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THE MATSUMOTO AND YOR PROCESS AND INFINITE DIMENSIONAL HYPERBOLIC SPACE

PHILIPPE BOUGEROL

A la mémoire de Marc Yor, avec admiration.

ABSTRACT. The Matsumoto–Yor process is $\int_0^t \exp(2B_s - B_t) ds$, where (B_t) is a Brownian motion. It is shown that it is the limit of the radial part of the Brownian motion at the bottom of the spectrum on the hyperbolic space of dimension q , when q tends to infinity. Analogous processes on infinite series of non compact symmetric spaces and on regular trees are described.

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1. INTRODUCTION

The aim of this paper is mainly to see that the Matsumoto–Yor process [26]

$$\eta_t = \int_0^t e^{2B_s - B_t} ds$$

where (B_t) is a standard Brownian motion, appears naturally as the radial part of the Brownian motion on the hyperbolic space \mathbb{H}_q at the bottom of the spectrum when the dimension $q \rightarrow +\infty$. Marc Yor [39] asked in 1999 whether there is a geometric interpretation of this process which provides a direct proof of its Markovianity. Notice that

$$\int_0^t e^{\mu B_s - B_t} ds$$

is a Markov process only for $\mu = 1$ and $\mu = 2$ (see [25]). The case $\mu = 1$ is easy since it follows from Ito’s formula that this process is solution of a stochastic differential equation. Things are more subtle for $\mu = 2$ because the process is not Markov for the Brownian filtration $\sigma(B_s, 0 \leq s \leq t)$.

In a second part, we describe the limit of the radial part of the Brownian motion for infinite series of symmetric spaces of higher rank, namely $SO(p, q)$, $SU(p, q)$, $Sp(p, q)$ when p is fixed and $q \rightarrow +\infty$. For $SU(p, q)$ we give the generator of the limit by studying the asymptotic behaviour of spherical functions (such an analysis is not available yet for the other cases).

In the higher rank case, these processes are different from the Whittaker processes obtained by O’Connell [29] and Chhaibi [10] who also generalize the Matsumoto–Yor process for real split semi simple groups, but in a different direction linked with representation theory and geometric crystals.

In the last part we show that in the q -adic case of rank one, or more generally on regular trees \mathbb{T}_q , the radial part of the simple random walk at the bottom of the spectrum converges when $q \rightarrow \infty$ to the discrete Pitman process

$$2 \max_{0 \leq k \leq n} \Sigma_k - \Sigma_n, n \geq 0,$$

where (Σ_n) is the simple random walk on \mathbb{Z} . Contrary to the real case we use here only elementary arguments and the treatment is self-contained.

In an appendix we describe how to modify Shimeno [37] in order to obtain the needed asymptotics for spherical functions in rank one.

2. THE MATSUMOTO–YOR PROCESS

Let us first recall a now classical theorem of Pitman [35]. Let $B_t, t \geq 0$, be a standard real Brownian motion starting at 0.

Theorem 2.1 (Pitman, [35]). *The process*

$$2 \max_{0 \leq s \leq t} B_s - B_t$$

is a Markov process on \mathbb{R}^+ . It is the Bessel(3) process with generator

$$\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}.$$

In 1999, Matsumoto and Yor [26, 27, 28], have found the following exponential generalization of Pitman's theorem.

Theorem 2.2 (Matsumoto and Yor, [26]). *The process*

$$\eta_t = \int_0^t e^{2B_s - B_t} ds$$

is a Markov process and the generator of $\log \eta_t$ is

$$\frac{1}{2} \frac{d^2}{dr^2} + \left(\frac{d}{dr} \log K_0(e^{-r}) \right) \frac{d}{dr}$$

where K_0 is the Macdonald function.

Recall that, for $\lambda \in \mathbb{R}$, the classical Macdonald function K_λ is

$$K_\lambda(x) = x^\lambda \int_0^\infty \frac{e^{-t-x^2/4t}}{(2t)^{1+\lambda}} dt.$$

A geometric intuition of the Markovianity of this process does not follow clearly from the rather intricate known proofs (either the original ones ([26, 27]) or Baudoin [2], see also [3]). Pitman's theorem can be recovered by Brownian scaling and Laplace's method.

3. BROWNIAN MOTION ON HYPERBOLIC SPACES

3.1. Hyperboloid model. There are many realizations of the hyperbolic spaces (see for instance Cannon et al. [7]). We will consider two of them: the hyperboloid and the upper half-space models. A very convenient reference for us is Franchi and Le Jan's book [15], which we will follow. A more Lie theoretic approach will be applied in Section 4 for the higher rank case (and of course could also be used here).

For $\xi, \xi' \in \mathbb{R}^{q+1}$ let

$$\langle \xi, \xi' \rangle = \xi_0 \xi'_0 - \sum_{k=1}^q \xi_k \xi'_k.$$

The hyperbolic space \mathbb{H}_q of dimension q is

$$\mathbb{H}_q = \{ \xi \in \mathbb{R}^{q+1}; \langle \xi, \xi \rangle = 1, \xi_0 > 0 \}$$

which is the upper sheet of an hyperboloid, with the Riemannian metric d defined by

$$\cosh d(\xi, \xi') = \langle \xi, \xi' \rangle.$$

The group $SO_0(1, q)$ is the connected component of the identity in

$$\{ g \in Gl(q+1, \mathbb{R}); \langle g\xi, g\xi' \rangle = \langle \xi, \xi' \rangle, \xi, \xi' \in \mathbb{R}^{q+1} \}.$$

It acts by isometry on \mathbb{H}_q by matrix multiplication on \mathbb{R}^{q+1} . Let $\{e_0, \dots, e_q\}$ be the canonical basis of \mathbb{R}^{q+1} . We consider e_0 as an origin in \mathbb{H}_q and we let $o = e_0$. The subgroup

$$K = \{g \in SO_o(1, q); ge_0 = e_0\}$$

is isomorphic to $SO(q)$, it is a maximal compact subgroup of $SO_o(1, q)$ and

$$\mathbb{H}_q = SO_o(1, q)/K.$$

Each $\xi \in \mathbb{H}_q$ can be written uniquely as

$$\xi = (\cosh r)e_0 + (\sinh r)\varphi$$

where $r \geq 0, \varphi \in S^{q-1} = \{\sum_{k=1}^q \varphi_k e_k; \sum_{k=1}^q \varphi_k^2 = 1\}$. One has $d(o, \xi) = r$ and (r, φ) is called the polar coordinates of ξ . The Laplace Beltrami operator Δ on \mathbb{H}_q is given in these coordinates by (see [15, Prop. 3.5.4])

$$(3.1) \quad \Delta = \frac{\partial^2}{\partial r^2} + (q-1) \coth r \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \Delta_{S^{q-1}}^\varphi$$

where $\Delta_{S^{q-1}}^\varphi$ is the Laplace operator on the sphere S^{q-1} acting on the φ -variable.

We denote (ξ_t) the Brownian motion on \mathbb{H}_q , defined as the diffusion process with generator $\Delta/2$, starting from the origin $o = e_0$. Since the group K acts transitively on the spheres $\{\xi \in \mathbb{H}_q; d(e_0, \xi) = r\}$, $d(o, \xi_t)$ is a Markov process, called the radial part of ξ_t . It follows from (3.1) that its generator is $\Delta_R/2$ where

$$(3.2) \quad \Delta_R = \frac{d^2}{dr^2} + (q-1) \coth r \frac{d}{dr}.$$

3.2. Upper half space model. We introduce the upper half space model following Franchi and Le Jan [15]: We consider the square matrices $\tilde{E}_j, 1 \leq j \leq 1+q$, of order $q+1$ given by the expression

$$t\tilde{E}_1 + \sum_{j=1}^{q-1} x_j \tilde{E}_{j+1} = \begin{pmatrix} 0 & t & x_1 & x_2 & \dots & x_{q-1} \\ t & 0 & x_1 & x_2 & \dots & x_{q-1} \\ x_1 & -x_1 & 0 & 0 & \dots & 0 \\ x_2 & -x_2 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_{q-1} & -x_{q-1} & 0 & 0 & \dots & 0 \end{pmatrix}$$

when $t, x_1, \dots, x_{q-1} \in \mathbb{R}$. Then

$$\mathfrak{S} = \left\{ t\tilde{E}_1 + \sum_{j=1}^{q-1} x_j \tilde{E}_{j+1}, t \in \mathbb{R}, x \in \mathbb{R}^{q-1} \right\}$$

is a solvable subalgebra of the Lie algebra of $SO_0(1, q)$. For $x \in \mathbb{R}^{q-1}$ and $y > 0$ let

$$T_{x,y} = \exp\left(\sum_{j=1}^{q-1} x_j \tilde{E}_{j+1}\right) \exp((\log y)\tilde{E}_1).$$

Then

$$S = \{T_{x,y}, (x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*\}$$

is the Lie subgroup of $SO_o(1, q)$ with Lie algebra \mathfrak{S} . Moreover,

$$T_{x,y}T_{x',y'} = T_{x+yx',yy'}$$

(see [15, Proposition 1.4.3]). Therefore the map $(x, y) \mapsto T_{x,y}$ is an isomorphism between the affine group of \mathbb{R}^{q-1} , namely the semi-direct product $\mathbb{R}^{q-1} \rtimes \mathbb{R}_+^*$, and the group S . One has ([15, Proposition 2.1.3, Corollary 3.5.3]),

Proposition 3.1. *The map $\tilde{T} : \mathbb{R}^{q-1} \times \mathbb{R}_+^* \rightarrow \mathbb{H}_q$ given by $\tilde{T}(x, y) = T_{x,y}e_0$ is a diffeomorphism. In these so-called horocyclic or Poincaré coordinates $(x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$, the hyperbolic distance is given by*

$$\cosh d(\tilde{T}(x, y), \tilde{T}(x', y')) = \frac{\|x - x'\|^2 + y^2 + y'^2}{2yy'}.$$

The pull back of the Laplace Beltrami operator Δ on \mathbb{H}_q is

$$y^2 \frac{\partial^2}{\partial y^2} + \Delta_{q-1}^x + (2 - q)y \frac{\partial}{\partial y}$$

where Δ_{q-1}^x is the Euclidean Laplacian of \mathbb{R}^{q-1} acting on the coordinate x .

In the horocyclic coordinates on \mathbb{H}_q , the hyperbolic Brownian motion ξ_t has a nice probabilistic representation which comes from the fact that it can be seen as a Brownian motion on the group S , i.e. a continuous process with independent stationary multiplicative increments. Let $(X_t, Y_t) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$ be the horocyclic coordinates of ξ_t , and let

$$\varrho = \frac{q-1}{2},$$

then (see [5], or [15, Theorem 7.6.5.1]):

Proposition 3.2. *One can write*

$$X_t = \int_0^t e^{B_s^{-\varrho}} dW_s^{(q-1)}, Y_t = e^{B_t^{-\varrho}}$$

where $B_t^{-\varrho} = B_t - t\varrho$ is a real Brownian motion on \mathbb{R} with drift $-\varrho$ and $W_s^{(q-1)}$ is a $q-1$ -dimensional standard Brownian motion, independent of $B^{-\varrho}$.

Proof. This follows immediately from Ito's formula and the expression of the generator given in Proposition 3.1.

3.3. Ground state processes.

3.3.1. *Ground state processes on a manifold.* We will need the notion of ground state process. In order to introduce it rapidly we use the set up presented in Pinsky [34] (see also Pinchover [33]). On a manifold M we consider an elliptic operator which can be written locally as

$$D = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i} + V(x)$$

where the coefficients are locally C^∞ and the matrix a is symmetric positive definite (i.e. hypothesis H_{loc} in [34, p. 124]).

Let

$$C_D = \{u \in C^\infty(M); Du(x) = 0, u(x) > 0, \text{ for all } x \in M\}$$

There exists $\lambda_0 \in (-\infty, +\infty]$ such that, for any $\lambda < \lambda_0$, $C_{D-\lambda}$ is empty, and for $\lambda > \lambda_0$, $C_{D-\lambda}$ is not empty ([34, Theorem 4.3.2]). It has several equivalent definitions, and for a self adjoint operator, $-\lambda_0$ coincide with the bottom of the spectrum on L^2 under general conditions ([34, Proposition 4.10.1]), for instance for the Brownian motion on a Riemannian manifold.

Definition 3.3. *When $\lambda_0 < +\infty$, λ_0 is called the generalized principal eigenvalue of D .*

A positive function $h \in C^\infty(M)$ such that $Dh = \lambda_0 h$ is called a ground state (it does not always exist and is in general not unique). The Doob h -transform of D is the operator D^h defined by

$$D^h f = \frac{1}{h} D(hf) - \lambda_0 f.$$

We call it the h -ground state process. When a is the identity matrix,

$$(3.3) \quad D^h = D + 2 \sum_i \frac{\partial \log(h)}{\partial x_i} \frac{\partial}{\partial x_i}$$

(see [34, Section 4.1]). For the Brownian motion on a Riemannian manifold, when $V = 0$ and once an origin is fixed, there is sometimes a canonical choice of a ground state for which the ground state process has a probabilistic interpretation as an infinite Brownian loop (see [1]). The infinite Brownian loop is the limit of the beginning of the Brownian bridge around the origin when its length goes to infinity.

3.3.2. *A radial ground state process on \mathbb{H}_q .* In Proposition 3.2 the component Y_t of the Brownian motion on \mathbb{H}_q depends on the dimension q only through the drift $\varrho = (q-1)/2$. In order to see what happens when $q \rightarrow +\infty$, it is natural to first kill this drift. That is the reason why we will consider the Brownian motion at the bottom of its spectrum.

The generalized principal eigenvalue of the Laplace Beltrami operator Δ is $-\varrho^2$ and there exists a unique radial function φ_0 on \mathbb{H}_q , called the basic Harish Chandra function, such that

$$\Delta \varphi_0 = -\varrho^2 \varphi_0$$

and $\varphi_0(o) = 1$ (see, e.g., Davies [13, 5.7.1], Gangolli and Varadarajan, [16], Helgason [18]).

Definition 3.4. *The infinite Brownian loop on \mathbb{H}_q is the φ_0 -ground state process (ξ_t^0) with generator*

$$\frac{1}{2}\Delta^{\varphi_0}f = \frac{1}{2\varphi_0}\Delta(\varphi_0f) + \frac{\varrho^2}{2}f.$$

The function φ_0 is radial, which means that $\varphi_0(\xi)$ is a function of $r = d(o, \xi)$; we define $\tilde{\varphi}_0 : \mathbb{R} \rightarrow \mathbb{R}$ by,

$$\tilde{\varphi}_0(r) = \varphi_0(\xi).$$

It follows from (3.2) and (3.3) that,

Proposition 3.5. *The process $\{d(o, \xi_t^0), t \geq 0\}$ is a Markov process on \mathbb{R}^+ with generator $\Delta_R^{\tilde{\varphi}_0}/2$ where*

$$\Delta_R^{\tilde{\varphi}_0} = \frac{d^2}{dr^2} + ((q-1)\coth r + 2\frac{\tilde{\varphi}_0'(r)}{\tilde{\varphi}_0(r)})\frac{d}{dr}.$$

3.3.3. *A non radial ground state process on \mathbb{H}_q .* Although our main interest is in ξ_t^0 we will actually need another ground state process for which computations are easier. It is not invariant under rotations but its radial part is the same as the one of ξ_t^0 (see Proposition 3.7 below). Let $\Psi_0 : \mathbb{H}_q \rightarrow \mathbb{R}$ be the function defined by, if $(x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$ are the horocyclic coordinates of $\xi \in \mathbb{H}_q$,

$$(3.4) \quad \Psi_0(\xi) = y^{(q-1)/2} = e^{\varrho \log y}.$$

Notice that $\Psi_0(o) = 1$ and

$$\Delta\Psi_0(\xi) = y^2\frac{\partial^2}{\partial y^2}y^\varrho - (q-2)y\frac{\partial}{\partial y}y^\varrho = -\varrho^2\Psi_0(\xi).$$

Thus Ψ_0 is, like φ_0 , a positive ground state of Δ . Let $\{S_t, t \geq 0\}$ be the Ψ_0 -ground state process of the Brownian motion (ξ_t) on \mathbb{H}_q . It is also a ground state process of (ξ_t) . By definition, for all $T > 0$, when $f : C([0, T], \mathbb{H}_q) \rightarrow \mathbb{R}^+$ is measurable,

$$(3.5) \quad \mathbb{E}(f(S_t, 0 \leq t \leq T)) = e^{\varrho^2 T/2}\mathbb{E}(f(\xi_t, 0 \leq t \leq T)\frac{\Psi_0(\xi_T)}{\Psi_0(\xi_0)})$$

where $\Psi_0(\xi_0) = 1$ since $\xi_0 = o$. Then, it is easy to see that:

Lemma 3.6 ([5]). *In horospherical coordinates,*

$$S_t = \left(\int_0^t e^{B_s} dW_s^{(q-1)}, e^{B_t} \right)$$

where B_t is a standard (i.e. driftless) real Brownian motion, and $W^{(q-1)}$ is a $q-1$ standard dimensional Brownian motion, independent of B .

The function

$$\int_K \Psi_0(k \cdot \xi) dk$$

is a positive radial eigenvector of Δ with eigenvalue $-\varrho^2$. Therefore, by uniqueness, we have the well known formula of Harish Chandra (see Helgason, [18])

$$\varphi_0(\xi) = \int_K \Psi_0(k \cdot \xi) dk,$$

where dk is the normalized Haar measure on K . The processes S_t and ξ_t^0 do not have the same law, and S_t is not rotation invariant. However,

Proposition 3.7. *The two processes $\{d(o, S_t), t \geq 0\}$ and $\{d(o, \xi_t^0), t \geq 0\}$ have the same law.*

Proof. By invariance under rotation of the Brownian motion on \mathbb{H}_q , for any $k \in K$, the processes (ξ_t) and $(k \cdot \xi_t)$ have the same law. Hence, after integration over K , for $T > 0$ and any measurable function $f : C([0, T], \mathbb{R}^+) \rightarrow \mathbb{R}^+$,

$$\begin{aligned} \mathbb{E}(f(d(o, S_t), 0 \leq t \leq T)) &= e^{\varrho^2 T/2} \mathbb{E}(f(d(o, \xi_t), 0 \leq t \leq T) \Psi_0(\xi_T)) \\ &= e^{\varrho^2 T/2} \mathbb{E}(f(d(o, k \cdot \xi_t), 0 \leq t \leq T) \Psi_0(k \cdot \xi_T)) \\ &= e^{\varrho^2 T/2} \mathbb{E}(f(d(o, \xi_t), 0 \leq t \leq T) \Psi_0(k \cdot \xi_T)) \\ &= e^{\varrho^2 T/2} \mathbb{E}(f(d(o, \xi_t), 0 \leq t \leq T) \int_K \Psi_0(k \cdot \xi_T) dk) \\ &= e^{\varrho^2 T/2} \mathbb{E}(f(d(o, \xi_t), 0 \leq t \leq T) \varphi_0(\xi_T)) \\ &= \mathbb{E}(f(d(o, \xi_t^0), 0 \leq t \leq T)). \quad \square \end{aligned}$$

Remark 3.8. In (3.4) the function $\log y$ is the Busemann function on \mathbb{H}_q associated with the point at infinity $y = \infty$ and Ψ_0 is a minimal eigenfunction. The process (S_t) can be interpreted as the Brownian motion on \mathbb{H}_q conditioned to have 0 speed and to exit at $y = \infty$ (see [17]).

3.4. Matsumoto–Yor process as a limit. Our main result is the following:

Theorem 3.9. *As $q \rightarrow +\infty$, the process $d(0, \xi_t^0) - \log q, t > 0$, converges to $\log \eta_t, t > 0$, where*

$$\eta_t = \int_0^t e^{2B_s - B_t} ds$$

is the Matsumoto–Yor process, which is therefore a Markov process.

Proof. By Proposition 3.7, it is enough to show that, almost surely,

$$(3.6) \quad \lim_{q \rightarrow \infty} d(o, S_t) - \log q = \log \int_0^t e^{2B_s - B_t} ds.$$

The origin o in \mathbb{H}_q is $e_0 = \tilde{T}(0, 1)$. By Proposition 3.1,

$$\cosh d(o, \tilde{T}(x, y)) = \frac{\|x\|^2 + y^2 + 1}{2y},$$

and

$$\cosh d(0, S_t) = \frac{\|\int_0^t e^{Bs} dW_s^{(q-1)}\|^2 + e^{2Bt} + 1}{2e^{Bt}}$$

thus

$$\frac{2 \cosh d(0, S_t)}{q} = \frac{e^{Bt} + e^{-Bt}}{q} + e^{-Bt} \frac{1}{q} \sum_{k=1}^{q-1} \left(\int_0^t e^{Bs} d\beta_s^{(k)} \right)^2,$$

where $\beta_s^{(k)}$, $k \geq 1$, are independent standard Brownian motions. Conditionally on the σ -algebra $\sigma(B_r, r \geq 0)$, the random variables $\int_0^t e^{Bs} d\beta_s^{(k)}$, $k \geq 1$, are independent with the same distribution and

$$\mathbb{E}\left(\left(\int_0^t e^{Bs} d\beta_s^{(k)}\right)^2 \mid \sigma(B_r, r \geq 0)\right) = \int_0^t e^{2Bs} ds.$$

Therefore, by the law of large numbers, a.s.

$$\lim_{q \rightarrow +\infty} \frac{1}{q} e^{d(0, S_t)} = \lim_{q \rightarrow +\infty} \frac{2}{q} \cosh d(0, S_t) = e^{-Bt} \int_0^t e^{2Bs} ds.$$

The limit is Markov as a limit of Markov processes. \square

Let us recover the generator of the Matsumoto–Yor process. Let

$$\delta(r) = \sinh^{q-1} r,$$

then, by Proposition 3.5,

$$\Delta_{\tilde{R}} \tilde{\varphi}_0 = \frac{d^2}{dr^2} + 2 \frac{d}{dr} \log(\delta^{1/2} \tilde{\varphi}_0)(r) \frac{d}{dr}$$

hence the generator of $d(o, \xi_t^0) - \log q$ is

$$(3.7) \quad \frac{1}{2} \frac{d^2}{dr^2} + \frac{d}{dr} \log(\delta^{1/2} \tilde{\varphi}_0)(r + \log q) \frac{d}{dr}.$$

Therefore the next proposition follows from Corollary 7.3 (since $m_\alpha = q - 1$ for $SO(1, q)$).

Proposition 3.10 ([26]). *The generator of the Matsumoto–Yor process is*

$$\frac{1}{2} \frac{d^2}{dr^2} + \left(\frac{d}{dr} \log K_0(e^{-r}) \right) \frac{d}{dr}.$$

Remark 3.11. We will clarify in Section 4 the occurrence of the ground state K_0 of the Toda operator $\frac{d^2}{dr^2} - e^{-2r}$.

3.5. A conditional law. The intertwining property which occurs in the proof of Matsumoto–Yor theorem can also be established by our approach.

Proposition 3.12 ([26]). *When $\eta_t = e^{-Bt} \int_0^t e^{2B_s} ds$, and $\lambda \in \mathbb{R}$,*

$$\mathbb{E}(e^{\lambda B_t} | \sigma(\eta_s, 0 \leq s \leq t)) = \frac{K_\lambda(1/\eta_t)}{K_0(1/\eta_t)}.$$

Proof. For $\xi \in \mathbb{H}_q$ with horocyclic coordinates $(x, y) \in \mathbb{R}^{q-1} \times \mathbb{R}_+^*$ and $\lambda \in \mathbb{R}$, let $\lambda(\xi) = \lambda \log y$. We consider the Harish Chandra function φ_λ (see [18, Theorem IV.4.3]) which of course depends on q ,

$$\varphi_\lambda(\xi) = \int_K e^{(\lambda+\rho)(k \cdot \xi)} dk,$$

and we write $\tilde{\varphi}_\lambda(r) = \varphi_\lambda(\xi)$ when $r = d(o, \xi)$. Since $\lambda(S_t) = \lambda B_t$ and $e^{\rho(\xi)} = \Psi_0(\xi)$, it follows from (3.5) that, when $f : C([0, t]) \rightarrow \mathbb{R}^+$ is measurable,

$$\begin{aligned} & e^{-\rho^2 t/2} \mathbb{E}(f(d(o, S_s), 0 \leq s \leq t) e^{\lambda B_t}) \\ &= e^{-\rho^2 t/2} \mathbb{E}(f(d(o, S_s), 0 \leq s \leq t) e^{\lambda(S_t)}) \\ &= \mathbb{E}(f(d(o, \xi_s), 0 \leq s \leq t) e^{(\lambda+\rho)(\xi_t)}) \\ &= \mathbb{E}(f(d(o, \xi_s), 0 \leq s \leq t) \int_K e^{(\lambda+\rho)(k \cdot \xi_t)} dk) \\ &= \mathbb{E}(f(d(o, \xi_s), 0 \leq s \leq t) \varphi_\lambda(\xi_t)) \\ &= \mathbb{E}(f(d(o, \xi_s), 0 \leq s \leq t) \frac{\varphi_\lambda(\xi_t)}{\varphi_0(\xi_t)} \varphi_0(\xi_t)) \\ &= \mathbb{E}(f(d(o, \xi_s), 0 \leq s \leq t) \frac{\varphi_\lambda(\xi_t)}{\varphi_0(\xi_t)} e^{\rho(\xi_t)}) \\ &= e^{-\rho^2 t/2} \mathbb{E}(f(d(o, S_s), 0 \leq s \leq t) \frac{\tilde{\varphi}_\lambda(d(o, S_t))}{\tilde{\varphi}_0(d(o, S_t))}). \end{aligned}$$

We have again used the fact that for any $k \in K$, $\{\xi_t, t \geq 0\}$ has the same law as $\{k \cdot \xi_t, t \geq 0\}$. For each q ,

$$\begin{aligned} & \mathbb{E}(f(d(0, S_s) - \log q, 0 \leq s \leq t) e^{\lambda B_t}) = \\ & \mathbb{E}(f(d(0, S_s) - \log q, 0 \leq s \leq t) \frac{\tilde{\varphi}_\lambda((d(o, S_t) - \log q) + \log q)}{\tilde{\varphi}_0((d(o, S_t) - \log q) + \log q)}) \end{aligned}$$

We choose for f a continuous bounded function with bounded support. Letting $q \rightarrow +\infty$, it then follows from Theorem 3.9 and Corollary 7.3 that

$$\mathbb{E}(f(\log \eta_s, 0 \leq s \leq t) e^{\lambda B_t}) = \mathbb{E}(f(\log \eta_s, 0 \leq s \leq t) \frac{K_\lambda(1/\eta_t)}{K_0(1/\eta_t)}). \quad \square$$

Remark 3.13. It is straightforward to deduce from this proposition and Girsanov's theorem that, as proved by Matsumoto–Yor [26], when $B_t^{(\lambda)} =$

$B_t + \lambda t$ is a Brownian motion with drift λ , then

$$\log \int_0^t e^{2B_s^{(\lambda)} - B_t^{(\lambda)}} ds$$

is a Markov process on \mathbb{R} with generator

$$\frac{1}{2} \frac{d^2}{dr^2} + \left(\frac{d}{dr} \log K_\lambda(e^{-r}) \right) \frac{d}{dr}.$$

4. INFINITE SERIES OF SYMMETRIC SPACES

We will now consider the same problem as above for the infinite series of symmetric spaces of non positive curvature, when the rank is fixed and the dimension goes to infinity. We will see that in the rank one case one finds the same result as for hyperbolic spaces, but that new phenomena occur in higher rank.

There are only three infinite series of (irreducible) Riemannian symmetric spaces of non positive (non zero) curvature namely the spaces G/K where $G = SO(p, q), SU(p, q), Sp(p, q)$ and K is a maximal compact subgroup (see [18]). We will suppose that $p \leq q$, then p is the rank of the symmetric space.

4.1. $SU(p, q)$. We first consider the symmetric spaces associated with the series $SU(p, q)$ with the rank p fixed, when $q \rightarrow +\infty$. As in the case of hyperbolic spaces, it is convenient to use two descriptions of $SU(p, q)$. The usual description of $SU(p, q)$ is the following (see, e.g., [19], [23]): it is the set of $(p + q) \times (p + q)$ matrices with entries in \mathbb{C} and with determinant 1, which conserve the quadratic form

$$\sum_{i=1}^p z_i \bar{z}_i - \sum_{j=1}^q z_{p+j} \bar{z}_{p+j}.$$

In other words let $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ then

$$SU(p, q) = \{M \in Sl(p + q, \mathbb{C}); MJM^* = J\}.$$

If we write M by block as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

where the size of the matrix M_1 is $p \times p$, M_2 is $p \times q$, M_3 is $q \times p$ and M_4 is $q \times q$, then $M \in SU(p, q)$ if and only if $M_1 M_1^* - M_2 M_2^* = I_p$, $M_4 M_4^* - M_3 M_3^* = I_q$, $M_1 M_3^* = M_2 M_4^*$, $\det(M) = 1$ and I_p is the identity matrix of order p .

4.2. Cartan Decomposition and radial part. Let \bar{K} be the following maximal compact subgroup of $SU(p, q)$ (we put the notation K aside for a later use)

$$\bar{K} = \left\{ \begin{pmatrix} K_1 & 0 \\ 0 & K_4 \end{pmatrix} \in Sl(p+q, \mathbb{C}), K_1 \in U(p), K_4 \in U(q) \right\}.$$

When $r = (r_1, \dots, r_p) \in \mathbb{R}^p$, let

$$D(r) = \begin{pmatrix} \cosh r & \sinh r & 0 \\ \sinh r & \cosh r & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}$$

where

$$\cosh r = \begin{pmatrix} \cosh r_1 & 0 & \cdots & 0 \\ 0 & \cosh r_2 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cosh r_p \end{pmatrix}$$

and $\sinh r$ is the same matrix with each $\cosh r_i$ replaced by $\sinh r_i$. We consider the closed Weyl chamber

$$\mathfrak{A}^+ = \{(r_1, \dots, r_p) \in \mathbb{R}^p, r_1 \geq r_2 \geq \cdots \geq r_p \geq 0\}.$$

The Cartan decomposition says that any M in $SU(p, q)$ can be written as $M = k_1 D(r) k_2$ with $k_1, k_2 \in \bar{K}$ and $r \in \mathfrak{A}^+$. Moreover such a r is unique and is called the radial part of M . We let $r = \text{Rad}(M)$. Recall that

Definition 4.1. Let N be a $p \times p$ square matrix, the vector of singular values of N is

$$\sigma(N) = (\sigma_1, \dots, \sigma_p) \in \mathfrak{A}^+$$

where the σ_i 's are the square roots of the eigenvalues of N^*N written in decreasing order.

Lemma 4.2. If $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \in SU(p, q)$, then

$$\cosh \text{Rad}(M) = \sigma(M_1).$$

Proof. By the Cartan decomposition, there exists $K_1, \tilde{K}_1 \in U(p)$ and $K_4, \tilde{K}_4 \in U(q)$ such that, if r is the radial part of M then

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} K_1 & 0 \\ 0 & K_4 \end{pmatrix} \begin{pmatrix} \cosh r & \sinh r & 0 \\ \sinh r & \cosh r & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix} \begin{pmatrix} \tilde{K}_1 & 0 \\ 0 & \tilde{K}_4 \end{pmatrix}.$$

Therefore $M_1 = K_1(\cosh r)\tilde{K}_1$ and $M_1 M_1^* = K_1(\cosh^2 r)K_1^*$ which proves the lemma. \square

4.3. Iwasawa Decomposition and horocyclic coordinates. It will be convenient to write matrices in $SU(p, q)$ in another basis to make more tractable the solvable part of the Iwasawa decomposition (see, e.g., Lu [24], Iozzi and Morris [22] or Sawyer [36]). We still suppose that $q \geq p$. Let

$$P = \left\{ \begin{pmatrix} I_p/\sqrt{2} & 0 & I_p/\sqrt{2} \\ I_p/\sqrt{2} & 0 & -I_p/\sqrt{2} \\ 0 & I_{q-p} & 0 \end{pmatrix} \right\}.$$

Then $P^* = P^{-1}$ and $P^{-1}JP = \bar{J}$ where

$$\bar{J} = \begin{pmatrix} 0 & 0 & -I_p \\ 0 & I_{q-p} & 0 \\ -I_p & 0 & 0 \end{pmatrix}.$$

We introduce

$$G = \{P^{-1}MP, M \in SU(p, q)\}$$

which is the set of matrices M in $Sl(p+q, \mathbb{C})$ such that $M\bar{J}M^* = \bar{J}$. The group G is obviously isomorphic to $SU(p, q)$. So, we will work with G instead of $SU(p, q)$.

Let l, b, c be $p \times p, p \times (q-p)$ and $p \times p$ complex matrices, respectively, where l is **lower** triangular and invertible, and define

$$(4.1) \quad S(l, b, c) = \begin{pmatrix} l & b & cl^{*-1} \\ 0 & I_{q-p} & b^*l^{*-1} \\ 0 & 0 & l^{*-1} \end{pmatrix}.$$

Let

$$S = \{S(l, b, c); l \text{ is lower triangular with positive diagonal, } c + c^* = bb^*\},$$

$$A = \{S(l, 0, 0) \in S; l \text{ is diagonal}\},$$

$$N = \{S(l, b, c) \in S; \text{ the diagonal elements of } l \text{ are equal to } 1\},$$

$$K = P^{-1}\bar{K}P.$$

Notice that in general l is lower triangular and hence neither N nor S is made of upper triangular matrices. If $D = \{D(r), r \in \mathbb{R}^p\}$ then $A = P^{-1}DP$. The Iwasawa decomposition is $G = NAK$, N is the nilpotent component and $S = NA$. Notice that $SU(p, q)/\bar{K}$ is isomorphic with G/K . For $M \in G$, PMP^{-1} is in $SU(p, q)$ and we let $Rad(M) = Rad(PMP^{-1})$. In G/K , we choose as origin $o = K$, then $Rad(M)$ plays the role of a generalized distance between the cosets $o = K$ and $\xi = MK$ in G/K .

Lemma 4.3 ([24]). *For $S(l, b, c) \in S$,*

$$\cosh Rad(S(l, b, c)) = \frac{1}{2}\sigma(l + l^{*-1} + cl^{*-1}).$$

Proof. This follows immediately from Lemma 4.2 since, for $S(l, b, c) \in G$, the corresponding element in $SU(p, q)$ can be written as

$$PS(l, b, c)P^{-1} = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

where $M_1 = (l + (I + c)l^{*-1})/2$. \square

By the Iwasawa decomposition each element $\xi \in G/K$ can be written uniquely as $\xi = S(l, b, c)K$ with $S(l, b, c) \in S$. We see that we can and will identify S and G/K . We call $S(l, b, c)$ the horocyclic coordinates of ξ . They generalize the horocyclic–Poincaré coordinates of \mathbb{H}_q .

4.4. Brownian motion on G/K and infinite Brownian loop. The Lie algebra \mathfrak{S} of S is

$$\mathfrak{S} = \left\{ \begin{pmatrix} l & b & c \\ 0 & 0 & b^* \\ 0 & 0 & -l^* \end{pmatrix} ; c \text{ skew-Hermitian, } l \text{ lower triangular} \right. \\ \left. \text{with real diagonal} \right\}.$$

The Killing form on the Lie algebra of G allows to define a scalar product on \mathfrak{S} by

$$\langle X, Y \rangle = \frac{1}{2} \text{Trace}(XY^*).$$

Then \mathfrak{S} is a real Euclidean space. Let \mathfrak{A} and \mathfrak{N} be the Lie algebras of A and N , then $\mathfrak{S} = \mathfrak{A} \oplus \mathfrak{N}$. Let X_1, \dots, X_p be an orthonormal basis of \mathfrak{A} and N_1, \dots, N_s be an orthonormal (real) basis of \mathfrak{N} adapted to the root space decomposition. In horospherical coordinates the Laplace Beltrami operator on $G/K = S$ is (e.g. [5], [11], [17])

$$\Delta = \sum_{i=1}^p X_i^2 + 2 \sum_{j=1}^s N_j^2 + 2 \sum_{i=1}^p \varrho(X_i) X_i$$

where ϱ is given by (5.1) and X_i and N_j are considered as left invariant vector fields. We consider the Brownian motion $\{\xi_t, t \geq 0\}$ on $G/K = S$. It is the process with generator $\Delta/2$ starting from the origin o .

As in the hyperbolic case one can consider the ground state process $\{\xi_t^0, t \geq 0\}$ of this Brownian motion associated with the basic Harish-Chandra spherical function φ_0 . By [18, Theorem IV.4.3],

$$(4.2) \quad \varphi_0(g) = \int_K e^{\varrho(A(kg))} dk,$$

where for $g \in G$, we write $A(g)$ for the element of the Lie algebra of A such that $g \in Ne^{A(g)}K$ in the Iwasawa decomposition $G = NAK$. The generator of ξ^0 is $\Delta\varphi_0/2$ and it corresponds to the infinite Brownian loop on G/K (see [1]). By invariance of Δ under K , the radial part of $\text{Rad}(\xi_t)$ and $\text{Rad}(\xi_t^0)$ are Markov processes with values in \mathfrak{A}^+ .

4.5. Distinguished Brownian motion on S . We define $\Psi_0 : G \rightarrow \mathbb{R}^+$ by

$$\Psi_0(g) = e^{\varrho(A(g))}.$$

Since $A(gk) = A(g)$ for any $k \in K$, Ψ_0 is well defined on G/K . We consider the Ψ_0 -ground state process $\{S_t, t \geq 0\}$ of the Brownian motion ξ on G/K .

Using the identification $G/K = S$, we see it as a process on S (it has the same interpretation as the one given in Remark 3.8).

Definition 4.4. *The distinguished Brownian motion on S is the process $S_t, t \geq 0$.*

The generator of (S_t) is

$$\frac{1}{2} \left(\sum_{i=1}^p X_i^2 + 2 \sum_{j=1}^s N_j^2 \right)$$

and called in Harmonic analysis the distinguished Laplacian on S (see, e.g., [5, Proposition 1.2], [11, 12]). One shows as in the hyperbolic case that (see also [1], proof of theorem 6.1),

Lemma 4.5. *The two processes $\{Rad(S_t), t \geq 0\}$ and $\{Rad(\xi_t^{(0)}), t \geq 0\}$ have the same law.*

The process (S_t) is a solution of a stochastic differential equation. Indeed, let

$$V_t = \begin{pmatrix} \lambda_t & \beta_t & \kappa_t \\ 0 & 0 & \beta_t^* \\ 0 & 0 & -\lambda_t^* \end{pmatrix} \in \mathfrak{S}$$

where the coefficients $\lambda_t^{r,r}, i\kappa_t^{r,r}/2, 1 \leq r \leq p$, and the real and imaginary parts of $\lambda_t^{r,s}/\sqrt{2}, r > s, \beta_t^{k,l}/\sqrt{2}, 1 \leq k \leq p, 1 \leq l \leq q-p$ and $\kappa_t^{r,s}/\sqrt{2}, 1 \leq r < s \leq q$ are standard real independent Brownian motions. Moreover, $\lambda_t^{r,s} = 0$ if $1 \leq r < s \leq p$ and $\kappa_t(r,s) = -\bar{\kappa}_t(s,r)$, when $q \geq r > s \geq 1$, (where $m^{i,j}$ denotes the coefficients of a matrix m).

When (X_t) is a continuous semimartingale, we use δX_t to denote its Stratonovich differential and dX_t to denote its Ito one (see, e.g., [21]).

Proposition 4.6. *The horocyclic Brownian motion (S_t) is the solution of the Stratonovich stochastic differential equation in the set of $(p+q) \times (p+q)$ complex matrices,*

$$\delta S_t = S_t \delta V_t, S_0 = I.$$

In order to compute the radial component we use a decomposition of S which is slightly different from the factorization $S = NA$ coming from the Iwasawa factorization. We write, using the notation (4.1)

$$S_t = S(l_t, b_t, c_t),$$

and

$$(4.3) \quad S_t = M_t L_t$$

where

$$M_t = \begin{pmatrix} I & b_t & c_t \\ 0 & I & b_t^* \\ 0 & 0 & I \end{pmatrix}, \quad L_t = \begin{pmatrix} l_t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & l_t^{*-1} \end{pmatrix}.$$

Recall that the matrix l_t is lower triangular. Its diagonal is the A component of S_t but it also have a part of the N component. We solve the equation satisfied by (S_t) . By Stratonovich calculus,

$$\delta S_t = M_t \delta L_t + (\delta M_t) L_t$$

and therefore,

$$M_t L_t \delta V_t = M_t \delta L_t + (\delta M_t) L_t$$

which implies that

$$\delta V_t = L_t^{-1} \delta L_t + L_t^{-1} M_t^{-1} (\delta M_t) L_t$$

hence

$$L_t^{-1} \delta L_t = \begin{pmatrix} \delta \lambda_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\delta \lambda_t^* \end{pmatrix}$$

and

$$L_t^{-1} M_t^{-1} (\delta M_t) L_t = \begin{pmatrix} 0 & \delta \beta_t & \delta \kappa_t \\ 0 & 0 & \delta \beta_t^* \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain that $\delta l_t = l_t \delta \lambda_t$ and

$$\delta M_t = M_t L_t \begin{pmatrix} 0 & \delta \beta_t & \delta \kappa_t \\ 0 & 0 & \delta \beta_t^* \\ 0 & 0 & 0 \end{pmatrix} L_t^{-1},$$

which gives

$$\begin{pmatrix} 0 & \delta b_t & \delta c_t \\ 0 & 0 & \delta b_t^* \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I & b_t & c_t \\ 0 & I & b_t^* \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} 0 & l_t \delta \beta_t & l_t \delta \kappa_t l_t^* \\ 0 & 0 & \delta \beta_t^* l_t^* \\ 0 & 0 & 0 \end{pmatrix},$$

hence,

Proposition 4.7.

$$\delta l_t = l_t \delta \lambda_t, b_t = \int_0^t l_s \delta \beta_s, c_t = \int_0^t l_s (\delta \kappa_s) l_s^* + \int_0^t b_s (\delta \beta_s^*) l_s^*.$$

The process (l_t) is itself a Brownian motion on the subgroup of $Gl(p, \mathbb{C})$ consisting of lower triangular matrices with positive diagonal. The linear equation $\delta l_t = l_t \delta \lambda_t$ is therefore easy to solve explicitly by induction on p . For instance, when $p = 2$, if

$$\lambda_t = \begin{pmatrix} \lambda_t^{(1)} & 0 \\ \lambda_t^{(3)} & \lambda_t^{(2)} \end{pmatrix}$$

we obtain that

$$l_t = \begin{pmatrix} e^{\lambda_t^{(1)}} & 0 \\ e^{\lambda_t^{(2)}} \int_0^t e^{\lambda_s^{(1)} - \lambda_s^{(2)}} \delta \lambda_s^{(3)} & e^{\lambda_t^{(2)}} \end{pmatrix}.$$

4.6. **Limit as $q \rightarrow +\infty$.** We study the asymptotic behaviour of the radial part $Rad(S_t)$ of the distinguished Brownian motion on S , using the decomposition (4.3). We first consider Ito's integral.

Lemma 4.8. *Almost surely,*

$$\lim_{q \rightarrow +\infty} \frac{1}{q} \int_0^t \left[\int_0^s l_u d\beta_u \right] (d\beta_s^*) l_s^* = 0.$$

Proof. For $1 \leq i, j \leq p$,

$$\begin{aligned} \left[\int_0^t \left(\int_0^s l_u d\beta_u \right) (d\beta_s^*) l_s^* \right]^{i,j} &= \sum_{r=1}^{q-p} \left[\int_0^t \int_0^s [l_u d\beta_u]^{i,r} [d\beta_s^* l_s^*]^{r,j} \right. \\ &= \sum_{r=1}^{q-p} \sum_{n=1}^p \sum_{m=1}^p \int_0^t \left(\int_0^s l_u^{i,n} d\beta_u^{n,r} \right) \bar{l}_s^{j,m} d\bar{\beta}_s^{m,r}. \end{aligned}$$

If we fix i, j, n, m , conditionally on the sigma-algebra $\sigma(l_s, s \geq 0)$ the random variables

$$\int_0^t \left(\int_0^s l_u^{i,n} d\beta_u^{n,r} \right) \bar{l}_s^{j,m} d\bar{\beta}_s^{m,r}$$

for $r = 1, 2, \dots$, are independent with the same law, and with expectation equal to 0 since they are martingales. Therefore, by the law of large numbers,

$$\frac{1}{q-p} \sum_{r=1}^{q-p} \int_0^t \left(\int_0^s l_u^{i,k} d\beta_u^{k,r} \right) \bar{l}_s^{j,l} d\bar{\beta}_s^{l,r}$$

converges a.s. to 0 when $q \rightarrow +\infty$, which proves the lemma. \square

Proposition 4.9. *Let $S_t, t \geq 0$, be the distinguished Brownian motion of $SU(p, q)$. Then, a.s.,*

$$\lim_{q \rightarrow +\infty} \frac{1}{q} \cosh Rad(S_t) = \sigma \left(\int_0^t l_s l_s^* l_t^{*-1} ds \right).$$

Proof. By Proposition 4.7,

$$c_t = \int_0^t l_s (\delta \kappa_s) l_s^* + \int_0^t \left(\int_0^s l_u \delta \beta_u \right) (\delta \beta_s^*) l_s^*.$$

Since the processes (l_t) and (κ_t) do not depend on q ,

$$\lim_{q \rightarrow +\infty} \frac{1}{q} c_t = \lim_{q \rightarrow +\infty} \frac{1}{q} \int_0^t \left(\int_0^s l_u \delta \beta_u \right) (\delta \beta_s^*) l_s^*.$$

Now, recall the link between Stratonovich and Ito integral: if X and Y are continuous semimartingales,

$$\int_0^t Y \delta X = \int_0^t Y dX + \frac{1}{2} \langle X, Y \rangle_t,$$

where, if X and Y are matrices, $\langle X, Y \rangle_t$ is the matrix with entries

$$\langle X, Y \rangle_t^{i,j} = \sum_k \langle X^{i,k}, Y^{k,j} \rangle_t.$$

Since (β_t) is independent of (l_t) , $\int_0^s l_u \delta\beta_u = \int_0^s l_u d\beta_u$, and

$$\int_0^t \left[\int_0^s l_u \delta\beta_u \right] (\delta\beta_s^*) l_s^* = \int_0^t \left[\int_0^s l_u d\beta_u \right] (d\beta_s^*) l_s^* + \frac{1}{2} \left\langle \int_0^t l_s d\beta_s, \int_0^t d\beta_s^* l_s^* \right\rangle_t,$$

so it follows from the preceding lemma that

$$\lim_{q \rightarrow +\infty} \frac{1}{q} c_t = \frac{1}{2} \lim_{q \rightarrow +\infty} \left\langle \int_0^t l_s d\beta_s, \int_0^t d\beta_s^* l_s^* \right\rangle_t.$$

On the other hand, for $1 \leq i, j \leq p$,

$$\begin{aligned} & \left[\left\langle \int_0^t l_s d\beta_s, \int_0^t d\beta_s^* l_s^* \right\rangle_t \right]^{i,j} \\ &= \sum_{r=1}^{q-p} \left\langle \left(\int_0^t l_s d\beta_s \right)^{i,r}, \left(\int_0^t d\beta_s^* l_s^* \right)^{r,j} \right\rangle_t \\ &= \sum_{r=1}^{q-p} \sum_{k=1}^p \sum_{l=1}^p \left\langle \int_0^t l_s^{i,k} d\beta_s^{k,r}, \int_0^t \bar{l}_s^{j,l} d\bar{\beta}_s^{l,r} \right\rangle_t \\ &= \sum_{r=1}^{q-p} \sum_{k=1}^p \left\langle \int_0^t l_s^{i,k} d\beta_s^{k,r}, \int_0^t \bar{l}_s^{j,k} d\bar{\beta}_s^{k,r} \right\rangle_t \\ &= \sum_{r=1}^{q-p} \sum_{k=1}^p \int_0^t l_s^{i,k} \bar{l}_s^{j,k} d\langle \beta^{k,r}, \bar{\beta}^{k,r} \rangle_s. \end{aligned}$$

Recall that $\beta_t^{k,r}/\sqrt{2}$ is a complex Brownian motions, hence

$$\langle \beta^{k,r}, \bar{\beta}^{k,r} \rangle_s = 4s,$$

therefore

$$\sum_{r=1}^{q-p} \sum_{k=1}^p \int_0^t l_s^{i,k} \bar{l}_s^{j,k} d\langle \beta^{k,r}, \bar{\beta}^{k,r} \rangle_s = 4(q-p) \sum_{k=1}^p \int_0^t l_s^{i,k} \bar{l}_s^{j,k} ds,$$

and

$$\left\langle \int_0^t l_s d\beta_s, \int_0^t d\beta_s^* l_s^* \right\rangle_t = 4(q-p) \int_0^t l_s l_s^* ds$$

Consequently, a.s.,

$$\lim_{q \rightarrow +\infty} \frac{1}{q} c_t = \lim_{q \rightarrow +\infty} \frac{2(q-p)}{q} \int_0^t l_s l_s^* ds = 2 \int_0^t l_s l_s^* ds$$

and the proposition follows from Lemma 4.3. \square

The following is a direct consequence of the previous proposition and Lemma 4.5 (recall that $\sigma(M)$ is the singular value vector of a matrix M). Let $\mathbf{1} \in \mathbb{R}^p$ denote the vector $\mathbf{1} = (1, 1, \dots, 1)$.

Theorem 4.10. *For $SU(p, q)$, in law as a process,*

$$\lim_{q \rightarrow +\infty} \text{Rad}(\xi_t^0) - \log(2q)\mathbf{1} = \log \sigma \left(\int_0^t l_s l_s^* l_t^{*-1} ds \right)$$

where l is the Brownian motion on the group of $p \times p$ complex lower triangular invertible matrices with positive diagonal, solution of $\delta l_t = l_t \delta \lambda_t$, $l_0 = I$.

The limit is a Markov process with values in $\{(r_1, \dots, r_p) \in \mathbb{R}^p, r_1 \geq r_2 \geq \dots \geq r_p\}$.

We will compute in Theorem 5.2 the generator L of the limit.

4.7. $SO(p, q)$ and $Sp(p, q)$. For $SO(p, q)$ the only difference with $SU(p, q)$ is that all the entries are real. Hence, in the preceding computation,

$$\langle \beta^{k,r}, \bar{\beta}^{k,r} \rangle_s = \langle \beta^{k,r}, \beta^{k,r} \rangle_s = 2s.$$

Similarly, for $Sp(p, q)$ the entries are quaternionic, therefore, in that case,

$$\langle \beta^{k,r}, \bar{\beta}^{k,r} \rangle_s = 8s.$$

So we obtain,

Theorem 4.11. *For $SO(p, q)$, resp. $Sp(p, q)$, the limit of*

$$\lim_{q \rightarrow +\infty} \text{Rad}(\xi_t^0) - \log(\alpha q)\mathbf{1} = \log \sigma \left(\int_0^t l_s l_s^{*-1} l_t^{*-1} ds \right)$$

where $\alpha = 1$ for $SO(p, q)$, resp. $\alpha = 4$ for $Sp(p, q)$, where l is the Brownian motion on the group of $p \times p$ lower triangular matrices with real, resp. quaternionic, entries and with positive diagonal, solution of $\delta l_t = l_t \delta \lambda_t$.

Corollary 4.12. *In rank one, i.e. $p = 1$,*

$$\lim_{q \rightarrow +\infty} \text{Rad}(\xi_t^0) - \log(\alpha q)\mathbf{1} = \log \int_0^t e^{2B_s - B_t} ds$$

where B is a standard Brownian motion.

5. GENERATOR OF THE LIMIT

5.1. Inozemtsev limit to Quantum Toda Hamiltonian. In order to compute the generator L of the limit process in Theorem 4.10 it is convenient to relate $\text{Rad}(\xi_t^0)$ to a Calogero Moser Sutherland model.

We consider the general case of a symmetric space G/K associated with one of the groups $G = SO(p, q)$, $SU(p, q)$ and $Sp(p, q)$, where K is a maximal compact subgroup. The root system is of the type BC_p . Recall that, if $\{e_1, \dots, e_p\}$ is an orthonormal basis of a real Euclidean space \mathfrak{a} of dimension p , then the positive roots Σ^+ of the root system BC_p are given by

$$\Sigma^+ = \{e_k, 2e_k, (1 \leq k \leq p), e_i + e_j, e_i - e_j, (1 \leq i < j \leq p)\}.$$

We associate to each considered group a triplet $m = (m_1, m_2, m_3)$ given by, when $p \geq 2$, for $SO(p, q)$, $m = (q - p, 0, 1)$, for $SU(p, q)$, $m = (2(q - p), 1, 2)$ and for $Sp(p, q)$, $m = (4(q - p), 3, 4)$. When $p = 1$, the only difference

is that $m_3 = 0$. The quantum Calogero-Moser-Sutherland trigonometric Hamiltonian of type BC_p is (see [31, 32])

$$H_{CMS} = H_{CMS}(p, q, r) = \sum_{k=1}^p \left(\frac{\partial^2}{\partial r_k^2} - \frac{m_1(m_1 + 2m_2 - 2)}{4 \sinh^2 r_k} - \frac{m_2(m_2 - 2)}{\sinh^2 2r_k} \right) - \sum_{1 \leq i < j \leq p} m_3(m_3 - 2) \left(\frac{1}{2 \sinh^2(r_i - r_j)} + \frac{1}{2 \sinh^2(r_i + r_j)} \right).$$

For $\alpha \in \Sigma^+$ we let $m_\alpha = m_1$ if $\alpha = e_k$, $m_\alpha = m_2$ if $\alpha = 2e_k$ and $m_\alpha = m_3$ if $\alpha = e_i + e_j$ or $\alpha = e_i - e_j$. Define

$$(5.1) \quad \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha, \quad \delta = \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha})^{m_\alpha}.$$

The radial part Δ_R of the Laplace Beltrami operator on G/K is equal (see [18, p.268]) to

$$\Delta_R = \sum_{k=1}^p \frac{\partial^2}{\partial r_k^2} + \sum_{k=1}^p \frac{\partial}{\partial r_k} (\log \delta) \frac{\partial}{\partial r_k}$$

which can be written as

$$\Delta_R = \delta^{-1/2} \circ \{H_{CMS} - (\rho, \rho)\} \circ \delta^{1/2}$$

(see [30, 31, 32]). On the other hand, we define $\tilde{\varphi}_0 : \mathbb{R}^p \rightarrow \mathbb{R}$ by $\tilde{\varphi}_0(r) = \varphi_0(D(r))$ where φ_0 is the basic Harish Chandra function. Then $\tilde{\varphi}_0(0) = 1$ and

$$\Delta_R \tilde{\varphi}_0 = -(\rho, \rho) \tilde{\varphi}_0$$

therefore

$$H_{CMS}(\delta^{1/2} \tilde{\varphi}_0) = 0.$$

The generator of the radial part $\text{Rad}(\xi_t^0)$ of the ground state process associated with φ_0 is $\Delta_R^{\tilde{\varphi}_0}/2$ where

$$\begin{aligned} \Delta_R^{\tilde{\varphi}_0} &= \sum_{k=1}^p \frac{\partial^2}{\partial r_k^2} + 2 \sum_{k=1}^p \frac{\partial}{\partial r_k} (\log \delta^{1/2} \tilde{\varphi}_0) \frac{\partial}{\partial r_k} \\ &= (\delta^{1/2} \tilde{\varphi}_0)^{-1} \circ H_{CMS} \circ (\delta^{1/2} \tilde{\varphi}_0). \end{aligned}$$

For $SU(p, q)$, since $m_3 = 2$, we have the following (called the Inozemtsev limit ([20]),

$$\lim_{q \rightarrow +\infty} H_{CMS}(p, q, r + \log(2q) \mathbf{1}) = \sum_{k=1}^p \frac{\partial^2}{\partial r_k^2} - e^{-2r_k}.$$

Similarly, for $SO(p, q)$,

$$\lim_{q \rightarrow +\infty} H_{CMS}(p, q, r + \log q \mathbf{1}) = \sum_{k=1}^p \left(\frac{\partial^2}{\partial r_k^2} - e^{-2r_k} \right) + \sum_{1 \leq i < j \leq p} \frac{1}{2 \sinh^2(r_i - r_j)},$$

and for $Sp(p, q)$,

$$\lim_{q \rightarrow +\infty} H_{CMS}(p, q, r + \log(4q)\mathbf{1}) = \sum_{k=1}^p \left(\frac{\partial^2}{\partial r_k^2} - e^{-2r_k} \right) - \sum_{1 \leq i < j \leq p} \frac{4}{\sinh^2(r_i - r_j)}.$$

We denote by H_T the respective limit. Since $\delta^{1/2}\tilde{\varphi}_0$ is a ground state of H_{CMS} , it is reasonable to infer that there is a ground state Ψ of H_T such that the generator L of the limit process in Theorem 4.10 is given by

$$L = \frac{1}{2}\Psi^{-1} \circ H_T \circ \Psi.$$

Oshima and Shimeno [32] goes into that direction but is not precise enough to obtain this conclusion. We will see that this hold true for $SU(p, q)$, but that the result is quite subtle. The reason is that on the one hand, there are many ground states for H_T and on the other hand that L is the generator of a process with values in a proper subcone of \mathbb{R}^p when $p > 1$.

5.2. Asymptotics for $SU(p, q)$. We compute the generator L for $SU(p, q)$, using that in this case the spherical functions are explicitly known for all (p, q) . In $SO(p, q)$ and $Sp(p, q)$, such an expression is not known and it seems difficult to compute the generator of the limit in Theorem 4.11.

For $\lambda \in \mathbb{R}^p$, the Harish Chandra spherical function $\varphi_\lambda^{(p,q)}$ of $SU(p, q)$, is defined by (see [18, Theorem IV.4.3]), for $g \in SU(p, q)$,

$$\varphi_\lambda^{(p,q)}(g) = \int_K e^{(\lambda + \varrho)(A(kg))} dk.$$

Then (Hoogenboom [19, Theorem 3]), for

$$A(p) = (-1)^{\frac{3}{2}(p(p-1))} 2^{2p(p-1)} \prod_{j=1}^{p-1} \{(k+j)^{p-j} j!\},$$

when for all i , $\lambda_i \notin \mathbb{Z}$, $r \in \mathbb{R}^p$,

$$\varphi_\lambda^{(p,q)}(D(r)) = \frac{A(p)}{\prod_{i < j} (\cosh 2r_i - \cosh 2r_j)} \frac{\det(\varphi_{\lambda_i}^{(1,q-p+1)}(D(r_j)))}{\prod_{i > j} (\lambda_i^2 - \lambda_j^2)}.$$

Lemma 5.1. *There is $c_p > 0$ depending only p such that, for $r \in \mathbb{R}^p$,*

$$\varphi_0^{(p,q)}(D(r)) = \frac{c_p}{\prod_{1 \leq i < j \leq p} (\cosh 2r_i - \cosh 2r_j)} \det(M_{(p,q)}(r))$$

where $M_{(p,q)}(r)$ is the $p \times p$ matrix with coefficients

$$M_{(p,q)}(r)^{i,j} = \frac{d^{2(j-1)}}{d\lambda^{2(j-1)}} \varphi_\lambda^{(1,q-p+1)}(D(r_i))_{\{\lambda=0\}}.$$

Proof. Since; for $\lambda \in \mathbb{R}$, $\varphi_\lambda^{(1,q-p+1)} = \varphi_{-\lambda}^{(1,q-p+1)}$, the lemma follows from Hua's lemma [19, Lemma 4.1] with $c_p = \frac{A_p(-1)^{\frac{1}{2}p(p-1)}}{2!4!\dots(2(p-1))!}$. \square

For $SU(p, q)$, $m_\alpha = 2(q - p)$ when $\alpha = e_k$, $m_\alpha = 1$ when $\alpha = 2e_k$ and $m_\alpha = 2$ when $\alpha = e_i + e_j$ or $\alpha = e_i - e_j$. Thus, by (5.1)

$$\delta(D(r)) = \prod_k (2 \sinh r_k)^{2(q-p)} (2 \sinh 2r_k) \prod_{i < j} 4(\cosh 2r_i - \cosh 2r_j)^2.$$

On the other hand, using the notation (7.1) adapted to the complex case, for $r \in \mathbb{R}$,

$$\delta_{q-p+1}(r) = 2^{2(q-p)+1} \sinh^{2(q-p)} r \sinh 2r.$$

So there is a constant $C_p > 0$ such that

$$(\delta^{1/2} \varphi_0^{(p,q)})(D(r)) = C_p \det M_{(p,q)}(r) \prod_{k=1}^p \delta_{q-p+1}(r_k) = C_p \det N_{(p,q)}(r)$$

where

$$(N_{(p,q)}(r))^{i,j} = \frac{d^{2(j-1)}}{d\lambda^{2(j-1)}} (\delta_{q-p+1}^{1/2} \tilde{\varphi}_\lambda^{q-p+1})(r_i)_{\{\lambda=0\}}$$

with $\tilde{\varphi}_\lambda^{q-p+1}(r) = \varphi_\lambda^{(1,q-p+1)}(D(r))$. Let (see (7.5),(7.6))

$$\alpha(q-p+1) = \frac{\Gamma(q-p)}{\Gamma(2(q-p))2^{5/2}}$$

and

$$g_{q-p+1}(\lambda, r) = \alpha(q-p+1) (\delta_{q-p+1}^{1/2} \tilde{\varphi}_\lambda^{q-p+1})(r + \log 2(q-p)) - K_\lambda(e^{-r}).$$

By Corollary 7.3, $g_{q-p+1}(\lambda, r)$ and all its derivatives converge to 0 for $\lambda = 0$ as $q \rightarrow +\infty$. The generator of $\text{Rad}(\xi_t^0)$ is

$$\frac{1}{2} \sum_{i=1}^p \frac{\partial^2}{\partial r_i^2} + \sum_{i=1}^p \frac{\partial}{\partial r_i} \log(\delta^{1/2} \varphi_0^{(p,q)})(D(r)) \frac{\partial}{\partial r_i}.$$

Hence, we deduce from Corollary 7.3 that

Theorem 5.2. *The generator L of the limit process in Theorem 4.10 is*

$$\frac{1}{2} \sum_{i=1}^p \frac{\partial^2}{\partial r_i^2} + \sum_{i=1}^p \frac{\partial}{\partial r_i} \log \det K^{(p)}(e^{-r}) \frac{\partial}{\partial r_i}$$

where $K^{(p)}(e^{-r})$ is the $p \times p$ matrix with entries

$$K^{(p)}(e^{-r})^{i,j} = \frac{d^{2(j-1)}}{d\lambda^{2(j-1)}} K_\lambda(e^{-r_i})_{\{\lambda=0\}}.$$

Remark 5.3. The limit $\det K^{(p)}(e^{-r})$ is a ground state of the Toda Hamiltonian

$$\sum_{k=1}^p \frac{\partial^2}{\partial r_k^2} - e^{-2r_k}.$$

It is equal to 0 on the walls $\{r_i = r_j\}$. This is linked to the fact that the correction term $(\log 2q) \mathbf{1}$ in Theorem 4.10 corresponds to a direction along

a wall of the Weyl chamber, when $p > 1$. Notice that $\prod_{k=1}^p K_0(e^{-r_k})$ is another ground state.

6. SERIES OF HOMOGENEOUS TREES

We now consider the same question for q -adic symmetric spaces of rank one. The most important series is given by the symmetric spaces associated with $Gl(2, \mathbb{Q}_q)$ where q is the sequence of prime numbers. The associated symmetric space is the homogeneous tree \mathbb{T}_q . The analogue of the Brownian motion is the simple random walk. Let us consider more generally the tree \mathbb{T}_q , where q is an arbitrary integer. We will deal to this case by an elementary treatment. By definition, \mathbb{T}_q is the connected graph without cycle whose vertices have exactly $q + 1$ edges. We choose an origin o in this tree. The simple random walk $W_n, n \geq 0$, starts at o and at each step goes to one of its $q + 1$ neighbours with uniform probability.

Let us recall some (well known) elementary facts about (W_n) ([14], [38]). For the convenience of the reader we give the simple proofs. Let us write $x \sim y$ when x and y are neighbours. A function $f : \mathbb{T}_q \rightarrow \mathbb{R}$ is called radial when $f(x)$ depends only on $d(o, x)$ where d is the standard distance on the tree. In this case one defines $\tilde{f} : \mathbb{N} \rightarrow \mathbb{R}$ by

$$\tilde{f}(n) = f(x), \text{ when } d(o, x) = n.$$

The probability transition of W_n is

$$P(x, y) = \frac{1}{q+1}, \text{ iff } x \sim y.$$

The radial part of W_n is the process $X_n = d(o, W_n)$, it is a Markov chain on \mathbb{N} with transition probability R given by $R(0, 1) = 1$, and if $n > 0$,

$$R(n, n-1) = \frac{1}{q+1}, R(n, n+1) = \frac{q}{q+1}.$$

Let us introduce the average operator A which associates to a function $f : \mathbb{T}_q \rightarrow \mathbb{R}$ the radial function

$$Af(x) = \frac{1}{S(o, x)} \sum_{y \in S(o, x)} f(y)$$

where $S(o, x)$ is the sphere $S(o, x) = \{y \in \mathbb{T}_q; d(o, y) = d(o, x)\}$. It is easy to see that P and A commute:

$$(6.1) \quad PA = AP.$$

Therefore

Lemma 6.1. *When a function $f : \mathbb{T}_q \rightarrow \mathbb{R}$ is a λ -eigenfunction of P (i.e. $Pf = \lambda f$), then Af is a radial λ -eigenfunction of P and \tilde{Af} is a λ -eigenfunction of R . Conversely, if $\tilde{g} : \mathbb{N} \rightarrow \mathbb{R}$ is a λ -eigenfunction of R , then $g(x) = \tilde{g}(d(o, x))$ is a radial λ -eigenfunction of P .*

Let $\lambda > 0$, $\tilde{g} : \mathbb{N} \rightarrow \mathbb{R}^+$ is a λ -eigenfunction of R when $\tilde{g}(1) = \lambda\tilde{g}(0)$ and

$$\tilde{g}(n-1) + q\tilde{g}(n+1) = \lambda(q+1)\tilde{g}(n), n \geq 1.$$

One sees easily that there exists a positive solution of this equation if and only if $\lambda \geq \frac{2\sqrt{q}}{q+1}$. Hence, the principal generalized eigenvalue ϱ of R is

$$\varrho = \frac{2\sqrt{q}}{q+1},$$

the associated eigenfunction $\tilde{\varphi}_0$ is

$$\tilde{\varphi}_0(n) = \left(1 + n \frac{q-1}{q+1}\right) \frac{1}{q^{n/2}}.$$

It is the only one if we suppose that $\tilde{\varphi}_0(0) = 1$. It follows from Lemma 6.1 that ϱ is also the principal generalized eigenvalue of P . The function $\varphi_0 : \mathbb{T}_q \rightarrow \mathbb{R}$ given by $\varphi_0(x) = \tilde{\varphi}_0(d(o, x))$ for $x \in \mathbb{T}_q$, is a radial ϱ -eigenfunction of P . By uniqueness of $\tilde{\varphi}_0$, φ_0 is the unique radial ground state of the random walk $W_n, n \geq 0$, equal to 1 at o . We consider the φ_0 -ground state process $W_n^{(0)}, n \geq 0$, on \mathbb{T}_q , defined as the Markov chain with transition probability $P^{(0)}$ given by

$$P^{(0)}(x, y) = \frac{1}{\varrho\phi_0(x)} P(x, y)\phi_0(y).$$

Its radial part is a Markov chain on \mathbb{N} with transition probability, for $m, n \in \mathbb{N}$,

$$R^{(0)}(n, m) = \frac{1}{\varrho\tilde{\varphi}_0(n)} R(n, m)\tilde{\varphi}_0(m).$$

It is clear that

$$\lim_{q \rightarrow +\infty} R^{(0)}(m, n) = B(m, n)$$

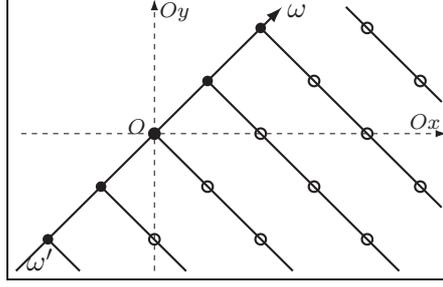
where B is transition probability of the so-called discrete Bessel(3) Markov chain on \mathbb{N} , namely,

$$B(0, 1) = 1, B(n, n+1) = \frac{n+2}{2(n+1)}, B(n, n-1) = \frac{n}{2(n+1)}.$$

Therefore

Proposition 6.2. *When $q \rightarrow +\infty$, the radial part of the φ_0 -ground state process $W_n^{(0)}$ of the simple random walk on the tree \mathbb{T}_q converges in distribution to the discrete Bessel(3) chain.*

Our aim is to obtain a path description of this limit process in term of the simple random walk on \mathbb{Z} . We use the analogue of the distinguished Brownian motion. A geodesic ray ω is an infinite sequence $x_n, n \geq 0$, of distinct vertices in \mathbb{T}_q such that $d(x_n, x_{n+1}) = 1$ for any $n \geq 0$. We fix such a geodesic ray $\omega = \{x_n, n \geq 0\}$ starting from $x_0 = o$. For any x in \mathbb{T}_q there is a unique geodesic ray $\pi(x, \omega)$ which starts at x and such that the symmetric difference between $\pi(x, \omega)$ and ω is finite. We let $x \wedge o$ be the

FIGURE 2. The graph G

on \mathbb{T}_q is, like φ_0 , a ground state of the Markov chain W_n . By Lemma 6.1 $A\varphi^\omega$ is a radial ground state of P . By uniqueness, this implies that

$$(6.2) \quad \varphi_0 = A\varphi^\omega.$$

Definition 6.3. Let $S_n, n \geq 0$, be the φ^ω -ground state process associated to (W_n) on \mathbb{T}_q starting from o . We denote by Q_ω its probability transition.

We have,

$$\begin{aligned} Q_\omega(x, y) &= P(x, y) \frac{\varphi^\omega(y)}{\varrho \varphi^\omega(x)} = P(x, y) \frac{q^{-h(y)/2}}{\varrho q^{-h(x)/2}} \\ &= P(x, y) \frac{(q+1)q^{-h(y)/2}}{2\sqrt{q}q^{-h(x)/2}} = P(x, y) \frac{(q+1)q^{-h(y)/2}}{2q^{(1-h(x))/2}}. \end{aligned}$$

Therefore, when $x \sim y$, since $P(x, y) = 1/(q+1)$,

$$Q_\omega(x, y) = \begin{cases} 1/2q, & \text{when } h(y) = h(x) + 1, \\ 1/2, & \text{when } h(y) = h(x) - 1. \end{cases}$$

As in Proposition 3.7,

Lemma 6.4. The process $d(o, S_n), n \geq 0$, has the same law as $d(o, W_n^{(0)})$, $n \geq 0$.

Proof. Let $N \in \mathbb{N}$ and $F : \mathbb{Z}^{N+1} \rightarrow \mathbb{R}^+$. For any isometry k of the tree \mathbb{T}_q which fixes o , $W_n, n \geq 0$, has the same law as $k.W_n, n \geq 0$. Hence, using also (6.2),

$$\begin{aligned} \mathbb{E}(F(d(o, S_n), n \leq N)) &= \varrho^{-N} \mathbb{E}(F(d(o, W_n), n \leq N) \varphi^\omega(W_N)) \\ &= \varrho^{-N} \mathbb{E}(F(d(o, k.W_n), n \leq N) \varphi^\omega(k.W_N)) \\ &= \varrho^{-N} \mathbb{E}(F(d(o, W_n), n \leq N) \varphi^\omega(k.W_N)) \\ &= \varrho^{-N} \mathbb{E}(F(d(o, W_n), n \leq N) A\varphi^\omega(W_N)) \\ &= \varrho^{-N} \mathbb{E}(F(d(o, W_n), n \leq N) \varphi_0(W_N)) \\ &= \mathbb{E}(F(d(o, W_n^{(0)}), n \leq N)). \end{aligned} \quad \square$$

We consider the following (unoriented) graph G embedded in \mathbb{Z}^2 (see Figure 2): the edges are the points with coordinates

$$(k + 2n, k), k \in \mathbb{Z}, n \in \mathbb{N},$$

and the vertices are the first diagonal segments joining (k, k) and $(k+1, k+1)$ and the segments joining $(k + 2n, k)$ and $(k + 2n + 1, k - 1)$ for $n \geq 0$. We choose in the tree \mathbb{T}_q another geodesic ray $\omega' = \{x_{-n}, n \geq 0\}$ such that $\omega \cap \omega' = \{o\}$. Then $\omega\omega' = \{x_n, n \in \mathbb{Z}\}$ is a two-sided geodesic. Let $\Psi : \mathbb{T}_q \rightarrow G$ defined as follows: for $x \in \mathbb{T}_q$,

$$\Psi(x) = (-h(x) + 2d(x, \omega'\omega), -h(x)).$$

Then $\Psi(S_n), n \geq 0$, is the nearest neighbour Markov chain on the graph G with probability transition P_G given by

$$\begin{aligned} P_G((k, k), (k - 1, k - 1)) &= 1/2q, \\ P_G((k, k), (k + 1, k + 1)) &= 1/2, \\ P_G((k, k), (k + 1, k - 1)) &= (q - 1)/2q, \end{aligned}$$

and, for $n > 0$ and $\varepsilon = 1$ or -1 ,

$$P_G((k + 2n, k), (k + 2n - \varepsilon, k + \varepsilon)) = 1/2$$

When $q \rightarrow +\infty$, the Markov chain $\Psi(S_n)$ converges to the Markov chain on the graph G with transition probability Q given by

$$\begin{aligned} Q((k, k), (k - 1, k - 1)) &= 0, \\ Q((k, k), (k + 1, k + 1)) &= 1/2, \\ Q((k, k), (k + 1, k - 1)) &= 1/2, \end{aligned}$$

and for $n > 0$ and $\varepsilon = 1$ or -1 ,

$$Q((k + 2n, k), (k + 2n - \varepsilon, k + \varepsilon)) = 1/2.$$

When the chain with kernel Q starts from $(0, 0)$ it cannot reach $(-1, -1)$ therefore it lives on the subgraph \tilde{G} of G described in Figure 3. It is the one which appears in the discrete time Pitman theorem (see Figure 8 in Biane [4]). It has the following simple description in terms of the simple symmetric random walk $\Sigma_n, n \geq 0$, on \mathbb{Z} . Let $\Sigma_n = \varepsilon_1 + \dots + \varepsilon_n$ where the (ε_n) are i.i.d. random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$ and

$$M_n = \max\{\Sigma_k, k \leq n\}.$$

The Markov chain on \tilde{G} starting from $(0, 0)$ is, in xOy coordinates,

$$(X_n, Y_n) = (2M_n - \Sigma_n, \Sigma_n),$$

(see Figure 3). The radial component of S_n is $d(o, S_n) = D((0, 0), \Psi(S_n))$ for the graph distance D on G . Since $D((0, 0), (a, b)) = a$, we obtain using Lemma 6.4 that,

Then (see (5.1))

$$\begin{aligned} H_{CMS} &= \delta_q^{1/2} \circ \{\Delta_R + (\rho_q, \rho_q)\} \circ \delta_q^{-1/2} \\ &= \frac{d^2}{dr^2} - \frac{m_\alpha(m_\alpha + 2m_{2\alpha} - 2)}{\sinh^2 r} - \frac{m_{2\alpha}(m_{2\alpha} - 2)}{\sinh^2 2r}. \end{aligned}$$

For $\lambda \in \mathbb{C}$, the spherical function φ_λ satisfies

$$\Delta_R \tilde{\varphi}_\lambda = (\lambda^2 - \varrho_q^2) \tilde{\varphi}_\lambda$$

where $\tilde{\varphi}_\lambda(r) = \varphi_\lambda(D(r))$, $r \in \mathbb{R}$, therefore,

$$H_{CMS}(\delta_q^{1/2} \tilde{\varphi}_\lambda) = \lambda^2 \delta_q^{1/2} \tilde{\varphi}_\lambda.$$

There exists a unique function $\Psi_{CMS}(\lambda, q, r)$, $r \in \mathbb{R}$, such that

$$(7.2) \quad H_{CMS} \Psi_{CMS} = \lambda^2 \Psi_{CMS},$$

of the form

$$(7.3) \quad \Psi_{CMS}(\lambda, q, r) = \sum_{n \in \mathbb{N}} b_n(\lambda, q) e^{(\lambda-n)r}, \quad b_0(\lambda, q) = 1,$$

(see [37, (17)]). When $q \rightarrow +\infty$,

$$\lim H_{CMS}(r + \log m_\alpha) = H_T$$

where H_T is the Toda type Hamiltonian

$$H_T = \frac{d^2}{dr^2} - e^{-2r}.$$

There is also a unique function $\Psi_T(\lambda, r)$, $r \in \mathbb{R}$, such that

$$H_T \Psi_T = \lambda^2 \Psi_T$$

of the form

$$(7.4) \quad \Psi_T(\lambda, r) = \sum_{n \in \mathbb{N}} b_n(\lambda) e^{(\lambda-n)r}, \quad b_0(\lambda) = 1,$$

([37], notice that this is the function denoted $\Psi_T(-\lambda, -r)$ in [37]).

Lemma 7.1. *If $\lambda \in \mathfrak{a}_\mathbb{C}^*$ and $2\lambda \notin \mathbb{Z}^*$, then*

$$\lim_{q \rightarrow +\infty} \frac{1}{m_\alpha^\lambda} \Psi_{CMS}(\lambda, q, r + \log m_\alpha) = \Psi_T(\lambda, r)$$

Proof. The proof is similar to the one of Proposition 1 in [37]. □

Let

$$(7.5) \quad \alpha(q) = \frac{\Gamma(\frac{m_\alpha}{2})^2}{\Gamma(m_\alpha) 2^{1+3m_{2\alpha}/2}}$$

and

$$(7.6) \quad g_q(\lambda, r) = \alpha(q) (\delta_q^{1/2} \tilde{\varphi}_\lambda)(\log m_\alpha + r) - K_\lambda(e^{-r}).$$

Proposition 7.2. *Uniformly on λ in a small neighborhood of 0 in \mathbb{C} , $\lambda g_q(\lambda, r)$ and its derivatives in λ converge to 0 as $q \rightarrow +\infty$.*

Proof. Using the Harish Chandra spherical function expansion of φ_λ (see [18, Theorems IV.5.5 and IV.6.4] or [16]), when λ is for instance in the ball $\{\lambda \in \mathbb{C}; |\lambda| \leq 1/4\}$ and $\lambda \neq 0$,

$$\delta_q^{1/2}(r)\tilde{\varphi}_\lambda(r) = c(\lambda)\Psi_{CMS}(\lambda, q, r) + c(-\lambda)\Psi_{CMS}(-\lambda, q, r)$$

where

$$c(\lambda) = \frac{2^{\frac{1}{2}m_\alpha + m_{2\alpha} - \lambda} \Gamma(\frac{1}{2}(m_\alpha + m_{2\alpha} + 1))}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + 1 + \lambda))\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha + m_{2\alpha} + \lambda))}.$$

It follows from Lemma 7.1 that

$$\lim_{q \rightarrow +\infty} \alpha(q)c(\lambda) \frac{1}{m_\alpha^\lambda} \Psi_{CMS}(\lambda, q, r + \log m_\alpha) = \Gamma(\lambda)2^{\lambda-1}\Psi_T(\lambda, r).$$

Hence

$$\lim_{q \rightarrow +\infty} \alpha(q)(\delta_q^{1/2}\tilde{\varphi}_\lambda)(r + \log m_\alpha) = \Gamma(\lambda)2^{\lambda-1}\Psi_T(\lambda, r) + \Gamma(-\lambda)2^{-\lambda-1}\Psi_T(-\lambda, r).$$

The limit can be expressed in terms of the Whittaker function for $Sl(2, \mathbb{R})$ as in [37, Theorem 3]. Due to the relation between Whittaker and Macdonald functions (see, e.g., Bump [6]), one finds that this limit is $K_\lambda(e^{-r})$, thus $g_q(\lambda, r)$ tends to 0 when $q \rightarrow +\infty$.

Now we remark that, for r fixed, the functions $\lambda g_q(\lambda, r)$ are analytic functions in λ in the ball $\{\lambda \in \mathbb{C}; |\lambda| \leq 1/4\}$ which are uniformly bounded in q by the computations of [37, Proposition 1]. Notice that we have to multiply by λ to avoid the singularity at 0 of the Harish Chandra c function. The uniform convergence of $\lambda g_q(\lambda, r)$ and its derivatives in λ thus follows from Montel's theorem. \square

Corollary 7.3. *As $q \rightarrow +\infty$, $g_q(\lambda, r)$ and all its derivatives converge to 0 at $\lambda = 0$.*

Proof. This follows from the proposition and the fact that

$$\frac{d^n}{d\lambda^n} g_q(\lambda, r)_{\{\lambda=0\}} = \frac{1}{n+1} \frac{d^{n+1}}{d\lambda^{n+1}} (\lambda g_q(\lambda, r))_{\{\lambda=0\}}.$$

Remark 7.4. A small mistake in the constant in [37, Example 1] is corrected in [32].

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