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Explicit linearization of one-dimensional germs through tree-expansions

Frédéric Fauvet, Frédéric Menous, David Sauzin

1st August 2014

Abstract

We explain Écalle’s “arbomould formalism” in its simplest instance, showing how it allows one to give explicit formulas for the operators naturally attached to a germ of holomorphic map in one dimension. When applied to the classical linearization problem of non-resonant germs, which contains the well-known difficulties due to the so-called small divisor phenomenon, this elegant and concise tree formalism yields compact formulas, from which one easily recovers the classical analytical results of convergence of the solution under suitable arithmetical conditions on the multiplier. We rediscover this way Yoccoz’s lower bound for the radius of convergence of the linearization and can even reach a global regularity result with respect to the multiplier (C^1 -holomorphy) which improves on Carminati-Marmi’s result.

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1 Introduction

The purpose of this article is twofold:

- to expound J. Écalle’s “arbomould formalism” by illustrating it on the linearization problem for holomorphic germs in one complex dimension—this amounts to a novel approach to *formal* linearization by means of a powerful and elegant combinatorial machinery,
- to show how this allows one to find again the classical *analytic* results on the convergence of the formal linearization by Koenigs, Siegel, Bruno, Yoccoz, and even improve on Carminati-Marmi’s result on the regularity with respect to the multiplier.

The classification of diffeomorphisms near fixed points is one of the starting points for Poincaré’s theory of normal forms and has its roots in 19th century mathematics, with E. Schröder and G. Koenigs’s works on the linearization problem in one complex dimension (see *e.g.* [Mil00]). The problem consists in finding a conjugacy between a map $g: z \mapsto qz + O(z^2)$ holomorphic near the origin and its linear part $z \mapsto qz$, assuming that the multiplier q is non-zero. One thus looks for an invertible map $z \mapsto h(z)$ such that $g \circ h(z) = h(qz)$ (*i.e.* the inverse of h should satisfy the Schröder functional equation). At a formal level, there is a solution as soon as q is not a root of unity, *i.e.* there is a formal linearization $h(z) \in \mathbb{C}[[z]]$ in that case, and the Koenigs linearization theorem asserts its convergence whenever $|q| \neq 1$.

When $|q| = 1$, things are much more delicate. In the so-called resonant case, namely when q is a root of unity, linearization is in fact generically not even possible at the level of formal series, and the classification of resonant diffeomorphisms leads to Gevrey divergent series and questions of summability and resurgence (see *e.g.* [Eca92]). For $q = e^{2\pi i\omega}$ with ω real and irrational, the recursive expressions available for the coefficients of the formal linearization h involve small denominators of the form $q^n - 1$ (with any $n \geq 1$), which call for a suitable number-theoretic hypothesis on ω in order to prove that $h(z)$ is analytic, as shown by H. Cremer, C. L. Siegel, A. D. Bruno and J.-C. Yoccoz.

The article [Eca92] proposed a totally new approach to deal with general singularities of analytic dynamical systems with discrete or continuous time, in any dimension, with an array of techniques to cover the most general situations where the complications due to resonances and small denominators coexist, but this work has not really been assimilated by the dynamical systems community. In that article, J. Écalle introduced the key concept of “arborification”, according to which the formal series first expressed as “mould expansions” have to be reencoded by expansions over families of trees.

In the present paper, we explain the basics of Écalle’s tree formalism and show how it leads to an explicit formula for the conjugacy $h(z)$. Writing the Taylor expansion of the holomorphic germ g in the form $g(z) = q(z + a_1z^2 + a_2z^3 + \dots)$, we shall obtain

$$h(z) = z + \sum_T \gamma_T \left(\prod_{\sigma \in V_T} \frac{a_{N_T(\sigma)}}{q^{\|\text{Tree}(\sigma, T)\|} - 1} \right) z^{\|T\|+1},$$

where the summation is performed over trees T whose vertices σ are decorated by positive integers $N_T(\sigma)$ and the coefficients γ_T are non-negative rational numbers to be defined in due time; the product is over all the vertices of the given tree T , the notation $\|\cdot\|$ indicating the sum of the decorations of a tree and $\text{Tree}(\sigma, T)$ denoting the subtree of T “rooted at σ ”.

In fact, Écalle’s formalism will give more: it is the composition operator itself $\varphi \in \mathbb{C}[[z]] \mapsto \varphi \circ h \in \mathbb{C}[[z]]$ which can be represented as the sum of a formally summable family of explicit

elementary differential operators. This is related to the idea, due to A. Cayley [Cay57], that trees are the relevant combinatorial objects to deal with the composition of differential operators.

All the precise definitions are given below in a self-contained way, and it is in fact one of the objectives of the present text to clarify the notions introduced by Écalle, connecting them with well-known combinatorial objects and constructions. An originality of our presentation is that we arrive directly at the tree representation of h or of its composition operator, without constructing a preliminary mould expansion and then passing through the process of arborification. In this sense, this part of the paper can be considered as a lightened introduction to Écalle's formalism, and the interested reader can pursue with [FM14] and the references therein to learn more about the algebraic structures underlying arborification.

Next, we show that the explicit expression of h can be efficiently used to prove its analyticity when $q = e^{2\pi i\omega}$ and ω satisfies Bruno's arithmetic condition (relying on an arithmetical lemma due to Davie, as in [CM00]), finding again Yoccoz's lower bound for the radius of convergence of h .

It also gives us access to a new result on the *monogenic dependence* of h with respect to the multiplier q , in the spirit of [He85]. The first result of that kind was proved by C. Carminati and S. Marmi in [CM08]. The idea consists in considering all the ω 's satisfying a uniform Bruno condition and constructing a closed subset K of \mathbb{C} such that the map $q \in K \mapsto h \in B$ is C^1 -holomorphic, where B is a suitable Banach space of functions of z . When it comes to C^1 -holomorphy, our method is quite different from that of [CM08] and gives an improvement for the radius of the disc in the z -plane which determines the Banach space B that one can take.

The paper is self-contained and we hope it will constitute an accessible entry to some of the beautiful and far reaching constructions of Écalle, while yielding original proofs of non trivial dynamical results and paving the way for further works.

2 Linearization of formal diffeomorphisms or germs of holomorphic diffeomorphisms in dimension 1

Let us denote by

$$\tilde{\mathcal{G}} := \left\{ g(z) = \sum_{n \geq 1} b_n z^n \in \mathbb{C}[[z]] \mid b_1 \neq 0 \right\} \quad (1)$$

the group of formal diffeomorphisms in one dimension, the group law being the composition of formal series without constant term, with the notation $g^{\circ(-1)}$ for the inverse of an element g . The coefficient b_1 of a given $g \in \tilde{\mathcal{G}}$ is called its *multiplier*; the formal diffeomorphisms with multiplier 1 form a subgroup of $\tilde{\mathcal{G}}$ that we denote by $\tilde{\mathcal{G}}_1$. The group of germs of holomorphic diffeomorphisms in one dimension can be identified with a subgroup \mathcal{G} of $\tilde{\mathcal{G}}$, in which tangent-to-identity convergent series also form a subgroup:

$$\mathcal{G} := \{ g \in \tilde{\mathcal{G}} \mid g \in \mathbb{C}\{z\} \}, \quad \mathcal{G}_1 := \mathcal{G} \cap \tilde{\mathcal{G}}_1. \quad (2)$$

The local theory of holomorphic dynamics is concerned with the iteration of elements of \mathcal{G} and the description of the conjugacy classes of \mathcal{G} . The rotations $R_q \in \mathcal{G}$ defined by

$$R_q(z) := qz, \quad \text{for } q \in \mathbb{C}^* \quad (3)$$

display the simplest possible dynamics: the k th iterate of R_q is R_{q^k} (for any $k \in \mathbb{Z}$). One is thus interested in the

Holomorphic Linearization Problem: Given $g \in \mathcal{G}$, find $h \in \mathcal{G}_1$ such that

$$g \circ h = h \circ R_q, \quad (4)$$

where q is the multiplier of g .

It is indeed clear that q cannot be anything else but $\frac{dg}{dz}(0)$ and that, if h solves (4), then the k th iterate of g is $h \circ R_{q^k} \circ h^{o(-1)}$. Notice that there is no loss of generality in imposing a priori $h \in \mathcal{G}_1$: if $h_0 \in \mathcal{G}$ is a solution of (4) with multiplier λ , then $h_0 \circ R_\lambda^{o(-1)}$ is a solution which belongs to \mathcal{G}_1 .

Similarly, we may consider the

Formal Linearization Problem: Given $g \in \tilde{\mathcal{G}}$, find $h \in \tilde{\mathcal{G}}_1$ which solves (4).

A solution h to this problem will be called a *formal linearization* of g .

Viewing $\tilde{\mathcal{G}}$ as a skew-product $\mathbb{C}^* \times \tilde{\mathcal{G}}_1$, we will systematically write g in the form

$$g = R_q \circ f, \quad q \in \mathbb{C}^*, \quad f \in \tilde{\mathcal{G}}_1, \quad (5)$$

so that Equation (4) takes the form $f \circ h = R_q^{o(-1)} \circ h \circ R_q$. We first recall the elementary

Lemma 2.1. *Let $f(z) = z + \sum_{n \geq 1} a_n z^{n+1} \in \tilde{\mathcal{G}}_1$ and $q \in \mathbb{C}^*$. Suppose that q is not a root of unity. Then the Formal Linearization Problem for $g = R_q \circ f$ has a unique solution*

$$h(z) = z + \sum_{n \geq 1} c_n z^{n+1} \in \tilde{\mathcal{G}}_1. \quad (6)$$

The coefficients of the formal linearization are inductively determined by the formula

$$c_n = \frac{1}{q^n - 1} \sum_{r=1}^n \sum_{\substack{(n_0, \dots, n_r) \in \mathbb{N}^{r+1} \text{ s.t.} \\ n_0 + \dots + n_r + r = n}} a_r c_{n_0} \cdots c_{n_r}, \quad n \geq 1, \quad (7)$$

with the convention $c_0 = 1$ and $\mathbb{N} = \{0, 1, 2, \dots\}$.

Proof. Write the conjugacy equation (4) as $h(qz) = qf(h(z))$ and expand it. \square

Our aim in this article is

- to show how Écalle's tree formalism leads to an explicit formula for h , with a clear separation of its dependence on f and its dependence on q (Theorems A in Section 4 and A' in Section 6),
- to recover from this explicit formula the classical result for the Holomorphic Linearization Problem, according to which $h \in \mathcal{G}_1$ whenever $f \in \mathcal{G}_1$ and $|q| \neq 1$ or q satisfies the *Bruno condition* (see Definition 7.1 and Theorem B),
- to study the global regularity properties of the dependence in q and obtain refined results of C^1 -holomorphic and monogenic dependence (see the definition in Section 8.2 and Theorem C).

FORMAL LINEARIZATION

3 \mathcal{N} -trees and \mathcal{N} -forests

The method we want to expound relies on expansions indexed by what we call \mathbb{N}^* -trees, which can informally be defined as non-plane rooted trees decorated by elements of \mathbb{N}^* up to isomorphism. Here and everywhere in the article, we use the notations

$$\mathbb{N}^* = \{1, 2, 3, \dots\}, \quad \mathbb{N} = \{0, 1, 2, \dots\}.$$

\mathbb{N}^* -trees are classical graph-theoretic objects, however we prefer to give a formal definition in terms of a special class of finite posets. Recall that a poset is a pair (V, \preceq) , where V is a set (possibly empty) and \preceq is an order relation on V , and that a poset isomorphism from (V, \preceq) to (V', \preceq') is a bijection $\Phi: V \rightarrow V'$ such that, for every $\sigma, \tau \in V$, $\sigma \preceq \tau \Leftrightarrow \Phi(\sigma) \preceq' \Phi(\tau)$.

Definition 3.1. A poset (V, \preceq) is said to be *arborescent* if the underlying set V is finite and any element of V has at most one direct predecessor.

Equivalently: a finite poset (V, \preceq) is arborescent if and only if, for every $\tau \in V$, the set of all its ancestors $\{\sigma \in V \mid \sigma \preceq \tau\}$ is totally ordered; or: if and only if, for any two incomparable elements $\sigma, \tau \in V$, the successors of σ and the successors of τ are incomparable.

The terminology is motivated by the fact that the Hasse diagram of an arborescent poset (V, \preceq) (*i.e.* the graph whose vertices are the elements of V and whose edges are defined from the cover relation of \preceq) is a disjoint union of trees; it is a tree if and only if (V, \preceq) has a unique minimal element [Sta11, Appendix]—see Figure 1.

Definition 3.2. Given a non-empty set \mathcal{N} , we call *\mathcal{N} -arborescent poset* any triple (V, \preceq, N) , where (V, \preceq) is an arborescent poset and $N: V \rightarrow \mathcal{N}$ is a map. If (V, \preceq, N) and (V', \preceq', N') are \mathcal{N} -arborescent posets, we call *\mathcal{N} -arborescent poset isomorphism from (V, \preceq, N) to (V', \preceq', N')* any poset isomorphism $\Phi: (V, \preceq) \rightarrow (V', \preceq')$ such that $N' \circ \Phi = N$. The \mathcal{N} -arborescent poset isomorphisms from (V, \preceq, N) to itself form the *automorphism group of (V, \preceq, N)* , denoted by $\text{Aut}(V, \preceq, N)$ (it is a subgroup of the group \mathfrak{S}_V of all permutations of V).

Definition 3.3. Given a non-empty set \mathcal{N} , we call *\mathcal{N} -forest* any isomorphy class of \mathcal{N} -arborescent posets; we denote by $\mathcal{F}(\mathcal{N})$ the set of all \mathcal{N} -forests. We call *\mathcal{N} -tree* any \mathcal{N} -forest for which a representative (and thus each representative) is of the form (V, \preceq, N) where (V, \preceq) has a unique minimal element; we denote by $\mathcal{T}(\mathcal{N})$ the set of all \mathcal{N} -trees.

Remark 3.4. *Abuse of language.* For a given $F \in \mathcal{F}(\mathcal{N})$, we will speak of the set V_F of its vertices or of its decoration map N_F , meaning that we choose a representative (V_F, \preceq_F, N_F) of F and consider the underlying finite set or the corresponding \mathcal{N} -valued map. Similarly we define the root $\rho_T \in V_T$ of a given $T \in \mathcal{T}(\mathcal{N})$ as its minimal element, although it depends on the chosen representative (V_T, \preceq_T, N_T) of T .

Definition 3.5. Let F be an \mathcal{N} -forest.

- The *size* of F , denoted by $\#F$, is defined as the cardinality of V_F .
- The *degree* of F , denoted by $\deg(F) \in \mathbb{N}$, is the number of minimal elements of (V_F, \preceq_F) .

- The *height* of F is the maximal cardinality of a chain¹ of (V_F, \preceq_F) .
- The *symmetry factor* of F is the number

$$\text{sym}(F) := \text{card}(\text{Aut}(V_F, \preceq_F, N_F)) \in \mathbb{N}^*, \quad (8)$$

where (V_F, \preceq_F, N_F) is any representative of F .

- For each vertex $\sigma \in V_F$, we denote by $S_F^+(\sigma)$ the set of its direct successors in (V_F, \preceq_F) and call *outdegree* of σ the cardinality of this set, with the notation

$$\text{deg}_F^+(\sigma) := \text{card}(S_F^+(\sigma)) \in \mathbb{N}.$$

Note that the empty set can be considered as an arborescent poset or as an \mathcal{N} -arborescent poset. In the latter case, its isomorphism class is called the *empty \mathcal{N} -forest* and denoted by ε (it is an \mathcal{N} -forest but not an \mathcal{N} -tree). Its size and degree are $\#\varepsilon = \text{deg}(\varepsilon) = 0$; its symmetry factor is $\text{sym}(\varepsilon) = 1$.

Definition 3.6. If \mathcal{N} is contained in a commutative monoid $\hat{\mathcal{N}}$ and $F \in \mathcal{F}(\mathcal{N})$, then the *weight* of F is defined as

$$\|F\| := \sum_{\sigma \in V_F} N_F(\sigma) \in \hat{\mathcal{N}} \quad (9)$$

and the *weights of the vertices of F* are defined by

$$\sigma \in V_F \mapsto \hat{\sigma} := \sum_{\mu \in V_F \text{ s.t. } \sigma \preceq_F \mu} N_F(\mu) \in \hat{\mathcal{N}}. \quad (10)$$

For the empty \mathcal{N} -forest, the weight is $\|\varepsilon\| = 0_{\hat{\mathcal{N}}}$ (unit of the monoid law).

Note that the abuse of language of Remark 3.4 was used to define the maps $\sigma \mapsto \text{deg}_F^+(\sigma)$ and $\sigma \mapsto \hat{\sigma}$ (they depend on the representative (V_F, \preceq_F, N_F) and not only on F), whereas $\#F$, $\text{deg}(F)$, $\text{sym}(F)$ and $\|F\|$ are independent of the chosen representative of F —see Figure 2.

Observe that an \mathcal{N} -forest F is an \mathcal{N} -tree if and only if $\text{deg}(F) = 1$ (in which case $\|F\| = \hat{\rho}_F$) and that, for $\sigma \in V_F$,

- $\hat{\sigma} = \|\text{Tree}(\sigma, F)\|$, where $\text{Tree}(\sigma, F)$ is the \mathcal{N} -subtree of F rooted at σ (whose set of vertices is $\{\mu \in V_F \mid \sigma \preceq_F \mu\}$, with the arborescent poset structure induced by \preceq_F),
- $\text{deg}_F^+(\sigma) = \text{deg}(\text{For}^+(\sigma, F))$, where $\text{For}^+(\sigma, F)$ is the \mathcal{N} -subforest obtained by removing the root of $\text{Tree}(\sigma, F)$ (whose set of vertices is $\{\mu \in V_F \mid \sigma \prec_F \mu\}$ and whose set of minimal elements is $S_F^+(\sigma)$).

Remark 3.7. If we fix a one-element set $\mathcal{N} = \{*\}$, so that the decoration maps is trivial, then $\{*\}$ -trees are in one-to-one correspondence with isomorphism classes of non-plane rooted trees via Hasse diagrams. They are generalizations of positive integers: given $n \in \mathbb{N}^*$, one can view n itself as a particular $\{*\}$ -tree called “ladder” and represented by any totally ordered set with n elements (*cf.* the first among the four examples of Figure 1 for $n = 3$), but for $n \geq 3$ there are other $\{*\}$ -trees of size n (*cf.* the second example of Figure 1).

Similarly, when \mathcal{N} is an arbitrary non-empty set, an \mathcal{N} -tree is a generalization of a non-empty word on the alphabet \mathcal{N} : any word $N_1 \cdots N_n$ can be identified with an “ \mathcal{N} -ladder” with n vertices decorated by the letters N_1, \dots, N_n in appropriate order, but for $n \geq 3$ there are other \mathcal{N} -trees with the same decorations (which can be viewed as “arborescent words”).

¹A chain of a poset is a subposet which is totally ordered.

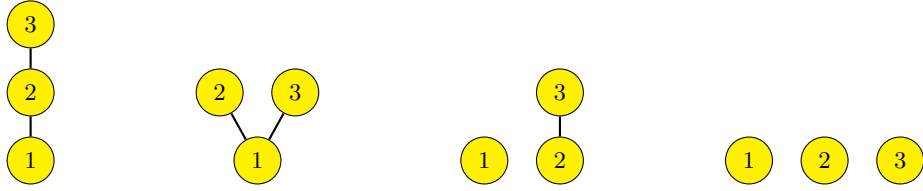


Figure 1: The Hasse diagrams of four different arborescent posets with the same underlying set $V = \{1, 2, 3\}$. Only the first two are trees. Any arborescent poset of size 3 is isomorphic to one of these four posets.



Figure 2: The left diagram shows an \mathbb{N}^* -tree T for which $\text{sym}(T) = 2$ and $\|T\| = 40$. If we choose to represent T by (V, \preceq, N) with $V = \{1, 2, 3, 4, 5\}$, $N(1) = 8$, $N(2) = N(3) = 7$, $N(4) = 12$, $N(5) = 6$ and \preceq determined by the Hasse diagram shown on the right (so $\rho_T = 1$ for this choice), then $\hat{1} = 40$, $\hat{2} = \hat{3} = 7$, $\hat{4} = 18$, $\hat{5} = 6$, $\text{deg}_T^+(1) = 3$, $\text{deg}_T^+(2) = 0$, etc.

4 Tree-expansion of the solution — Theorem A

In practice, in this article, we shall use $\mathcal{N} = \mathbb{N}^*$ and $\hat{\mathcal{N}} = \mathbb{N}$ most of the time. Given an \mathbb{N}^* -forest F , represented by (V_F, \preceq_F, N_F) , we associate with each vertex $\sigma \in V_F$ a non-negative integer:

$$k_F(\sigma) := \frac{m!}{(m-s)!} \in \mathbb{N}^* \text{ if } m := N_F(\sigma) + 1 \geq s := \text{deg}_F^+(\sigma), \quad k_F(\sigma) = 0 \text{ else,} \quad (11)$$

so that $z^s \left(\frac{d}{dz}\right)^s z^m = k_F(\sigma) z^m$.

The first central result that we wish to explain in this paper is due to J. Écalle [Eca92]:

Theorem A. *Let $f(z) = z + \sum_{n \geq 1} a_n z^{n+1} \in \tilde{\mathcal{G}}_1$ and $q \in \mathbb{C}^*$. Suppose that q is not a root of unity. Then the formal linearization (6) of $g = R_q \circ f$ is given by*

$$h(z) = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} \frac{1}{\text{sym}(T)} \left(\prod_{\sigma \in V_T} \frac{k_T(\sigma) a_{N_T(\sigma)}}{q^{\hat{\sigma}} - 1} \right) z^{\|T\|+1}. \quad (12)$$

The proof of Theorem A is spread over Sections 5–6. Observe that there are only finitely many \mathbb{N}^* -trees with given weight, thus formula (12) makes sense and yields the solution of the induction (7) in the form

$$c_n = \sum_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} \frac{1}{\text{sym}(T)} \left(\prod_{\sigma \in V_T} \frac{k_T(\sigma) a_{N_T(\sigma)}}{q^{\hat{\sigma}} - 1} \right), \quad n \in \mathbb{N}^*. \quad (13)$$

Remark 4.1. Let us set, for each² \mathbb{N}^* -forest F ,

$$S^F(q) := \prod_{\sigma \in V_F} \frac{1}{q^{\widehat{\sigma}} - 1}, \quad \beta_F(f) := \frac{1}{\text{sym}(F)} \prod_{\sigma \in V_F} k_F(\sigma) a_{N_F(\sigma)}, \quad (14)$$

where we emphasize that the first number depends only on the multiplier q , while the second depends only on the tangent-to-identity part f of the formal diffeomorphism g (observe that none of them depends on the chosen representative of F , so the abuse of language of Remark 3.4 is innocuous here—*e.g.* in the example of Figure 2, $S^T(q) = 1/(q^{40} - 1)(q^7 - 1)^2(q^{18} - 1)(q^6 - 1)$ and $\beta_T(f) = \frac{1}{2} \cdot \frac{9!}{6!} a_8 \cdot a_7^2 \cdot \frac{13!}{12!} a_{12} \cdot a_6$). Formula (12) is then equivalent to

$$h(z) = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} S^T(q) \beta_T(f) z^{\|T\|+1}. \quad (15)$$

Theorem A thus yields an explicit formula which achieves a clear separation between q -dependence and f -dependence for the solution of the formal linearization problem. We shall see later how easily this type of expansion lends itself to majorant series arguments.

In Écalle’s terminology, a function defined on the set $\mathcal{F}(\mathbb{N}^*)$, like $F \mapsto S^F(q)$ for a fixed value of q , is called an “arbomould” (all the arbomoulds in this article will be scalar functions, but one could as well consider functions taking their values in rings more general than \mathbb{C}). Dually, the coefficients $\beta_F(f)$ stem from an operator-valued map $F \in \mathcal{F}(\mathbb{N}^*) \mapsto D_F$ which depends on f and is called “coarbomould” (see Definition 5.3); formula (15) will appear as a kind of projection of an identity between operators (formula (24)) involving an “arbomould-coarbomould contraction”, identity whose proof in the end amounts to little more than a few lines of computation.

Remark 4.2. We mentioned in Remark 3.7 that \mathcal{N} -trees (and thus \mathcal{N} -forests) can be viewed as generalizations of words on the alphabet \mathcal{N} , arbomoulds can correspondingly be considered as generalizations of “moulds”, *i.e.* scalar functions on the set of words (see *e.g.* [Sau09] for an introduction to mould calculus). The arbomould-coarbomould contraction (24) that we just alluded to appears in [Eca92] as a refinement of a “mould-comould contraction”; Écalle passes from the latter to the former by the “arborification” process, which we will not use in this article—the reader is referred to [Eca92] or [FM14].

5 Operator formulation

In fact we shall prove more than Theorem A: we shall give an explicit formula for the operator

$$\varphi \in \mathbb{C}[[z]] \mapsto \varphi \circ h \in \mathbb{C}[[z]]$$

from which (12) will follow by choosing $\varphi(z) = z$.

5.1 Composition operators

The *order* of a non-zero formal series $\varphi(z) = \sum_{n \geq 0} \alpha_n z^n \in \mathbb{C}[[z]]$ is defined as the least integer n such that $\alpha_n \neq 0$ and denoted by $\text{ord } \varphi$, while by convention $\text{ord } 0 = \infty$. For each $k \in \mathbb{N}$, the ideal formed by all formal series of order $\geq k$ is denoted by $z^k \mathbb{C}[[z]]$.

²The value of an empty product is 1 by convention, thus $S^\varepsilon(q) = \beta_\varepsilon(f) = 1$.

We call *operators* the \mathbb{C} -linear endomorphisms of $\mathbb{C}[[z]]$ and denote by $\text{End}_{\mathbb{C}} \mathbb{C}[[z]]$ the space they form. An operator Θ is said to be *tangent-to-identity* if $\Theta - \text{Id}$ increases order by at least one unit, *i.e.* $\text{ord}(\Theta\varphi - \varphi) \geq \text{ord} \varphi + 1$ for all $\varphi \in \mathbb{C}[[z]]$. An operator Θ is said to be an *algebra endomorphism* if $\Theta 1 = 1$ and $\Theta(\varphi\psi) = (\Theta\varphi)(\Theta\psi)$ for all $\varphi, \psi \in \mathbb{C}[[z]]$. Given any $v \in z\mathbb{C}[[z]]$ (*i.e.* any power series without constant term), its *composition operator* is defined to be

$$C_v: \varphi \in \mathbb{C}[[z]] \mapsto \varphi \circ v \in \mathbb{C}[[z]]. \quad (16)$$

Composition operators clearly are examples of algebra endomorphisms of $\mathbb{C}[[z]]$; in fact these are the only ones:

Lemma 5.1. *The map $v \mapsto C_v$ is a bijection between $z\mathbb{C}[[z]]$ and the set of all algebra endomorphisms Θ of $\mathbb{C}[[z]]$, whose inverse is $\Theta \mapsto v = \Theta z$ and which satisfies $C_v \circ C_w = C_{w \circ v}$ for all $v, w \in z\mathbb{C}[[z]]$. Moreover,*

$$v \in \tilde{\mathcal{G}} \iff C_v \text{ algebra automorphism of } \mathbb{C}[[z]],$$

$$v \in \tilde{\mathcal{G}}_1 \iff C_v \text{ tangent-to-identity algebra automorphism of } \mathbb{C}[[z]].$$

Proof. Let Θ be an algebra endomorphism of $\mathbb{C}[[z]]$. We content ourselves with explaining why $v := \Theta z$ has no constant term and why $\Theta = C_v$. The first point stems from the fact that

$$z\mathbb{C}[[z]] = \{ \psi \in \mathbb{C}[[z]] \mid \forall \alpha \in \mathbb{C}^*, \alpha + \psi \text{ has a multiplicative inverse in } \mathbb{C}[[z]] \}.$$

Indeed, for any $\alpha \in \mathbb{C}^*$, since $\alpha + z$ admits a multiplicative inverse in $\mathbb{C}[[z]]$, so does its image by Θ , which is $\alpha + v$, hence $v \in z\mathbb{C}[[z]]$.

Clearly, the action of Θ on polynomials coincides with that of C_v . For an arbitrary $\varphi \in \mathbb{C}[[z]]$, we show that $\Theta\varphi = C_v\varphi$ as follows: for every $N \in \mathbb{N}$ we can find a polynomial P_N and a formal series χ_N such that $\varphi = P_N + z^N\chi_N$, this implies $\Theta\varphi - C_v\varphi = v^N(\Theta\chi_N - C_v\chi_N)$, hence $\Theta\varphi - C_v\varphi \in z^N\mathbb{C}[[z]]$, but the only way for this to hold for every N is that $\Theta\varphi - C_v\varphi = 0$.

The other statements are left as an exercise. \square

Remark 5.2. By a similar argument one can prove that the *derivations* of $\mathbb{C}[[z]]$ (*i.e.* the operators D such that $D(\varphi\psi) = (D\varphi)\psi + \varphi(D\psi)$ for all $\varphi, \psi \in \mathbb{C}[[z]]$) are all of the form $D = u \frac{d}{dz}$ with arbitrary $u \in \mathbb{C}[[z]]$ (which is obtained from D by $u = Dz$).

Given $q \in \mathbb{C}^*$ not a root of unity and $f \in \tilde{\mathcal{G}}_1$, we are looking for a formal linearization of $g = R_q \circ f$, *i.e.* for $h \in \tilde{\mathcal{G}}_1$ such that $f \circ h = R_q^{\circ(-1)} \circ h \circ R_q$. According to Lemma 5.1, *it is equivalent to look for a tangent-to-identity algebra automorphism Θ such that*

$$\Theta \circ C_f = C_{R_q} \circ \Theta \circ C_{R_q^{\circ(-1)}}. \quad (17)$$

Indeed, one goes from the solution h to Θ by the relation $\Theta = C_h$ and, vice versa from Θ to h by $h = \Theta z$. The advantage of Equation (17) is that it is a linear equation, the solution of which can be sought in a huge linear space, namely $\text{End}_{\mathbb{C}} \mathbb{C}[[z]]$.

The idea of looking for the composition operator of h rather than h itself is reminiscent of the classical Lagrange reversion formula: given $f(z) = z + u(z)$ with $u(z) \in z^2\mathbb{C}[[z]]$, so that the Taylor formula yields

$$C_f = C_{\text{id}+u} = \text{Id} + \sum_{d \geq 1} \frac{1}{d!} u^d \partial^d \quad \text{where } \partial := \frac{d}{dz}, \quad (18)$$

not only do we know that

$$f^{\circ(-1)}(z) = (\text{id} + u)^{\circ(-1)}(z) = z + \sum_{d \geq 1} \frac{(-1)^d}{d!} \partial^{d-1}(u^d), \quad (19)$$

but in fact there is also a closed formula for the corresponding composition operator:

$$C_{f^{\circ(-1)}} = \text{Id} + \sum_{d \geq 1} \frac{(-1)^d}{d!} \partial^{d-1} \circ (u^d \partial). \quad (20)$$

Similarly, we shall obtain a closed formula for C_h and formula (12) for h will follow by letting C_h act on z .

Notice that the right-hand side of (19) is an infinite series of formal series, which is convergent for the topology of the *formal convergence*. This simply means that, given $n \in \mathbb{N}$, only finitely many summands contribute to the coefficient of z^n (similarly to (12)). A simple criterium for the formal convergence of a series of formal series $\sum \varphi_d$ is that $\text{ord } \varphi_d$ should tend to infinity as $d \rightarrow \infty$. Here $\text{ord } \partial^{d-1}(u^d) \geq d + 1$ because $\text{ord } u \geq 2$.

Similarly, the right-hand sides of (18) and (20) must be considered as *formally convergent series of operators*, in the sense that, when evaluated on a formal series φ , they yield formally convergent series of formal series; indeed, both $u^d \partial^d \varphi$ and $\partial^{d-1}(u^d \partial \varphi)$ have order $\geq \text{ord } \varphi + d$.

5.2 The coarbomould associated with $f \in \tilde{\mathcal{G}}_1$ and the contraction of an arbomould

Given a non-empty set \mathcal{N} , we call *arbomould* any map $A^\bullet: \mathcal{F}(\mathcal{N}) \rightarrow \mathbb{C}$ and *coarbomould* any map $B_\bullet: \mathcal{F}(\mathcal{N}) \rightarrow \text{End}_{\mathbb{C}} \mathbb{C}[[z]]$. Here the big dots represent the arguments which the arbomould or the coarbomould may take, and it is customary to denote their value on an \mathcal{N} -forest F by A^F or B_F . From now, we take $\mathcal{N} = \mathbb{N}^*$.

Definition 5.3. Given $f \in \tilde{\mathcal{G}}_1$, we define the *coarbomould* $D_\bullet(f)$ associated with f by the formula

$$D_F(f) := \beta_F(f) z^{\|F\| + \deg(F)} \partial^{\deg(F)} \quad \text{for each } F \in \mathcal{F}(\mathbb{N}^*), \quad (21)$$

with $\beta_F(f) \in \mathbb{C}$ as in (14).

Remark 5.4. The coefficients $\beta_F(f)$ still look mysterious at this stage. Their true origin will be uncovered later in Proposition 6.7 (a)–(b).

We will often omit the explicit dependence on f and simply write D_\bullet , D_F or β_F when f is clear from the context.

Notice that $D_\varepsilon = \text{Id}$ and that D_T is a derivation for any \mathcal{N} -tree T . For any $F \in \mathcal{F}(\mathbb{N}^*)$, the operator D_F is homogeneous of degree $\|F\|$, in the sense that it maps $\mathbb{C}z^k$ to $\mathbb{C}z^{k + \|F\|}$. Since there are only finitely many \mathbb{N}^* -forests with given weight, this implies that, for any arbomould A^\bullet , the series of operators $\sum_{F \in \mathcal{F}(\mathbb{N}^*)} A^F D_F$ is formally convergent: when evaluated on a formal series φ , it yields a formally convergent series³ of formal series, because $\text{ord}(A^F D_F \varphi) \geq \text{ord } \varphi + \|F\|$. We can thus define an operator

$$\sum A^\bullet D_\bullet := \sum_{F \in \mathcal{F}(\mathbb{N}^*)} A^F D_F, \quad (22)$$

³Strictly speaking, since we have not chosen any specific bijection from \mathbb{N} to $\mathcal{F}(\mathbb{N}^*)$, it is rather the sum of a “summable family”—which is meaningful since the formal topology can be induced by a distance which makes $\mathbb{C}[[z]]$ a complete metric space and a topological ring (see [Sau09]).

called *the contraction of the arbomould A^\bullet in the coarbomould D_\bullet* (or simply *the contraction of A^\bullet*).

Example 5.5. The simplest possible example is $\sum 1^\bullet D_\bullet = \text{Id}$, where the *unit arbomould 1^\bullet* is defined by $1^F = 1$ if $F = \varepsilon$ and $1^F = 0$ else.

Example 5.6. Let us define the arbomoulds I^\bullet , J^\bullet and K^\bullet by

- (i) $I^F = 1$ if any two distinct elements of (V_F, \preceq_F) are incomparable and $I^F = 0$ else,
- (ii) $J^F = (-1)^{\#F}$,
- (iii) $K^F = 1$.

Then

$$\sum I^\bullet D_\bullet = C_f, \quad \sum J^\bullet D_\bullet = C_{f \circ (-1)}, \quad \sum K^\bullet D_\bullet = C_{(\text{id} - u) \circ (-1)}, \quad (23)$$

with the notation $f = \text{id} + u$, *i.e.* $u(z) := \sum a_n z^{n+1}$ (hence $z - u(z) = 2z - f(z)$). Very simple proofs of these identities will be given in Section 6.2, with the help of the concept of “separativity” (the first identity can also be proved by a direct computation from (18) with a little combinatorial argument).

6 Tree-expansion for the composition operator of the solution — Theorem A’

Here is the closed formula for the composition operator of the formal linearization h that was alluded to at the beginning of Section 5:

Theorem A’. *Let $f \in \tilde{\mathcal{G}}_1$ and $q \in \mathbb{C}^*$, and suppose that q is not a root of unity. Let D_\bullet denote the coarbomould associated with f and let $S^\bullet(q)$ denote the arbomould defined in (14). Then the composition operator of the formal linearization (6) of $g = R_q \circ f$ coincides with the contraction of $S^\bullet(q)$ in D_\bullet :*

$$C_h = \sum S^\bullet(q) D_\bullet. \quad (24)$$

Theorem A’ implies Theorem A. Observe that, according to (21),

$$T \in \mathcal{T}(\mathbb{N}^*) \implies D_T z = \beta_T z^{\|T\|+1}, \quad \varepsilon \neq F \in \mathcal{F}(\mathbb{N}^*) \setminus \mathcal{T}(\mathbb{N}^*) \implies D_F z = 0, \quad (25)$$

thus (24) implies that $h = C_h z = \sum S^\bullet(q) D_\bullet z$ is given by (15). \square

The rest of this section is dedicated to the proof of Theorem A’. It will require a description of some natural structures available in $\mathcal{F}(\mathcal{N})$, and then the definition of a class of arbomoulds whose contractions are tangent-to-identity algebra automorphisms, before the actual proof of (24) which takes few lines by itself.

6.1 The set of \mathcal{N} -forests as a free commutative monoid

Let \mathcal{N} be a non-empty set.

Definition 6.1. For $F_1, F_2 \in \mathcal{F}(\mathcal{N})$, the *disjoint union* of F_1 and F_2 , denoted by $F_1 F_2$, is defined as follows: choose any \mathcal{N} -arborescent posets (V_1, \preceq_1, N_1) and (V_2, \preceq_2, N_2) representing F_1 and F_2 such that $V_1 \cap V_2$ is empty and form their disjoint union (V, \preceq, N) (*i.e.* $V := V_1 \cup V_2$, V_1 and V_2 are incomparable for \preceq and the restriction to V_i of \preceq , resp. N , is \preceq_i , resp. N_i); (V, \preceq, N) clearly is an \mathcal{N} -arborescent poset and its isomorphism class $F_1 F_2$ depends only on F_1 and F_2 .

We thus obtain a commutative monoid law

$$(F_1, F_2) \in \mathcal{F}(\mathcal{N}) \times \mathcal{F}(\mathcal{N}) \mapsto F_1 F_2 \in \mathcal{F}(\mathcal{N})$$

for which ε is a unit element. In fact, it is easily seen that $\mathcal{F}(\mathcal{N})$ is the free commutative monoid on $\mathcal{T}(\mathcal{N})$, denoted by

$$\mathcal{F}(\mathcal{N}) = \text{MSET}(\mathcal{T}(\mathcal{N})), \quad (26)$$

i.e. any non-empty forest F can be written in a unique way as a product $\prod T^{d(T)}$ over all \mathcal{N} -trees T , with a finitely supported function $d: \mathcal{T}(\mathcal{N}) \rightarrow \mathbb{N}$; we usually will omit the trivial factors corresponding to \mathcal{N} -trees outside the support of d and rather write

$$F = T_1^{d_1} \cdots T_r^{d_r}, \quad T_1, \dots, T_r \in \mathcal{T}(\mathcal{N}) \text{ pairwise distinct, } d_1, \dots, d_r \in \mathbb{N}^*, \quad (27)$$

a decomposition which is unique up to a permutation of the pairs (T_i, d_i) . Notice that $\deg(F) = d_1 + \cdots + d_r$, the sum of the multiplicities.⁴

Definition 6.2. Let $n \in \mathcal{N}$ and $F \in \mathcal{F}(\mathcal{N})$. We define an \mathcal{N} -tree, denoted by $n \triangleleft F$, which is said to be obtained by *attaching n to F* , as follows: choose any representative (V_F, \preceq_F, N_F) of F , choose any one-element set $\{\rho\}$ disjoint from V_F , consider the ordinal sum (V, \preceq) of $\{\rho\}$ and (V_F, \preceq_F) (*i.e.* $V := \{\rho\} \cup V_F$, the restriction of \preceq to V_F is \preceq_F and $\rho \preceq \sigma$ for every $\sigma \in V_F$) and define $N: V \rightarrow \mathcal{N}$ as the extension of N_F such that $N(\rho) = n$; (V, \preceq, N) clearly is an \mathcal{N} -arborescent poset with a unique minimal element and its isomorphism class $n \triangleleft F$ depends only on n and F .

The map thus defined

$$\begin{aligned} \mathcal{N} \times \mathcal{F}(\mathcal{N}) &\rightarrow \mathcal{T}(\mathcal{N}) \\ (n, F) &\mapsto n \triangleleft F \end{aligned} \quad (28)$$

is a bijection⁵, with inverse $T \in \mathcal{T}(\mathcal{N}) \mapsto (N_T(\rho_T), \text{For}^+(\rho_T, T))$ (notations of the end of Section 3).

Lemma 6.3. For any non-empty forest $T_1^{d_1} \cdots T_r^{d_r}$ decomposed as in (27),

$$\text{sym}(T_1^{d_1} \cdots T_r^{d_r}) = d_1! \cdots d_r! (\text{sym}(T_1))^{d_1} \cdots (\text{sym}(T_r))^{d_r}. \quad (29)$$

For any $n \in \mathcal{N}$ and $F \in \mathcal{F}(\mathcal{N})$,

$$\text{sym}(n \triangleleft F) = \text{sym}(F). \quad (30)$$

⁴We use the notation MSET because an element of the free commutative monoid on a set A can be viewed as a “multiset”, *i.e.* a finite subset of A possibly with “repetitions”: each element a has a multiplicity $d_a \geq 1$.

⁵Note that (26) and (28) can be used as a recursive definition of $\mathcal{F}(\mathcal{N})$ and $\mathcal{T}(\mathcal{N})$. More precisely, we then obtain filtrations by height: $\mathcal{F}(\mathcal{N}) = \bigcup_{\ell \geq 0} \mathcal{F}_{\leq \ell}(\mathcal{N})$ and $\mathcal{T}(\mathcal{N}) = \bigcup_{\ell \geq 1} \mathcal{T}_{\leq \ell}(\mathcal{N})$, where $\mathcal{F}_{\leq 0}(\mathcal{N}) := \{\varepsilon\}$ and, for $\ell \geq 1$, $\mathcal{T}_{\leq \ell}(\mathcal{N}) := \{n \triangleleft F \mid n \in \mathcal{N}, F \in \mathcal{F}_{\leq \ell-1}(\mathcal{N})\}$, $\mathcal{F}_{\leq \ell}(\mathcal{N}) := \text{MSET}(\mathcal{T}_{\leq \ell}(\mathcal{N}))$; the latter set coincides with the set of all \mathcal{N} -forests of height $\leq \ell$. For example, the \mathbb{N}^* -tree of height 3 of Figure 2 is $8 \triangleleft ((7 \triangleleft \varepsilon)^2 (12 \triangleleft (6 \triangleleft \varepsilon)))$.

Proof. Let us choose representatives (V_i, \preceq_i, N_i) for the T_i 's with pairwise distinct underlying finite sets and consider, for each i , d_i disjoint copies of (V_i, \preceq_i, N_i) , say

$$(V_i \times \{1\}, \preceq_i, N_i), \dots, (V_i \times \{d_i\}, \preceq_i, N_i), \quad (31)$$

so that we can represent $F = T_1^{d_1} \cdots T_r^{d_r}$ by the disjoint union (V_F, \preceq_F, N_F) of these $d_1 + \cdots + d_r$ \mathcal{N} -arborescent posets. Denoting by \mathfrak{S}_d the permutation group of $\{1, \dots, d\}$ for any $d \in \mathbb{N}^*$, we see that we can define a bijective map

$$\text{Aut}(V_1, \preceq_1, N_1)^{d_1} \times \cdots \times \text{Aut}(V_r, \preceq_r, N_r)^{d_r} \times \mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_r} \rightarrow \text{Aut}(V_F, \preceq_F, N_F),$$

because an automorphism of (V_F, \preceq_F, N_F) must send a subset $V_i \times \{m\}$ of V_F belonging to the i th collection (31) (thus with $1 \leq m \leq d_i$) onto a subset of the same collection, thus inducing an automorphism of (V_i, \preceq_i, N_i) and a permutation of $\{1, \dots, d_i\}$, and (29) follows. The construction of a bijection which yields (30) is obvious. \square

Lemma 6.4. *The coarbomould B_\bullet defined by*

$$B_\varepsilon := \text{Id} \quad \text{and} \quad B_F := \frac{1}{d_1! \cdots d_r!} \partial^{d_1 + \cdots + d_r} \quad \text{for } F = T_1^{d_1} \cdots T_r^{d_r} \text{ as in (27)} \quad (32)$$

satisfies

$$B_F(\varphi\psi) = \sum_{\substack{(F', F'') \in \mathcal{F}(\mathcal{N}) \times \mathcal{F}(\mathcal{N}) \\ \text{such that } F = F'F''}} (B_{F'}\varphi)(B_{F''}\psi) \quad (33)$$

for any $F \in \mathcal{F}(\mathcal{N})$ and $\varphi, \psi \in \mathbb{C}[[z]]$.

Proof. We proceed by induction on the number r of distinct \mathcal{N} -trees which appear in the decomposition (27) of F . The formula obviously holds for $r = 0$, i.e. $F = \varepsilon$. Let $F = GT^d$, where $G \in \mathcal{F}(\mathcal{N})$, $d \geq 1$ and T is an \mathcal{N} -tree distinct from any \mathcal{N} -tree which appears in the decomposition of G (a requirement that we omit if $G = \varepsilon$). Assuming that (33) holds for B_G and writing $B_F = B_G \circ \frac{\partial^d}{d!}$, we get from the Leibniz rule

$$B_F(\varphi\psi) = \sum_{\substack{(G', G'') \in \mathcal{F}(\mathcal{N}) \times \mathcal{F}(\mathcal{N}) \\ \text{such that } G = G'G''}} \sum_{\substack{(d', d'') \in \mathbb{N} \times \mathbb{N} \\ \text{such that } d = d' + d''}} \frac{1}{d'!d''!} (B_{G'}\partial^{d'}\varphi)(B_{G''}\partial^{d''}\psi),$$

which can be recognized as a summation over all pairs (F', F'') such that $F = F'F''$ (setting $F' = G'T^{d'}$ and $F'' = G''T^{d''}$) since we have assumed that T does not belong to the list of factors of G , and the summands can then be written $(B_{F'}\varphi)(B_{F''}\psi)$ (because T is not a factor of G' nor of G'') hence (33) holds for B_F . \square

Remark 6.5. See [FM14] for more on the structures available on the monoid algebra of $\mathcal{F}(\mathcal{N})$ over \mathbb{C} , $\mathbb{C}\mathcal{F}(\mathcal{N})$, and its relationship with the Connes-Kreimer Hopf algebra.

6.2 Separative arbomoulds and their contractions

Definition 6.6. An arbomould $A^\bullet: \mathcal{F}(\mathcal{N}) \rightarrow \mathbb{C}$ is called *separative* if it is a monoid homomorphism when considering the disjoint union of \mathcal{N} -forests at the source and the multiplication of complex numbers at the target, i.e. if

$$A^\varepsilon = 1 \quad \text{and} \quad A^{F'F''} = A^{F'}A^{F''} \quad \text{for every } F', F'' \in \mathcal{F}(\mathcal{N}). \quad (34)$$

Observe that, because of (26), the restriction to $\mathcal{T}(\mathcal{N})$ induces a one-to-one correspondence between separative arbomoulds and arbitrary functions on $\mathcal{T}(\mathcal{N})$.

From now on we take $\mathcal{N} = \mathbb{N}^*$. We have already encountered five separative arbomoulds: 1^\bullet , I^\bullet , J^\bullet , K^\bullet in Section 5.2, and $S^\bullet(q)$ in the statement of Theorem A'. The separativity of the first four is obvious, the separativity of the fifth one is easy too and will be crucial to the proof of Theorem A'. In fact, everything relies on the following properties of the coarbomoulds $D_\bullet(f)$ and of separative arbomoulds:

Proposition 6.7. *The coarbomould $D_\bullet := D_\bullet(f)$ of any $f \in \tilde{\mathcal{G}}_1$ satisfies*

(a) $D_{n \triangleleft F} = (D_F(a_n z^{n+1}))\partial$ for every $n \in \mathbb{N}^*$ and $F \in \mathcal{F}(\mathbb{N}^*)$,

(b) $D_F = \frac{1}{d_1! \cdots d_r!} (D_{T_1} z)^{d_1} \cdots (D_{T_r} z)^{d_r} \partial^{d_1 + \cdots + d_r}$ for $F = T_1^{d_1} \cdots T_r^{d_r} \in \mathcal{F}(\mathbb{N}^*)$ as in (27),

(c) $D_F(\varphi\psi) = \sum_{\substack{(F', F'') \in \mathcal{F}(\mathcal{N}) \times \mathcal{F}(\mathcal{N}) \\ \text{such that } F = F'F''}} (D_{F'}\varphi)(D_{F''}\psi)$ for any $F \in \mathcal{F}(\mathcal{N})$ and $\varphi, \psi \in \mathbb{C}[[z]]$.

Proposition 6.8. *The contraction of any separative arbomould in the coarbomould D_\bullet is a tangent-to-identity algebra automorphism of $\mathbb{C}[[z]]$. In fact, if A^\bullet is a separative arbomould, then $\sum A^\bullet D_\bullet = C_\theta$ with $\theta \in \tilde{\mathcal{G}}_1$ defined by*

$$\theta(z) = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} A^T D_T z = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} A^T \beta_T z^{\|T\|+1}. \quad (35)$$

Proof of Proposition 6.7. (a) Let $d := \deg(F)$. According to (14), using the fact that $k_{n \triangleleft F}(\sigma) = k_F(\sigma)$ for $\sigma \in V_F$, we can write $\beta_{n \triangleleft F} = \frac{1}{\text{sym}(n \triangleleft F)} k^* a_n \prod_{\sigma \in V_F} k_F(\sigma) a_{N_F(\sigma)}$ with

$$k^* := k_{n \triangleleft F}(\rho_{n \triangleleft F}), \quad \text{i.e.} \quad k^* = \frac{(n+1)!}{(n+1-d)!} \text{ if } n+1 \geq d, \quad k^* = 0 \text{ else.}$$

By (30), we get $\beta_{n \triangleleft F} = k^* a_n \beta_F$, hence

$$D_{n \triangleleft F} = k^* a_n \beta_F z^{\|F\|+n+1} \partial.$$

On the other hand, $D_F(a_n z^{n+1}) = \beta_F z^{\|F\|+d} \partial^d (a_n z^{n+1})$. The conclusion stems from the identity $z^d \partial^d (z^{n+1}) = k^* z^{n+1}$.

(b) For $F = T_1^{d_1} \cdots T_r^{d_r}$ decomposed as in (27), the second part of (14) gives

$$\beta_F = \frac{1}{\text{sym}(F)} Q \quad \text{with} \quad Q = \prod_{\sigma \in V_F} k_F(\sigma) a_{N_F(\sigma)}, \quad \beta_{T_i} = \frac{1}{\text{sym}(T_i)} P_i \quad \text{with} \quad P_i = \prod_{\tau \in V_{T_i}} k_{T_i}(\tau) a_{N_{T_i}(\tau)}.$$

In view of (11), $\tau \in V_{T_i} \implies k_{T_i}(\tau) = k_F(\tau)$, hence $Q = P_1^{d_1} \cdots P_r^{d_r}$ and (29) yields

$$\beta_F = \frac{P_1^{d_1} \cdots P_r^{d_r}}{d_1! \cdots d_r! (\text{sym}(T_1))^{d_1} \cdots (\text{sym}(T_r))^{d_r}} = \frac{1}{d_1! \cdots d_r!} (\beta_{T_1})^{d_1} \cdots (\beta_{T_r})^{d_r}.$$

With the help of the coarbomould B_\bullet defined by (32), Definition 5.3 can thus be rewritten $D_F = M_F B_F$, with $M_F := (\beta_{T_1} z^{\|T_1\|+1})^{d_1} \cdots (\beta_{T_r} z^{\|T_r\|+1})^{d_r} = (D_{T_1} z)^{d_1} \cdots (D_{T_r} z)^{d_r}$.

(c) Use the property $M_{F'F''} = M_{F'} M_{F''}$ and (33). □

Proof of Proposition 6.8. Let A^\bullet be a separative arbomould, $\Theta := \sum A^\bullet D_\bullet$, and $\varphi, \psi \in \mathbb{C}[[z]]$. For each $F \in \mathcal{F}(\mathbb{N}^*)$, putting together (34) and Proposition 6.7.c we get

$$A^F D_F(\varphi\psi) = \sum_{\substack{(F', F'') \in \mathcal{F}(\mathbb{N}) \times \mathcal{F}(\mathbb{N}) \\ \text{such that } F = F'F''}} (A^{F'} D_{F'}\varphi)(A^{F''} D_{F''}\psi).$$

With a Fubini-like manipulation for series (granted by the standard properties of the topology of formal convergence), this implies that $\Theta(\varphi\psi) = (\Theta\varphi)(\Theta\psi)$. On the other hand $\Theta 1 = 1$ because $A^\varepsilon = 1$, thus Θ is an algebra automorphism.

From (25) we see that Θz coincides with the formal series θ defined by (35), whence $\Theta = C_\theta$ by Lemma 5.1. \square

Note that points (a) and (b) of Proposition 6.7, together with $D_\varepsilon = \text{Id}$, provide a recursive definition⁶ of the coarbomould D_\bullet . The first point is particularly interesting when using (35), because one may parametrize the set of all \mathbb{N}^* -trees by $(n, F) \in \mathbb{N}^* \times \mathcal{F}(\mathbb{N}^*)$, by means of the bijection (28).

Remark 6.9. Since a derivation of $\mathbb{C}[[z]]$ is determined by its action on z and we know that $D_\varepsilon = \text{Id}$, point (a) of Proposition 6.7 is equivalent to

$$D_{n \triangleleft F} \text{ is a derivation, } \quad D_{n \triangleleft \varepsilon} = a_n z^{n+1} \partial, \quad D_{n \triangleleft F} z = D_F D_{n \triangleleft \varepsilon} z$$

for all $n \in \mathbb{N}^*$ and $F \in \mathcal{F}(\mathbb{N}^*)$.

Before moving on to the proof that the contraction of $S^\bullet(q)$ yields the composition operator we search for, let us illustrate the previous concepts by proving (23).

- (i) Since I^\bullet is separative, by Proposition 6.8 we know that $\Theta_I := \sum I^\bullet D_\bullet$ coincides with C_{f^*} , where $f^*(z) := z + \sum I^T D_T z$ and the summation is over all \mathbb{N}^* -trees T . But $I^T = 1$ if T is of the form $n \triangleleft \varepsilon$ and $I^T = 0$ else, therefore $f^*(z) = z + \sum_{n \in \mathbb{N}^*} D_{n \triangleleft \varepsilon} z = z + \sum a_n z^{n+1}$ by Proposition 6.7.a, *i.e.* $f^* = f$.
- (ii) Since J^\bullet is separative, we know that $\Theta_J := \sum J^\bullet D_\bullet$ coincides with C_v , where $v(z) := z + \sum J^T D_T z$. Using the bijection (28), the relation $J^{n \triangleleft F} = -J^F$ and Proposition 6.7.a, we get

$$v(z) = z + \sum_{F \in \mathcal{F}(\mathbb{N}^*)} \sum_{n \in \mathbb{N}^*} J^{n \triangleleft F} D_{n \triangleleft F} z = z - \sum_{F \in \mathcal{F}(\mathbb{N}^*)} \sum_{n \in \mathbb{N}^*} J^F D_F (a_n z^{n+1}).$$

The last term is nothing but the action of the operator Θ_J on the formal series $-\sum a_n z^{n+1} = z - f(z)$, hence $v(z) = z + C_v(z - f) = z + v(z) - f \circ v(z)$, which entails $f \circ v(z) = z$, *i.e.* $v = f^{\circ(-1)}$.

- (iii) Since K^\bullet is separative, we know that $\Theta_K := \sum K^\bullet D_\bullet$ coincides with C_w , where $w(z) := z + \sum_T K^T D_T z = z + \sum_F \sum_n D_{n \triangleleft F} z = z + \sum_F \sum_n D_F (a_n z^{n+1}) = z + \Theta_K u$, *i.e.* $w = \text{id} + u \circ w$, which yields $(\text{id} - u) \circ w = \text{id}$, hence $w = (\text{id} - u)^{\circ(-1)}$.

⁶Using the recursive definition of $\mathcal{F}(\mathbb{N}^*)$ and $\mathcal{T}(\mathbb{N}^*)$ based on (26) and (28) explained in footnote 5. Notice that, although for a given $F \in \mathcal{F}(\mathbb{N}^*) \setminus \mathcal{T}(\mathbb{N}^*)$ the pairs (T_i, d_i) are only defined up to permutation, the right-hand side of point (b) can be used as a definition of D_F because it is invariant under permutation. When $F \in \mathcal{T}(\mathbb{N}^*)$, point (b) says that $D_F = (D_F z) \partial$, *i.e.* merely that D_F is a derivation, which can be defined by point (a) in terms of $N_F(\rho_F)$ and $D_{\text{For}^+(\rho_F, F)}$ (notations of the end of Section 3).

6.3 Proof of Theorem A'.

Let $\Theta := \sum S^\bullet(q)D_\bullet$. Since $S^\bullet(q)$ is separative, by Proposition 6.8 we have $\Theta = C_{h^*}$ with

$$h^*(z) := \Theta z = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} S^T(q)D_T z \in \tilde{\mathcal{G}}_1. \quad (36)$$

We want to show that this h^* is the solution of the formal linearization problem, *i.e.* that $f \circ h^* = R_q^{\circ(-1)} \circ h^* \circ R_q$.

Lemma 6.10. *If $n \in \mathbb{N}$ and $\varphi(z)$ is a constant multiple of z^{n+1} , then $R_q^{\circ(-1)} \circ \varphi \circ R_q = q^n \varphi$.*

Proof. Obvious. □

Since each $D_T z \in \mathbb{C}z^{\|T\|+1}$, we thus have

$$R_q^{\circ(-1)} \circ h^* \circ R_q(z) = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} q^{\|T\|} S^T(q)D_T z. \quad (37)$$

On the other hand,

$$f \circ h^* = \Theta f = \Theta z + \sum_{n \in \mathbb{N}^*} \Theta(a_n z^{n+1}) = \Theta z + \sum_{F \in \mathcal{F}(\mathbb{N}^*)} \sum_{n \in \mathbb{N}^*} S^F(q)D_F(a_n z^{n+1}).$$

But $D_F(a_n z^{n+1}) = D_{n \triangleleft F} z$ by Proposition 6.7.a, and (14) entails

$$S^F(q) = (q^{n+\|F\|} - 1)S^{n \triangleleft F}(q), \quad (38)$$

thus

$$f \circ h^* = \Theta z - \sum_{F \in \mathcal{F}(\mathbb{N}^*)} \sum_{n \in \mathbb{N}^*} S^{n \triangleleft F}(q)D_{n \triangleleft F} z + \sum_{F \in \mathcal{F}(\mathbb{N}^*)} \sum_{n \in \mathbb{N}^*} q^{n+\|F\|} S^{n \triangleleft F}(q)D_{n \triangleleft F} z.$$

The bijection (28) allows us to rewrite the summations in the last two terms as summations over $T \in \mathcal{T}(\mathbb{N}^*)$; then the first two terms of the right-hand side yield only z (because of (36)) and we end up with

$$f \circ h^* = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} q^{\|T\|} S^T(q)D_T z.$$

Comparing with (37) we conclude that $f \circ h^* = R_q^{\circ(-1)} \circ h^* \circ R_q$.

7 Convergence of the formal linearization — Theorem B

7.1 Solution of the Holomorphic Linearization Problem

We move on to the Holomorphic Linearization Problem described in Section 2: we want to show how the explicit tree-representation (15) that we have obtained for the formal linearization h allows one to recover certain classical results on the radius of convergence of h when one starts from a convergent $g = R_q \circ f$, *i.e.* when one assumes $g \in \mathcal{G}$ (or, equivalently, $f \in \mathcal{G}_1$) and not only $g \in \tilde{\mathcal{G}}$.

We shall see that the analysis is elementary when the multiplier q has modulus different from 1. When $|q| = 1$, we have already seen the necessity of assuming that q is not a root of unity in the Formal Linearization Problem; it turns out that an arithmetical condition is needed for the Holomorphic Linearization Problem:

Definition 7.1. Given a complex number q of modulus 1, we define $\mathcal{B}(q) \in \mathbb{R}^+ \cup \{\infty\}$ as follows: if q is a root of unity, then we set $\mathcal{B}(q) := \infty$; if not, then we take any $\omega \in \mathbb{R} \setminus \mathbb{Q}$ such that $q = e^{2\pi i \omega}$ and consider the numerical series

$$\mathcal{B}(q) := \sum_{n \geq 0} \frac{\ln Q_{n+1}}{Q_n} \in \mathbb{R}^+ \cup \{\infty\}, \quad (39)$$

where $(Q_n)_{n \in \mathbb{N}}$ is the sequence of the denominators of the convergents of ω . If $\mathcal{B}(q) < \infty$, then we say that q satisfies the *Bruno condition* and write $q \in \mathcal{B}$.

Notice that the sequence $(Q_n)_{n \in \mathbb{N}}$ and thus the value of the series (39) depend only on q , not on ω , hence the notation $\mathcal{B}(q)$ is legitimate. For the definition of the convergents $\frac{P_n}{Q_n}$ of a real number and the theory of continued fractions, the reader is referred to [HW79] or [Khi64].

Remark 7.2. It is well known that, for any $\omega \in \mathbb{R} \setminus \mathbb{Q}$, the denominators of its convergents satisfy $Q_n \geq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ for all $n \geq 1$, hence the series $\sum \frac{\ln Q_n}{Q_n}$ is convergent and bounded by a universal constant. The idea is that, if an irrational ω is “abnormally well” approximated by rationals, and hence $q = e^{2\pi i \omega}$ is abnormally well approximated by roots of unity, then this is reflected in the growth of the sequence $(Q_n)_{n \in \mathbb{N}}$. The Bruno condition discards such a possibility: if $\mathcal{B}(q) < \infty$, then one can check that the roots of unity do not accumulate q “too fast”; this idea is made precise by Davie’s lemma (Lemma 7.6 below).

An example is provided by the Diophantine numbers: given $\tau \geq 2$, a real number ω is said to be Diophantine of exponent τ if there exists $M > 0$ such that $|\omega - \frac{P}{Q}| \geq \frac{1}{MQ^\tau}$ for every rational $\frac{P}{Q}$; this implies that the denominators of the convergents of ω satisfy $Q_{n+1} < MQ_n^{\tau-1}$ for all $n \geq 0$, hence $\mathcal{B}(e^{2\pi i \omega}) < \infty$. It is easy to see that, for $\tau > 2$, the set of all Diophantine numbers of exponent τ has full measure in \mathbb{R} . It follows that \mathcal{B} has full Haar measure in the unit circle.

The condition $\mathcal{B}(q) < \infty$ was first introduced by Bruno around 1965 (and published in [Bru71]).

We now gather the classical results on the convergence of h in

Theorem B. *Let $g \in \mathcal{G}$. If its multiplier q has modulus different from 1 or satisfies the Bruno condition, then the formal linearization h of g is convergent.*

More precisely, there exist universal constants $\kappa, \kappa' > 0$ such that, for any $R > 0$, for any $f(z) = z + \sum_{n \geq 1} a_n z^{n+1} \in \mathcal{G}_1$ such that

$$|a_n| \leq R^{-n}, \quad n \geq 1, \quad (40)$$

and for any $q \in \mathbb{C}^$,*

(i) if $|q| \neq 1$, then the radius of convergence of h is at least $\kappa R \min \left\{ 1, \left| |q| - 1 \right| \right\}$,

(ii) if $q \in \mathcal{B}$, then the radius of convergence of h is at least $\kappa' R e^{-\mathcal{B}(q)}$.

Our proof is in Section 7.2, based on the tree-representation (15) and on two propositions proved in Sections 9 and 10. We shall see that one can take $\kappa = 3 - \sqrt{8}$ and $\kappa' = \kappa e^{-\mu}$, where μ comes from Davie's lemma.

Remark 7.3. It is clear that, given $f \in \mathcal{G}_1$, there always exists $R > 0$ such that the inequalities (40) are satisfied. If $r > 0$ is small enough, then f is univalent in the disc $\{|z| < r\}$ and, by de Branges's theorem, one can take $R = r/2$.

The case (i) of Theorem B is the Koenigs linearization theorem, which dates back to 1884. It is easily obtained by a majorant method from the induction formulas (7). Here it will appear as a by-product of our analysis of the tree-representation (15).

The case (ii) of Theorem B, which is much less elementary, is the result of a long history. The difficulty is the so-called ‘‘small denominator problem’’: the induction formulas (7) contain in their denominators the expressions $q^n - 1$, $n \geq 1$, which can be arbitrarily close to 0 when $|q| = 1$, thus making dubious the convergence of h .

In fact, in 1927, H. Cremer showed that, for every q belonging to a certain dense G_δ subset of the unit circle, there exists $f \in \mathcal{G}_1$ such that the formal linearization h of $g = R_q \circ f$ is not convergent. It is only in 1942 that the first positive result for multipliers of modulus 1 came: C. L. Siegel [Sie42] showed that, if ω is Diophantine (*cf.* Remark 7.2), then the formal linearization of $R_{e^{2\pi i \omega}} \circ f$ is convergent for any $f \in \mathcal{G}_1$. Siegel's result was improved by A. D. Bruno, who showed that the weaker condition $\mathcal{B}(e^{2\pi i \omega}) < \infty$ is sufficient [Bru71].

The lower bound for the radius of convergence of h in the case (ii) of Theorem B is part of J.-C. Yoccoz's celebrated 1987 work published in [Yoc95] (Bruno's proof only affords a lower bound of the form $C_1 \mathcal{B}(q) + C_2$, with $C_1 = 2$ according to [He86]). Yoccoz also gave an upper bound for the infimum of the possible radii of convergence as follows: if $q = e^{2\pi i \omega}$ with $\omega \in \mathbb{R} \setminus \mathbb{Q}$, then there exists $f \in \mathcal{G}_1$ univalent in the unit disc such that the radius of convergence of the formal linearization of $R_{e^{2\pi i \omega}} \circ f$ is at most $C e^{-\mathcal{B}(q)}$, where $C > 0$ is a universal constant; in particular, one can have a divergent formal linearization whenever $q \notin \mathcal{B}$.

Yoccoz's proof is based on his novel renormalization theory for holomorphic germs. Later on, it was realized by T. Carletti and S. Marmi [CM00] that Yoccoz's lower bound can be recovered by a classical majorant method based on (7), exploiting the Bruno condition through Davie's lemma (Lemma 7.6 below).

7.2 Proof of Theorem B

Our proof Theorem B relies on three ingredients:

- the explicit formula (15) for h obtained in the first part of this article,
- a majorant series argument to bound the coefficients $\beta_T(f)$ involved in the coarbo mould associated with an $f \in \mathcal{G}_1$,
- direct estimates to bound the coefficients of the mould $S^\bullet(q)$.

Proposition 7.4. *Let $\kappa := 3 - \sqrt{8}$. Then, for any $R > 0$ and $f(z) = z + \sum_{n \geq 1} a_n z^{n+1} \in \mathcal{G}_1$ satisfying (40), the coefficients $\beta_T(f)$ defined by the second part of (14) satisfy*

$$\sum_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} |\beta_T(f)| \leq \kappa^{-n-1} R^{-n}, \quad n \in \mathbb{N}^*. \quad (41)$$

The proof of Proposition 7.4 is in Section 9.

Proposition 7.5. (i) *If $q \in \mathbb{C}$ has modulus different from 1, then*

$$|S^T(q)| \leq \left| \frac{1}{|q| - 1} \right|^{\#T}, \quad T \in \mathcal{T}(\mathbb{N}^*). \quad (42)$$

(ii) *There exists a universal constant $\mu > 0$ such that, for any $q \in \mathcal{B}$,*

$$|S^T(q)| \leq e^{(\mu + \mathcal{B}(q))\|T\|}, \quad T \in \mathcal{T}(\mathbb{N}^*). \quad (43)$$

The proof of Proposition 7.5 is in Section 10. The second part relies on Davie's lemma, which we quote here without proof:

Lemma 7.6. *There exist a universal constant $\mu > 0$ and, for each $q \in \mathcal{B}$, a positive real function K_q on \mathbb{N}^* such that*

(i) *for all $n \geq 1$,*

$$K_q(n) \leq (\mu + \mathcal{B}(q))n,$$

(ii) *for all $n_1, n_2 \geq 1$,*

$$K_q(n_1) + K_q(n_2) \leq K_q(n_1 + n_2),$$

(iii) *for all $n \geq 1$,*

$$\frac{1}{|q^n - 1|} \leq e^{K_q(n) - K_q(n-1)}$$

with the convention $K_q(0) = 0$.

For the proof of this lemma, see [Dav94] or [CM00]. We shall see in Section 10 that one can take the same μ in Proposition 7.5 as in Lemma 7.6.

Propositions 7.4 and 7.5 imply Theorem B. Let $q \in \mathbb{C}^*$ be such that either $|q| \neq 1$ or $q \in \mathcal{B}$. We set

$$\chi := \max \left\{ 1, \left| \frac{1}{|q| - 1} \right| \right\} \text{ if } |q| \neq 1, \quad \chi := e^{\mu + \mathcal{B}(q)} \text{ if } q \in \mathcal{B},$$

with μ as in Lemma 7.6, so that Proposition 7.5 yields

$$|S^T(q)| \leq \chi^{\|T\|}, \quad T \in \mathcal{T}(\mathbb{N}^*) \quad (44)$$

(using $\#T \leq \|T\|$).

Let R and f be as in the statement, and let $h(z) = z + \sum_{n \geq 1} c_n z^n$ be the formal linearization of $g = R_q \circ f$. Formula (15) yields, for each $n \geq 1$,

$$c_n = \sum_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} S^T(q) \beta_T(f),$$

whence, by (44),

$$|c_n| \leq \chi^n \sum_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} |\beta_T(f)|,$$

and $|c_n| \leq \chi^n \kappa^{-n-1} R^{-n}$ with $\kappa = 3 - \sqrt{8}$ by (41). It follows that $h(z)$ converges at least for $|z| < \kappa R \chi^{-1}$, which amounts to the desired conclusion, with $\kappa' = \kappa e^{-\mu}$ in the second case. \square

8 Regularity of the solution with respect to the multiplier — Theorem C

8.1 Position of the problem

Theorem B can be enhanced into a statement about the regularity of the solution h as a function of the multiplier q . Let us fix $f \in \mathcal{G}_1$. From now on, for q not a root of unity, we denote by

$$h_q(z) = z + \sum_{n \geq 1} c_n(q) z^n \quad (45)$$

the formal linearization of $R_q \circ f$.

In view of (14), for each $F \in \mathcal{F}(\mathbb{N}^*)$, the coefficient $S^F(q)$ is a rational function of q with poles among the roots of unity of order less than $\|F\|$, and it extends meromorphically to the Riemann sphere

$$\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$

with $S^F(0) = (-1)^{\#F}$ and $S^F(\infty) = 1$ or 0 according as F is empty or not. We thus set $S^\bullet(0) = J^\bullet$ and $S^\bullet(\infty) = 1^\bullet$ (cf. Examples 5.6 (ii) and 5.5), and accordingly

$$h_0 = f^{o(-1)}, \quad h_\infty = \text{id},$$

so that formulas (15) and (24) still hold for $q = 0$ or $q = \infty$, even though the linearization problem is then meaningless.

Each coefficient

$$c_n(q) = \sum_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} \beta_T(f) S^T(q) \quad (46)$$

is a rational function of q , with poles among the roots of unity of order less than n , but what is left of this regularity in q when it comes to h_q itself, thus taking into account all the roots of unity at the same time? Theorem B has provided us with a *pointwise* convergence result, according to which $h_q(z) \in \mathbb{C}\{z\}$ for each $q \in \widehat{\mathbb{C}}$ of modulus different from 1 and for each $q \in \mathcal{B}$; we now wish to discuss the *global* regularity of the map $q \mapsto h_q$.

Even if we may hope analyticity in the usual sense at any q of modulus different from 1, we must expect in general a natural barrier on the unit circle for the classical Weierstrass notion of analytic continuation. Indeed, it is proved in [BMS00] that, at least when $f \in \mathcal{G}_1$ extends to an entire function, the radius of convergence of h_q tends to 0 as q tends radially from the inside to any root of unity $q_* = e^{2\pi i n/m}$ such that $(R_{q_*} \circ f)^{om} \neq \text{id}$ (see ‘‘Corollary 2.1’’ in [BMS00]).

8.2 C^1 -holomorphic and monogenic functions

A. N. Kolmogorov was the first, in 1954, to raise this kind of question in a small divisor problem (in his case, it was in the realm of what has later become KAM theory); he explicitly asked whether the regularity of the solution could be investigated using Borel's theory of monogenic functions. Following subsequent work of V. Arnold, M. Herman [He85] proved the first result of this nature in the context of circle maps in 1985, reformulating Borel's ideas in a modern terminology.

Like [MS03], [CM08], [MS11] or [CMS14], we will follow Herman to construct compact subsets of $\widehat{\mathbb{C}}$ on which regularity will be investigated. The strategy consists in defining, for each $M > 0$, a compact K_M of $\widehat{\mathbb{C}}$ which intersects the unit circle along $\{q \in \mathcal{B} \mid \mathcal{B}(q) \leq M\}$ and a complex Banach space B_M such that $q \in K_M \mapsto h_q \in B_M$ can be proved to be tame enough.

Lemma 8.1. *Define $E: \omega \in \mathbb{C} \mapsto e^{2\pi i \omega} \in \mathbb{C}^*$ and, for each real $M > 0$,*

$$A_M^{\mathbb{R}} := \{\omega \in \mathbb{R} \mid \mathcal{B}(E(\omega)) \leq M\}, \quad (47)$$

$$A_M^{\mathbb{C}} := \{\omega \in \mathbb{C} \mid \exists \omega_* \in A_M^{\mathbb{R}} \text{ such that } |\Im m \omega| \geq |\omega_* - \Re e \omega|\}, \quad (48)$$

$$K_M := E(A_M^{\mathbb{C}}) \cup \{0, \infty\} \subset \widehat{\mathbb{C}}. \quad (49)$$

Then $A_M^{\mathbb{R}}$ and $A_M^{\mathbb{C}}$ are closed, and K_M is compact and perfect. Moreover

$$\bigcup_{M>0} K_M = \mathbb{D} \cup \mathcal{B} \cup \mathbb{E},$$

with $\mathbb{D} := \{|q| < 1\}$ and $\mathbb{E} := \{|q| > 1 \text{ or } q = \infty\}$.

Proof. The function $\mathcal{B} \circ E$ is lower semi-continuous on \mathbb{R} (see the appendix), the rest follows or is obvious. \square

The sets K_M are very similar to the sets called “complex multiplier domains $K_M^{(S)}$ ” in [MS11] (p. 61) or the sets “ K_M ” of [CMS14]; see the pictures there. Observe that the Haar measure of $\{|q| = 1\} \setminus K_M$ in the unit circle tends to 0 as $M \rightarrow \infty$, and that the two-dimensional Lebesgue measure of $\mathbb{C} \setminus K_M$ also tends to 0.

The corresponding target spaces B_M will be of the form $H^\infty(D_r)$ for some real $r > 0$, with the notations

$$D_r := \{z \in \mathbb{C} \mid |z| < r\}$$

and, for any open subset D of \mathbb{C} , $H^\infty(D) := \{\text{bounded holomorphic functions of } D\}$ (Banach space for the sup norm).

Given a compact $K \subset \widehat{\mathbb{C}}$ and a Banach space B , we denote by $\mathcal{O}(K, B)$ the Banach space consisting of continuous maps from K to B which are holomorphic in the interior of K . If K is perfect, we denote by $\mathcal{C}_{\text{hol}}^1(K, B)$ the space of all C^1 -holomorphic maps from K to B (this means that they are Whitney-differentiable for the underlying real structure of $\widehat{\mathbb{C}}$ and that their partial derivatives satisfy the Cauchy-Riemann equations); this is also a Banach space (see *e.g.* [He85], [MS03] § 2.1 or [CMS14] § 2.1), the definition of a possible norm will be given in Section 8.3. As a set, $\mathcal{C}_{\text{hol}}^1(K, B) \subset \mathcal{O}(K, B)$.

Theorem C. *There exist universal constants $\kappa_1, \kappa_2 > 0$ such that, for any $R > 0$, for any $f \in \mathcal{G}_1$ whose Taylor coefficients satisfy (40), and for any $M > 0$, if we consider the map $q \mapsto h_q = \text{linearization of } R_q \circ f$, then*

(i) this map induces an element of $\mathcal{O}(K_M, H^\infty(D_{\kappa_1 R e^{-M}}))$,

(ii) this map induces an element of $\mathcal{C}_{\text{hol}}^1(K_M, H^\infty(D_{\kappa_2 R e^{-2M}}))$.

The proof is in Section 8.3, based on a proposition proved in Section 11.

The statement (i) of Theorem C is essentially in [BMS00] (see the first footnote on p. 966). The factor e^{-M} is optimal, in view of Yoccoz's upper bound result.

As for the C^1 -holomorphy of the map $q \mapsto h_q$, such a property was proved for the first time in 2008 by C. Carminati and S. Marmi in [CM08], but with a less precise control of the target space due to the technique they employed which is less direct than ours⁷: the factor e^{-2M} in part (ii) of Theorem C is a significant improvement with respect to [CM08] which has e^{-5M} .

The monogenic character is obtained by considering a sequence $M_j \uparrow \infty$: the space of monogenic functions \mathcal{M} associated with this sequence is the projective limit of the spaces $\mathcal{C}_{\text{hol}}^1(K_{M_j}, H^\infty(D_{\kappa_2 R e^{-2M_j}}))$ (see [MS11]). We thus get that

the map $q \mapsto h_q$ considered on $\bigcup_j K_{M_j} = \mathbb{D} \cup \mathcal{B} \cup \mathbb{E}$ belongs to the space of monogenic functions \mathcal{M} .

An interesting feature of the spaces \mathcal{M} , $\mathcal{C}_{\text{hol}}^1(K_M, B)$ and $\mathcal{O}(K_M, B)$ (with any Banach space B) is the \mathcal{H}^1 -quasianalyticity property, which holds as soon as M is larger than a universal constant in the last two cases, so that $K_M \cap \{|q| = 1\}$ have positive Haar measure—see [MS11]. This means that the map $q \mapsto h_q$ is determined by its restriction to any subset Γ of positive linear Hausdorff measure, however small it may be. This is remarkable even in the case of $\mathcal{O}(K_M, B)$, since the interior of K_M has two connected components (one is contained in \mathbb{D} , the other is contained in \mathbb{E}) and Γ may be entirely contained in one of them; still, the unit circle may be a natural barrier for the analytic continuation of an element of $\mathcal{O}(K_M, B)$.

8.3 Proof of Theorem C

Our proof of Theorem C is similar to the proof of Theorem B: it relies on

- the formula (46) for the Taylor coefficients of h_q , deduced from its tree-representation (15),
- the very same Proposition 7.4 to control the coefficients $\beta_T(f)$ involved in the coarbo mould associated with f ,
- direct estimates for the norms of the functions $q \mapsto S^T(q)$ contained in Proposition 8.2 (the role of which is analogous to that of Proposition 7.5).

Let us now recall the definition of the norms in the spaces $\mathcal{O}(K, B)$ and $\mathcal{C}_{\text{hol}}^1(K, B)$, for a compact $K \subset \widehat{\mathbb{C}}$ and a Banach space B . If $\varphi \in \mathcal{O}(K, B)$, we just set

$$\|\varphi\|_{\mathcal{O}(K, B)} := \max_{q \in K} \|\varphi(q)\|_B. \quad (50)$$

For $\mathcal{C}_{\text{hol}}^1(K, B)$, we assume that K is perfect so as to ensure the uniqueness of the derivative. We cover $\widehat{\mathbb{C}}$ with two charts, using q as a complex coordinate in \mathbb{C} and $\xi = \frac{1}{q}$ in $\widehat{\mathbb{C}} \setminus \{0\}$; a function $\varphi: K \rightarrow B$ belongs to $\mathcal{C}_{\text{hol}}^1(K, B)$ if its restriction $\varphi|_{K \cap \mathbb{C}}$ belongs to $\mathcal{C}_{\text{hol}}^1(K \cap \mathbb{C}, B)$

⁷and with a slightly different definition of K_M : their construction is based on a variant of the Bruno function, but the difference is immaterial.

and the function $\tilde{\varphi}: \xi \mapsto \varphi(1/\xi)$ belongs to $\mathcal{E}_{\text{hol}}^1(\tilde{K}, B)$, where $\tilde{K} := \{\xi \in \mathbb{C} \mid 1/\xi \in K\}$ (with the convention $1/0 = \infty$); we set

$$\|\varphi\|_{\mathcal{E}_{\text{hol}}^1(K, B)} := \max \left\{ \|\varphi|_{K \cap \mathbb{C}}\|_{\mathcal{E}_{\text{hol}}^1(K \cap \mathbb{C}, B)}, \|\tilde{\varphi}\|_{\mathcal{E}_{\text{hol}}^1(\tilde{K}, B)} \right\}, \quad (51)$$

where, for any perfect closed $C \subset \mathbb{C}$, the Banach space $\mathcal{E}_{\text{hol}}^1(C, B)$ and its norm are defined as follows: a function $\psi: C \rightarrow B$ is in $\mathcal{E}_{\text{hol}}^1(C, B)$ if it is continuous and bounded, and there is a bounded continuous function from C to B , which we denote by ψ' , such that the function $\Omega\psi: C \times C \rightarrow B$ defined by the formula

$$\Omega\psi(q, q') := \begin{cases} \psi'(q) & \text{if } q = q', \\ \frac{\psi(q') - \psi(q)}{q' - q} & \text{if } q \neq q', \end{cases} \quad (52)$$

is continuous and bounded, the function ψ' is then unique⁸ and we set

$$\|\psi\|_{\mathcal{E}_{\text{hol}}^1(C, B)} := \max \left\{ \sup_{q \in C} \|\psi(q)\|_B, \sup_{(q, q') \in C \times C} \|\Omega\psi(q, q')\|_B \right\}. \quad (53)$$

This is a Banach space norm equivalent to the one indicated in [He85] or [MS03] (or to the one indicated in [CMS14], which is designed to be a Banach algebra norm whenever B is a Banach algebra⁹).

As usual, we simply denote by $\mathcal{O}(K)$ and $\mathcal{E}_{\text{hol}}^1(K)$ the spaces obtained when $B = \mathbb{C}$. Here are the direct estimates of the norms of the functions $q \mapsto S^T(q)$ we have alluded to earlier:

Proposition 8.2. *Let $\nu := \mu + 2$, where μ is as in Lemma 7.6. Then, for any $M > 0$ and $T \in \mathcal{T}(\mathbb{N}^*)$, the function $q \mapsto S^T(q)$ induces a C^1 -holomorphic function on K_M and*

$$(i) \quad \|S^T\|_{\mathcal{O}(K_M)} \leq e^{(\nu+M)\|T\|},$$

$$(ii) \quad \|S^T\|_{\mathcal{E}_{\text{hol}}^1(K_M)} \leq (\#T)\|T\| e^{2(\nu+M)\|T\|} \leq \|T\|^2 e^{2(\nu+M)\|T\|}.$$

The proof of Proposition 8.2 is in Section 11.

Proof of Theorem C. Let R , f and M be as in the statement. The rational functions c_n defined by (46) can be considered as elements of $\mathcal{E}_{\text{hol}}^1(K_M)$; for each $r > 0$ and $n \in \mathbb{N}^*$, c_n gives rise to a C^1 -holomorphic function $q \in K_M \mapsto c_n(q)z^n \in H^\infty(D_r)$, with

$$\|q \mapsto c_n(q)z^n\|_{\mathcal{O}(K_M, H^\infty(D_r))} \leq r^n \|c_n\|_{\mathcal{O}(K_M)}, \quad \|q \mapsto c_n(q)z^n\|_{\mathcal{E}_{\text{hol}}^1(K_M, H^\infty(D_r))} \leq r^n \|c_n\|_{\mathcal{E}_{\text{hol}}^1(K_M)}.$$

Now, for each $n \in \mathbb{N}^*$, setting $\kappa := 3 - \sqrt{8}$ and putting together (41) and (46), we get

$$\|c_n\|_{\mathcal{O}(K_M)} \leq \kappa^{-n-1} R^{-n} \max_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} \|S^T\|_{\mathcal{O}(K_M)} \leq \kappa^{-n-1} R^{-n} e^{(\nu+M)n},$$

where we used Proposition 8.2 (i) in the last step, and similarly

$$\|c_n\|_{\mathcal{E}_{\text{hol}}^1(K_M)} \leq \kappa^{-n-1} R^{-n} \max_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} \|S^T\|_{\mathcal{E}_{\text{hol}}^1(K_M)} \leq \kappa^{-n-1} R^{-n} n^2 e^{2(\nu+M)n}$$

thanks to part (ii) of the same proposition. It follows that $q \mapsto \sum c_n(q)z^n$ is a convergent series in $\mathcal{O}(K_M, H^\infty(D_r))$ as soon as $r < \kappa R e^{-(\nu+M)}$, so we can take $\kappa_1 := \kappa e^{-\nu}/2$, and the same expression is a convergent series in $\mathcal{E}_{\text{hol}}^1(K_M, H^\infty(D_r))$ as soon as $r < \kappa R e^{-2(\nu+M)}$, so we can take $\kappa_2 := \kappa e^{-2\nu}/2$. \square

⁸Moreover, for any interior point q_0 of C , the complex derivative of ψ at q_0 exists and coincides with $\psi'(q_0)$.

⁹One gets a slightly simpler Banach algebra norm than in [CMS14] by taking a sum instead of a max in (53).

9 The majorant series argument to bound $\beta_T(f)$ (proof of Proposition 7.4)

Let us give ourselves $R > 0$ and $f(z) = z + \sum_{n \geq 1} a_n z^{n+1} \in \mathcal{G}_1$ such that $|a_n| \leq R^{-n}$ for $n \geq 1$. Let $\tilde{u}(z) := \sum_{n \geq 1} \tilde{a}_n z^{n+1}$ and $\tilde{f}(z) = z + \tilde{u}(z)$ with $\tilde{a}_n := 1$ for all $n \geq 1$, *i.e.*

$$\tilde{u}(z) = z^2(1 - z)^{-1}.$$

The definition (14) of the coefficients β_F can be rephrased as

$$\beta_F(f) = \beta_F(\tilde{f}) \prod_{\sigma \in V_F} a_{N_F(\sigma)}, \quad \beta_F(\tilde{f}) = \frac{1}{\text{sym}(F)} \prod_{\sigma \in V_F} k_F(\sigma)$$

for every $F \in \mathcal{F}(\mathbb{N}^*)$. We observe that $\beta_F(\tilde{f}) \geq 0$, thus

$$|\beta_F(f)| \leq \beta_F(\tilde{f}) \prod_{\sigma \in V_F} |a_{N_F(\sigma)}| \leq \beta_F(\tilde{f}) R^{-\|F\|}.$$

In particular, we have

$$\sum_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} |\beta_T(f)| \leq \tilde{c}_n R^{-n}, \quad \tilde{c}_n := \sum_{T \in \mathcal{T}(\mathbb{N}^*) \text{ s.t. } \|T\|=n} \beta_T(\tilde{f}), \quad n \geq 1. \quad (54)$$

It turns out that we can easily compute the generating series

$$\theta(z) := z + \sum_{n \geq 1} \tilde{c}_n z^{n+1} = z + \sum_{T \in \mathcal{T}(\mathbb{N}^*)} \beta_T(\tilde{f}) z^{\|T\|+1}.$$

Indeed, Example 5.6 (iii) and Proposition 6.8 yield $C_\theta = \sum K^\bullet D_\bullet(\tilde{f})$ (because K^\bullet is a separative arbomould), and (23) yields $\sum K^\bullet D_\bullet(\tilde{f}) = C_{(\text{id} - \tilde{u})^{\circ(-1)}}$, hence

$$\theta = (\text{id} - \tilde{u})^{\circ(-1)}.$$

We invert $\text{id} - \tilde{u}$ by solving a second-order algebraic equation and get

$$\theta(z) = \frac{1}{4}(1 + z - (1 - 6z + z^2)^{1/2}).$$

Let $\kappa := 3 - \sqrt{8}$ and $\kappa^* := 3 + \sqrt{8}$. Since $(1 - 6z + z^2)^{1/2} = (\kappa - z)(\kappa^* - z)$, the function θ is holomorphic in the open disc D_κ and extends continuously to the closure of this disc; the Cauchy inequalities thus entail

$$\tilde{c}_n = \frac{1}{(n+1)!} \theta^{(n+1)}(0) \leq \kappa^{-n-1} \max_{|z|=\kappa} |\theta(z)|, \quad n \geq 1.$$

One can easily check that $\max_{|z|=\kappa} |\theta(z)| < 1$. In view of (54), this completes the proof.

Remark 9.1. This proof is in essence the majorant series argument used in the articles [Men06] and [Men07], regarding respectively non-linear q -difference equations and the Birkhoff decomposition in spaces of Gevrey series. The same argument is used in [FM14].

10 Pointwise bounds on the arbomould $S^\bullet(q)$ (proof of Proposition 7.5)

The definition (14) says that, for each q not a root of unity,

$$S^T(q) = \prod_{\sigma \in V_T} \frac{1}{q^{\hat{\sigma}} - 1}, \quad T \in \mathcal{T}(\mathbb{N}^*).$$

Part (i) of Proposition 7.5 is obtained by observing that, if $|q| \neq 1$, then

$$|q^n - 1| \geq ||q| - 1|, \quad n \in \mathbb{N}^*,$$

hence $|S^T(q)| \leq \left| \frac{1}{|q|-1} \right|^{\#T}$.

We thus focus on part (ii). With a view to later purposes, we prove a slightly more general inequality:

Lemma 10.1. *Let $\mu > 0$ as in Lemma 7.6. Then, for any $T \in \mathcal{T}(\mathbb{N}^*)$ and for any $\tilde{V} \subset V_T$,*

$$\prod_{\sigma \in \tilde{V}} \left| \frac{1}{q^{\hat{\sigma}} - 1} \right| \leq e^{(\mu + \mathcal{B}(q))\|T\|} \quad \text{for all } q \in \mathcal{B}$$

(with the usual convention that the value of an empty product is 1).

Proof of Lemma 10.1. We fix $q \in \mathcal{B}$ and take $K_q: \mathbb{N}^* \rightarrow \mathbb{R}^+$ as in Davie's lemma 7.6, setting $K_q(0) = 0$. We will prove that, for all $T \in \mathcal{T}(\mathbb{N}^*)$,

$$\prod_{\sigma \in \tilde{V}} \left| \frac{1}{q^{\hat{\sigma}} - 1} \right| \leq e^{K_q(\|T\|)} \quad \text{for all } \tilde{V} \subset V_T \quad (55)$$

by induction on the size $\#T$. In view of Lemma 7.6 (i), this will be sufficient to get the conclusion.

When $\#T = 1$, inequality (55) results from Lemma 7.6 (iii). Let $m \in \mathbb{N}$ and assume that (55) holds for all \mathbb{N}^* -trees of size not larger than m . Suppose that $T \in \mathcal{T}(\mathbb{N}^*)$ has size $\#T \leq m + 1$ and let $\tilde{V} \subset V_T$. We have $T = n \triangleleft (T_1 \cdots T_d)$ with $n \in \mathbb{N}^*$, $d = \deg_T^+(\rho_T) \geq 1$, $T_i \in \mathcal{T}(\mathbb{N}^*)$ and $\#T_i \leq m$ for $i = 1, \dots, d$. We can write

$$\prod_{\sigma \in \tilde{V}} \left| \frac{1}{q^{\hat{\sigma}} - 1} \right| = \prod_{i=0}^d S_i, \quad S_i := \prod_{\sigma \in \tilde{V}_i} \left| \frac{1}{q^{\hat{\sigma}} - 1} \right| \quad \text{for } i = 0, \dots, d$$

with $\tilde{V}_0 := \tilde{V} \cap \{\rho_T\}$ and $\tilde{V}_i := \tilde{V} \cap V_{T_i}$ for $i = 1, \dots, d$.

For $i \geq 1$, each $\sigma \in \tilde{V}_i$ can be viewed as a vertex of T as well as a vertex of T_i , and the meaning of the symbol $\hat{\sigma}$ is the same in both cases, thus the induction hypothesis applied to T_i entails $S_i \leq e^{K_q(\|T_i\|)}$. For $i = 0$, we have $S_0 \leq e^{K_q(\|T\|) - K_q(\|T\| - 1)}$ by Lemma 7.6 (iii).

Now, by Lemma 7.6 (ii), we have $K_q(\|T_1\|) + \cdots + K_q(\|T_d\|) \leq K_q(\|T_1\| + \cdots + \|T_d\|)$, but $\|T_1\| + \cdots + \|T_d\| \leq \|T\| - 1$ and the function K_q is increasing, hence $S_0 S_1 \cdots S_d \leq e^{K_q(\|T\|)}$, i.e. (55) holds for T . \square

Lemma 10.1 clearly implies part (ii) of Proposition 7.5 by taking $\tilde{V} = V_T$. The proof of Proposition 7.5 is thus complete. At this stage of the article, the proof of Theorem B is complete, since all that was left behind after Section 7.2 was the proof of Propositions 7.4 and 7.5.

11 Global bounds on the arbomould S^\bullet (proof of Proposition 8.2)

11.1 Part (i)

We start with

Lemma 11.1. *For every $q \in K_M$, there exists $q_* \in K_M \cap \mathcal{B}$ such that*

$$\left| \frac{q^\ell}{q^n - 1} \right| \leq \frac{e^2}{|q_*^n - 1|} \quad \text{for all } n \geq 1 \text{ and } \ell \in \{0, \dots, n\}. \quad (56)$$

Proof of Lemma 11.1. Let $q \in K_M$. If $q = 0$ or ∞ then any $q_* \in K_M \cap \mathcal{B}$ will do. If not, we pick $\omega \in A_M^{\mathbb{C}}$ such that $q = e^{2\pi i \omega}$, and then $\omega_* \in A_M^{\mathbb{R}}$ such that $|\Im m \omega| \geq |\omega_* - \Re e \omega|$. We will prove that $q_* := e^{2\pi i \omega_*}$ (which belongs to $K_M \cap \mathcal{B}$) satisfies

$$|q^n - 1| \geq e^{-2} |q_*^n - 1| \quad \text{for all } n \in \mathbb{Z}^*. \quad (57)$$

This will imply (56) by distinguishing the possibilities $|q| \leq 1$ (in which case (56) follows from $|q^\ell| \leq 1$) and $|q| > 1$ (in which case one can write $\frac{q^\ell}{q^n - 1} = -\frac{q^{-(n-\ell)}}{q^{-n-1}}$ and use $|q^{-(n-\ell)}| \leq 1$ and $|q_*^n| = 1$).

Let $n \in \mathbb{Z}^*$. If $|\Im m(n\omega)| \geq \frac{1}{2}$, then $|q^n - 1| \geq \frac{3}{4}$ (because $|q^n| = e^{-2\pi \Im m(n\omega)}$ is either $\leq e^{-\pi}$ or $\geq e^\pi$) and $|q_*^n - 1| \leq 2$, therefore $\left| \frac{q_*^n - 1}{q^n - 1} \right| \leq \frac{8}{3} < e^2$. If $|\Im m(n\omega)| \leq \frac{1}{2}$, then (57) follows from the inequalities

$$\forall z \in \mathbb{C}, \quad |\Im m z| \leq \frac{1}{2} \Rightarrow |e^{2\pi i z} - 1| \geq \left(\frac{3}{2\pi} + \frac{1}{2\sqrt{2}}\right)^{-1} \text{dist}(z, \mathbb{Z}) \quad (58)$$

$$\forall z \in \mathbb{C}, \forall x \in \mathbb{R}, \quad |\Im m z| \geq |x - \Re e z| \Rightarrow \text{dist}(z, \mathbb{Z}) \geq (2\pi\sqrt{2})^{-1} |e^{2\pi i x} - 1|. \quad (59)$$

applied to $z = n\omega$ and $x = n\omega_*$ and from $\left(\frac{3}{2\pi} + \frac{1}{2\sqrt{2}}\right)^{-1} (2\pi\sqrt{2})^{-1} \geq e^{-2}$.

Proof of (58): By periodicity, we may suppose $|\Re e z| \leq \frac{1}{2}$, hence $\text{dist}(z, \mathbb{Z}) = |z|$. It is then sufficient to bound the modulus of $F(z) := \frac{z}{e^{2\pi i z} - 1} = \frac{z}{2}(\coth(\pi i z) - 1)$. A classical identity yields

$$F(z) = \frac{1}{2\pi i} - \frac{1}{2}z + \frac{1}{\pi i} \sum_{\ell \geq 1} \frac{z^2}{z^2 - \ell^2}.$$

We have $|z|^2 \leq \frac{1}{2}$ and $|z^2 - \ell^2| = |\ell - z| \cdot |\ell + z| \geq |\ell - \Re e z| \cdot |\ell + \Re e z| \geq \ell^2 - \frac{1}{4}$, whence $|F(z)| \leq \frac{1}{2\pi} + \frac{1}{2\sqrt{2}} + \frac{1}{2\pi} \sum_{\ell \geq 1} \left(\frac{1}{\ell - \frac{1}{2}} - \frac{1}{\ell + \frac{1}{2}} \right) = \frac{3}{2\pi} + \frac{1}{2\sqrt{2}}$.

Proof of (59): Let $d > 0$ and $p \in \mathbb{Z}$ and suppose $|z - p| \leq d$; then

$$\begin{aligned} \text{dist}(x, \mathbb{Z}) &\leq |x - p| = |\Re e z - p + x - \Re e z| \leq |\Re e(z - p)| + |x - \Re e z| \\ &\leq |\Re e(z - p)| + |\Im m z| = |\Re e(z - p)| + |\Im m(z - p)| \leq \sqrt{2}d. \end{aligned}$$

Hence $\text{dist}(x, \mathbb{Z}) \leq \sqrt{2} \text{dist}(z, \mathbb{Z})$. The result follows because $|e^{2\pi i x} - 1| \leq 2\pi \text{dist}(x, \mathbb{Z})$. \square

For each $M > 0$ and $T \in \mathcal{T}(\mathbb{N}^*)$, the rational function

$$S^T(q) = \prod_{\sigma \in V_T} \frac{1}{q^{\hat{\sigma}} - 1}$$

(cf. (14)) induces an element of $\mathcal{O}(K_M)$, whose norm we are to estimate: part (i) of Proposition 8.2 amounts to

$$|S^T(q)| \leq e^{(\nu+M)\|T\|} \quad \text{for all } q \in K_M, \quad (60)$$

with $\nu := \mu + 2$, where μ is as in Proposition 7.5 and Lemma 7.6. Inequalities (60) are clearly a particular case of the following ones (which will be used also in the second part of the proof):

Lemma 11.2. *Let $\nu := \mu + 2$. Let $M > 0$ and $T \in \mathcal{T}(\mathbb{N}^*)$. Suppose that we are given $\tilde{V} \subset V_T$ and a family of integers $(n_\sigma)_{\sigma \in \tilde{V}}$. If $0 \leq n_\sigma \leq \hat{\sigma}$ for each $\sigma \in \tilde{V}$, then*

$$\prod_{\sigma \in \tilde{V}} \left| \frac{q^{n_\sigma}}{q^{\hat{\sigma}} - 1} \right| \leq e^{(\nu+M)\|T\|} \quad \text{for all } q \in K_M. \quad (61)$$

Proof. For each $q \in K_M$, Lemma 11.1 yields $q_* \in K_M \cap \mathcal{B}$ such that $\left| \frac{q^\ell}{q^n - 1} \right| \leq \frac{e^2}{|q_*^\ell - 1|}$ whenever $n \in \mathbb{N}^*$ and $0 \leq \ell \leq n$. Thus the left-hand side of (61) is $\leq e^{2\text{card}(\tilde{V})} \prod_{\sigma \in \tilde{V}} \frac{1}{|q_*^{\hat{\sigma}} - 1|}$, which is itself $\leq e^{2\text{card}(\tilde{V})} e^{(\mu+\mathcal{B}(q_*))\|T\|}$ by Lemma 10.1. Since $\text{card}(\tilde{V}) \leq \#T \leq \|T\|$ and $\mathcal{B}(q_*) \leq M$, we end up with (61). \square

This completes the proof of part (i) of Proposition 8.2.

11.2 Part (ii)

Let $M > 0$ and $T \in \mathcal{T}(\mathbb{N}^*)$. As a rational function which is regular outside the roots of unity, S^T induces an element of $\mathcal{C}_{\text{hol}}^1(K_M)$. In view of (51)–(53), since $\{\xi \in \mathbb{C} \mid 1/\xi \in K_M\} = K_M \cap \mathbb{C}$,

$$\|S^T\|_{\mathcal{C}_{\text{hol}}^1(K_M)} \leq \max \left\{ \|S^T\|_{\mathcal{O}(K_M)}, \sup_{K_M \times K_M} |\Omega S^T|, \sup_{K_M \times K_M} |\Omega \tilde{S}^T| \right\},$$

with $\tilde{S}^T(q) := S^T(1/q)$ for all $q \in K_M$.

By part (i) of Proposition 8.2, we already have

$$\|S^T\|_{\mathcal{O}(K_M)} \leq e^{(\nu+M)\|T\|} \leq (\#T)\|T\| e^{2(\nu+M)\|T\|}. \quad (62)$$

Let $r := \#T$. To end the proof of Proposition 8.2, it is thus sufficient to prove

$$\sup_{K_M \times K_M} |\Omega S^T|, \sup_{K_M \times K_M} |\Omega \tilde{S}^T| \leq r\|T\| e^{2(\nu+M)\|T\|}. \quad (63)$$

Let us introduce the elementary functions

$$\psi_{n,\ell}(q) := \frac{q^\ell}{q^n - 1} \quad \text{for } q \in K_M, n \in \mathbb{N}^*, \ell \in \mathbb{N}. \quad (64)$$

Numbering the vertices as $V_T = \{\sigma_1, \dots, \sigma_r\}$, we can write

$$S^T = \chi_1 \cdots \chi_r, \quad \tilde{S}^T = \tilde{\chi}_1 \cdots \tilde{\chi}_r \quad \text{with } \chi_i := \psi_{\hat{\sigma}_i, 0}, \quad \tilde{\chi}_i := -\psi_{\hat{\sigma}_i, \hat{\sigma}_i} \quad \text{for } i = 1, \dots, r. \quad (65)$$

Lemma 11.3. *Suppose $\chi_1, \dots, \chi_r \in \mathcal{C}_{hol}^1(K_M)$. Then*

$$\Omega(\chi_1 \cdots \chi_r) = \sum_{i=1}^r \left(\prod_{j<i} A\chi_j \right) (\Omega\chi_i) \left(\prod_{j>i} B\chi_j \right), \quad (66)$$

where, for any $\chi: K_M \rightarrow \mathbb{C}$, we define $A\chi, B\chi: K_M \times K_M \rightarrow \mathbb{C}$ by

$$A\chi(q, q') := \chi(q), \quad B\chi(q, q') := \chi(q').$$

Proof. Induction on r . □

Simple computations show that, for every $n \in \mathbb{N}^*$, $\psi_{n,n} = 1 + \psi_{n,0}$ and

$$\Omega\psi_{n,0} = \Omega\psi_{n,n} = - \sum_{a,b \geq 0 \text{ s.t. } a+b=n-1} (A\psi_{n,a})(B\psi_{n,b}),$$

whence, by applying Lemma 11.3 to the products (65),

$$\begin{aligned} \Omega S^T &= - \sum_{i=1}^r \sum_{a,b \geq 0 \text{ s.t. } a+b=\hat{\sigma}_i-1} A \left(\left(\prod_{j<i} \psi_{\hat{\sigma}_j,0}^\wedge \right) \psi_{\hat{\sigma}_i,a}^\wedge \right) B \left(\psi_{\hat{\sigma}_i,b}^\wedge \prod_{j>i} \psi_{\hat{\sigma}_j,0}^\wedge \right), \\ \Omega \tilde{S}^T &= (-1)^{r+1} \sum_{i=1}^r \sum_{a,b \geq 0 \text{ s.t. } a+b=\hat{\sigma}_i-1} A \left(\left(\prod_{j<i} \psi_{\hat{\sigma}_j,\hat{\sigma}_j}^\wedge \right) \psi_{\hat{\sigma}_i,a}^\wedge \right) B \left(\psi_{\hat{\sigma}_i,b}^\wedge \prod_{j>i} \psi_{\hat{\sigma}_j,\hat{\sigma}_j}^\wedge \right). \end{aligned}$$

By virtue of Lemma 11.2, whenever $0 \leq a, b \leq \hat{\sigma}_i$,

$$\sup_{K_M} \left(\prod_{j<i} \left| \psi_{\hat{\sigma}_j,\ell_j}^\wedge \right| \left| \psi_{\hat{\sigma}_i,a}^\wedge \right| \right), \sup_{K_M} \left(\left| \psi_{\hat{\sigma}_i,b}^\wedge \right| \prod_{j>i} \left| \psi_{\hat{\sigma}_j,\ell_j}^\wedge \right| \right) \leq e^{(\nu+M)\|T\|}$$

provided $0 \leq \ell_j \leq \hat{\sigma}_j$ for $j = 1, \dots, r$, hence

$$\sup_{K_M \times K_M} |\Omega S^T|, \sup_{K_M \times K_M} |\Omega \tilde{S}^T| \leq \left(\sum_{i=1}^r \hat{\sigma}_i \right) e^{2(\nu+M)\|T\|},$$

which entails (63) because $\sum_{i=1}^r \hat{\sigma}_i \leq r\|T\|$. The proof of Proposition 8.2 is thus complete.

At this stage of the article, the proof of Theorem C is complete, since all that was left behind after Section 8.3 was the proof of Proposition 8.2.

Appendix: lower semicontinuity of the Bruno function

Let $B := \mathcal{B} \circ E$ with the notations of Definition 7.1 and Lemma 8.1, *i.e.*

$$B: \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{\infty\}, \quad B(\omega) = \begin{cases} \infty & \text{if } \omega \in \mathbb{Q}, \\ \sum_{n \geq 0} \frac{\ln Q_{n+1}(\omega)}{Q_n(\omega)} & \text{if } \omega \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

where $(Q_n(\omega))_{n \in \mathbb{N}}$ is the sequence of the denominators of the convergents of ω .

In this appendix, we prove that B is lower semicontinuous on \mathbb{R} , *i.e.* that

$$U_M := \{ \omega \in \mathbb{R} \mid B(\omega) > M \}$$

is open for every $M > 0$, a fact which was used in the proof of Lemma 8.1. The arguments are similar to the ones used in [CM08] to prove the lower semicontinuity of a close variant of B .

We use the standard notation for continued fractions: given $k \geq 0$, $A_0 \in \mathbb{Z}$ and (if $k \geq 1$) $A_1, \dots, A_k \in \mathbb{N}^*$,

$$[A_0, A_1, \dots, A_k] := A_0 + \frac{1}{A_1 + \frac{1}{\ddots + \frac{1}{A_k}}}.$$

For $0 \leq n \leq k$, the reduced expression P_n/Q_n of $[A_0, \dots, A_n]$ can be obtained inductively from the formulas

$$(P_{-1}, Q_{-1}) := (1, 0), \quad (P_0, Q_0) = (A_0, 1), \quad (P_n, Q_n) = (A_n P_{n-1} + P_{n-2}, A_n Q_{n-1} + Q_{n-2}).$$

Every $\omega \in \mathbb{R} \setminus \mathbb{Q}$ can be represented in a unique way as an infinite continued fraction

$$\omega = \lim_{k \rightarrow \infty} [A_0(\omega), \dots, A_k(\omega)],$$

whereas every rational number ω has exactly two distinct representations as a finite continued fraction, one shorter than the other:

$$[A_0(\omega), \dots, A_{k-1}(\omega), A_k(\omega)] = [A_0(\omega), \dots, A_{k-1}(\omega), A_k(\omega) - 1, 1], \quad \text{with } k = k(\omega) \in \mathbb{N}.$$

We shall need the following elementary property:

Let $k \geq 0$, $A_0 \in \mathbb{Z}$ and $A_1, \dots, A_k \in \mathbb{N}^*$. Let $J(A_0, \dots, A_k)$ denote the set of all real numbers which can be represented by a finite or infinite continued fraction starting with the string A_0, \dots, A_k . Then $J(A_0, \dots, A_k)$ is the closed interval whose endpoints are $[A_0, \dots, A_{k-1}, A_k]$ and $[A_0, \dots, A_{k-1}, A_k + 1] = [A_0, \dots, A_{k-1}, A_k, 1]$.

The reader is referred to [HW79] or [Khi64] for the previous facts. We now start the proof of the lower semicontinuity of B and give ourselves $M > 0$ and $\omega_* \in U_M$. It is enough to show that U_M is a neighbourhood of ω_* .

– If $\omega_* \notin \mathbb{Q}$, then we can choose k large enough so that $\sum_{n=0}^{k-1} \frac{\ln Q_{n+1}(\omega_*)}{Q_n(\omega_*)} > M$, and clearly

$J(A_0(\omega_*) \dots, A_k(\omega_*))$ is a neighbourhood of ω_* contained in U_M .

– If $\omega_* \in \mathbb{Q}$, then we denote its reduced expression by $\omega_* = P/Q$. There are two continued fraction representations $\omega_* = [A_0, \dots, A_k]$, one with k even, the other with k odd. Using the first representation, we set $\omega_n^+ := [A_0, \dots, A_k, n]$ for $n \geq 1$, thus defining a decreasing sequence to the right of ω_* . One easily checks that

$$\omega \in (\omega_*, \omega_n^+) \quad \Rightarrow \quad B(\omega) \geq \frac{\ln(nQ)}{Q}, \tag{67}$$

because if $\omega \in (\omega_*, \omega_n^+) \setminus \mathbb{Q}$, then $\omega \in [\omega_*, \omega_1^+] = J(A_0, \dots, A_k)$, thus $Q_k(\omega) = Q$, while $A_{k+1}(\omega) \geq n$ and $Q_{k+1}(\omega) = A_{k+1}(\omega)Q_k(\omega) + Q_{k-1}(\omega) \geq nQ$. Therefore $(\omega_*, \omega_n^+) \subset U_M$ for n large enough. Similarly, $(\omega_n^-, \omega_*) \subset U_M$ for n large enough, where $(\omega_n^-)_{n \geq 1}$ is the increasing sequence to the left of ω_* defined from the second continued fraction representation of ω_* .

CONCLUDING REMARKS

Écalle’s arbomould formalism and arborification mechanism, though available for more than twenty years ([Eca92]), has of all evidence not been digested until now by the mathematical communities most susceptible of using it. The algebraic structures underlying arborification, *e.g.* combinatorial Hopf algebras, pre-Lie products, have lately been the object of several works (see for example [FM14] and the references therein). In the present article, however, we have tried to illustrate how properly introduced notions of arborification theory enable, in a very natural way, to reach difficult results without recourse to any heavy machinery.

We have reexamined a classical problem of analytic classification of dynamical systems in one complex dimension and shown the explicit tree-indexed formulas that some of the concepts and structures of arbomould calculus enable to obtain. Next, we have been able to recover in a direct way non-trivial analyticity results and to obtain a refined result on the regularity of the linearizing transformations with respect to the multiplier.

In a forthcoming article, we will address the linearization problem for multidimensional dynamical systems with discrete or continuous time: indeed, one of the striking features of Écalle’s formalism is to make possible a unified treatment of diffeomorphisms and vector fields and to yield compact formulas in arbitrary dimension; the formulas are slightly more complicated but as explicit as in dimension 1, and they eventually lead to analyticity results under the multidimensional multiplicative or additive Bruno condition.

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