1 + O(\lambda^{-1})) |dV_M| \\ &= \frac{1}{\sqrt{2\pi}^r} \left(\frac{\lambda}{\sqrt{n+2}} \right)^r \int_M \mathbb{E}[|\det^\perp(L)|] + O(\lambda^{-1}) |dV_M|, \text{ by (6.1) and lemma 7.1,} \\ &= \left(\frac{\lambda}{\sqrt{n+2}} \right)^r \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)} \text{Vol}(M) + O(\lambda^{r-1}). \end{aligned}$$

In the last equality we applied lemma 4.14 to L , and used the uniformity in x of the error term. This proves theorem 1.1.

Remark 7.2. The same proof shows that for any open set $U \subset M$ we have:

$$\mathbb{E}[\text{Vol}(Z_f \cap U)] = \left(\frac{\lambda}{n+2} \right)^r \text{Vol}(U) \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)} + O(\lambda^{r-1}).$$

7.2 Proof of theorem 1.3

We now consider the real algebraic setting described in section 5.6. The proof goes along the same lines as above. Recall that \mathcal{X} is a complex projective manifold of dimension n , equipped with a rank r holomorphic vector bundle \mathcal{E} and an ample holomorphic line bundle

\mathcal{L} , and that \mathcal{X} , \mathcal{E} and \mathcal{L} are endowed with compatible real structures. Let E_d denote the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$.

By equation (6.19), the matrix of $E_d(x, x)$ is $d^n \pi^{-n} (I_r + O(d^{-1}))$ in any orthonormal basis of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L})_x$, and $\det(E_d(x, x)) = d^{rn} \pi^{-rn} (1 + O(d^{-1}))$. Since the estimates are uniform in $x \in \mathbb{R}\mathcal{X}$, this shows that, for d large enough, $E_d(x, x)$ is positive-definite for all x . Hence $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ is 0-ample for d large enough. Applying proposition 2.4 for d large enough, and using the partial differentials given by equation (5.16) we get:

$$\mathbb{E}[\text{Vol}(Z_s)] = \frac{1}{\sqrt{2\pi^r}} \int_{x \in \mathbb{R}\mathcal{X}} \frac{1}{\sqrt{\det(E_d(x, x))}} \mathbb{E}[|\det^\perp(\nabla_x^d s)| \mid s(x) = 0] |dV_{\mathbb{R}\mathcal{X}}|,$$

where $s \sim \mathcal{N}(0, \text{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and ∇^d any real connection on $\mathcal{E} \otimes \mathcal{L}^d$. Let $(\zeta_1^d, \dots, \zeta_r^d)$ be any orthonormal basis of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$. We can assume that this basis is realized as the value at x of a real holomorphic frame $(\zeta_1 \otimes \zeta_0^d, \dots, \zeta_r \otimes \zeta_0^d)$ satisfying the conditions of section 6.3. We will compute the variance matrices in an orthonormal basis similar to (7.2), with ζ_p replaced by $\zeta_p \otimes \zeta_0^d$.

Let $x \in \mathbb{R}\mathcal{X}$, $\mathbb{E}[|\det^\perp(\nabla_x^d s)| \mid s(x) = 0]$ does not depend on the choice of ∇^d , since $\nabla_x^d s$ does not depend on ∇^d when $s \in \ker(j_x^0)$. We choose a connection that satisfies the conditions of section 6.3. That is, the frame $(\zeta_1 \otimes \zeta_0^d, \dots, \zeta_r \otimes \zeta_0^d)$ is flat in a neighborhood of x .

We consider the rescaled variables $t_d = \sqrt{\frac{\pi^n}{d^n}} s(x)$ and $L_d = \sqrt{\frac{\pi^n}{d^{n+1}}} \nabla_x^d s$. Using the estimates of proposition 6.8, we get $\Lambda(d) = I_{r(n+1)} + O(d^{-1})$, where $\Lambda(d)$ denotes the variance matrix of (t_d, L_d) and $I_{r(n+1)}$ stands for the identity matrix of size $r(n+1)$. More precisely, using notations similar to (7.3) and (7.4), one gets:

$$(7.8) \quad \Lambda_{00}(d) = \left(\frac{\pi^n}{d^n} \langle E_d(x, x), \zeta_p^d \otimes \zeta_q^d \rangle \right)_{1 \leq p, q \leq r} = I_r + O(d^{-1}),$$

$$(7.9) \quad \forall i \in \{1, \dots, n\}, \quad \Lambda_{10}^i(d) = \left(\frac{\pi^n}{d^{n+\frac{1}{2}}} \langle \partial_{x_i} E_d(x, x), \zeta_p \otimes \zeta_q \rangle \right)_{1 \leq p, q \leq r} = O(d^{-1}),$$

and, $\forall i, j \in \{1, \dots, n\}$,

$$(7.10) \quad \Lambda_{11}^{i,j}(d) = \left(\frac{\pi^n}{d^{n+1}} \langle \partial_{x_i} \partial_{y_j} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle \right)_{1 \leq p, q \leq r} = \begin{cases} I_r + O(d^{-1}) & \text{if } i = j, \\ O(d^{-1}) & \text{otherwise.} \end{cases}$$

As in lemma 7.1, $\mathbb{E}[|\det^\perp(L_d)| \mid t_d = 0] = \mathbb{E}[|\det^\perp(L)|] + O(d^{-1})$, where L is a standard Gaussian vector in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d) \otimes T_x^*(\mathbb{R}\mathcal{X})$. Then, as above:

$$\begin{aligned} \mathbb{E}[\text{Vol}(Z_s)] &= \frac{1}{\sqrt{2\pi^r}} \left(\frac{1}{\pi^n d^n} \right)^{\frac{r}{2}} \int_{\mathbb{R}\mathcal{X}} \mathbb{E}[|\det^\perp(\pi^n d^{n+1} L_d)| \mid t_d = 0] (1 + O(d^{-1})) |dV_{\mathbb{R}\mathcal{X}}| \\ &= \sqrt{d}^r \frac{1}{\sqrt{2\pi^r}} \int_{\mathbb{R}\mathcal{X}} \mathbb{E}[|\det^\perp(L)|] + O(d^{-1}) |dV_{\mathbb{R}\mathcal{X}}| \\ &= \sqrt{d}^r \text{Vol}(\mathbb{R}\mathcal{X}) \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)} + O(\sqrt{d}^{r-2}). \end{aligned}$$

Remark 7.3. In the same way, we can compute $\mathbb{E}[\text{Vol}(Z_s \cap U)]$ where U is an open subset of $\mathbb{R}\mathcal{X}$. The result is the same with $\text{Vol}(\mathbb{R}\mathcal{X})$ replaced by $\text{Vol}(U)$.

7.3 Proof of theorem 1.2

Consider first the general setting of section 5.1. For almost every $f \in V^r$, Z_f is a closed submanifold of dimension $n - r$. The Euler characteristic of a closed manifold of odd dimension is 0, see [18, corollary 3.37]. Thus if $n - r$ is odd, $\chi(Z_f)$ is almost surely 0. From now on, we

assume that $n - r$ is even and set $m = \frac{n-r}{2}$. If $n = r$, then Z_f is almost surely a finite set, and $\chi(Z_f) = \text{Vol}(Z_f)$ is just the cardinal of Z_f . In this case theorems 1.2 and 1.1 coincide, so we only need to consider the case $r < n$ in the sequel.

For any $f \in V^r$ we denote by R_f the Riemann curvature tensor of Z_f , and by Π_f its second fundamental form. By the Chern-Gauss-Bonnet theorem 3.3, the double-fibration trick of proposition 2.4, and (5.3):

$$\begin{aligned} \mathbb{E}[\chi(Z_f)] &= \mathbb{E} \left[\frac{1}{(2\pi)^m m!} \int_{Z_f} \text{Tr}((R_f)^{\wedge m}) |dV_f| \right] \\ &= \frac{1}{m! \sqrt{2\pi}^{N+2m}} \int_{f \in V^r} \left(\int_{Z_f} e^{-\frac{1}{2}\|f\|^2} \text{Tr}(R_f(x)^{\wedge m}) |dV_f| \right) df \\ &= \frac{1}{m! \sqrt{2\pi}^{N+2m}} \int_{x \in M} \left(\int_{\ker(j_x^0)} e^{-\frac{1}{2}\|f\|^2} \text{Tr}(R_f(x)^{\wedge m}) \frac{|\det^\perp(\partial_2 F(f, x))|}{|\det^\perp(\partial_1 F(f, x))|} df \right) |dV_M| \\ &= \frac{1}{m! \sqrt{2\pi}^{r+2m}} \int_{x \in M} \left(\int_{f \in \ker(j_x^0)} \frac{|\det^\perp(d_x f)|}{|\det^\perp(j_x^0)|} \text{Tr}(R_f(x)^{\wedge m}) d\mu_{(N-r)} \right) |dV_M|. \end{aligned}$$

Recall that $n = 2m + r$ and $|\det^\perp(j_x^0)| = \sqrt{\det(E(x, x))}$. As in the volume case, we can express the innermost integral as a conditional expectation, using corollary 4.6:

(7.11)

$$\mathbb{E}[\chi(Z_f)] = \frac{1}{m! \sqrt{2\pi}^n} \int_{x \in M} \frac{1}{\sqrt{\det(E(x, x))}} \mathbb{E} [|\det^\perp(d_x f)| \text{Tr}(R_f(x)^{\wedge m}) \mid f(x) = 0] |dV_M|.$$

Let $x \in M$ and let $U \sim \mathcal{N}(0, \text{Id})$ in $T_x M$ be independent of f . We denote by R the Riemann curvature tensor of M . By lemmas 4.16 and 3.7,

$$R_f(x) = R(x) + \frac{1}{2} \mathbb{E}_U [\langle \Pi_f(x), U \rangle^2] = R(x) + \frac{1}{2} \mathbb{E}_U [\langle d_x f^\dagger \circ \nabla_x^2 f, U \rangle^2].$$

The notation $\mathbb{E}_U [\cdot]$ means that we take the expectation with respect to the variable U only.

Remark 7.4. Here and in everything that follows, $R(x)$ and $\nabla_x^2 f$ are implicitly restricted to $\ker(d_x f) = T_x Z_f$.

As in the volume case, let $(t, L, S) \in \mathbb{R}^r \otimes (\mathbb{R} \oplus T_x^* M \oplus \text{Sym}(T_x^* M))$ be a random vector distributed as the 2-jet, $j_x^2(f)$, of f at x . It is a centered Gaussian vector whose variance is given by lemma 5.4. Then, the conditional expectation in (7.11) is:

$$(7.12) \quad \mathbb{E}_{L,S} \left[|\det^\perp(L)| \text{Tr} \left(\left(R(x) + \frac{1}{2} \mathbb{E}_U [\langle L^\dagger S, U \rangle^2] \right)^{\wedge m} \right) \mid t = 0 \right].$$

As above, $R(x)$ and S are implicitly restricted to $\ker(L)$. This shows that the expected Euler characteristic only depends on R and the values of E and its derivatives (up to order 2 in each variable) along the diagonal in $M \times M$. It turns out that the asymptotics no longer depend on R , both in the harmonic and the algebraic case.

In the case of random eigenfunctions of the Laplacian, we will consider the following rescaled variables:

$$t_\lambda = \frac{1}{\sqrt{\gamma_0 \lambda^n}} f(x), \quad L_\lambda = \frac{1}{\sqrt{\gamma_1 \lambda^{n+2}}} d_x f \quad \text{and} \quad S_\lambda = \frac{1}{\sqrt{\gamma_2 \lambda^{n+4}}} \nabla_x^2 f.$$

As in 7.1 we choose (x_1, \dots, x_n) normal coordinates centered at x and $(\zeta_1, \dots, \zeta_r)$ an orthonormal basis of \mathbb{R}^r . For any i and j such that $1 \leq i < j \leq n$, we set $dx_{ij} = dx_i \otimes dx_j$

and $dx_{ij} = (dx_i \otimes dx_j + \overline{dx_j} \otimes dx_i)$. We complete the basis of $\mathcal{J}_x^1(\mathbb{R}^r)$ given in (7.2) into an orthonormal basis of $\mathcal{J}_x^2(\mathbb{R}^r)$ by adding the following elements at the end of the list:

$$(7.13) \quad \begin{aligned} & \zeta_1 \otimes dx_{11}, \dots, \zeta_r \otimes dx_{11}, \zeta_1 \otimes dx_{22}, \dots, \zeta_r \otimes dx_{22}, \dots, \zeta_1 \otimes dx_{nn}, \dots, \zeta_r \otimes dx_{nn}, \\ & \zeta_1 \otimes dx_{12}, \dots, \zeta_r \otimes dx_{12}, \zeta_1 \otimes dx_{13}, \dots, \zeta_r \otimes dx_{13}, \dots, \zeta_1 \otimes dx_{1n}, \dots, \zeta_r \otimes dx_{1n}, \\ & \dots, \zeta_1 \otimes dx_{(n-1)n}, \dots, \zeta_r \otimes dx_{(n-1)n}. \end{aligned}$$

The matrix of $\text{Var}(t_\lambda, L_\lambda, S_\lambda)$ with respect to this basis is:

$$\Lambda(\lambda) = \begin{pmatrix} \Lambda_{00}(\lambda) & \Lambda_{01}(\lambda) & \Lambda_{02}(\lambda) \\ \Lambda_{10}(\lambda) & \Lambda_{11}(\lambda) & \Lambda_{12}(\lambda) \\ \Lambda_{20}(\lambda) & \Lambda_{21}(\lambda) & \Lambda_{22}(\lambda) \end{pmatrix}.$$

Here $\Lambda_{00}(\lambda)$, $\Lambda_{10}(\lambda) = {}^t\Lambda_{01}(\lambda)$ and $\Lambda_{11}(\lambda)$ are defined similarly to (7.3), for the rescaled variables. Besides, $\Lambda_{22}(\lambda)$ is the matrix of $\text{Var}(S_\lambda)$, $\Lambda_{20}(\lambda) = {}^t\Lambda_{02}(\lambda)$ is the matrix of $\text{Cov}(S_\lambda, t_\lambda)$ and $\Lambda_{21}(\lambda) = {}^t\Lambda_{12}(\lambda)$ is the matrix of $\text{Cov}(S_\lambda, L_\lambda)$. We can decompose further each of these matrices in blocks of size $r \times r$ as in (7.4). Then Λ_{00} , Λ_{10} and Λ_{11} satisfy (7.5), (7.6) and (7.7) respectively. Moreover,

$$\Lambda_{20} = (\Lambda_{20}^{ik})_{1 \leq i \leq k \leq n}, \quad \Lambda_{21} = (\Lambda_{21}^{ik,j})_{\substack{1 \leq i \leq k \leq n \\ 1 \leq j \leq n}} \quad \text{and} \quad \Lambda_{22} = (\Lambda_{22}^{ik,jl})_{\substack{1 \leq i \leq k \leq n \\ 1 \leq j \leq l \leq n}},$$

where for all i, j, k and $l \in \{1, \dots, n\}$, with $i \leq k$ and $j \leq l$,

$$(7.14) \quad \Lambda_{20}^{ik}(\lambda) = \left(\frac{\langle \partial_{x_i, x_k} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle}{\sqrt{\gamma_0 \gamma_2} \lambda^{n+2}} \right)_{1 \leq p, q \leq r} = \begin{cases} -\sqrt{\frac{\gamma_1^2}{\gamma_0 \gamma_2}} I_r + O(\lambda^{-1}) & \text{if } i = k, \\ O(\lambda^{-1}) & \text{if } i \neq k, \end{cases}$$

$$(7.15) \quad \Lambda_{21}^{ik,j}(\lambda) = \left(\frac{\langle \partial_{x_i, x_k} \partial_{y_j} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle}{\sqrt{\gamma_1 \gamma_2} \lambda^{n+3}} \right)_{1 \leq p, q \leq r} = O(\lambda^{-1}),$$

$$(7.16) \quad \Lambda_{22}^{ik,jl}(\lambda) = \left(\frac{\langle \partial_{x_i, x_k} \partial_{y_j, y_l} E_\lambda(x, x), \zeta_p \otimes \zeta_q \rangle}{\gamma_2 \lambda^{n+4}} \right)_{1 \leq p, q \leq r} = \begin{cases} 3I_r + O(\lambda^{-1}) & \text{if } i = j = k = l, \\ I_r + O(\lambda^{-1}) & \text{if } i = j \neq k = l \\ & \text{or } i = k \neq j = l, \\ O(\lambda^{-1}) & \text{otherwise.} \end{cases}$$

Here I_r denotes the identity matrix of size r and we used (6.10), (6.12) and (6.13). Recalling (6.1), we set: $\gamma = -\sqrt{\frac{\gamma_1^2}{\gamma_0 \gamma_2}} = -\sqrt{\frac{n+4}{n+2}}$. Then $\Lambda(\lambda)$ writes, by blocks of size $r \times r$:

$$\Lambda(\lambda) = \left(\begin{array}{c|ccc|cccc|ccc|ccc} I_r & 0 & \cdots & 0 & \gamma I_r & \gamma I_r & \cdots & \gamma I_r & 0 & \cdots & 0 \\ \hline 0 & I_r & & & & & & & & & \\ \vdots & & \ddots & & & & & & & & \\ 0 & & & I_r & & & & & & & \\ \hline \gamma I_r & & & & 3I_r & I_r & \cdots & I_r & & & \\ \gamma I_r & & & & I_r & 3I_r & \ddots & \vdots & & & \\ \vdots & & & & \vdots & \ddots & \ddots & I_r & & & \\ \gamma I_r & & & & I_r & \cdots & I_r & 3I_r & & & \\ \hline 0 & & & & & & & & I_r & & \\ \vdots & & & & & & & & & \ddots & \\ 0 & & & & & & & & & & I_r \end{array} \right) + O(\lambda^{-1}),$$

Lemma 7.5. For all $q \in \{1, \dots, m\}$,

$$\begin{aligned} \mathbb{E}_{U, \tilde{L}_\lambda, \tilde{S}_\lambda} \left[\left| \det^\perp(\tilde{L}_\lambda) \right| \operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{L}_\lambda^\dagger \tilde{S}_\lambda, U \rangle^{\wedge 2q} \right) \right] \\ = \mathbb{E}_{U, \tilde{L}, \tilde{S}} \left[\left| \det^\perp(\tilde{L}) \right| \operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, \tilde{L}^{\dagger*} U \rangle^{\wedge 2q} \right) \right] + O(\lambda^{-1}). \end{aligned}$$

Proof. First, we prove that the right-hand side is finite. Let $\zeta \in \mathbb{R}^r$, then,

$$(7.20) \quad \mathbb{E}_{\tilde{S}} \left[\operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, \zeta \rangle^{\wedge 2q} \right) \right]$$

is finite since it is the expectation of some polynomial in \tilde{S} . Again \tilde{S} is implicitly restricted to some $2m$ -dimensional subspace V of $T_x M$, and it is not obvious that (7.20) depends only on ζ and not on V . This is the case because the distribution of \tilde{S} is invariant under orthogonal transformations of $T_x M$. Thus (7.20) only depends on ζ , and it is a homogeneous polynomial in ζ of degree $2q$.

We assumed that U was independent of (\tilde{L}, \tilde{S}) , and the expression (7.17) of $\tilde{\Lambda}(\infty)$ shows that \tilde{L} and \tilde{S} are independent. Let $U' \sim \mathcal{N}(0, \operatorname{Id})$ in \mathbb{R}^r and X_{n-r+1}, \dots, X_n be standard Gaussian vectors with $X_p \in \mathbb{R}^p$ such that $U', \tilde{S}, X_{n-r+1}, \dots, X_n$ are globally independent. Applying proposition 4.9, we have:

$$\begin{aligned} (7.21) \quad \mathbb{E}_{U, \tilde{L}, \tilde{S}} \left[\left| \det^\perp(\tilde{L}) \right| \operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, \tilde{L}^{\dagger*} U \rangle^{\wedge 2q} \right) \right] \\ = \mathbb{E}_{U, \tilde{L}} \left[\left| \det^\perp(\tilde{L}) \right| \mathbb{E}_{\tilde{S}} \left[\operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, \tilde{L}^{\dagger*} U \rangle^{\wedge 2q} \right) \right] \right] \\ = \mathbb{E}_{U', X_p} \left[\|X_n\| \cdots \|X_{n-r+1}\| \mathbb{E}_{\tilde{S}} \left[\operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, \frac{U'}{\|X_{n-r+1}\|} \rangle^{\wedge 2q} \right) \right] \right] \\ = \mathbb{E}_{U', X_p} \left[\frac{\|X_n\| \cdots \|X_{n-r+2}\|}{\|X_{n-r+1}\|^{2q-1}} \mathbb{E}_{\tilde{S}} \left[\operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, U' \rangle^{\wedge 2q} \right) \right] \right] \\ = \mathbb{E} \left[\frac{1}{\|X_{n-r+1}\|^{2q-1}} \right] \mathbb{E}_{(U', \tilde{S})} \left[\operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, U' \rangle^{\wedge 2q} \right) \right] \prod_{p=n-r+2}^n \mathbb{E}[\|X_p\|]. \end{aligned}$$

Since $2q - 1 \leq 2m - 1 \leq n - r - 1$ and X_{n-r+1} is a standard Gaussian in \mathbb{R}^{n-r+1} , $\mathbb{E} \left[\frac{1}{\|X_{n-r+1}\|^{2q-1}} \right] < +\infty$ (see lemma 4.8). The other factors on the right-hand side of (7.21) are all finite since they are expectations of polynomials in some Gaussian variables.

Then, $(\tilde{\Lambda}(\lambda)^{-1} - \tilde{\Lambda}(\infty)^{-1})$ is bounded by $\frac{C}{\lambda}$ for some positive C . Hence, as in the proof of lemma 7.1:

$$(7.22) \quad \int \left| \det^\perp(L) \right| \operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle L^\dagger S, U' \rangle^{\wedge 2q} \right) \times \\ \left(e^{-\frac{1}{2} \langle \tilde{\Lambda}(\lambda)^{-1}(L, S), (L, S) \rangle} - e^{-\frac{1}{2} \langle \tilde{\Lambda}(\infty)^{-1}(L, S), (L, S) \rangle} \right) dL dS d\mu_n(U')$$

is bounded by:

$$\frac{C}{2\lambda} \int \left| \det^\perp(L) \right| \left| \operatorname{Tr} \left(R(x)^{\wedge m-q} \wedge \langle L^\dagger S, U' \rangle^{\wedge 2q} \right) \right| \| (L, S) \|^2 \times \\ e^{-\frac{1}{4} \langle \tilde{\Lambda}(\infty)^{-1}(L, S), (L, S) \rangle} dL dS d\mu_n(U'),$$

for λ large enough. As for (7.21) above, this integral is finite, hence (7.22) is $O(\lambda^{-1})$. Moreover, $\det(\tilde{\Lambda}(\lambda)) = \det(\tilde{\Lambda}(\infty)) + O(\lambda^{-1})$, and this is enough to establish lemma 7.5. \square

Using lemma 7.5 and equation (7.19), the sum in (7.18) equals:

$$(7.23) \quad (\gamma_1 \lambda^{n+2})^{\frac{r}{2}} \frac{\lambda^{2m}}{(n+4)^m} \frac{m!}{(2m)!} \mathbb{E} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(\left\langle \tilde{S}, \tilde{L}^\dagger * U \right\rangle^{\wedge 2m} \right) \right] (1 + O(\lambda^{-1})).$$

Then, for a standard Gaussian vector f in $(V_\lambda)^r$, equations (7.11), (7.12), (7.18) and (7.23) yield:

$$\begin{aligned} \mathbb{E}[\chi(Z_f)] &= (\gamma_1 \lambda^{n+2})^{\frac{r}{2}} \frac{\lambda^{2m}}{(n+4)^m} \frac{1}{(2m)! \sqrt{2\pi}^n} \times \\ &\quad \int_{x \in M} \frac{1}{\sqrt{\det(E_\lambda(x, x))}} \mathbb{E} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(\left\langle \tilde{S}, \tilde{L}^\dagger * U \right\rangle^{\wedge 2m} \right) \right] |dV_M| (1 + O(\lambda^{-1})), \end{aligned}$$

where we used the fact that all our estimates are uniform in $x \in M$. Recall from section 7.1 that $\det(E_\lambda(x, x)) = (\gamma_0 \lambda^n)^r (1 + O(\lambda^{-1}))$. Then, by (6.1),

$$(7.24) \quad \begin{aligned} \mathbb{E}[\chi(Z_f)] &= \left(\frac{\gamma_1}{\gamma_0} \right)^{\frac{r}{2}} \frac{\lambda^{2m+r}}{(n+4)^m} \frac{\text{Vol}(M)}{(2m)! \sqrt{2\pi}^n} \mathbb{E} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(\left\langle \tilde{S}, \tilde{L}^\dagger * U \right\rangle^{\wedge 2m} \right) \right] (1 + O(\lambda^{-1})) \\ &= \frac{\lambda^n \text{Vol}(M)}{\sqrt{2\pi}^n (2m)!} \frac{1}{(n+2)^{\frac{r}{2}} (n+4)^m} \mathbb{E} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(\left\langle \tilde{S}, \tilde{L}^\dagger * U \right\rangle^{\wedge 2m} \right) \right] + O(\lambda^{-1}). \end{aligned}$$

We still have to compute the expectation on the right-hand side. Recall from (7.21) that:

$$\begin{aligned} &\mathbb{E} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(\left\langle \tilde{S}, \tilde{L}^\dagger * U \right\rangle^{\wedge 2m} \right) \right] \\ &= \mathbb{E} \left[\frac{1}{\|X_{n-r+1}\|^{2m-1}} \right] \mathbb{E}_{(U', \tilde{S})} \left[\text{Tr} \left(\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right) \right] \prod_{p=n-r+2}^n \mathbb{E}[\|X_p\|], \end{aligned}$$

where $U' \sim \mathcal{N}(0, \text{Id})$ in \mathbb{R}^r and $X_p \sim \mathcal{N}(0, \text{Id})$ in \mathbb{R}^p for all p . Then, by lemma 4.8, this equals:

$$(7.25) \quad \sqrt{2\pi}^{r-2m} \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^1)} \frac{\text{Vol}(\mathbb{S}^{n-r+1})}{\text{Vol}(\mathbb{S}^n)} \mathbb{E}_{(U', \tilde{S})} \left[\text{Tr} \left(\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right) \right],$$

where \tilde{S} is restricted to any $2m$ -dimensional subspace of $T_x M$. We will compute this term for \tilde{S} restricted to the span of $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}} \right)$.

Lemma 7.6.

$$\mathbb{E}_{\tilde{S}, U'} \left[\text{Tr} \left(\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right) \right] = (-1)^m \left(\frac{n+4}{n+2} \right)^m \pi^m \frac{((2m)!)^2}{m!} \frac{\text{Vol}(\mathbb{S}^{r-1})}{\text{Vol}(\mathbb{S}^{n-1})}.$$

Proof. First we keep U' fixed and apply lemma 4.15 to the random bilinear form $\langle \tilde{S}, U' \rangle$. We have:

$$(7.26) \quad \mathbb{E}_{\tilde{S}} \left[\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right] = \frac{(2m)!}{2^m m!} \mathbb{E}_{\tilde{S}} \left[\left\langle \tilde{S}, U' \right\rangle^{\wedge 2} \right]^{\wedge m}.$$

Assume that $\tilde{S}_{/V} = \sum_{1 \leq i, k \leq 2m} \tilde{S}_{ik} dx_i \otimes dx_k$ with $\tilde{S}_{ik} \in \mathbb{R}^r$. Then,

$$\left\langle \tilde{S}_{/V}, U' \right\rangle = \sum_{1 \leq i, k \leq 2m} \left\langle \tilde{S}_{ik}, U' \right\rangle dx_i \otimes dx_k,$$

and

$$\begin{aligned}
(7.27) \quad \mathbb{E}_{\tilde{S}} \left[\left\langle \tilde{S}_{/V}, U' \right\rangle^{\wedge 2} \right] &= \sum_{1 \leq i, j, k, l \leq 2m} \mathbb{E}_{\tilde{S}} \left[\left\langle \tilde{S}_{ik}, U' \right\rangle \left\langle \tilde{S}_{jl}, U' \right\rangle \right] (dx_i \wedge dx_j) \otimes (dx_k \wedge dx_l) \\
&= \sum_{1 \leq i, j, k, l \leq 2m} \left\langle U', \left(\tilde{\Lambda}^{ik, jl} U' \right) \right\rangle (dx_i \wedge dx_j) \otimes (dx_k \wedge dx_l),
\end{aligned}$$

where we denoted by $\tilde{\Lambda}^{ik, jl}$ the covariance operator of \tilde{S}_{ik} and \tilde{S}_{jl} and applied lemma 4.3 to these random variables. The matrix of $\tilde{\Lambda}^{ik, jl}$ is the limit of $\tilde{\Lambda}_{22}^{ik, jl}(\lambda)$ as $\lambda \rightarrow +\infty$, hence is given by the leading term of (7.17). By (7.26) and (7.27), we get:

$$\begin{aligned}
\mathbb{E}_{\tilde{S}} \left[\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right] &= \frac{(2m)!}{2^m m!} \sum_{\substack{1 \leq i_1, \dots, i_m \leq 2m \\ 1 \leq j_1, \dots, j_m \leq 2m \\ 1 \leq k_1, \dots, k_m \leq 2m \\ 1 \leq l_1, \dots, l_m \leq 2m}} \left(\prod_{p=1}^m \left\langle U', \left(\tilde{\Lambda}^{i_p k_p, j_p l_p} \right) U' \right\rangle \right) \times \\
&\quad (dx_{i_1} \wedge dx_{j_1} \wedge \dots \wedge dx_{i_m} \wedge dx_{j_m}) \otimes (dx_{k_1} \wedge dx_{l_1} \wedge \dots \wedge dx_{k_m} \wedge dx_{l_m}) \\
&= \frac{(2m)!}{2^m m!} \sum_{\sigma, \sigma' \in \mathfrak{S}_{2m}} \varepsilon(\sigma) \varepsilon(\sigma') \prod_{p=1}^m \left\langle U', \left(\tilde{\Lambda}^{\sigma(2p-1)\sigma'(2p-1), \sigma(2p)\sigma'(2p)} \right) U' \right\rangle (dx \otimes dx),
\end{aligned}$$

where \mathfrak{S}_{2m} is the set of permutations of $\{1, \dots, 2m\}$, $\varepsilon : \mathfrak{S}_{2m} \rightarrow \{-1, 1\}$ denotes the signature morphism and $dx = dx_1 \wedge \dots \wedge dx_{2m}$. We get the last line by setting $\sigma(2p-1) = i_p$, $\sigma(2p) = j_p$, $\sigma'(2p-1) = k_p$ and $\sigma'(2p) = l_p$ and by reordering the wedge products. Now, $\text{Tr}(dx \otimes dx) = \|dx\|^2 = 1$, because our local coordinates are such that $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}} \right)$ is orthonormal at x . Thus:

$$(7.28) \quad \text{Tr} \left(\mathbb{E}_{\tilde{S}} \left[\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right] \right) = \frac{(2m)!}{2^m m!} \sum_{\sigma, \sigma' \in \mathfrak{S}_{2m}} \varepsilon(\sigma) \varepsilon(\sigma') \prod_{p=1}^m \left\langle U', \left(\tilde{\Lambda}^{\sigma(2p-1)\sigma'(2p-1), \sigma(2p)\sigma'(2p)} \right) U' \right\rangle.$$

By equation (7.17), for any $\sigma, \sigma' \in \mathfrak{S}_{2m}$ and for any $p \in \{1, \dots, m\}$,

$$\tilde{\Lambda}^{\sigma(2p-1)\sigma'(2p-1), \sigma(2p)\sigma'(2p)} = K(p, \sigma, \sigma') \text{Id},$$

where,

$$(7.29) \quad K(p, \sigma, \sigma') = \begin{cases} \beta & \text{if } \sigma(2p-1) = \sigma'(2p-1) \text{ and } \sigma(2p) = \sigma'(2p), \\ 1 & \text{if } \sigma(2p-1) = \sigma'(2p) \text{ and } \sigma(2p) = \sigma'(2p-1), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $K(p, \sigma, \sigma') = K(p, \text{id}, \sigma^{-1} \circ \sigma')$, where id stands for the identity permutation. Then, setting $\tau = \sigma^{-1} \circ \sigma'$:

$$\begin{aligned}
(7.30) \quad \sum_{\sigma, \sigma' \in \mathfrak{S}_{2m}} \varepsilon(\sigma) \varepsilon(\sigma') \prod_{p=1}^m \left\langle U', \left(\tilde{\Lambda}^{\sigma(2p-1)\sigma'(2p-1), \sigma(2p)\sigma'(2p)} \right) U' \right\rangle \\
&= \sum_{\sigma, \sigma' \in \mathfrak{S}_{2m}} \varepsilon(\sigma) \varepsilon(\sigma') \|U'\|^{2m} \prod_{p=1}^m K(p, \sigma, \sigma') \\
&= (2m)! \|U'\|^{2m} \sum_{\tau \in \mathfrak{S}_{2m}} \varepsilon(\tau) \prod_{p=1}^m K(p, \text{id}, \tau).
\end{aligned}$$

From the definition (7.29) of $K(p, \text{id}, \tau)$, we get that $\prod_{p=1}^m K(p, \text{id}, \tau) \neq 0$ if and only if τ is a product of transpositions of the type $((2p-1) (2p))$. Now, if $I \subset \{1, \dots, m\}$ and $\tau = \prod_{p \in I} ((2p-1) (2p))$, then:

$$\prod_{p=1}^m K(p, \text{id}, \tau) = \beta^{m-|I|} \quad \text{and} \quad \varepsilon(\tau) = (-1)^{|I|},$$

where $|I|$ stands for the cardinal of I . Thus,

$$(7.31) \quad \sum_{\tau \in \mathfrak{S}_{2m}} \varepsilon(\tau) \prod_{p=1}^m K(p, \text{id}, \tau) = \sum_{I \subset \{1, \dots, m\}} (-1)^{|I|} \beta^{m-|I|} = \sum_{p=1}^m \binom{m}{p} (-1)^p \beta^{m-p} \\ = (\beta - 1)^m = (-1)^m \left(\frac{n+4}{n+2} \right)^m.$$

Finally, by lemma 4.8, (7.28), (7.30) and (7.31):

$$\mathbb{E}_{\tilde{S}, U'} \left[\text{Tr} \left(\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right) \right] = \frac{((2m)!)^2}{2^m m!} \mathbb{E} \left[\|U'\|^{2m} \right] (-1)^m \left(\frac{n+4}{n+2} \right)^m \\ = (-1)^m \left(\frac{n+4}{n+2} \right)^m \pi^m \frac{((2m)!)^2 \text{Vol}(\mathbb{S}^{r-1})}{m! \text{Vol}(\mathbb{S}^{n-1})}. \quad \square$$

By equations (7.24), (7.25) and by lemma 7.6,

$$(7.32) \quad \mathbb{E}[\chi(Z_f)] = \frac{(-1)^m \lambda^n \text{Vol}(M) (2m)! \text{Vol}(\mathbb{S}^{n-r}) \text{Vol}(\mathbb{S}^{r-1}) \text{Vol}(\mathbb{S}^{n-r+1})}{(n+2)^{\frac{r}{2}} \text{Vol}(\mathbb{S}^n) m! 2^{n-r+1} \pi^{\frac{n-r}{2}+1} \text{Vol}(\mathbb{S}^{n-1})} + O(\lambda^{n-1}).$$

Then,

$$\frac{(2m)!}{m!} \text{Vol}(\mathbb{S}^{n-r}) = \frac{2\Gamma(n-r+1) \sqrt{\pi}^{n-r+1}}{\Gamma\left(\frac{n-r+2}{2}\right) \Gamma\left(\frac{n-r+1}{2}\right)} = 2^{n-r+1} \sqrt{\pi}^{n-r},$$

where we used that $\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi} 2^{1-2z} \Gamma(2z)$. This concludes the proof of theorem 1.2.

7.4 Proof of theorem 1.4

Let us now adapt the proof of the previous section to the case of real algebraic submanifolds. Once again we need only consider the case where $r < n$ and $n-r$ is even. Recall that \mathcal{X} is a complex projective manifold of dimension n , equipped with a rank r holomorphic vector bundle \mathcal{E} and an ample holomorphic line bundle \mathcal{L} , and that \mathcal{X} , \mathcal{E} and \mathcal{L} are endowed with compatible real structures. Let E_d denote the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Applying the Chern-Gauss-Bonnet theorem 3.3 and the double fibration trick yields:

$$(7.33) \quad \mathbb{E}[\chi(Z_s)] = \frac{1}{m! \sqrt{2\pi}^n} \int_{x \in \mathbb{R}\mathcal{X}} \frac{1}{\sqrt{\det(E_d(x, x))}} \mathbb{E} \left[|\det^\perp(\nabla_x^d s)| \text{Tr}(R_s(x)^{\wedge m}) \mid s(x) = 0 \right] |dV_{\mathbb{R}\mathcal{X}}|,$$

where ∇^d is any real connection on $(\mathcal{E} \otimes \mathcal{L}^d)$, s is a standard Gaussian in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ and R_s denotes the Riemann tensor of Z_s . We need to compute the conditional expectation in (7.33) for some fixed $x \in \mathbb{R}\mathcal{X}$. Since it does not depend on our choice of connection, we will use one that is adapted to x as in sections 6.3 and 7.2.

Let $x \in \mathbb{R}\mathcal{X}$ and $U \sim \mathcal{N}(0, \text{Id})$ in $T_x \mathbb{R}\mathcal{X}$. By lemmas 4.16 and 3.7 and by (6.18),

$$R_s(x) = R(x) + \frac{1}{2} \mathbb{E}_U \left[\langle \Pi_s(x), U \rangle^{\wedge 2} \right] = R(x) + \frac{1}{2} \mathbb{E}_U \left[\langle (\nabla_x^d s)^\dagger \circ \nabla_x^{2,d} s, U \rangle^{\wedge 2} \right],$$

where R is the Riemann tensor of $\mathbb{R}\mathcal{X}$ and Π_s stands for the second fundamental form of $Z_s \subset \mathbb{R}\mathcal{X}$. We consider the rescaled variables:

$$t_d = \sqrt{\frac{\pi^n}{d^n}} s(x), \quad L_d = \sqrt{\frac{\pi^n}{d^{n+1}}} \nabla_x^d s \quad \text{and} \quad S_d = \sqrt{\frac{\pi^n}{d^{n+2}}} \nabla_x^{2,d} s.$$

As in 7.2 let $(\zeta_1^d, \dots, \zeta_r^d)$ be an orthonormal basis of $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ that we assume to be the value at x of a real holomorphic frame $(\zeta_1 \otimes \zeta_0^d, \dots, \zeta_r \otimes \zeta_0^d)$ satisfying the conditions of section 6.3. Then we get an orthonormal basis of $\mathbb{R}\mathcal{J}_x^2(\mathcal{E} \otimes \mathcal{L}^d)$ similar to the one defined by (7.2) and (7.13) and we compute the matrix $\Lambda(d)$ of $\text{Var}(t_d, L_d, S_d)$ in this basis. For this we use (7.8), (7.9), (7.10), and similar estimates for the blocks implying second derivatives given by proposition 6.8. Namely, for all i, j, k and $l \in \{1, \dots, n\}$, with $i \leq k$ and $j \leq l$,

$$(7.34) \quad \Lambda_{20}^{ik}(d) = O(d^{-1}),$$

$$(7.35) \quad \Lambda_{21}^{ik,j}(d) = O(d^{-1}),$$

$$(7.36) \quad \Lambda_{22}^{ik,jl}(d) = \begin{cases} 2I_r + O(d^{-1}) & \text{if } i = j = k = l, \\ I_r + O(d^{-1}) & \text{if } i = j \neq k = l, \\ O(d^{-1}) & \text{otherwise.} \end{cases}$$

Note that in this case t_d , L_d and S_d are asymptotically independent. By corollary 4.6, the distribution of (L_d, S_d) given $t_d = 0$ is then a centered Gaussian whose variance matrix is, by blocks of size $r \times r$,

$$(7.37) \quad \tilde{\Lambda}(d) = \begin{pmatrix} I_r & & & & & \\ & \ddots & & & & \\ & & I_r & & & \\ \hline & & & 2I_r & & \\ & & & & \ddots & \\ & & & & & 2I_r \\ \hline & & & & & & I_r \\ & & & & & & & \ddots \\ & & & & & & & & I_r \end{pmatrix} + O(d^{-1}),$$

where the empty blocks are zeros. Let $(\tilde{L}_d, \tilde{S}_d)$ be distributed as (L_d, S_d) given $t_d = 0$ and $(\tilde{L}, \tilde{S}) \sim \mathcal{N}(0, \tilde{\Lambda}(\infty))$ where $\tilde{\Lambda}(\infty)$ is the limit of $\tilde{\Lambda}(d)$ as $d \rightarrow +\infty$. The proof of lemma 7.5 adapts immediately in this setting.

Lemma 7.7. *For all $q \in \{1, \dots, m\}$,*

$$\begin{aligned} \mathbb{E}_{U, \tilde{L}_d, \tilde{S}_d} \left[\left| \det^\perp(\tilde{L}_d) \right| \text{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{L}_d^\dagger \tilde{S}_d, U \rangle^{\wedge 2q} \right) \right] \\ = \mathbb{E}_{U, \tilde{L}, \tilde{S}} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}, \tilde{L}^\dagger U \rangle^{\wedge 2q} \right) \right] + O(d^{-1}). \end{aligned}$$

Then the same proof as in 7.3 gives:

$$\begin{aligned} \mathbb{E} \left[\left| \det^\perp(\nabla_x^d s) \right| \text{Tr}(R_s(x)^{\wedge m}) \mid s(x) = 0 \right] \\ = \left(\frac{d^{n+1}}{\pi^n} \right)^{\frac{r}{2}} \sum_{q=1}^m \binom{m}{q} \frac{d^q q!}{(2q)!} \mathbb{E} \left[\left| \det^\perp(\tilde{L}_d) \right| \text{Tr} \left(R(x)^{\wedge m-q} \wedge \langle \tilde{S}_d, \tilde{L}_d^\dagger U \rangle^{\wedge 2q} \right) \right] \\ = \left(\frac{d^{n+1}}{\pi^n} \right)^{\frac{r}{2}} \frac{d^m m!}{(2m)!} \mathbb{E} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(\langle \tilde{S}, \tilde{L}^\dagger U \rangle^{\wedge 2m} \right) \right] (1 + O(d^{-1})). \end{aligned}$$

We denote by U' a standard Gaussian vector in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ and by X_p a standard Gaussian in \mathbb{R}^p for every $p \in \{n-r+1, \dots, n\}$. Then:

$$\begin{aligned} & \mathbb{E} \left[\left| \det^\perp(\tilde{L}) \right| \text{Tr} \left(\left\langle \tilde{S}, \tilde{L}^\dagger U' \right\rangle^{\wedge 2m} \right) \right] \\ &= \mathbb{E} \left[\frac{1}{\|X_{n-r+1}\|^{2m-1}} \right] \mathbb{E}_{(U', \tilde{S})} \left[\text{Tr} \left(\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right) \right] \prod_{p=n-r+2}^n \mathbb{E}[\|X_p\|] \\ &= \sqrt{2\pi}^{r-2m} \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^1)} \frac{\text{Vol}(\mathbb{S}^{n-r+1})}{\text{Vol}(\mathbb{S}^n)} \mathbb{E}_{(U', \tilde{S})} \left[\text{Tr} \left(\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right) \right]. \end{aligned}$$

Lemma 7.8.

$$\mathbb{E}_{\tilde{S}, U'} \left[\text{Tr} \left(\left\langle \tilde{S}, U' \right\rangle^{\wedge 2m} \right) \right] = (-1)^m \pi^m \frac{((2m)!)^2}{m!} \frac{\text{Vol}(\mathbb{S}^{r-1})}{\text{Vol}(\mathbb{S}^{n-1})}.$$

Proof. The proof is the same as for lemma 7.6. The only difference comes from the fact that the $\tilde{\Lambda}^{ik, jl}$ are now given by (7.37). Hence we have to change the definition (7.29) of K . In this setting:

$$K(p, \sigma, \sigma') = \begin{cases} 1 & \text{if } \sigma(2p-1) = \sigma'(2p) \text{ and } \sigma(2p) = \sigma'(2p-1), \\ 0 & \text{otherwise,} \end{cases}$$

so that $\prod_{p=1}^m K(p, \text{id}, \tau)$ is 0, unless τ is the permutation $\tau_0 = \prod_{p=1}^m ((2p-1)(2p))$. Once again, id denotes the identity element of \mathfrak{S}_{2m} . Then (7.31) becomes:

$$\sum_{\tau \in \mathfrak{S}_{2m}} \varepsilon(\tau) \prod_{p=1}^m K(p, \text{id}, \tau) = \varepsilon(\tau_0) \prod_{p=1}^m K(p, \text{id}, \tau_0) = (-1)^m.$$

The same computations as in lemma 7.6 give the result. \square

Recalling from section 7.2 that $\sqrt{\det(E_d(x, x))} = \left(\frac{d^n}{\pi^n}\right)^{\frac{r}{2}} (1 + O(d^{-1}))$, we get:

$$\mathbb{E}[\chi(Z_s)] = (-1)^{\frac{n-r}{2}} \left(\sqrt{d}\right)^n \text{Vol}(\mathbb{R}\mathcal{X}) \frac{\text{Vol}(\mathbb{S}^{n-r+1}) \text{Vol}(\mathbb{S}^{r-1})}{\pi \text{Vol}(\mathbb{S}^n) \text{Vol}(\mathbb{S}^{n-1})} + O\left(\sqrt{d}^{n-2}\right)$$

by the same kind of computations as in the harmonic case.

7.5 Two special cases

In some special cases, the covariance kernel is known explicitly. It is then possible to prove more precise results. In this section, we sketch what happens on the flat torus, and in the projective space.

The flat torus. Let $\mathbb{T}^n = \mathbb{R}^n / (2\pi\mathbb{Z})^n$ denote the torus of dimension n that we equip with the quotient of the Euclidean metric on \mathbb{R}^n . We have $\text{Vol}(\mathbb{T}^n) = (2\pi)^n$. In our notations, we identify functions on \mathbb{T}^n and $(2\pi\mathbb{Z})^n$ -periodic functions on \mathbb{R}^n . Then, the Laplacian is $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, and it is known that its eigenvalues are the integers of the form $\|p\|^2$, with $p = (p_1, \dots, p_n) \in \mathbb{N}^n$. The eigenspace associated to 0 is spanned by the constant

function $x \mapsto \frac{1}{\sqrt{2\pi}^n}$. For $\lambda > 0$, the eigenspace associated to λ^2 is spanned by the normalized functions of the form:

$$x \mapsto \frac{2}{\sqrt{2\pi}^n} \sin(\langle p, x \rangle) \quad \text{and} \quad x \mapsto \frac{2}{\sqrt{2\pi}^n} \cos(\langle p, x \rangle),$$

where $p \in \mathbb{Z}^n$ is such that $\|p\| = \lambda$, and $\langle \cdot, \cdot \rangle$ is the canonical scalar product on \mathbb{R}^n .

We set $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\| \leq \lambda\}$. Then, after some computations, we get e_λ , the spectral function of the Laplacian on \mathbb{T}^n :

$$\forall \lambda \geq 0, \forall x, y \in \mathbb{T}^n, \quad e_\lambda(x, y) = \frac{1}{(2\pi)^n} \sum_{p \in \mathbb{B}_\lambda} \cos(\langle p, x - y \rangle).$$

Let $\lambda \geq 0$ and $r \in \{1, \dots, n\}$, let V_λ be spanned by the eigenfunctions of Δ associated to eigenvalues smaller than λ^2 , and let E_λ denote the Schwarz kernel of $(V_\lambda)^r$. Then E_λ is determined by the above expression for e_λ , see (5.8). Let $f \sim \mathcal{N}(0, \text{Id})$ in $(V_\lambda)^r$. Going back to equation (7.1) we have:

$$\mathbb{E}[\text{Vol}(Z_f)] = \frac{1}{\sqrt{2\pi}^r} \int_{\mathbb{T}^n} \frac{1}{\sqrt{\det(E_\lambda)}} \mathbb{E}[|\det^\perp(L_\lambda)| \mid t_\lambda = 0] |dV_{\mathbb{T}^n}|,$$

where (t_λ, L_λ) is distributed as $j_x^1(f)$. The variance of (t_λ, L_λ) is explicit, and it only depends on E_λ and its derivatives along the diagonal in $\mathbb{T}^n \times \mathbb{T}^n$ (see lemma 5.4). Moreover, E_λ and its derivatives are constant along the diagonal. By computations similar to those of section 7.1 we get the following.

Proposition 7.9. *On the flat torus \mathbb{T}^n , let $\lambda \geq 0$ and let $f \sim \mathcal{N}(0, \text{Id})$ in $(V_\lambda)^r$. We have:*

$$\mathbb{E}[\text{Vol}(Z_f)] = \left(\frac{1}{|\mathbb{B}_\lambda|} \sum_{(p_1, \dots, p_n) \in \mathbb{B}_\lambda} (p_1)^2 \right)^{\frac{r}{2}} (2\pi)^n \frac{\text{Vol}(\mathbb{S}^{n-r})}{\text{Vol}(\mathbb{S}^n)},$$

where $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\| \leq \lambda\}$ and $|\mathbb{B}_\lambda|$ denotes the cardinal of \mathbb{B}_λ .

This result was already known, see [29], where Rudnick and Wigman compute the variance of $\text{Vol}(Z_f)$ when $r = 1$, and the references therein.

Assuming that $n - r$ is even and equals $2m$, equations (7.11) and (7.12) yield:

$$\mathbb{E}[\chi(Z_f)] = \frac{1}{2^m m! \sqrt{2\pi}^n} \int_{\mathbb{T}^n} \frac{1}{\sqrt{\det(E_\lambda)}} \mathbb{E} \left[|\det^\perp(L_\lambda)| \text{Tr} \left(\mathbb{E}_U \left[\left\langle L_\lambda^\dagger S_\lambda, U \right\rangle^{\wedge 2} \right]^{\wedge m} \right) \mid t_\lambda = 0 \right] |dV_{\mathbb{T}^n}|,$$

where $(t_\lambda, L_\lambda, S_\lambda)$ is distributed as $j_x^2(f)$ and $U \sim \mathcal{N}(0, \text{Id})$ in $T_x \mathbb{T}^n$. Indeed, the Riemann curvature of \mathbb{T}^n is uniformly 0. Once again, the variance of $(t_\lambda, L_\lambda, S_\lambda)$ only depends on E_λ and its derivatives along the diagonal, and it does not depend on $x \in \mathbb{T}^n$. Computations similar to those of 7.3 give the following.

Proposition 7.10. *On the flat torus \mathbb{T}^n , let $\lambda \geq 0$ and let $f \sim \mathcal{N}(0, \text{Id})$ in $(V_\lambda)^r$. Then, if $n - r$ is even, we have:*

$$\mathbb{E}[\chi(Z_f)] = (-1)^{\frac{n-r}{2}} \left(\frac{1}{|\mathbb{B}_\lambda|} \sum_{(p_1, \dots, p_n) \in \mathbb{B}_\lambda} (p_1)^2 \right)^{\frac{r}{2}} (2\pi)^n \frac{\text{Vol}(\mathbb{S}^{n-r+1}) \text{Vol}(\mathbb{S}^{r-1})}{\pi \text{Vol}(\mathbb{S}^n) \text{Vol}(\mathbb{S}^{n-1})},$$

where $\mathbb{B}_\lambda = \{p \in \mathbb{Z}^n \mid \|p\| \leq \lambda\}$ and $|\mathbb{B}_\lambda|$ denotes the cardinal of \mathbb{B}_λ .

The projective space. We consider the algebraic case with $\mathcal{X} = \mathbb{C}P^n$, $\mathcal{E} = \mathbb{C}^r \times \mathbb{C}P^n$ with the standard Hermitian metric on each fiber, and $\mathcal{L} = \mathcal{O}(1)$ with its usual metric. Then, ω is the standard Fubini-Study form. We consider the real structures induced by the standard conjugation in \mathbb{C} . Let e_d denote the Bergman kernel of \mathcal{L}^d and E_d denote the Bergman kernel of $\mathcal{E} \otimes \mathcal{L}^d$. Since \mathcal{E} is trivial, E_d and e_d satisfy a relation similar to (5.8), that is:

$$\forall x, y \in \mathbb{C}P^n, \forall \zeta, \zeta' \in \mathbb{C}^r, \quad \langle E_d(x, y), \zeta \otimes \zeta' \rangle = e_d(x, y) \langle \zeta, \zeta' \rangle.$$

As we said in section 1, we have $\mathbb{R}\mathcal{X} = \mathbb{R}P^n$, and elements of $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d)$ are just r -tuples of real homogeneous polynomials of degree d in $n+1$ variables. We denote by $\mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n]$ the space of real homogeneous polynomials of degree d in X_0, \dots, X_n . For any $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, we set $|\alpha| = \alpha_0 + \dots + \alpha_n$ and $X^\alpha = X_0^{\alpha_0} \dots X_n^{\alpha_n}$. If $|\alpha| = d$, we also set $\binom{d}{\alpha} = \frac{d!}{\alpha_0! \dots \alpha_n!}$. It is well-known, see [5, 8, 10, 23], that an orthonormal basis of $\mathbb{R}H^0(\mathbb{C}P^n, \mathcal{O}(d)) \simeq \mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n]$ for the inner product (5.15) is given by the sections:

$$s_\alpha = \sqrt{\frac{(n+d)!}{\pi^n d!}} \binom{d}{\alpha} X^\alpha, \quad \text{with } |\alpha| = d.$$

Then, formally,

$$e_d = \frac{(n+d)!}{\pi^n d!} \sum_{|\alpha|=d} \binom{d}{\alpha} X^\alpha Y^\alpha = \frac{(n+d)!}{\pi^n d!} \langle X, Y \rangle^d.$$

More precisely, we consider the chart $(x_1, \dots, x_n) \mapsto [1 : x_1 : \dots : x_n]$ around the point of homogeneous coordinates $[1 : 0 : \dots : 0]$, and the real holomorphic frame $s_{(d,0,\dots,0)}$ for $\mathcal{O}(d)$ around this point. In these coordinates:

$$e_d(x, y) = \frac{(n+d)!}{\pi^n d!} (1 + \langle x, y \rangle)^d (s_{(d,0,\dots,0)}(x) \otimes s_{(d,0,\dots,0)}(y)),$$

with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Since everything is invariant under unitary transformations of $\mathbb{C}P^n$, this totally describes e_d , and E_d .

Let $d \in \mathbb{N}$ and $r \in \{1, \dots, n\}$. Let $s \sim \mathcal{N}(0, \text{Id})$ in $\mathbb{R}H^0(\mathcal{X}, \mathcal{E} \otimes \mathcal{L}^d) \simeq (\mathbb{R}_d^{\text{hom}}[X_0, \dots, X_n])^r$. Then, as in 7.2:

$$\mathbb{E}[\text{Vol}(Z_s)] = \frac{1}{\sqrt{2\pi}^r} \int_{\mathbb{R}P^n} \frac{1}{\sqrt{\det(E_d)}} \mathbb{E}[|\det^\perp(L_d)| \mid t_d = 0] |dV_{\mathbb{R}P^n}|,$$

where (t_d, L_d) is distributed as $j_x^{1,d}(s)$. Since E_d is explicit, we can compute the variance of (t_d, L_d) . Some further computations yield Kostlan's result [23]:

$$\mathbb{E}[\text{Vol}(Z_s)] = (\sqrt{d})^r \text{Vol}(\mathbb{R}P^{n-r}).$$

Assuming that $n-r$ is even and equals $2m$, we have:

$$\mathbb{E}[\chi(Z_s)] = \frac{1}{m! \sqrt{2\pi}^n} \int_{\mathbb{R}P^n} \frac{1}{\sqrt{\det(E_d)}} \mathbb{E}\left[|\det^\perp(L_d)| \text{Tr}\left(\left(R + \frac{1}{2} \mathbb{E}_U \left[\left\langle L_d^\dagger S_d, U \right\rangle^{\wedge 2}\right]^{\wedge m}\right)\right)\right] |dV_{\mathbb{R}P^n}|,$$

where R is the Riemann curvature of $\mathbb{R}P^n$, (t_d, L_d, S_d) is distributed as $j_x^{2,d}$ and $U \sim \mathcal{N}(0, \text{Id})$ in $T_x \mathbb{R}P^n$. In this expression we already used the fact that t_d is independent of (L_d, S_d) , this is why the expectation is not conditioned on $t_d = 0$.

Since everything is invariant under orthogonal transformations of $\mathbb{R}P^n$, we only need to compute the expectation on the right-hand side at the point $x = [1 : 0 : \dots : 0]$. In the same chart as above we get:

$$R(x) = \frac{1}{2} \sum_{1 \leq i, j \leq n} dx_i \wedge dx_j \otimes dx_i \wedge dx_j.$$

Then, following the computations of section 7.3, we have:

$$\mathbb{E} \left[\left| \det^\perp(L_d) \right| \operatorname{Tr} \left(\left(R(x) + \frac{1}{2} \mathbb{E}_U \left[\langle L_d^\dagger S_d, U \rangle^{\wedge 2} \right] \right)^{\wedge m} \right) \right] = \sqrt{d}^r \sum_{q=0}^m (d-1)^q \binom{m}{q} \times \\ \frac{\sqrt{2\pi}^{r-2q} \operatorname{Vol}(\mathbb{S}^{n-r}) \operatorname{Vol}(\mathbb{S}^{n-r+1})}{2^q \operatorname{Vol}(\mathbb{S}^n) \operatorname{Vol}(\mathbb{S}^{n-r+1-2q})} \mathbb{E}_{U'} \left[\operatorname{Tr} \left(\mathbb{R}(x)^{\wedge m-q} \wedge \mathbb{E}_S \left[\langle S, U' \rangle^{\wedge 2} \right]^{\wedge q} \right) \right],$$

where U' is a standard Gaussian vector in $\mathbb{R}(\mathcal{E} \otimes \mathcal{L}^d)_x$ and $S = \frac{1}{\sqrt{d(d-1)}} S_d$. Using equation (7.27) and the values of the variance of S_d , we get:

$$\mathbb{E}_S \left[\langle S, U' \rangle^{\wedge 2} \right] = -\|U'\|^2 \sum_{1 \leq i, j \leq n} dx_i \wedge dx_j \otimes dx_i \wedge dx_j.$$

Then, restricting R and S to the span of $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{2m}} \right)$, we get:

$$\forall q \in \{0, \dots, m\}, \quad \operatorname{Tr} \left(\mathbb{R}(x)^{\wedge m-q} \wedge \mathbb{E}_S \left[\langle S, U' \rangle^{\wedge 2} \right]^{\wedge q} \right) = (-1)^q \frac{1}{2^{m-q}} (2m)! \|U'\|^{2q}.$$

Some further computations, similar to those of section (7.3), yield Bürgisser's result [10]:

$$\mathbb{E}[\chi(Z_s)] = \left(\sqrt{d} \right)^r \sum_{p=1}^{\frac{n-r}{2}} (1-d)^p \frac{\Gamma(p + \frac{r}{2})}{p! \Gamma(\frac{r}{2})}.$$

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