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► **To cite this version:**

Samuel Petite, Fabien Durand. CONJUGACY OF UNIMODULAR PISOT SUBSTITUTIONS SUBSHIFTS TO DOMAIN EXCHANGES. 2020. hal-01053723v2

HAL Id: hal-01053723

<https://hal.science/hal-01053723v2>

Preprint submitted on 10 Jul 2020 (v2), last revised 10 Nov 2023 (v4)

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CONJUGACY OF UNIMODULAR PISOT SUBSTITUTIONS SUBSHIFTS TO DOMAIN EXCHANGES

FABIEN DURAND, SAMUEL PETITE

ABSTRACT. We prove that any unimodular Pisot substitution subshift is measurably conjugate to a domain exchange in Euclidean spaces which factorizes onto a minimal rotation on a torus. This generalizes the pioneer works of Rauzy and Arnoux-Ito providing geometric realizations to any unimodular Pisot substitution without any additional combinatorial condition.

1. INTRODUCTION

A classical way to tackle problems of diffeomorphism dynamics is to code the orbits of the points through a well-chosen finite partition to obtain a "nice" subshift which is easier to study (see the emblematic works [17] and [23]). The interesting dynamical properties of the subshift may translate into relevant properties of the original dynamical system.

In the seminal paper [24], G. Rauzy proposed to go in the other way round: take your favorite subshift and try to give it a geometrical representation. He took what is now called the Tribonacci substitution given by

$$1 \mapsto 12, 2 \mapsto 13 \text{ and } 3 \mapsto 1,$$

and proved that the subshift it generates is measure theoretically conjugate to a rotation on the torus \mathbb{T}^2 . A similar result was already known for substitutions of constant length under some necessary and sufficient conditions [9]. Later, in [2], the authors showed that subshifts whose block complexity is $2n+1$, and satisfy some technical conditions, are measure theoretically conjugate to an interval exchange on 3 intervals. This family of subshifts includes the subshift generated by the Tribonacci substitution.

1991 *Mathematics Subject Classification.* Primary: 54H20; Secondary: 37B20.

Key words and phrases. minimal Cantor systems, finite rank Bratteli-Vershik dynamical systems, eigenvalues, Pisot conjecture, substitutions.

Both authors acknowledge the ANR Program SubTile. Fabien Durand also acknowledges the ANR Programs DynA3S and FAN.

The Tribonacci substitution has the specificity to be a unimodular and Pisot substitution, that is, its incidence matrix has determinant 1, its characteristic polynomial is irreducible and its dominant eigenvalue is a Pisot number (all its algebraic conjugates are, in modulus, strictly less than 1). These properties provide key arguments to prove the main result in [24] mentioned above. It naturally leads to what is now called the **Pisot conjecture** for symbolic dynamics:

Let σ be a Pisot substitution. Then, the subshift it generates has purely discrete spectrum, i.e., it is measure theoretically conjugate to a rotation on a torus.

Many attempts have been done to solve this conjecture. The usual strategy is the same as the Rauzy's one in [24]: show first that the substitution subshift is measurably conjugate to a domain exchange (see Definition 4), then that this system is measurably conjugate to a rotation on a torus.

In a widely cited, but unpublished, manuscript [21] (see also Section 6.3.3 in [?]), Host proved that the Pisot conjecture is true for unimodular substitutions defined on two letters, provided a condition called *strong coincidence condition* holds. This combinatorial condition first appeared in [9]. Then, it was shown in [4] that this condition is satisfied for any unimodular Pisot substitution on two letters. So the Pisot conjecture is true in this case [18].

Following the Rauzy's strategy, but in a different way from the Host's approach, in [1] the authors associate a self-affine domain exchange called *Rauzy fractal* to any unimodular Pisot substitution. They proved, this domain exchange is measurably conjugate to the substitution subshift provided the substitution satisfies a combinatorial condition. Few time later, Host's results were generalized by Canterini and Siegel in [8] to any unimodular Pisot substitution and to the non-unimodular case [25, 26], but without avoiding the strong coincidence condition. These works led to the development of a huge number of techniques to study the Rauzy fractals (see for instance [16] and references therein). Let us also mention other fruitful geometrical approaches by using tilings in [5, 3] and more recently in [?] for the one-dimensional case.

In this paper, we show a similar result to the main results in [1] and [8] but skipping the combinatorial conditions: any unimodular Pisot substitution subshift is measurably conjugate to a self-affine domain exchange. It answers the 30-years old problem of finding geometric realizations to any unimodular Pisot substitution subshifts. Notice the domain exchange may, a priori, be different from the usual Rauzy fractal.

Let $\pi: \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ denote the canonical projection. We obtain the following result.

Theorem 1. *Let σ be a unimodular Pisot substitution on d letters and let (Ω, S) be the associated substitution subshift. Then, there exist a self-affine domain exchange transformation $(E, \mathcal{B}, \tilde{\lambda}, T)$ in \mathbb{R}^{d-1} and a continuous onto map $\Phi: \Omega \rightarrow E \subset \mathbb{R}^{d-1}$ which is a measurable conjugacy map between the two systems.*

Moreover, there is a measurable map $\Psi: E \rightarrow \mathbb{R}^{d-1}$ such that

- the map $\Psi \circ \Phi - \Phi$ is locally constant.
- The map $\pi \circ \Psi \circ \Phi$ defines a (continuous) factor map from (Ω, S) to the dynamical system associated with a minimal rotation on the torus $\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$.
- There is a constant $R_\xi \geq 1$ such that the map $\pi \circ \Psi \circ \Phi$ is a.e. R_ξ -to-one (i.e. $\#(\pi \circ \Psi \circ \Phi)^{-1}(\{y\}) = R_\xi$ for a.e. $y \in \mathbb{T}^{d-1}$ with respect to the Lebesgue measure).

The toral rotation is explicitly described in [8] (see also [16]). If in addition the substitution σ is *proper* (i.e., having a nice combinatorial property implying, in particular, the strong coincidence condition, see Definition 2.3), the function $\Psi \circ \Phi - \Phi$ takes only integer vector values so that $\pi \circ \Phi$ is a (continuous) factor map onto a minimal rotation on the torus.

To prove the Pisot conjecture, one still have to prove this domain exchange is measurably conjugate to the toral rotation.

We postpone to the next section the basic definitions and notions we use for dynamical systems, substitution subshifts and Pisot substitutions. In Section 3, we prove by using the notion of return words, that any substitution subshift is conjugate to a proper substitution subshift. But, such a substitution may not be irreducible since the spectrum of its matrix contains the spectrum of a power of the older one but may also contain the values 0 and 1. We then show, in Section 4, that such a subshift, having enough multiplicatively independent eigenvalues (precised later), is measurably conjugate to a self-affine domain exchange. A byproduct of these two results gives us Theorem 1. The proof follows the same strategy as in [8].

We would like to emphasize that the irreducibility of Pisot substitutions implies that the number of multiplicatively independent non trivial eigenvalues equals $\sum_{0 < |\lambda| < 1} \dim E_\lambda$, where E_λ denotes the eigenspace associated with the eigenvalue λ of the substitution matrix. This property is crucial in our proof and might be defined not only for substitution subshifts. This suggests a possible extension of our main result to linearly recurrent subshifts, or S -adic subshifts, like in [6].

2. BASIC DEFINITIONS

2.1. Words and sequences. An *alphabet* A is a finite set of elements called *letters*. Its cardinality is $\#A$. A *word* over A is an element of the free monoid generated by A , denoted by A^* . Let $x = x_0x_1 \cdots x_{n-1}$ (with $x_i \in A$, $0 \leq i \leq n-1$) be a word, its *length* is n and is denoted by $|x|$. The *empty word* is denoted by ϵ , $|\epsilon| = 0$. The set of nonempty words over A is denoted by A^+ . The elements of $A^{\mathbb{Z}}$ are called *sequences*. If $x = \cdots x_{-1}x_0x_1 \cdots$ is a sequence (with $x_i \in A$, $i \in \mathbb{Z}$) and $I = [k, l]$ an interval of \mathbb{Z} we set $x_I = x_kx_{k+1} \cdots x_l$ and we say that x_I is a *factor* of x . The set of factors of length n of x is written $\mathcal{L}_n(x)$ and the set of factors of x , or the *language* of x , is denoted by $\mathcal{L}(x)$. The *occurrences* in x of a word u are the integers i such that $x_{[i, i+|u|-1]} = u$. If u has an occurrence in x , we also say that u *appears* in x . When x is a word or an element of $A^{\mathbb{N}}$ (called *one-sided sequence*), we use the same terminology with similar notations. The words $x_{[0, n]}$ are called *prefixes* of x .

Let x be an element of $A^{\mathbb{Z}}$ or $A^{\mathbb{N}}$. A word u is *recurrent* in x if it appears in x infinitely many times. A sequence x is *uniformly recurrent* if all $u \in \mathcal{L}(x)$ is recurrent in x and the difference of two consecutive occurrences of u in x is bounded.

2.2. Morphisms and matrices. Let A and B be two finite alphabets. Let σ be a *morphism* from A^* to B^* . When $\sigma(A) = B$, we say σ is a *coding*. If $\sigma(A)$ is included in B^+ , we say σ is *non erasing*. In this case, it induces by concatenation two maps also denoted σ : One from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$ and the other from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ satisfying $\sigma(x) = y$ with $\sigma(x_{[0, +\infty)}) = y_{[0, +\infty)}$ and $\sigma(x_{(-\infty, 0]}) = y_{(-\infty, 0]}$.

We naturally associate to the morphism σ an *incidence matrix* $M_\sigma = (m_{i,j})_{i \in B, j \in A}$, where $m_{i,j}$ is the number of occurrences of i in the word $\sigma(j)$. Notice that when σ is an endomorphism, for any positive integer n we get $M_{\sigma^n} = M_\sigma^n$.

We say that an endomorphism is *primitive* whenever its incidence matrix is primitive (*i.e.*, when it has a power with strictly positive coefficients). The Perron's theorem tells that the dominant eigenvalue is a real simple root of the characteristic polynomial and is strictly greater than the modulus of any other eigenvalue.

2.3. Substitutions. Let us recall the definition of *substitution* as it appears in [?]: It is a non erasing endomorphism.

A substitution σ is *left proper* (resp. *right proper*) if all words $\sigma(b)$, $b \in A$, starts (resp. ends) with the same letter. Notice that all word $\sigma^n(b)$, $b \in A$ for every power $n \geq 1$, starts (resp. ends) also with the

same letter. For short, we say that a left and right proper substitution is *proper*.

The *language* of $\sigma : A^* \rightarrow A^*$, denoted by $\mathcal{L}(\sigma)$, is the set of words having an occurrence in $\sigma^n(b)$ for some $n \in \mathbb{N}$ and $b \in A$. Notice that when σ is primitive, then $\mathcal{L}(\sigma^n) = \mathcal{L}(\sigma)$ for any positive integer n .

A *fixed point* of σ is an element x of $A^{\mathbb{Z}}$ such that $\sigma(x) = x$. We say it is *admissible* whenever $\mathcal{L}(x)$ is a subset of $\mathcal{L}(\sigma)$. It could happen that a fixed point is not admissible as shown by the substitution defined by $1 \mapsto 121$ and $2 \mapsto 212$ and its non-admissible fixed point $x = \cdots 21212.21212 \cdots$. It could also happen that σ has no fixed point. Nevertheless, if σ is primitive, there always exists some $p > 0$ such that σ^p has an admissible fixed point. Indeed, there exist some $p > 0$ and letters l, r such that

- (1) r is the last letter of $\sigma^p(r)$;
- (2) l is the first letter of $\sigma^p(l)$;
- (3) rl belongs to $\mathcal{L}(\sigma)$.

It is then standard to show that there is an admissible fixed point $x \in A^{\mathbb{Z}}$ of σ^p verifying $x_{-1}x_0 = rl$. Moreover, x is uniformly recurrent [?].

We will mainly focus on substitutions with specific arithmetic properties. Let us recall that an algebraic number β is called a *Pisot-Vijayaraghan number* if all its algebraic conjugates have a modulus strictly smaller than 1.

Definition 2. *Let σ be a primitive substitution and let P_σ denote the characteristic polynomial of the incidence matrix M_σ . We say that the substitution σ is*

- of Pisot type (or Pisot for short) if P_σ has a dominant root $\beta > 1$ and any other root β' satisfies $0 < |\beta'| < 1$;
- of weakly irreducible Pisot type (or W. I. Pisot for short) whenever P_σ has a real Pisot-Vijayaraghan number as dominant root, its algebraic conjugates, with possibly 0 or roots of the unity as other roots;
- an irreducible substitution whenever P_σ is irreducible over \mathbb{Q} ;
- unimodular if $\det M_\sigma = \pm 1$.

For instance the Fibonacci substitution $0 \mapsto 01, 1 \mapsto 0$ and the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ are unimodular substitutions of Pisot type. Whereas the Thue-Morse substitution $0 \mapsto 01, 1 \mapsto 10$ is a W. I. Pisot substitution. Notice that the notions of Pisot, W. I. Pisot, irreducible, unimodular depend only on the properties of

the incidence matrix. So starting from a Pisot (resp. W. I. Pisot, irreducible, unimodular) substitution σ , we get many examples of Pisot (resp. W. I. Pisot, irreducible, unimodular) substitutions by permuting the letters in the letter images of σ .

Standard algebraic arguments ensure that a Pisot substitution is an irreducible substitution, and of course, a Pisot substitution is of weakly irreducible Pisot type.

In the following we will strongly use the fact that for any substitution σ of (resp. W. I. Pisot, irreducible, unimodular) Pisot type and for every integer $n \geq 1$, the substitutions σ^n are also of (resp. W. I. Pisot, irreducible, unimodular) Pisot type.

2.4. Dynamical systems and subshifts. A *measurable dynamical system* is a quadruple (X, \mathcal{B}, μ, S) , where (X, \mathcal{B}, μ) is a probability space and $S : X \rightarrow X$ is a measurable map that preserves the measure μ , *i.e.*, $\mu(S^{-1}B) = \mu(B)$ for any $B \in \mathcal{B}$. This system is called *ergodic* if any S -invariant measurable set has measure 0 or 1. Two measurable dynamical systems (X, \mathcal{B}, μ, S) and $(Y, \mathcal{B}', \nu, T)$ are *measure theoretically conjugate* if we can find invariant subsets $X_0 \subset X$, $Y_0 \subset Y$ with $\mu(X_0) = \nu(Y_0) = 1$ and a bimeasurable bijective map $\psi : X_0 \rightarrow Y_0$ such that $T \circ \psi = \psi \circ S$ and $\mu(\psi^{-1}B) = \nu(B)$ for any $B \in \mathcal{B}'$.

By a *topological dynamical system*, or dynamical system for short, we mean a pair (X, S) , where X is a compact metric space and S a continuous map from X to itself. It is well-known that such a system endowed with the Borel σ -algebra admits a probability measure μ preserved by the map S , and then form a measurable dynamical system. If (X, S) admits a unique measure preserved by S , then the system is said *uniquely ergodic*.

The system (X, S) is *minimal* whenever X and the empty set are the only S -invariant closed subsets of X . We say that a minimal system (X, S) is *periodic* whenever X is finite.

A dynamical system (Y, T) is called a *factor* of (X, S) if there is a continuous and onto factor map $\phi : X \rightarrow Y$ such that $\phi \circ S = T \circ \phi$. The map ϕ is a *factor map*. If ϕ is one-to-one we say that ϕ is a *conjugacy*, and, that (X, S) and (Y, T) are *conjugate*.

A *Cantor system* is a dynamical system (X, S) , where the space X is a Cantor space, *i.e.*, X has a countable basis of its topology which consists of closed and open sets and does not have isolated points.

For a finite alphabet A , we endow $A^{\mathbb{Z}}$ with the product topology. A *subshift* on A is a pair $(X, S|_X)$, where X is a closed S -invariant subset of $A^{\mathbb{Z}}$ ($S(X) = X$) and S is the *shift transformation*

$$\begin{aligned} S : A^{\mathbb{Z}} &\rightarrow A^{\mathbb{Z}} \\ (x_n)_{n \in \mathbb{Z}} &\mapsto (x_{n+1})_{n \in \mathbb{Z}}. \end{aligned}$$

We call *language* of X the set $\mathcal{L}(X) = \{x_{[i,j]}; x \in X, i \leq j\}$. A set defined with two words u and v of A^* by

$$[u.v]_X = \{x \in X; x_{[-|u|, |v|-1]} = uv\}$$

is called a *cylinder set*. When u is the empty word we set $[u.v]_X = [v]_X$. The family of cylinder sets is a base of the induced topology on X . When no confusion is possible we write $[u]$ and S instead of $[u]_X$ and $S|_X$.

For $x \in A^{\mathbb{Z}}$, let $\Omega(x)$ be the set $\{y \in A^{\mathbb{Z}}; y_{[i,j]} \in \mathcal{L}(x), \forall [i,j] \subset \mathbb{Z}\}$. It is clear that $(\Omega(x), S)$ is a subshift, it is called the *subshift generated by x* . Notice that $\Omega(x) = \overline{\{S^n x; n \in \mathbb{Z}\}}$. For a subshift (X, S) on A , the following are equivalent:

- (1) (X, S) is minimal;
- (2) For all $x \in X$ we have $X = \Omega(x)$;
- (3) For all $x \in X$ we have $\mathcal{L}(X) = \mathcal{L}(x)$.

We also have that $(\Omega(x), S)$ is minimal if and only if x is uniformly recurrent. Note that if (Y, S) is another subshift then $\mathcal{L}(X) = \mathcal{L}(Y)$ if and only if $X = Y$.

2.5. Substitution subshifts. Let σ be a primitive substitution. We call *substitution subshift generated by σ* the topological dynamical system $(\Omega(\sigma), S)$ where $\Omega(\sigma)$ is the set of sequences $x = (x_n)_{n \in \mathbb{Z}}$ such that $\mathcal{L}(x) \subset \mathcal{L}(\sigma)$. Notice that each power p of σ generates the same subshift: $\Omega(\sigma) = \Omega(\sigma^p)$.

When σ is proper we say $(\Omega(\sigma), S)$ is a *proper substitution subshift*. If the set $\Omega(\sigma)$ is not finite, the substitution σ is called *aperiodic*.

In the literature, substitution subshifts are often defined by a different (but equivalent) method, using fixed points. As observed in [14], the subshift generated by any admissible fixed point of a power of σ is the substitution subshift generated by σ . Hence $\Omega(\sigma)$ is not empty. Moreover this subshift is minimal and uniquely ergodic (for more details see [?]).

In [19] is shown that the subshift generated by a unimodular substitution of Pisot type is non-periodic for the shift, and in [8] that the substitution should be primitive. Thus the generated subshift is a non-periodic minimal Cantor system.

2.6. Dynamical spectrum of substitution subshifts. For a measurable dynamical system (X, \mathcal{B}, μ, T) , a complex number λ is an *eigenvalue* of the dynamical system (X, \mathcal{B}, μ, T) with respect to μ if there

exists $f \in L^2(X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called an *eigenfunction* (associated with λ). The value 1 is the *trivial eigenvalue* associated with a constant eigenfunction. If the system is ergodic, then every eigenvalue is of modulus 1, and, every eigenfunction has a constant modulus μ -almost surely. For a topological dynamical system, if the eigenfunction f is continuous, λ is called a *continuous eigenvalue*. The collection of eigenvalues is called the *spectrum* of the system, and form a multiplicative subgroup of the circle $\mathbb{S} = \{z \in \mathbb{C}; |z| = 1\}$.

An important result for the spectral theory of substitution subshifts is due to B. Host [20]. It states that any eigenvalue of a substitution subshift is a continuous eigenvalue. The following proposition, claimed in [21] (see Proposition 7.3.29 in [16] for a proof), shows that the spectrum of a unimodular substitution of Pisot type is not trivial.

Proposition 3. *Let σ be a unimodular substitution of Pisot type and let α be a frequency of a letter in any infinite word of $\Omega(\sigma)$. Then $\exp(2i\pi\alpha)$ is a continuous eigenvalue of the dynamical system $(\Omega(\sigma), S)$.*

Recall that these frequencies are the coordinates of the right normalized eigenvector associated with the dominant eigenvalue of the incidence matrix of the substitution [?], and, moreover, for a unimodular Pisot substitution they are multiplicatively independent (Proposition 3.1 in [8]).

2.7. Domain exchange. As we did not find in the litterature a formal definition of domain exchange, we propose below a definition that fit for the standard properties satisfied by the interval exchange transformations and the Rauzy fractal. Let us recall that a subset of a topological space is said *regular* if it equals the closure of its interior.

Definition 4. *We call domain exchange transformation a measurable dynamical system $(E, \mathcal{B}, \tilde{\lambda}, T)$, where E is a regular compact subset of an Euclidean space, $\tilde{\lambda}$ denotes the normalized Lebesgue measure on E and \mathcal{B} denotes the Borel σ -algebra, such that:*

- *There exist compact regular subsets E_1, \dots, E_n such that $E = E_1 \cup \dots \cup E_n$;*
- *The sets E_i are disjoint in measure for the Lebesgue measure λ :*

$$\lambda(E_i \cap E_j) = 0 \quad \text{when } i \neq j;$$

- *$\tilde{\lambda}(T(E)) = \tilde{\lambda}(E)$;*
- *For any index i , T restricted to E_i is a translation.*

Basics results in measure theory imply that for a domain exchange transformation, the map T is one-to-one except on a set of measure zero, and the map T^{-1} is measurable.

The domain exchange is said *self-affine*, if there is a finite number of affine maps f_1, \dots, f_ℓ , sharing the same linear part, such that $E = \bigcup_{i=1}^\ell f_i(E)$.

3. MATRIX EIGENVALUES AND RETURN SUBSTITUTIONS

In this section we recall the notion of return substitution introduced in [11] and that any substitution subshift is conjugate to an explicit primitive and proper substitution subshift without changing too much the eigenvalues of the associated substitution matrix [12].

Let A be an alphabet, x be an element of $A^{\mathbb{Z}}$ and u be a word of x . We call *return word* to u of x every factor $x_{[i,j-1]}$, where i and j are two successive occurrences of u in x . We denote by $\mathcal{R}_{x,u}$ the set of return words to u of x . Notice that for a return word v , vu belongs to $\mathcal{L}(x)$ and u is a prefix of the word vu . Suppose x is uniformly recurrent. It is easy to check that for any word u of x , the set $\mathcal{R}_{x,u}$ is finite. Moreover, for any sequence $y \in \Omega(x)$, we have $\mathcal{R}_{y,u} = \mathcal{R}_{x,u}$. The sequence x can be written naturally as a concatenation

$$x = \cdots w_{-1}w_0w_1 \cdots, \quad w_i \in \mathcal{R}_{x,u}, \quad i \in \mathbb{Z},$$

of return words to u , and this decomposition is unique. By enumerating the elements of $\mathcal{R}_{x,u}$ in the order of their first appearance in $(w_i)_{i \geq 0}$, we get a bijective map

$$\Theta_{x,u}: R_{x,u} \rightarrow \mathcal{R}_{x,u} \subset A^*,$$

where $R_{x,u} = \{1, \dots, \text{Card}(\mathcal{R}_{x,u})\}$. This map defines a morphism from $R_{x,u}^*$ to A^* . We denote by $D_u(x)$ the unique sequence on the alphabet $R_{x,u}$ characterized by

$$\Theta_{x,u}(D_u(x)) = x.$$

We call it the *derived sequence* of x on u .

A finite subset $\mathcal{R} \subset A^+$ is a *code* if every word $u \in A^+$ admits at most one decomposition as a concatenation of elements of \mathcal{R} .

We say that a code \mathcal{R} is a *circular code* if for any words

$$w_1, \dots, w_j, w, w'_1, \dots, w'_k \in \mathcal{R}; s \in A^+ \text{ and } t \in A^*$$

such that

$$w = ts \text{ and } w_1 \cdots w_j = sw'_1 \cdots w'_k t$$

then t is the empty word. It follows that $j = k + 1$, $w_{i+1} = w'_i$ for $1 \leq i \leq k$ and $w_1 = s$.

Proposition 5 ([14] Lemma 17). *Let $x \in A^{\mathbb{Z}}$ be a uniformly recurrent sequence and u a prefix of $x_{[0,+\infty)}$. The set $\mathcal{R}_{x,u}$ is a circular code.*

The next four propositions are usually stated for one-sided sequences, but they are still true for uniformly recurrent $x \in A^{\mathbb{Z}}$ as these statements only depend on $x^+ = x_{[0,+\infty)}$. Indeed, x^+ is also a uniformly recurrent element of $A^{\mathbb{N}}$. We can define in the same way the derived sequence $D_u(x^+)$, the sets $\mathcal{R}_{x^+,u}$ and $R_{x^+,u}$, and, the map $\Theta_{x^+,u}$ when u is a prefix of x^+ . Moreover we clearly have:

$$D_u(x^+) = D_u(x)_{[0,+\infty)}, \mathcal{R}_{x^+,u} = \mathcal{R}_{x,u}, R_{x^+,u} = R_{x,u} \text{ and } \Theta_{x^+,u} = \Theta_{x,u}.$$

The following proposition enables to associate to a substitution another substitution on the alphabet $R_{x,u}$. We recall that admissible fixed points of primitive substitutions are uniformly recurrent. So the return word notion are meaningful.

Proposition 6 ([11] proof of Proposition 5.1). *Let $x \in A^{\mathbb{Z}}$ be an admissible fixed point of the primitive substitution σ and u be a nonempty prefix of $x_{[0,+\infty)}$. There exists a primitive substitution σ_u , defined on the alphabet $R_{x,u}$, characterized by*

$$\Theta_{x,u} \circ \sigma_u = \sigma \circ \Theta_{x,u}.$$

For a prefix u of $x_{[0,+\infty)}$, where $x \in A^{\mathbb{Z}}$ is an admissible fixed point of a primitive substitution σ , the derived sequence $D_u(x)$ is an admissible fixed point of the substitution σ_u . Indeed, we have

$$\Theta_{x,u} \circ \sigma_u(D_u(x)) = \sigma \circ \Theta_{x,u}(D_u(x)) = \sigma(x) = x = \Theta_{x,u} \circ D_u(x).$$

Then, the unicity of the concatenation into return words implies $D_u(x)$ is a fixed point of σ_u . It is clearly admissible as it is uniformly recurrent. This substitution, defined in the previous proposition, is called the *return substitution (to u)*. Moreover, we observe that for any integer $l > 0$

$$(\sigma^l)_u = (\sigma_u)^l.$$

Furthermore the incidence matrix of the return substitution has almost the same spectrum as the initial substitution. More precisely, we have:

Proposition 7 ([12] Proposition 9). *Let σ be a primitive substitution and let u be a prefix of $x_{[0,+\infty)}$ where $x \in A^{\mathbb{Z}}$ is an admissible fixed point of σ . The incidence matrices M_σ and M_{σ_u} have the same eigenvalues, except perhaps zero and roots of the unity.*

For instance, for the Tribonacci substitution τ , the return substitution τ_1 is the same as τ . On the other hand, if we consider the substitution

$$\sigma: 1 \mapsto 1123, 2 \mapsto 211, \text{ and } 3 \mapsto 21,$$

it is also a substitution of Pisot type and the incidence matrix of the return substitution τ_{11} has 0 as eigenvalue.

The next proposition is a key statement in the proof of our main result Theorem 1. It allows to work with a proper substitution.

Proposition 8. *Let $y = (y_i)_{i \in \mathbb{Z}}$ be an admissible fixed point of a primitive substitution τ on the alphabet R . Let $\Theta : R^* \rightarrow A^*$ be a non-erasing morphism, $x = \Theta(y)$ and (X, S) be the subshift generated by x . Then, there exist a primitive substitution ξ on an alphabet B , an admissible fixed point z of ξ , and a map $\phi : B \rightarrow A$ such that:*

- (1) $\phi(z) = x$;
- (2) If $\Theta(R)$ is a circular code, then ϕ is a conjugacy from $(\Omega(\xi), S)$ to (X, S) ;
- (3) If τ is proper (resp. right or left proper), then ξ is proper (resp. right or left proper);
- (4) There exists a prefix $u \in B^+$ of $z_{[0, +\infty)}$ such that $R_{y, y_0} = R_{z, u}$ and there is an integer $l \geq 1$ such that the return substitutions $\tau_{y_0}^l$ and ξ_u are the same.

Actually the first three statements of this proposition correspond to Proposition 23 in [14]. The substitution ξ is explicit in the proof.

Proof. The statements (1), (2), (3), and the fact that ξ is primitive, have been proven in [14]. We will just give the proof of the first statement because we need it to prove the fourth statement.

Considering a power of τ instead of τ if needed, we can assume that $|\tau(j)| \geq |\Theta(j)|$ for any $j \in R$. For all $j \in R$, let us denote $m_j = |\tau(j)|$ and $n_j = |\Theta(j)|$. We define

- An alphabet $B := \{(j, p); j \in R, 1 \leq p \leq n_j\}$;
- A morphism $\phi : B^* \rightarrow A^*$ by $\phi(j, p) = (\Theta(j))_p$;
- A morphism $\psi : R^* \rightarrow B^*$ by $\psi(j) = (j, 1)(j, 2) \cdots (j, n_j)$.

Clearly, we have $\phi \circ \psi = \Theta$. We define a substitution ξ on B by

$$\forall j \in R, 1 \leq p \leq n_j; \xi(j, p) = \begin{cases} \psi((\tau(j))_p) & \text{if } 1 \leq p < n_j \\ \psi((\tau(j))_{[n_j, m_j]}) & \text{if } p = n_j. \end{cases}$$

Thus for every $j \in R$, we have $\xi(\psi(j)) = \xi(j, 1) \cdots \xi(j, n_j) = \psi(\tau(j))$, *i.e.*,

$$(3.1) \quad \xi \circ \psi = \psi \circ \tau.$$

For $z = \psi(y)$ we obtain $\xi(z) = \psi(\tau(y)) = \psi(y) = z$, that is z is a fixed point of ξ . Moreover, $\phi(z) = \phi(\psi(y)) = \Theta(y) = x$ and we get the point (1).

Let us prove the fourth statement.

Let $u = \psi(y_0) \in B^*$. First, notice the morphism ψ is one-to-one and then we have $\psi(\mathcal{R}_{y,y_0}) = \mathcal{R}_{\psi(y),\psi(y_0)}$. It follows that

$$R_{y,y_0} = R_{\psi(y),\psi(y_0)} = R_{z,u},$$

and

$$\psi \circ \Theta_{y,y_0} = \Theta_{\psi(y),\psi(y_0)} = \Theta_{z,u}.$$

Therefore for the return substitution τ_{y_0} to y_0 , Proposition 6 and Relation (3.1) give

$$\Theta_{z,u} \circ \tau_{y_0} = \psi \circ \Theta_{y,y_0} \circ \tau_{y_0} = \psi \circ \tau \circ \Theta_{y,y_0} = \xi \circ \psi \circ \Theta_{y,y_0} = \xi \circ \Theta_{z,u}.$$

Consequently, we have $\tau_{y_0} = \xi_u$. \square

As a corollary of propositions 5, 6, 7 and 8, we get

Corollary 9. *Let σ be a primitive substitution. Then there exists a proper primitive substitution ξ on an alphabet B , such that*

- (1) $(\Omega(\sigma), S)$ is conjugate to $(\Omega(\xi), S)$;
- (2) there exists $l \geq 1$ such that the substitution matrices M_σ^l and M_ξ have the same eigenvalues, except perhaps 0 and 1.

Proof. Considering a power of σ instead of σ if needed, we can assume it has an admissible fixed point x . Let us fix a nonempty prefix u of $x_{[0,+\infty)}$. We can also assume that the word $\Theta_{x,u}(1)u$ is a prefix of $\sigma(u)$. By the very definition of return word, for any letter $i \in R_{x,u}$, the word $\Theta_{x,u}(i)u$ has the word u as a prefix. Then $\Theta_{x,u}(1)u$ is a prefix of the word $\sigma(\Theta_{x,u}(i)u)$. It follows from the equality in Proposition 6, that $\Theta_{x,u}(1)u$ is also a prefix of the word $\Theta_{x,u} \circ \sigma_u(i)$. The uniqueness of the coding by $\Theta_{x,u}(R_{x,u})$, implies that the word $\sigma_u(i)$ starts with 1, and the substitution σ_u is left proper.

Proposition 8 (together with Proposition 5), applied to the admissible fixed point $y = D_u(x)$ of σ_u and to $\Theta = \Theta_{x,u}$, gives the existence of a left proper primitive substitution $\xi' : C^* \rightarrow C^*$ such that $(\Omega(\sigma), S)$ is conjugate to $(\Omega(\xi'), S)$. Moreover, by Proposition 7 there exists an integer $l > 0$ such that the incidence matrices M_σ^l and $M_{\xi'}$ share the same eigenvalues, except perhaps 0 and 1.

Let us recall some morphism relations we have from the definitions and the proof of Proposition 8 taking σ_u for τ , Θ_u for Θ :

$$\Theta_u \circ \sigma_u = \sigma \circ \Theta_u, \quad \xi' \circ \psi = \psi \circ \sigma_u, \quad \phi \circ \psi = \Theta_u$$

and thus

$$(3.2) \quad \phi \circ \xi' \circ \psi = \sigma \circ \Theta_u.$$

To obtain a proper substitution we need to modify ξ' . Let $a \in C$ be the letter such that for all letter $b \in C$, $\xi'(b) = aw(b)$ for some

word $w(b)$. Now consider the substitution $\xi'' : C^* \rightarrow C^*$ defined by $\xi'' : b \mapsto w(b)a$. Observe that ξ'' is primitive and, for all n , one gets $\xi''^n(b)a = a\xi''^n(b)$. Then, ξ' and ξ'' define the same language, so we have $\Omega(\xi') = \Omega(\xi'') = \Omega(\xi)$, where $\xi = \xi' \circ \xi''$. Thus ξ is clearly proper. We conclude observing that $M_\xi = M_{\xi'} M_{\xi''} = M_{\xi'}^2$. \square

In terms of Pisot substitutions, Corollary 9 becomes:

Corollary 10. *Let σ be a substitution of Pisot type, then the substitution subshift associated with σ is conjugate to a substitution subshift $(\Omega(\xi), S)$, where ξ is a proper primitive substitution of weakly irreducible Pisot type.*

The example after Proposition 7 shows that the use of return substitutions seems to force to deal with W. I. Pisot substitutions. In fact, it is unavoidable to consider W. I. Pisot substitution to represent a Pisot substitution subshift by a proper substitution. For instance, consider the non-proper substitution $\sigma : 0 \mapsto 001, 1 \mapsto 10$. The dimension group of the associated subshift, computed in [10], is of rank 3. As a consequence, any proper substitution ξ representing the subshift $\Omega(\sigma)$ should be, at least, on 3 letters (see [14] for the details). Moreover, Cobham's theorem (see Theorem 14 in [13]) for minimal substitution subshifts implies that, taking powers if needed, ξ and σ share the same dominant eigenvalue. So, the substitution ξ can not be irreducible.

4. CONJUGACY WITH A DOMAIN EXCHANGE

In this section we give sufficient conditions on a primitive proper substitution so that the associated substitution subshift is measurably conjugate to a domain exchange in an Euclidean space.

4.1. Using Kakutani-Rohlin partitions. In this subsection, we will assume that ξ is a primitive proper substitution on a finite alphabet A equipped with a fixed order.

First let us recall a structural property of the system $(\Omega(\xi), S)$ in terms of Kakutani-Rohlin partitions.

Proposition 11 ([14]). *Let ξ be a primitive proper substitution on a finite alphabet A . Then for every $n > 0$,*

$$\mathcal{P}_n = \{S^{-k}\xi^{n-1}([a]); a \in A, 0 \leq k \leq |\xi^{n-1}(a)| - 1\}$$

is a clopen partition of $\Omega(\xi)$ defining a nested sequence of Kakutani-Rohlin partitions of $\Omega(\xi)$, that is to say:

- The sequence $(\xi^n(\Omega(\xi)))_{n \geq 0}$ is decreasing and the intersection is only one point;
- For every $n > 0$, \mathcal{P}_{n+1} is finer than \mathcal{P}_n ;
- The sequence $(\mathcal{P}_n)_{n > 0}$ spans the topology of $\Omega(\xi)$.

The fact that the family of clopen sets \mathcal{P}_n is a partition is an immediate consequence Mossé's theorem asserting that primitive substitutions are recognizable (see [?, ?]).

To be coherent with the notations in [7], we take the conventions $\mathcal{P}_0 = \{\Omega(\xi)\}$ and for an integer $n \geq 1$, $r_n(x)$ denotes the *entrance time* of a point $x \in \Omega(\xi)$ in $\xi^{n-1}(\Omega(\xi))$, that is

$$r_n(x) = \min\{k \geq 0; S^k x \in \xi^{n-1}(\Omega(\xi))\}.$$

By minimality, this value is finite for any $x \in \Omega(\xi)$ and the function r_n is continuous.

The homeomorphism $S_{\xi(\Omega(\xi))}: \xi(\Omega(\xi)) \ni x \mapsto S^{r_2(Sx)}(Sx) \in \xi(\Omega(\xi))$ is what is usually called the induced map of the system (X, S) on the clopen set $\xi(\Omega(\xi))$. Since we have the relation

$$(4.1) \quad \xi \circ S = S_{\xi(\Omega(\xi))} \circ \xi,$$

the induced system $(\xi(\Omega(\xi)), S_{\xi(\Omega(\xi))})$ is a factor of $(\Omega(\xi), S)$ via the map ξ (and in fact a conjugacy).

Note that for any integer $n > 0$,

$$(4.2) \quad r_n(Sx) - r_n(x) = \begin{cases} -1 & \text{if } x \notin \xi^{n-1}(\Omega(\xi)) \\ |\xi^{n-1}(a)| - 1 & \text{if } x \in \xi^{n-1}([a]), a \in A. \end{cases}$$

More precisely, we can relate the entrance time and the incidence matrix by the following equality (see Lemma 4 in [7]): For a primitive proper substitution ξ , we have for any $x \in \Omega(\xi)$ and $n \geq 2$

$$(4.3) \quad r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^{k-1} H(1) \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product, M_ξ^t is the transpose of the incidence matrix, $H(1) = (1, \dots, 1)^t$ and $s_k: \Omega(\xi) \rightarrow \mathbb{Z}^{\#A}$ is a continuous function defined by

$$s_k(x)_a = \#\{r_k(x) < i \leq r_{k+1}(x); S^i x \in \xi^{k-1}([a])\}, \quad \text{for } a \in A.$$

In other words, the vector $s_k(x)$ counts, in each coordinate $a \in A$, the number of time that the positive iterates of x meet the clopen set $\xi^{k-1}([a])$ until meeting, for the first time, the clopen set $\xi^k(\Omega(\xi))$ and after meeting the clopen set $\xi^{k-1}(\Omega(\xi))$.

The proof of the following lemma is direct from the definitions of s_k and r_k and Proposition 11.

Lemma 12. *For ξ a primitive proper aperiodic substitution, we have, for any $x \in \Omega(\xi)$,*

$$s_1(\xi(x)) = 0 \quad \text{and} \quad \forall k > 1, \quad s_k(\xi(x)) = s_{k-1}(x).$$

For every letter $a \in A$, $k \in \mathbb{N}^*$, we also have $s_k(x)_a \leq \sup_{b \in A} |\xi(b)|$.

From the ergodic point of view, it is well-known (see [?]) that subshifts generated by primitive substitutions are uniquely ergodic. We call μ the unique probability shift-invariant measure of $(\Omega(\xi), S)$. We have the following relations, for any positive integer n ,

$$(4.4) \quad \vec{\mu}(n) = M_\xi \vec{\mu}(n+1), \quad \text{and} \quad \langle H(1), \vec{\mu}(1) \rangle = 1,$$

where $\vec{\mu}(n) \in \mathbb{R}^{\#A}$ is the vector defined by

$$\vec{\mu}(n)_a = \mu(\xi^{n-1}([a])), \quad \text{for every letter } a \in A.$$

4.2. On the spectrum of a substitution subshift. From this subsection, we assume that ξ is a primitive proper substitution on a finite alphabet A .

Set ρ_1 be the maximum of modulus of eigenvalues of M_ξ smaller than 1 in modulus. Taking a power of ξ if needed, from classical results of linear algebra, there are M_ξ^t -invariant \mathbb{R} -vectorial subspaces E^0, E^u, E^b and E^s , a norm $\|\cdot\|_\xi$ and constants $C_s, C_{u,b}, > 0$ such that for any $n \in \mathbb{N}$

- (1) $\mathbb{R}^{\#A} = E^0 \oplus E^s \oplus E^u \oplus E^b$,
- (2) $M_\xi^t v = 0$ for all $v \in E^0$,
- (3) $0 < \|(M_\xi^t)^n v\|_\xi \leq C_s \rho_1^n \|v\|_\xi$, for all $v \in E^s \setminus \{0\}$,
- (4) $\|(M_\xi^t)^n v\|_\xi \geq C_{u,b} ((1 + \rho_1)/2)^n \|v\|_\xi$, for all $v \in E^b \oplus E^u$.

Let us apply some well-known arguments (see [20] or [15] for substitutions and [7] for a wider context). Let r_n and s_n be as defined in Section 4.1.

Proposition 13. *Let ξ be a primitive proper substitution on an alphabet A . If $\lambda \in \mathbb{S}$ is an eigenvalue of the system $(\Omega(\xi), S)$, then $(\lambda^{-r_n})_{n \geq 1}$ converges uniformly to a continuous eigenfunction associated with λ . Moreover, $\sum_{n \geq 1} \max_{a \in A} |\lambda^{|\xi^n(a)|} - 1|$ converges.*

So if $\exp(2i\pi\alpha)$ is an eigenvalue of the substitution subshift $(\Omega(\xi), S)$, for any letter a of the alphabet $|\xi^n(a)|\alpha$ converges to 0 mod \mathbb{Z} as n goes to infinity. In an equivalent way the vector $(M_\xi^t)^n \alpha(1, \dots, 1)^t$ tends to 0 mod $\mathbb{Z}^{\#A}$. The next lemma precises this for the usual convergence.

Lemma 14. *Let ξ be a primitive proper substitution on an alphabet A and let $\lambda = \exp(2i\pi\alpha)$ be an eigenvalue of the substitution subshift $(\Omega(\xi), S)$. Then, there exist $m \in \mathbb{N}$, $v, w \in \mathbb{R}^{\#A}$ such that*

$$(4.5) \quad \alpha H(1) = v + w, \quad (M_\xi^t)^m w \in \mathbb{Z}^{\#A} \text{ and } (M_\xi^t)^n v \xrightarrow{n \rightarrow \infty} 0,$$

where all entries of $H(1)$ are equal to 1. Moreover, for every vectors v, w satisfying (4.5)

i) *The convergence is geometric: there exist $0 < \rho \leq \rho_1 < 1$ and a constant C such that*

$$\|(M_\xi^t)^n v\| \leq C\rho^n, \text{ for any } n \in \mathbb{N}.$$

ii) *For any positive integer n ,*

$$\langle v, \vec{\mu}(n) \rangle = 0 \quad \text{and} \quad \alpha = \langle (M_\xi^t)^{n-1} w, \vec{\mu}(n) \rangle.$$

Proof. The first claim and item i) comes from the observation we made before the statement of the lemma and from [20]. We have just to show the item ii). Notice that the relations (4.4) give us for any positive integer

$$\langle v, \vec{\mu}(n) \rangle = \langle v, M_\xi^p \vec{\mu}(n+p) \rangle = \langle (M_\xi^t)^p v, \vec{\mu}(n+p) \rangle \xrightarrow{p \rightarrow +\infty} 0.$$

Then, we deduce

$$\alpha = \alpha \langle H(1), \vec{\mu}(1) \rangle = \langle v, \vec{\mu}(1) \rangle + \langle w, \vec{\mu}(1) \rangle = \langle w, \vec{\mu}(1) \rangle = \langle (M_\xi^t)^{n-1} w, \vec{\mu}(n) \rangle.$$

□

Observe that the decomposition $\alpha H(1) = v + w$ is not unique since for every $u \in E^0$, we have $\alpha H(1) = (v - u) + (w + u)$ is another decomposition fulfilling the properties (4.5).

If $\exp(2i\pi\alpha_1), \dots, \exp(2i\pi\alpha_{d-1})$ are $d-1$ eigenvalues of the substitution subshift $(\Omega(\xi), S)$, from Proposition 13 and Lemma 14 there exist $m \in \mathbb{N}$, $v(1), \dots, v(d-1)$, $w(1), \dots, w(d-1) \in \mathbb{R}^{\#A}$ such that for all $i \in \{1, \dots, d-1\}$:

$$(4.6) \quad \alpha_i H(1) = v(i) + w(i), \quad (M_\xi^t)^m w(i) \in \mathbb{Z}^{\#A} \text{ and } \sum_{n \geq 1} (M_\xi^t)^n v(i) \text{ converges.}$$

Notice that up to take a power of ξ , we can assume below that the constant m is equal to 1. By the observation made after Lemma 14, we also assume that each $v(i)$ has no component in E^0 .

Let us recall Proposition 3: a unimodular Pisot substitution subshift on d letters admits $d-1$ non trivial eigenvalues $\exp(2i\pi\alpha_1), \dots, \exp(2i\pi\alpha_{d-1})$ that are *multiplicatively independent*, i.e., $1, \alpha_1, \dots, \alpha_{d-1}$ are rationally

independent. This motivates the next proposition that interprets the arithmetical properties of the eigenvalues in terms of the vectors $v(i)$ and $w(i)$.

Proposition 15. *Let ξ be a primitive proper substitution. If $\exp(2i\pi\alpha_1), \dots, \exp(2i\pi\alpha_{d-1})$ are $d-1$ multiplicatively independent eigenvalues of the substitution subshift $(\Omega(\xi), S)$. Then, both families of vectors $\{M_\xi^t v(1), \dots, M_\xi^t v(d-1)\}$ and $\{M_\xi^t H(1), M_\xi^t w(1), \dots, M_\xi^t w(d-1)\}$ are linearly independent.*

Notice it also implies that both family of vectors $\{v(1), \dots, v(d-1)\}$ and $\{H(1), w(1), \dots, w(d-1)\}$ are linearly independent.

Proof. This proof is similar to the proof of Proposition 10 in [?]. We adapt it to our context, since it does not straightforwardly apply.

Assume there exist real numbers $\delta_0, \delta_1, \dots, \delta_{d-1}$, one being different from 0, such that $\delta_0 M_\xi^t H(1) + \sum_{i=1}^{d-1} \delta_i M_\xi^t w(i) = 0$. Since all the vectors are in $\mathbb{Z}^{\sharp A}$, we can assume that every δ_i is an integer. Taking the inner product of this sum with the vector $\mu(\vec{2})$, the normalization and recurrence relations of this vector (Relation (4.4)) together with the normalization with respect to each $w(i)$ in item *ii*) of Lemma 14, give us $\delta_0 + \sum_{i=1}^{d-1} \delta_i \alpha_i = 0$. The rational independence of the numbers $1, \alpha_1, \dots, \alpha_{d-1}$ implies that each δ_i equals 0. So the vectors $M_\xi^t H(1), M_\xi^t w(1), \dots, M_\xi^t w(d-1)$ are independent.

Now, assume that there exist real numbers λ_i such that $\sum_{i=1}^{d-1} \lambda_i M_\xi^t v(i) = 0$. We obtain $(\sum_{i=1}^{d-1} \lambda_i \alpha_i) M_\xi^t H(1) - \sum_{i=1}^{d-1} \lambda_i M_\xi^t w(i) = 0$. The independence of the vectors $M_\xi^t H(1), M_\xi^t w(1), \dots, M_\xi^t w(d-1)$ implies that $\lambda_i = 0$ for any i . So the vectors $M_\xi^t v(1), \dots, M_\xi^t v(d-1)$ are independent. \square

The following property gives a bound on the number of multiplicatively independent eigenvalues for a substitution subshift. When $u(1), \dots, u(n)$ are elements of the same vectorial space, we denote by $\text{Vect}(u(1), \dots, u(n))$ the vectorial subspace they span.

Proposition 16. *Let ξ be a proper primitive substitution. If the substitution subshift $(\Omega(\xi), S)$ admits $d-1$ eigenvalues $\exp(2i\pi\alpha_1), \dots, \exp(2i\pi\alpha_{d-1})$, then the vectorial space spanned by the vectors $v(i)$, $E_\xi = \text{Vect}(v(1), \dots, v(d-1))$, is a subspace of E^s .*

Moreover, if the eigenvalues are multiplicatively independent, then $d-1 \leq \dim E^s$.

Proof. For $i \in \{1, \dots, d-1\}$, the vector $v(i)$ can be decomposed using the \mathbb{R} -vectorial subspaces E^0, E^u, E^b and E^s . From Lemma 14 and

since the norms $\|\cdot\|$ and $\|\cdot\|_\xi$ are equivalent, it has no component in E^u and E^b . From the choice we made in (4.6), it has no component in E^0 . Thus $v(i)$ belongs to E^s . So we get $E_\xi \subset E^s$. The bound is obtained with Proposition 15. \square

To construct the domain exchange of a Pisot substitution subshift we will need the following direct corollary.

Corollary 17. *Let ξ be a proper primitive substitution. If the substitution subshift $(\Omega(\xi), S)$ admits $\dim E^s$ multiplicatively independent eigenvalues, then $E_\xi := \text{Vect}(v(1), \dots, v(\dim E^s)) = E^s$. In particular, we have that $M_\xi^t(E_\xi)$ equals E_ξ .*

Notice that for a unimodular Pisot substitution σ , $\dim E^s + 1$ equals the degree of the associated Pisot number, or the number of letters in the alphabet, and the eigenvalues of M_σ are all simple [8]. Thus, by Proposition 3, the proper W. I. Pisot substitution ξ associated to σ in Corollary 10, fulfills the conditions of Corollary 17.

4.3. Domain exchange factorization. In this section we prove Theorem 1. For this, we start recalling the hypotheses we need to get the result.

Hypotheses P. *Let ξ be a primitive proper substitution on a finite alphabet A such that:*

- i) The characteristic polynomial P_ξ admits a unique root greater than one in modulus.*
- ii) The substitution subshift $(\Omega(\xi), S)$ admits $\dim E^s = d - 1$ eigenvalues $\exp(2i\pi\alpha_1), \dots, \exp(2i\pi\alpha_{d-1})$ such that $1, \alpha_1, \dots, \alpha_{d-1}$ are rationally independent.*
- iii) Its Perron number β satisfies $\beta |\det M_{\xi|E^s}^t| = 1$.*

For instance, all these hypotheses apply to the proper substitution ξ of Corollary 10 associated with a unimodular Pisot substitution on d letters: The statement *i*) is obvious. The others come from the fact that the space E^s is spanned by the eigenspaces associated with the algebraic conjugates $\beta_1, \dots, \beta_{d-1}$ of the leading eigenvalue β of M_ξ that it is a Pisot number. The unimodular hypothesis implies $|\beta\beta_1 \cdots \beta_{d-1}| = 1$.

The strategy to prove Theorem 1 is to construct a projection map $\tilde{\Phi}$ from the subshift to an Euclidean space, thanks to the approximations of the eigenfunctions given by Proposition 13. After checking this map has good properties with respect to the shift map and the substitution, we follow the same strategy as in [8].

Let us assume Hypothesis P *ii*). By Formula (4.3) on the entrance time r_n , we get for each $i \in \{1, \dots, d-1\}$ and $x \in \Omega(\xi)$:

$$\alpha_i r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^{k-1} \alpha_i H(1) \rangle \pmod{\mathbb{Z}}.$$

Notice this formula is also true for every power of ξ instead of ξ . Then, with Formula (4.6) on the vectors $v(i)$, up to consider a power of ξ , we obtain

$$\begin{aligned} \alpha_i r_n(x) &= \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^{k-1} (v(i) + w(i)) \rangle \pmod{\mathbb{Z}} \\ &= \langle s_1(x), w(i) \rangle + \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^{k-1} v(i) \rangle \pmod{\mathbb{Z}}. \end{aligned}$$

Let $\tilde{\Phi}_n = ((\langle s_1(x), w(i) \rangle)_{1 \leq i \leq d-1}^t + \Phi_n(x))$ where

$$\Phi_n = \left(\sum_{k=1}^{n-1} \langle s_k, (M_\xi^t)^{k-1} v(i) \rangle \right)_{1 \leq i \leq d-1}^t.$$

The Proposition 13 and Lemma 14 ensure the sequence $(\tilde{\Phi}_n)_{n \geq 1}$ uniformly converges to a continuous function $\tilde{\Phi}: \Omega(\xi) \rightarrow \mathbb{R}^{d-1}$, explicitly defined for $x \in \Omega(\xi)$ by

$$\tilde{\Phi}(x) = \left(\langle s_1(x), w(i) \rangle + \sum_{k=1}^{+\infty} \langle s_k(x), (M_\xi^t)^{k-1} v(i) \rangle \right)_{1 \leq i \leq d-1}^t.$$

Let V be the matrix with rows $v(1)^t, \dots, v(d-1)^t$. Then, the map $\tilde{\Phi}$ may be written as

$$\tilde{\Phi}(x) = ((\langle s_1(x), w(i) \rangle)_{1 \leq i \leq d-1}^t + \Phi(x)) \quad \text{where} \quad \Phi(x) = V \sum_{k=1}^{+\infty} M_\xi^{k-1} s_k(x).$$

Lemma 18. *Assume Hypotheses P *i*), *ii*). There exist a continuous map $\Delta: \Omega(\xi) \rightarrow \mathbb{R}^{\#A}$ and a bijective linear map $N: \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ such that for $\alpha = (\alpha_1, \dots, \alpha_{d-1})^t$ and for any $x \in \Omega(\xi)$,*

- (1) $\tilde{\Phi} \circ S(x) = \tilde{\Phi}(x) + \alpha \pmod{\mathbb{Z}^{d-1}}$;
- (2) $\Phi(x) = V \Delta(x)$;
- (3) $M_\xi^t V^t = V^t N$;
- (4) *the matrix N is conjugate to the matrix $M_\xi^t|_{E^s}$ restricted to the space E^s ;*
- (5) $\Phi \circ \xi(x) = N^t(\Phi(x))$.

Proof. By the approximation property of the eigenfunctions in Proposition 13 (see also Relation (4.2)), we get $\tilde{\Phi} \circ S(x) = \tilde{\Phi}(x) + \alpha \pmod{\mathbb{Z}^{d-1}}$. Let us prove Statement (2). We have

$$(4.7) \quad \Phi_n(x) = V \left(\sum_{k=1}^{n-1} M_\xi^{k-1} s_k(x) \right)$$

$$(4.8) \quad = V \text{Proj} \left(\sum_{k=1}^{n-1} M_\xi^{k-1} s_k(x) \right),$$

where $\text{Proj}: \mathbb{R}^{\#A} \rightarrow E_\xi = \text{Vect}(v(1), \dots, v(d-1))$ denotes the orthogonal projection onto E_ξ . Recall that by Corollary 17, E_ξ has dimension $d-1$. Since $(\Phi_n)_{n \geq 1}$ uniformly converges (see Proposition 13 and Lemma 14), the projection $\text{Proj}(\sum_{k=1}^{n-1} M_\xi^{k-1} s_k(x))$ converges when n goes to infinity to the vector $\Delta(x)$ belonging to E_ξ for any $x \in \Omega(\xi)$. Therefore, we obtain Statement (2).

Let us prove the other statements. The basic properties of $s_n \circ \xi$ (Lemma 12) give for any $x \in \Omega(\xi)$ and $n > 2$,

$$(4.9) \quad \Phi_n \circ \xi = VM_\xi \left(\sum_{k=1}^{n-2} M_\xi^{k-1} s_k \right).$$

By the \mathbb{R} -independence of the vectors $v(i)$ (Proposition 15), the linear map $V^t: \mathbb{R}^{d-1} \rightarrow E_\xi$ is bijective and since $M_\xi^t(E_\xi) = E_\xi$ (Corollary 17), there exists a bijective linear map $N: \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ such that

$$(4.10) \quad M_\xi^t V^t = V^t N.$$

This shows Statement (4). Therefore, using (4.7) and (4.9), we obtain for $n > 2$,

$$\Phi_n \circ \xi = VM_\xi \sum_{k=1}^{n-2} M_\xi^{k-1} s_k = N^t \Phi_{n-1}.$$

Passing through the limit in n , we get (5) and this achieves the proof. \square

The proof we present below follows the same scheme that was used and that appeared explicitly in [8] and in other works as [21]. The main difference here is that the factor map $\tilde{\Phi}$ is not built in the same way. In the context of [8] their function $\tilde{\Phi}$ itself satisfies the relation (5) in Lemma 18 with the matrix N being diagonal. Here we need to decompose the factor map $\tilde{\Phi}$ into a sum $\Phi + \Upsilon$ where Φ satisfies (5) in Lemma 18.

Recall that μ denotes the unique probability shift-invariant measure of the system $(\Omega(\xi), S)$, and λ denotes the Lebesgue measure on \mathbb{R}^{d-1} . As the map Φ is continuous with values in \mathbb{R}^{d-1} , it is useful to observe

that for every Borel set B of $\Omega(\xi)$ the image $\Phi(B)$ is analytic, hence is Lebesgue measurable in \mathbb{R}^{d-1} .

Lemma 19. *Assume Hypotheses P i) – iii). There exists a constant C such that for every letter $a \in A$ we have:*

- (1) $\lambda(\Phi([a])) = C\mu([a])$,
- (2) for all integer n large enough, $\Phi([a])$ is the union of the measure theoretically disjoint sets

$$\Phi(S^{-k}\xi^n([b])), \text{ with } 0 \leq k < |\xi^n(b)|, [a] \cap S^{-k}\xi^n([b]) \neq \emptyset,$$

- (3) for any Borel set $B \subset [a]$,

$$\lambda(\Phi(B)) = C\mu(B).$$

Proof. The Item (1) of Lemma 18 implies $\Phi \circ S(x) - \Phi(x) = \Upsilon(x) - \Upsilon(S(x)) + \alpha \pmod{\mathbb{Z}^{d-1}}$, where $\Upsilon(x) = (\langle s_1(x), w(i) \rangle)_{1 \leq i \leq d-1}$ for any $x \in \Omega(\xi)$. Since the function s_1 takes finitely many values and Φ is bounded, $\Psi_{diff} = \Phi \circ S - \Phi$ is a continuous function taking values into a finite set. So the function Ψ_{diff} is locally constant. Hence, there exists some integer $n_0 \geq 0$ such that Ψ_{diff} is constant on each set $S^{-k}\xi^n([b])$, with $n > n_0$, $b \in A$ and $0 \leq k < |\xi^n(b)|$ (see Proposition 11).

Therefore, from Lemma 12 and Item (5) of Lemma 18, for any such b and k , there exists a vector $\delta(k, b) \in \mathbb{R}^{d-1}$ such that

$$\Phi(S^{-k}\xi^n([b])) = \delta(k, b) + \Phi(\xi^n([b])) = \delta(k, b) + (N^t)^n \Phi([b]).$$

By the very hypothesis P iii) and Item (4) of Lemma 18, we have $|\det N^t| = 1/\beta$, so we get

$$\begin{aligned} \lambda(\Phi(S^{-k}\xi^n([b]))) &= \lambda((N^t)^n \Phi([b])) = |\det(N^t)^n| \lambda(\Phi([b])) \\ &= \frac{1}{\beta^n} \lambda(\Phi([b])). \end{aligned}$$

Let $a \in A$, the partitions of $\Omega(\xi)$ in Proposition 11 provide

$$[a] = \bigcup_{\substack{0 \leq k < |\xi(j)|, b \in A \\ [a] \cap S^{-k}\xi^n([b]) \neq \emptyset}} S^{-k}\xi^n([b]).$$

Consequently,

(4.11)

$$\lambda(\Phi([a])) \leq \sum_{\substack{k, b; 0 \leq k < |\xi^n(b)|, \\ [a] \cap S^{-k}\xi^n([b]) \neq \emptyset}} \frac{1}{\beta^n} \lambda(\Phi([b])) = \frac{1}{\beta^n} (M_\xi^n(\lambda(\Phi([b])))_{b \in A}^t)_a.$$

From the Perron's Theorem, the above inequality is an equality and $(\lambda(\Phi([b])))_{b \in A}^t$ is a multiple of the eigenvector $(\mu([a]))_{a \in A}^t = \vec{\mu}(1)$ of the

dominant eigenvalue β^n of M_ξ^n . This shows Item (1). Notice that the equality in (4.11) also implies Item (2).

To prove Item (3), it is enough to use the partitions of $\Omega(\xi)$ given in Proposition 11 and the ideas at the beginning of this proof. We left it to the reader (see the proof of Proposition 4.3 in [8]). \square

With the next two propositions, we continue to follow the approach (and the proofs) in [8].

Proposition 20. *Assume Hypotheses P i) – iii). The map Φ is μ -a.e. one-to-one on each cylinder set $[a]$: there exists a μ -negligeable borelian subset $\mathcal{M} \subset \Omega(\xi)$ such that for any x and y in $[a] \setminus \mathcal{M}$ satisfying $\Phi(x) = \Phi(y)$, we have $x = y$.*

Proof. Let $a \in A$. From Lemma 19, the sets

$$\mathcal{N}_a^{(\ell)} = \bigcup_{\substack{(k_1, j_1) \neq (k_2, j_2); \\ 0 \leq k_1 < |\xi^\ell(b_1)|, [a] \cap S^{-k_1} \xi^\ell([b_1]) \neq \emptyset \\ 0 \leq k_2 < |\xi^\ell(b_2)|, [a] \cap S^{-k_2} \xi^\ell([b_2]) \neq \emptyset}} \Phi(S^{-k_1} \xi^\ell([b_1])) \cap \Phi(S^{-k_2} \xi^\ell([b_2]))$$

have zero λ -measure, for any $\ell \in \mathbb{N}$ large enough. Item (3) of Lemma 19 gives furthermore that the sets $\mathcal{M}_a^{(\ell)} = (\Phi)^{-1}(\mathcal{N}_a^{(\ell)})$ have zero measure with respect to μ .

Let x_1 and x_2 be two distinct elements of $[a]$ such that $\Phi(x_1) = \Phi(x_2)$. It suffices to show that they belong to some $\mathcal{M}_a^{(\ell)}$. Considering the partitions \mathcal{P}_ℓ , $\ell \geq 0$, of Proposition 11, there exist infinitely many $\ell \in \mathbb{N}$ with two distinct couples (k_1, b_1) and (k_2, b_2) , such that $0 \leq k_1 < |\xi^\ell(b_1)|$, $0 \leq k_2 < |\xi^\ell(b_2)|$, $x_1 \in S^{-k_1} \xi^\ell([b_1])$ and $x_2 \in S^{-k_2} \xi^\ell([b_2])$. Then, x_1 and x_2 belong to $\mathcal{M}_a^{(\ell)}$ for infinitely many ℓ , which achieves the proof. \square

Proposition 21. *Assume Hypotheses P i) – iii). The sets \mathcal{M} and \mathcal{N} are negligible borelian sets and the map $\Phi : \Omega(\xi) \setminus \mathcal{M} \rightarrow \Phi(\Omega(\xi)) \setminus \mathcal{N}$ is a bi-mesurable map.*

Proof. As ξ is proper, there exists a letter a such that $\xi(\Omega(\xi))$ is included in $[a]$. Therefore, from Proposition 20, Φ is one-to-one on $\xi(\Omega(\xi))$ except on a set \mathcal{M} of zero measure. By the basic properties of the map Φ (precisely Item (5) of Lemma 18), if two points $x, y \in \Omega(\xi)$ have the same image through Φ , then $\Phi(\xi(x)) = \Phi(\xi(y))$, and hence x and y belong to $\xi^{-1}(\mathcal{M})$.

Recall that the induced system on $\xi(\Omega(\xi))$ is a factor of $(\Omega(\xi), S)$ via the map ξ (see Relation (4.1)). This implies that the measure $\mu(\xi^{-1}(\cdot))$ is invariant for the induced system $(\xi(\Omega(\xi)), S_{\xi(\Omega(\xi))})$. It is a classical fact that induced dynamical systems of uniquely ergodic dynamical

systems are uniquely ergodic. Thus, $(\xi(\Omega(\xi)), S_{\xi(\Omega(\xi))})$ is uniquely ergodic. Hence its unique invariant probability measure is $\mu/\mu(\xi(\Omega(\xi)))$ and $\mu(\xi^{-1}(\mathcal{M}))$ is proportional to $\mu(\mathcal{M})$, so it is null. This achieves the proof. \square

The following proposition is a modification of the arguments in [22] Lemma 2.1.

Proposition 22. *Assume Hypotheses P i), ii). For any clopen set C in $\Omega(\xi)$, the set $\Phi(C)$ is regular, i.e.,*

$$\overline{\text{int } \Phi(C)} = \Phi(C),$$

where $\text{int } A$ denotes the interior of the set A for the usual Euclidean topology.

Proof. First we show that $\text{int } \Phi(\Omega(\xi))$ is not empty. Since $1, \alpha_1, \dots, \alpha_{d-1}$ are rationally independent, by Lemma 18, denoting by π the canonical projection $\mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}/\mathbb{Z}^{d-1} = \mathbb{T}^{d-1}$, the continuous map $\pi \circ \tilde{\Phi}: \Omega(\xi) \rightarrow \mathbb{T}^{d-1}$ has a dense image hence is onto. It follows that for any small $\epsilon > 0$, there exists a finite family \mathcal{V} of integer vectors such that

$$B_\epsilon(0) \subset \bigcup_{p \in \mathcal{V}} \tilde{\Phi}(\Omega(\xi)) + p.$$

As in the proof of Lemma 19 we consider the function $\Upsilon: \Omega(\xi) \rightarrow \mathbb{R}^{d-1}$ defined by $\Upsilon(x) = (\langle s_1(x), w(i) \rangle)_{1 \leq i \leq d-1}$ and recall that $\tilde{\Phi}(x) = \Upsilon(x) + \Phi(x)$. Set \mathcal{G} to be the finite set of values of the map Υ , thus we obtain

$$B_\epsilon(0) \subset \bigcup_{p \in \mathcal{V}} \tilde{\Phi}(\Omega(\xi)) + p \subset \bigcup_{g \in \mathcal{G}, p \in \mathcal{V}} \Phi(\Omega(\xi)) + p + g.$$

By the Baire Category Theorem, the set $\Phi(\Omega(\xi))$ has a nonempty interior.

Now let $\Omega^* = \Omega(\xi) \setminus U$ where U is the union of all open sets O such that $\text{int } \Phi(O) = \emptyset$. From the previous remark it is a nonempty compact set. Notice that $\Omega(\xi) \setminus \Omega^*$ is the union of countably many open (and then σ -compact) subsets. The image $\Phi(\Omega(\xi) \setminus \Omega^*)$ is then a countable union of compact sets each of those with an empty interior. Thus, again by the Baire Category Theorem, the set $\Phi(\Omega^*)$ is dense in $\Phi(\Omega(\xi))$ and since Ω^* is compact, $\Phi(\Omega^*) = \Phi(\Omega(\xi))$.

Let us show that Ω^* is S -invariant. Let O be an open set in $\Omega(\xi)$ such that $\text{int } \Phi(O)$ is empty. By Lemma 18, the function $\Phi \circ S - \Phi$ takes finitely many values, hence it is constant on a partition by clopen sets \mathcal{P} of $\Omega(\xi)$. For any atom C of \mathcal{P} the set $\text{int } \Phi(C \cap O)$ is empty and then $\text{int } \Phi(S(C \cap O))$ is also empty. As $\Phi(SO) = \cup_{C \in \mathcal{P}} \Phi(S(C \cap O))$, the set

$\Phi(SO)$ is a countable union of compact sets with empty interiors. Again by the Baire Category Theorem, the set $\Phi(SO)$ has empty interior, and Ω^* is S -invariant.

The minimality implies that $\Omega^* = \Omega(\xi)$, so the image by Φ of any open set has a nonempty interior.

Finally, let C be a clopen set, and assume that $A := \Phi(C) \setminus \overline{\text{int } \Phi(C)}$ is not empty. From the previous assertion, the set $\Phi(\Phi^{-1}(A) \cap C) = A$ contains a ball and then A intersects $\text{int } \Phi(C)$: a contradiction. This shows the statement of the proposition. \square

Recall the map $\Upsilon(x) = (\langle s_1(x), w(i) \rangle)_{1 \leq i \leq d-1}$ (defined in the proof of Lemma 19) is constant on the sets $S^{-k}\xi([a])$, $0 \leq k < |\xi(a)|$, $a \in A$. From Proposition 20 one can define a map $\Psi : \Phi(\Omega(\xi)) \rightarrow \mathbb{R}^{d-1}$ almost everywhere by $\Psi(y) = y + (\langle s_1(x), w(i) \rangle)_{1 \leq i \leq d-1}$ where $y = \Phi(x)$. It follows it satisfies $\Psi \circ \Phi = \Phi + \Upsilon = \tilde{\Phi}$ μ -a.e.. Recall that $\pi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}/\mathbb{Z}^{d-1} = \mathbb{T}^{d-1}$ denotes the canonical projection.

Proposition 23. *Assume Hypotheses P i) – iii).*

There exists a uniformly discrete set Λ such that for λ -a.e. $y \in \Phi(\Omega(\xi))$,

$$\{y_1 - y_2; \pi \circ \Psi(y_1) = \pi \circ \Psi(y_2) = y\} \subset \Lambda.$$

Moreover the maps $Z : \Omega(\xi) \rightarrow \mathbb{Z} \cup \{\infty\}$ and $Z_\Psi : \Phi(\Omega(\xi)) \rightarrow \mathbb{Z} \cup \{\infty\}$ defined by $Z(x) = \#(\pi \circ \Psi \circ \Phi)^{-1}(\{\pi \circ \Psi \circ \Phi(x)\})$ and $Z_\Psi(y) = \#(\pi \circ \Psi)^{-1}(\{\pi \circ \Psi(y)\})$ are finite and constant μ -a.e. and λ -a.e with the same constant.

Proof. For any $z \in \mathbb{Z}^{d-1}$, set A_z to be the set $\{y \in \Phi(\Omega(\xi)); \exists y' \in \Phi(\Omega(\xi)), y = y' + z\}$. We have $A_z = \Phi(\Omega(\xi)) \cap (\Phi(\Omega(\xi)) + z)$, so it is a Borel set. Notice that for any integer n , $Z_\Psi^{-1}(\{n\})$ is a finite intersection of such sets, so the map Z_Ψ is measurable. Since $Z = Z_\Psi \circ \Phi$ a.e., the map Z is also measurable. Recall that the map Φ is a.e. one-to-one (Proposition 21) and $\Psi \circ \Phi - \Phi = \Upsilon$, where Υ is defined in the proof of Lemma 19. Moreover the map Υ takes values in the finite set $\mathcal{H} = \{(\langle s_1(x), w(i) \rangle)_{1 \leq i \leq d-1}; x \in \Omega(\xi)\}$.

Hence for two elements $\Phi(x)$ and $\Phi(x')$, $x, x' \in \Omega(\xi)$ in the same $\pi \circ \Psi$ fiber (i.e. $\pi \circ \Psi \circ \Phi(x) = \pi \circ \Psi \circ \Phi(x')$), the difference $\Phi(x) - \Phi(x')$ a.s. belongs to $\mathbb{Z}^{d-1} + \mathcal{H} - \mathcal{H}$, a uniformly discrete set. This proves the first claim for $\Lambda = \mathbb{Z}^{d-1} + \mathcal{H} - \mathcal{H}$.

It follows, by the compactness of the set $\Phi(\Omega(\xi))$, that the maps Z_Ψ is a.e. finite. Hence it is also the case for the map Z . Moreover since it is T -invariant, we conclude by ergodicity that the map Z is a.e. constant. Since $Z = Z_\Psi \circ \Phi$, we also deduce the map Z_Ψ is a.e. constant with the same constant. \square

Proof of Theorem 1. Let ξ be a unimodular Pisot substitution. By Corollary 10 and Proposition 3, we can assume that ξ is proper and satisfies the hypotheses P i) – iii) (Subsection 4.3). Let E be the compact set $\Phi(\Omega(\xi))$. Proposition 11 and Lemma 18 on the properties of the map Φ both ensure the existence of an integer n such that the map $\Phi \circ S - \Phi$ is constant on each set $E_{n,a,k} := \Phi(S^{-k}\xi^n([a]))$ with $a \in A$ and $0 \leq k < |\xi^n(a)|$. Let T be the transformation such that its restriction on $E_{n,a,k}$ is the translation of the vector $(\Phi \circ S - \Phi)|_{E_{n,a,k}}$. Lemma 19 ensures it is well defined almost everywhere on E with respect to the Lebesgue measure λ . We have to show that this map defines a domain exchange.

First, notice, from Proposition 22, that each compact set $E_{n,a,k}$ is regular. From Lemma 19, the sets $(E_{n,a,k})_{a,k}$ are pairwise disjoint in measure for the measure λ . Notice, moreover, from the definition of T , that it satisfies $\Phi \circ S = T \circ \Phi$ μ -a.e. Hence we get $\lambda(T(E)) = \lambda(E)$ and $T(E) \subset E$ modulo λ null set. Thus, it remains to show that T preserves the Lebesgue measure λ .

By Proposition 21 the map T is one-to-one except on a set of zero measure. As a consequence, the sets $(T(E_{n,a,k}))_{a,k}$ are also disjoint in measure. Furthermore, the map T^{-1} is defined λ -a.e. and is a translation on a subset of full λ -measure of $T(E_{n,a,k})$. Basic measure theoretical arguments ensure that the map T preserves the Lebesgue measure, and T defines a domain exchange transformation.

Let us also mention that Item (5) of Lemma 18 provides this domain exchange is self-affine with respect to the sets $E_{n,a,k}$ and the linear part $(N^t)^n$. Finally, Proposition 21 shows this domain exchange is measurably conjugate to the subshift $(\Omega(\xi), S)$.

We recall $\Psi: E \rightarrow \tilde{\Phi}(\Omega(\xi)) \subset \mathbb{R}^{d-1}$ is the measurable map defined by $\Psi \circ \Phi = \tilde{\Phi}$ μ -a.e.. Proposition 23 gives the map $\pi \circ \Psi \circ \Phi: \Omega(\xi) \rightarrow \mathbb{T}^{d-1}$ is λ -a.e. R_ξ -to-one for some integer R_ξ and is μ -a.e. equal to $\pi \circ \tilde{\Phi}$ which is a (continuous) factor map onto a minimal rotation on the torus \mathbb{T}^{d-1} . Hence $\pi \circ \Psi \circ \Phi$ is a measurable a.e. R_ξ -to-one factor map from $(\Omega(\xi), S)$ onto a minimal rotation on \mathbb{T}^{d-1} . \square

4.4. Precisions on Theorem 1. Let us summarize the proof of Theorem 1. Figure 1 might be useful to follow our strategy.

Due to the use of Bratteli-Vershik representation there is a "rather" evident continuous map $\tilde{\Phi}$ to consider which, modulo \mathbb{Z}^{d-1} , commutes with the expected rotation $(\mathbb{T}^{d-1}, \text{Rot})$ (Lemma 18). This map is analogous to the one classically used to tackle the Pisot conjecture for substitutions [8]. By unique ergodicity of the rotation, the pullback by

$\pi \circ \tilde{\Phi}$ of the Lebesgue measure $\lambda_{\mathbb{T}^{d-1}}$ on the torus is the S -invariant measure μ .

But unfortunately it is not clear that this map is almost everywhere one-to-one. Nevertheless we can decompose $\tilde{\Phi}$ as $\tilde{\Phi} = \Psi \circ \Phi$ a.e. by two measurable maps having the following properties.

The function $\Phi: \Omega(\xi) \rightarrow E \subset \mathbb{R}^{d-1}$ is continuous and maps by pull-back the normalized Lebesgue measure $\lambda_E = \frac{1}{\lambda(E)}\lambda$ on E to the S -invariant measure μ (Lemma 19). It is one-to-one almost everywhere (Proposition 21).

The map $\Psi: E \rightarrow \mathbb{R}^{d-1}$ is a.e. a translation.

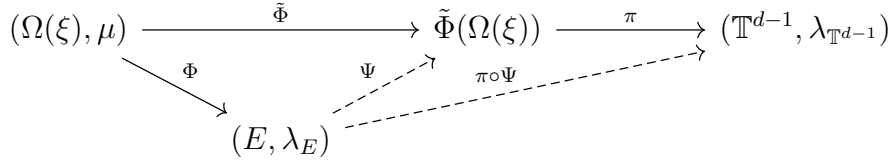


FIGURE 1. Sketch of the maps involved in the proof of Theorem 1: black arrows correspond to continuous maps and dashed arrows to almost everywhere defined maps.

Below we show that $\pi \circ \Psi$ is a.s. $\lambda(\Phi(\Omega(\xi)))$ -to-one, where $\lambda(\Phi(\Omega(\xi)))$ is the volume of the set $\Phi(\Omega(\xi)) = E$. Let R_ξ denote the constant of Proposition 23.

Proposition 24. *Assume Hypotheses P i)–iii). We have $\lambda(\Phi(\Omega(\xi))) = R_\xi$.*

Proof. Recall that the map Ψ satisfies $\Psi \circ \Phi = \tilde{\Phi} = \Phi + \Upsilon$ where Υ is defined in the proof of Lemma 19, so that Ψ restricted to a set $\Phi(S^{-k}\xi^n([a]))$, $a \in \mathcal{A}$, $0 \leq k < |\xi(a)|$, for n large enough, is defined on a subset $E'_{n,a,k}$ of full measure and is a.s. the translation by the constant value $\Upsilon|_{S^{-k}\xi^n([a])}$. Thus the closure of $\Psi(E'_{n,a,k})$ is a translated of $\Phi(S^{-k}\xi^n([a]))$ and Ψ admits a local continuous right inverse: a translation $\Psi_{E'_{n,a,k}}^{-1}: \overline{\Psi(E'_{n,a,k})} \rightarrow \Phi(S^{-k}\xi^n([a]))$ such that $\Psi \circ \Psi_{E'_{n,a,k}}^{-1} = \text{Id}$ a.s..

Since the map π is open, by Proposition 22, the sets $\pi(\overline{\Psi(E'_{n,a,k})})$ are regular. Moreover, taking n large enough, we can assume that the diameter of each set $\overline{\Psi(E'_{n,a,k})}$ is smaller than 1 so that π restricted to $\overline{\Psi(E'_{n,a,k})}$ is one-to-one.

Let I be a maximal (for the inclusion) subset of $\{E'_{n,a,k}; a \in \mathcal{A}, 0 \leq k < |\xi(a)|\}$ such that $V = \cap_{E \in I} \text{int}(\pi(\overline{\Psi(E)}))$ is not empty. Since the

sets are regular, if a set $\pi(\overline{\Psi(E'_{n,a,k})}) \subset \mathbb{T}^{d-1}$ intersects an open set O , then the interior of $\pi(\overline{\Psi(E'_{n,a,k})})$ also intersects O . It follows from the maximality that:

(H₁) The open set $(\pi \circ \Psi)^{-1}(V)$ intersects no set $\overline{E'_{n,a,k}}$ where $E'_{n,a,k}$ does not belong to I .

So, the map π restricted to $\overline{\Psi(E'_{n,a,k})}$ being one-to-one, almost every element in V admits $\#I$ pre-images in $\cup_{a,k} E'_{n,a,k}$ with respect to the map $\pi \circ \Psi$. Since V has a positive Lebesgue measure, Proposition 23 ensures $\#I = R_\xi$.

Moreover, by the definition of V , for any open subset $U \subset]0, 1[^{d-1}$ such that $\pi(U) \subset V$, one has

(H₂) for any set $E \in I$, there exists an integer vector $z \in \mathbb{Z}^{d-1}$ such that $U + z \subset \overline{\Psi(E)}$.

Denoting $\lambda_{\mathbb{T}^{d-1}}$ the Lebesgue measure on the torus \mathbb{T}^{d-1} , it follows

$$\begin{aligned} \lambda(U) &= \lambda_{\mathbb{T}^{d-1}}(\pi(U)) \\ &= \frac{1}{\lambda(\Phi(\Omega(\xi)))} \lambda((\pi \circ \Psi)^{-1}(\pi(U))) \\ &= \frac{1}{\lambda(\Phi(\Omega(\xi)))} \lambda(\Psi^{-1}(\bigcup_{E \in I, z \in \mathbb{Z}^{d-1}} \overline{\Psi(E)} \cap (U + z))) \\ &= \frac{1}{\lambda(\Phi(\Omega(\xi)))} \lambda(\bigcup_{E \in I, z \in \mathbb{Z}^{d-1}} \Psi_E^{-1}(U + z) \cap E) \\ &= \frac{R_\xi}{\lambda(\Phi(\Omega(\xi)))} \lambda(U). \end{aligned}$$

The first equality comes from properties of the Lebesgue measure on the torus. The second one comes from the fact that, by unique ergodicity, the pullback image of the Lebesgue measure on the torus by the measure theoretical factor map $\pi \circ \Psi$ is the normalized measure. The third equality follows from (H₁) and (H₂). The properties of the maps Ψ_E^{-1} provide the fourth equality. The last one follows from the facts the sets $E'_{n,a,k}$ form a measurable partition of $\Phi(\Omega(\xi))$, the maps Ψ_E^{-1} are translations and $\#I = R_\xi$.

This provides the conclusion. \square

The function $\pi \circ \tilde{\Phi}: \Omega(\xi) \rightarrow \mathbb{T}^{d-1}$ is a continuous factor map. We provide here a topological characterization of the constant R_ξ .

Proposition 25. *Assume Hypotheses P i) – iii). We have*

$$R_\xi = \inf_{x \in \Omega(\xi)} \#(\pi \circ \tilde{\Phi})^{-1}(\{\pi \circ \tilde{\Phi}(x)\}).$$

Proof. Let $\tilde{\Omega}(\xi) \subset \Omega(\xi)$ be a S -invariant set of full measure such that the map $x \mapsto \#(\pi \circ \Psi)^{-1}(\{\pi \circ \tilde{\Phi}(x)\})$ is constant, equal to R_ξ and any $\pi \circ \Psi$ fiber is uniformly discrete (Proposition 23). Let $x \in \tilde{\Omega}(\xi)$ and for any integer n , let $y_1^{(n)}, \dots, y_{R_\xi}^{(n)} \in \Phi(\Omega(\xi))$ be R_ξ elements in

$(\pi \circ \Psi)^{-1}(\{\pi \circ \tilde{\Phi}(S^n x)\})$ and let $x_1^{(n)}, \dots, x_{R_\xi}^{(n)} \in \Omega(\xi)$ be their pre-image by $\tilde{\Phi}$: $\tilde{\Phi}(x_i^{(n)}) = y_i^{(n)}$. Since the set of $y_j^{(n)}$ is uniformly discrete and the map $\tilde{\Phi}$ is uniformly continuous, the set of points $\{x_j^{(n)}; j = 1, \dots, R_\xi\}$ is uniformly (in n) separated.

Let $x_0 \in \Omega(\xi)$ be such that $\#(\pi \circ \tilde{\Phi})^{-1}(\{\pi \circ \tilde{\Phi}(x_0)\}) = \inf_{x \in \Omega(\xi)} \#(\pi \circ \tilde{\Phi})^{-1}(\{\pi \circ \tilde{\Phi}(x)\})$. Taking a sequence of integer $(n_i)_i$ such that $S^{n_i} x$ goes to x_0 when i goes to infinity, and taking accumulation points of $x_j^{(n_i)}$, provides R_ξ different points in $(\pi \circ \tilde{\Phi})^{-1}(\{\pi \circ \tilde{\Phi}(x_0)\})$. This proves the claim. \square

As a consequence of Proposition 25, to prove the Pisot conjecture it is enough to show the existence of a point in the torus that admits only one preimage by the continuous map $\pi \circ \tilde{\Phi}$. A similar observation was done in [3].

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