UNIMODULAR PISOT SUBSTITUTIONS AND DOMAIN EXCHANGES

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ABSTRACT. We show that any Pisot substitution on a finite alphabet is conjugate to a primitive proper substitution (satisfying then a coincidence condition) whose incidence matrix has the same eigenvalues as the original one, with possibly 0 and 1. Then, we prove also substitutive systems sharing this property and admitting "enough" multiplicatively independent eigenvalues (like for unimodular Pisot substitutions) are measurably conjugate to domain exchanges in Euclidean spaces which factorize onto minimal translations on tori. The combination of these results generalizes a well-known result of Arnoux-Ito to any unimodular Pisot substitution.

1. INTRODUCTION

A classical way to tackle problems in geometric dynamics is to code the dynamics through a well-chosen finite partition to obtain a "nice" subshift which is easier to study (see the emblematic works [Had98] and [Mor21]). The interesting aspects of the subshift could then be lifted back to the dynamical systems.

In the seminal paper [Rau82], G. Rauzy proposed to go in the other way round: take your favorite subshift and try to give it a geometrical representation. He took what is now called the Tribonacci substitution given by

$$\tau : 1 \mapsto 12, 2 \mapsto 13 \text{ and } 3 \mapsto 1,$$

and proved that the subshift it generates is measure theoretically conjugate to a rotation on the torus $\mathbb{T}^2$. A similar result was already known for substitutions of constant length under some necessary and sufficient conditions [Dek78]. Later, in [AR91], the author show that subshifts whose block complexity is $2n+1$, and satisfy what is called the Condition (*) (which includes the subshift generated by τ), are measure theoretically conjugate to an interval exchange on 3 intervals.

The substitution τ has the specificity to be a unimodular (and irreducible) Pisot substitution, that is, its incidence matrix has determinant 1, its characteristic polynomial is irreducible and its dominant eigenvalue is a Pisot number (all its algebraic conjugates are, in modulus, strictly less than 1). These properties provide key arguments to prove the main result in [Rau82]. It naturally leads to what is now called the **Pisot conjecture** for symbolic dynamics:

Let σ be a Pisot substitution. Then, the subshift it generates has purely discrete spectrum, i.e., *is measure theoretically conjugate to a translation on a group.*

Many attempts have been done in this direction. The usual strategy is the same as the Rauzy's one in [Rau82]: show first that the substitutive system is measurably conjugate to a domain exchange (see Definition 4). Then prove this system is measurably conjugate to a translation on a group.

A first important rigidity result, due to Host [Hos86], is that any eigenfunction of a primitive substitution is continuous. In a widely cited, but unpublished manuscript, Host also proved that the Pisot conjecture is true for unimodular substitutions defined on two letters, provided a condition called *strong coincidence condition* holds. This combinatorial condition first appeared in [Dek78] cited above. Barge an Diamond in [BD02], show then this condition is satisfied for any unimodular Pisot substitution on two letters. So the Pisot conjecture is true in this case [HS03]. Following the Rauzy's strategy, but in a different way from the Host's approach, Arnoux and Ito in [AI01], associate a self-affine domain exchange called *Rauzy fractal* to any unimodular Pisot substitution. They proved, this system is measurably conjugate to the substitutive system provided the substitution satisfies a combinatorial condition. Few time later, Host's results were generalized by Canterini and Siegel in [CS01] to any unimodular Pisot substitution and to the non-unimodular case [Sie03, Sie04], but without avoiding the strong coincidence condition. These works led to the development of a huge number of techniques to study the Rauzy fractals (see for instance [Fog02] and references therein). Let us mention also other fruitful geometrical approaches by using tilings in [BK06, BBJK06] and more recently in [Bar14] for the one-dimensional case.

In this paper, we show a similar result to [AI01] and [CS01] but skipping the combinatorial condition: any unimodular Pisot substitution is measurably conjugate to a self-affine domain exchange. Notice the domain exchange may, a priori, be different from the usual Rauzy fractal.

Theorem 1. *Let σ be a unimodular Pisot substitution on d letters and let (Ω, S) be the associated substitutive dynamical system. Then, there exist a self-affine domain exchange transformation $(E, \mathcal{B}, \tilde{\lambda}, T)$ in $\mathbb{R}^{d-1}$ and a continuous onto map $F\colon \Omega \to E$ which is a measurable conjugacy map between the two systems.*
If $\pi\colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$ denotes the canonical projection, then the map $\pi \circ F$ defines, for some constant $r \geq 1$, an a.e. r-to-one factor map from (Ω, S) to the dynamical system associated with a minimal translation on the torus $\mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$.

The toral translation is explicitly described in [CS01] (see also [Fog02]). To show the Pisot conjecture, one still have to show this domain exchange is conjugate to the toral translation.

We postpone to the next section the basic definitions and notions we use for dynamical systems, substitutive dynamics and Pisot substitutions. In Section 3, we prove by using the notion of return words, that any substitutive subshift is conjugate to a *proper* substitution (*i.e.*, having a nice combinatorial property implying, in particular, the strong coincidence condition). But, this new substitution may not be irreducible since the spectrum of its matrix contain the spectrum of a power of the older one but may also contain the values 0 and 1. We show then, in Section 4, that a such subshift, having enough multiplicatively independent eigenvalues (precised later), is measurably conjugate to a self-affine domain exchange. A byproduct of these two results gives us Theorem 1. The proof follows the same strategy as in [CS01]. However, here, the standard property of irreducibility of Pisot substitutions are not used. We strongly need, instead, a condition on the eigenvalues which is

precisely: the number of multiplicatively independent non trivial eigenvalues equals $\sum_{0<|\lambda|<1} \dim E_\lambda$ where E_λ denotes the eigenspace associated with the eigenvalue λ of the substitution matrix. This suggests a possible extension of these results to linearly recurrent symbolic systems like in [BJS12].

2. BASIC DEFINITIONS

2.1. **Words and sequences.** An *alphabet A* is a finite set of elements called *letters*. Its cardinality is $|A|$. A *word* over A is an element of the free monoid generated by A, denoted by A^*. Let $x = x_0 x_1 \cdots x_{n-1}$ (with $x_i \in A$, $0 \leq i \leq n-1$) be a word, its *length* is n and is denoted by $|x|$. The *empty word* is denoted by ϵ, $|\epsilon| = 0$. The set of non-empty words over A is denoted by A^+. The elements of $A^{\mathbb{Z}}$ are called *sequences*. If $x = \ldots x_{-1} x_0 x_1 \ldots$ is a sequence (with $x_i \in A$, $i \in \mathbb{Z}$) and $I = [k, l]$ an interval of $\mathbb{Z}$ we set $x_I = x_k x_{k+1} \cdots x_l$ and we say that x_I is a *factor* of x. If $k = 0$, we say that x_I is a *prefix* of x. The set of factors of length n of x is written $\mathcal{L}_n(x)$ and the set of factors of x, or the *language* of x, is denoted by $\mathcal{L}(x)$. The *occurrences* in x of a word u are the integers i such that $x_{[i,i+|u|-1]} = u$. If u has an occurrence in x, we also say that u *appears* in x. When x is a word, we use the same terminology with similar definitions.

A word u is *recurrent* in x if it appears in x infinitely many times. A sequence x is *uniformly recurrent* if it is recurrent and for each factor u, the difference of two consecutive occurrences of u in x is bounded.

2.2. **Morphisms and matrices.** Let A and B be two finite alphabets. Let σ be a *morphism* from A^* to B^*. When $\sigma(A) = B$, we say σ is a *coding*. We say σ is *non erasing* if there is no $b \in A$ such that $\sigma(b)$ is the empty word. If $\sigma(A)$ is included in B^+, it induces by concatenation a map from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$: $\sigma(\ldots x_{-1}.x_0 x_1 \ldots) = \ldots \sigma(x_{-1}).\sigma(x_0)\sigma(x_1)\ldots$, also denoted by σ. With the morphism σ is naturally associated its *incidence matrix* $M_\sigma = (m_{i,j})_{i \in B, j \in A}$ where $m_{i,j}$ is the number of occurrences of i in the word $\sigma(j)$. Notice that for any positive integer n we get $M_{\sigma^n} = M_\sigma^n$.

We say that an endomorphism is *primitive* whenever its incidence matrix is primitive (*i.e.*, when it has a power with strictly positive coefficients). The Perron's theorem tells that the dominant eigenvalue is a real simple root of the characteristic polynomial and is strictly greater than the modulus of any other eigenvalue.

2.3. **Substitutions and substitutive sequences.** We say that an endomorphism $\sigma : A^* \to A^*$ is a *substitution* if there exists a letter $a \in A$ such that the word $\sigma(a)$ begins with a and $\lim_{n \to +\infty} |\sigma^n(b)| = +\infty$ for any letter $b \in A$. In this case, for any positive integer n, $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$. Since $|\sigma^n(a)|$ tends to infinity with n, the sequence $(\sigma^n(\cdots aaa \cdots))_{n \geq 0}$ converges (for the usual product topology on $A^{\mathbb{Z}}$) to a sequence denoted by $\sigma^\infty(a)$. The substitution σ being continuous for the product topology, $\sigma^\infty(a)$ is a fixed point of σ: $\sigma(\sigma^\infty(a)) = \sigma^\infty(a)$.

A substitution σ is *left proper* (resp. *right proper*) if all words $\sigma(b)$, $b \in A$, starts (resp. ends) with the same letter. For short, we say that a left and right proper substitution is *proper*.

The *language* of $\sigma : A^* \to A^*$, denoted by $\mathcal{L}(\sigma)$, is the set of words having an occurrence in $\sigma^n(b)$ for some $n \in \mathbb{N}$ and $b \in A$. Notice that we have $\mathcal{L}(\sigma^n) = \mathcal{L}(\sigma)$ for any positive integer n.

2.4. **Dynamical systems and subshifts.** A *measurable dynamical system* is a quadruple $(X, \mathcal{B}, \mu, T)$ where X is a space endowed with a σ-algebra $\mathcal{B}$, a probability measure μ and measurable map $T : X \to X$ that preserves the measure μ, *i.e.*, $\mu(T^{-1}B) = \mu(B)$ for any $B \in \mathcal{B}$. This system is called *ergodic* if any T-invariant measurable set has measure 0 or 1. Two measurable dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{B}', \nu, S)$ are *measure theoretically conjugate* if we can find invariant subsets $X_0 \subset X$, $Y_0 \subset Y$ with $\mu(X_0) = \nu(Y_0) = 1$ and a bimeasurable bijective map $\psi \colon X_0 \to Y_0$ such that $S \circ \psi = \psi \circ T$ and $\mu(\psi^{-1}B) = \nu(B)$ for any $B \in \mathcal{B}'$.

By a *topological dynamical system*, or dynamical system for short, we mean a pair (X, S) where X is a compact metric space and S a continuous map from X to itself. It is well-known that such a system endowed with the Borel σ-algebra admits a probability measure μ preserved by the map S, and then form a measurable dynamical system. If the probability measure μ is unique, the system is said *uniquely ergodic*.

A *Cantor system* is a dynamical system (X, S) where the space X is a Cantor space, *i.e.*, X has a countable basis of its topology which consists of closed and open sets and does not have isolated points. The system (X, S) is *minimal* whenever X and the empty set are the only S-invariant closed subsets of X. We say that a minimal system (X, S) is *periodic* whenever X is finite.

A dynamical system (Y, T) is called a *factor* of, or is *semi-conjugate* to, (X, S) if there is a continuous and onto map $\phi \colon X \to Y$ such that $\phi \circ S = T \circ \phi$. The map ϕ is a *factor map*. If ϕ is one-to-one we say that ϕ is a *conjugacy*, and, that (X, S) and (Y, T) are *conjugate*.

For a finite alphabet A, we endow $A^{\mathbb{Z}}$ with the product topology. A *subshift* on A is a pair $(X, S_{|X})$ where X is a closed S-invariant subset of $A^{\mathbb{Z}}$ $(S(X) = X)$ and S is the *shift transformation*

$$S \quad : \quad \begin{array}{ccc} A^{\mathbb{Z}} & \to & A^{\mathbb{Z}} \\ (x_n)_{n \in \mathbb{Z}} & \mapsto & (x_{n+1})_{n \in \mathbb{Z}}. \end{array}$$

We call *language* of X the set $\mathcal{L}(X) = \{x_{[i,j]}; x \in X, i \leq j\}$. A set defined with two words u and v of A^* by

$$[u.v]_X = \{x \in X; x_{[-|u|, |v|-1]} = uv\}$$

is called a *cylinder set*. When u is the empty word we set $[u.v]_X = [v]_X$. The family of cylinder sets is a base of the induced topology on X. As it will not create confusion we will write $[u]$ and S instead of $[u]_X$ and $S_{|X}$.

For x a sequence on A, let $\Omega(x)$ be the set $\{y \in A^{\mathbb{N}}; y_{[i,j]} \in \mathcal{L}(x), \forall \, [i,j] \subset \mathbb{Z}\}$. It is clear that $(\Omega(x), S)$ is a subshift, it is called the *subshift generated by x*. Notice that $\Omega(x) = \overline{\{S^n x; n \in \mathbb{Z}\}}$. For a subshift (X, S) on A, the following are equivalent:

    (1) (X, S) is minimal;
    (2) For all $x \in X$ we have $X = \Omega(x)$;
    (3) For all $x \in X$ we have $\mathcal{L}(X) = \mathcal{L}(x)$.

We also have that $(\Omega(x), S)$ is minimal if and only if x is uniformly recurrent. Note that if (Y, S) is another subshift then, $\mathcal{L}(X) = \mathcal{L}(Y)$ if and only if $X = Y$.

2.5. **Substitutive subshifts.** For primitive substitutions σ, all the fixed points are uniformly recurrent and generate the same minimal and uniquely ergodic subshift (for more details see [Que87]). We call it the *substitutive subshift generated by σ* and we denote it (Ω_σ, S).

There is another useful way to generate subshifts. For $\mathcal{L}$ a language on the alphabet A, define $X_{\mathcal{L}} \subset A^{\mathbb{Z}}$ to be the set of sequences $x = (x_n)_{n \in \mathbb{Z}}$ such that $\mathcal{L}(x) \subset \mathcal{L}$. The pair $(X_{\mathcal{L}}, T)$ is a subshift and we call it the *subshift generated by* $\mathcal{L}$. If σ is a primitive substitution, then $\Omega_{\sigma} = X_{\mathcal{L}_{\sigma}}$ where $\mathcal{L}_{\sigma}$ denotes the language of σ [Que87]. It follows that for any positive integer n, σ^n and σ define the same subshift, that is $\Omega_{\sigma} = \Omega_{\sigma^n}$.

If the set Ω_{σ} is not finite, the substitution σ is called *aperiodic*.

An algebraic number β is called a *Pisot-Vijayaraghan number* if all its algebraic conjugates have a modulus strictly smaller than 1.

Definition 2. *Let σ be a primitive substitution and let P_{σ} denote the characteristic polynomial of the incidence matrix M_{σ}. We say that the substitution σ is*

- *of* Pisot type *(or* Pisot *for short) if P_{σ} has a dominant root $\beta > 1$ and any other root β' satisfies $0 < |\beta'| < 1$;*
- *of* weakly irreducible Pisot type *(or* W. I. Pisot *for short) whenever P_{σ} has a real Pisot-Vijayaraghan number as dominant root, its algebraic conjugates, with possibly 0 or roots of the unity as other roots;*
- *an* irreducible *substitution whenever P_{σ} is irreducible over $\mathbb{Q}$;*
- unimodular *if* $\det M_{\sigma} = \pm 1$.

For instance the Fibonacci substitution $0 \mapsto 01, 1 \mapsto 0$ and the Tribonacci substitution $1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ are unimodular substitutions of Pisot type. Whereas the Thue-Morse substitution $0 \mapsto 01$, $1 \mapsto 10$ is a W. I. Pisot substitution. Notice that the notions of Pisot, W. I. Pisot, irreducible, unimodular depend only on the properties of the incidence matrix. So starting from a Pisot (resp. W. I. Pisot, irreducible, unimodular) substitution, we get many examples of Pisot (resp. W. I. Pisot, irreducible, unimodular) substitutions by permuting the letters of the initial one.

Standard algebraic arguments ensure that a Pisot substitution is an irreducible substitution, and of course, a Pisot substitution is of weakly irreducible Pisot type. In the following we will strongly use the fact that for any substitution of (resp. W. I. Pisot, irreducible, unimodular) Pisot type σ and for every integer $n \geq 1$, the substitutions σ^n are also of (resp. W. I. Pisot, irreducible, unimodular) Pisot type. In [HZ98], the authors prove that the fixed point of a unimodular substitution of Pisot type is non-periodic for the shift, thus the subshift generated is a non-periodic minimal Cantor system.

2.6. **Dynamical spectrum of substitutive subshifts.** For a measurable dynamical system $(X, \mathcal{B}, \mu, T)$, a complex number λ is an *eigenvalue* of the dynamical system $(X, \mathcal{B}, \mu, T)$ with respect to μ if there exists $f \in L^2(X, \mu)$, $f \neq 0$, such that $f \circ T = \lambda f$; f is called an *eigenfunction* (associated with λ). The value 1 is the *trivial eigenvalue* associated with a constant eigenfunction. If the system is ergodic, then every eigenvalue is of modulus 1, and every eigenfunction has a constant modulus μ-almost surely. For a topological dynamical system, if the eigenfunction f is continuous, λ is called a *continuous eigenvalue*. The collection of eigenvalues is called the *spectrum* of the system, and form a multiplicative subgroup of the circle $\mathbb{S} = \{z \in \mathbb{C};\ |z| = 1\}$.

An important result for the spectrum is due to B. Host [Hos86]. It states that any eigenvalue of a substitutive subshift is a continuous eigenvalue. The following

proposition, claimed in [Hos92] (see Proposition 7.3.29 in [Fog02] for a proof), shows that the spectrum of a unimodular substitution of Pisot type is not trivial.

Proposition 3. *Let σ be a unimodular substitution of Pisot type and let α be a frequency of a letter in any infinite word of Ω_σ. Then $\exp(2i\pi\alpha)$ is a continuous eigenvalue of the dynamical system (Ω_σ, S).*

Recall that these frequencies are the coordinates of the right normalized eigenvector associated with the dominant eigenvalue of the incidence matrix of the substitution [Que87], and moreover for a unimodular Pisot substitution they are multiplicatively independent (Proposition 3.1 in [CS01]).
Notice the converse of the proposition is also true [BK06]. For a proof, see the remark below Lemma 14 or Proposition 11 in [CDHM03]. Actually, this is a general fact for any minimal Cantor system observed in [IO07]: given any continuous eigenvalue $\exp(2i\pi\alpha)$, α belongs to the additive subgroup of $\mathbb{R}$ generated by the intersection of sets of measures of clopen subsets for all the invariant probability measures. An other proof of that can be found in [CDHM03] (Proposition 11) but it was not pointed out.

2.7. **Domain exchange.** Let us recall that a compact Euclidean set is said *regular* if it equals the closure of its interior.

Definition 4. *We call* domain exchange transformation *a measurable dynamical system $(E, \mathcal{B}, \tilde{\lambda}, T)$ where E is a compact regular subset of an Euclidean space, $\tilde{\lambda}$ denotes the normalized Lebesgue measure on E and $\mathcal{B}$ denotes the Borel σ-algebra, such that:*

- *there exist compact regular subsets $E_1, \ldots, E_n$ such that $E = E_1 \cup \cdots \cup E_n$.*
- *The sets E_i are disjoint in measure for the Lebesgue measure λ:*

$$\lambda(E_i \cap E_j) = 0 \qquad when\ i \neq j.$$

- *For any index i, the map T restricted to the set E_i, is a translation such that $T(E_i) \subset E$.*

The domain exchange is said *self-affine*, if there is a finite number of affine maps $f_1, \ldots, f_\ell$ such that $E = \bigcup_{i=1}^{\ell} f_i(E)$ and sharing the same linear part.

3. Matrix eigenvalues and return substitutions

In this section, we recall the notion of return substitution introduced in [Dur98a] and that any primitive substitutive subshift is conjugate to an explicit primitive and proper substitutive subshift without changing too much the eigenvalues of the associated substitution matrix [Dur98b].

Let A be an alphabet and $x \in A^{\mathbb{Z}}$ and let u be a word of x. We call *return word* to u of x every factor $x_{[i,j-1]}$ where i and j are two successive occurrences of u in x. We denote by $\mathcal{R}_{x,u}$ the set of return words to u of x. Notice that for a return word v, vu belongs to $\mathcal{L}(x)$ and u is a prefix of the word vu. Suppose x is uniformly recurrent. It is easy to check that for any word u of x, the set $\mathcal{R}_{x,u}$ is finite. Moreover, for any sequence $y \in \Omega(x)$, we have $\mathcal{R}_{y,u} = \mathcal{R}_{x,u}$. The sequence x can be written naturally as a concatenation

$$x = \cdots m_{-1} m_0 m_1 \cdots, \qquad m_i \in \mathcal{R}_{x,u},\ i \in \mathbb{Z},$$

of return words to u, and this decomposition is unique. By enumerating the elements of $\mathcal{R}_{x,u}$ in the order of their first appearence in $(m_i)_{i \geq 0}$, we get a bijective map

$$\Theta_{x,u} \colon R_{x,u} \to \mathcal{R}_{x,u} \subset A^*,$$

where $R_{x,u} = \{1, \ldots, \mathrm{Card}\,(\mathcal{R}_{x,u})\}$. This map defines a morphism. We denote by $D_u(x)$ the unique sequence on the alphabet $R_{x,u}$ characterized by

$$\Theta_{x,u}(D_u(x)) = x.$$

We call it the *derived sequence of x on u*. Actually this sequence enables to code the dynamics of the induced system on the cylinder $[u]$. To be more precise, we need to introduce the following notions. A finite subset $\mathcal{R} \subset A^+$ is a *code* if every word $u \in A^+$ admits at most one decomposition in a concatenation of elements of $\mathcal{R}$.

We say that a code $\mathcal{R}$ is a *circular code* if for any words

$$w_1, \ldots, w_j, w, w'_1, \ldots, w'_k \in \mathcal{R}; s \in A^+ \text{ and } t \in A^*$$

such that

$$w = ts \text{ and } w_1 \ldots w_j = sw'_1 \ldots w'_k t$$

then t is the empty word. It follows that $j = k + 1$, $w_{i+1} = w'_{i'}$ for $1 \leq i \leq k$ and $w_1 = s$.

Proposition 5 ([Dur98a] Proposition 6)**.** *Let x be a uniformly recurrent sequence and let u be a non empty prefix of x.*

(1) *The set $\mathcal{R}_{x,u}$ is a circular code.*
(2) *If v is a prefix of u, then each return word on u belongs to $\Theta_{x,v}(R^*_{x,v})$, i.e., it is a concatenation of return words on v.*
(3) *Let v be a nonempty prefix of $D_u(x)$ and $w = \Theta_{x,u}(v)u$ then*
- *w is a prefix of x,*
- *$D_v(D_u(x)) = D_w(x)$.*
- *$\Theta_{x,u} \circ \Theta_{D_u(x),v} = \Theta_{x,w}.$*

The following proposition enables to associate to a substitution an other substitution on the alphabet $R_{x,u}$.

Proposition 6 ([Dur98a])**.** *Let $x \in A^{\mathbb{N}}$ be a fixed point of the primitive substitution σ which is not periodic for the shift and u be a nonempty prefix of x. There exists a primitive substitution σ_u, defined on the alphabet $R_{x,u}$, characterized by*

$$\Theta_{x,u} \circ \sigma_u = \sigma \circ \Theta_{x,u}.$$

Even if this proposition is not stated for bi-infinite sequences, it follows that each derived sequence $D_u(x)$, where u is a prefix of an aperiodic sequence $x \in A^{\mathbb{Z}}$ fixed by a primitive substitution σ, is a fixed point of the primitive substitution σ_u. To show this it is enough to check that

$$\Theta_{x,u} \circ \sigma_u(D_u(x)) = \sigma \circ \Theta_{x,u}(D_u(x)) = \sigma x = x = \Theta_{x,u} \circ D_u(x).$$

Since $\Theta_{x,u}(R_{x,u})$ is a circular code, we get that the sequence $D_u(x)$ is fixed by the substitution σ_u. This substitution, defined in the previous proposition, is called the *return substitution (to u)*. Moreover, we observe that for any integer $l > 0$

$$(\sigma^l)_u = (\sigma_u)^l.$$

Furthermore the incidence matrix of the return substitution has almost the same spectrum as the initial substitution. More precisely, we have:

Proposition 7 ([Dur98b]). *Let σ be a primitive substitution and let u be a prefix of a fixed point x which is not shift periodic. The incidence matrices M_σ and M_{σ_u} have the same eigenvalues, except perhaps zero and roots of the unity.*

For instance for the Tribonacci substitution τ, the induced substitution τ_1 is the same as τ. On the other hand, if we consider the substitution

$$\sigma \colon 1 \mapsto 1123, 2 \mapsto 211, \text{ and } 3 \mapsto 21,$$

it is also a substitution of Pisot type and the incidence matrix of the induced substitution σ_{11} has 0 as eigenvalue.

With the next property we obtain that if an induced system of a subshift (X, S) is a proper substitutive subshift (Ω, S), then the system (X, S) is conjugate to a proper substitutive subshift. The system (X, S) is called an *exduction* of the system (Ω, S).

Proposition 8. *Let $y = (y_i)_{i \in \mathbb{Z}}$ be a fixed point of an aperiodic primitive substitution σ on the alphabet R. Let $\Theta : R^* \to A^+$ be a non-erasing morphism, $x = \Theta(y)$ and (X, S) be the subshift generated by x.*
Then, there exist a primitive substitution ξ on an alphabet B, an admissible fixed point z of ξ, and a map $\phi : B \to A$ such that:

   (1) $\phi(z) = x$;
   (2) *If $\Theta(R)$ is a circular code, then ϕ is a conjugacy from (Ω_ξ, S) to (X, S);*
   (3) *If σ is proper (resp. right or left proper), then ξ is proper (resp. right or left proper);*
   (4) *There exists a prefix $u \in B^+$ of z such that $R_{y,y_0} = R_{z,u}$ and there is an integer $l \geq 1$ such that the return substitutions $\sigma_{y_0}^l$ and ξ_u are the same.*

Actually the first three statements of this proposition, correspond to Proposition 23 in [DHS99]. The substitution ξ is explicit in the proof.

Proof. The statements 1), 2), 3), and the fact that ξ is primitive, have been proven in [DHS99]. We will just give the proof of the first statement because we need it to prove the fourth statement.

Substituting a power of σ for σ if needed, we can assume that $|\sigma(j)| \geq |\Theta(j)|$ for any $j \in R$. For all $j \in R$, let us denote $m_j = |\sigma(j)|$ and $n_j = |\Theta(j)|$. We define

   • An alphabet $B := \{(j, p); j \in R, 1 \leq p \leq n_j\}$;
   • A morphism $\phi \colon B^* \to A^*$ by $\phi(j, p) = (\Theta(j))_p$;
   • A morphism $\psi \colon R^* \to B^*$ by $\psi(j) = (j, 1)(j, 2) \cdots (j, n_j)$.

Clearly, we have $\phi \circ \psi = \Theta$. We define a substitution ξ on B by

$$\forall j \in R, \ 1 \leq p \leq n_j; \ \xi(j, p) = \begin{cases} \psi((\sigma(j))_p) & \text{if } 1 \leq p < n_j \\ \psi((\sigma(j))_{[n_j, m_j]}) & \text{if } p = n_j. \end{cases}$$

Thus for every $j \in R$, we have $\xi(\psi(j)) = \xi(j, 1) \ldots \xi(j, n_j) = \psi(\sigma(j))$, *i.e.*,

$$(3.1) \qquad\qquad\qquad\qquad \xi \circ \psi = \psi \circ \sigma.$$

For $z = \psi(y)$ we obtain $\xi(z) = \psi(\sigma(y)) = \psi(y) = z$, that is z is a fixed point of ξ. Moreover $\phi(z) = \phi(\psi(y)) = \Theta(y) = x$ and we get the point (1).

Let us prove the fourth statement.

Let $u = \psi(y_0) \in B^*$ where $y = \dots y_{-1}.y_0 y_1 \dots$, $y_i \in B$, $i \in \mathbb{Z}$. First, notice the morphism ψ is one-to-one and then we have $\psi(\mathcal{R}_{y,y_0}) = \mathcal{R}_{\psi(y),\psi(y_0)}$. It follows that

$$R_{y,y_0} = R_{\psi(y),\psi(y_0)} = R_{z,u},$$

and

$$\psi \circ \Theta_{y,y_0} = \Theta_{\psi(y),\psi(y_0)} = \Theta_{z,u}.$$

Therefore for the return substitution σ_{y_0} to y_0, Proposition 6 and Relation (3.1) give

$$\Theta_{z,u} \circ \sigma_{y_0} = \psi \circ \Theta_{y,y_0} \circ \sigma_{y_0} = \psi \circ \sigma \circ \Theta_{y,y_0} = \xi \circ \psi \circ \Theta_{y,y_0} = \xi \circ \Theta_{z,u}.$$

Consequently, we have $\sigma_{y_0} = \xi_u$. $\qquad \square$

As a straightforward corollary of the propositions 6, 5, 8 and 7, we get

Corollary 9. *Let σ be a primitive aperiodic substitution. Then there exists a proper primitive substitution ξ on an alphabet B, such that*

(1) *(Ω_σ, S) is conjugate to (Ω_ξ, S);*
(2) *there exists $l \geq 1$ such that the substitution matrices M_σ^l and M_ξ have the same eigenvalues, except perhaps 0 and 1.*

Proof. Let us fix a nonempty prefix u of a fixed point x of σ. Thus x is not shift periodic. Substituting a power of σ for σ if needed, we can assume that the word $\Theta_{x,u}(1)u$ is a prefix of $\sigma(u)$. By the very definition of return word, for any letter $i \in R_{x,u}$, the word $\Theta_{x,u}(i)u$ has the word u as a prefix. Then $\Theta_{x,u}(1)u$ is a prefix of the word $\sigma(\Theta_{x,u}(i)u)$. It follows from the equality in Proposition 6, that $\Theta_{x,u}(1)u$ is also a prefix of the word $\Theta_{x,u} \circ \sigma_u(i)$. The uniqueness of the coding by $\Theta_{x,u}(R_{x,u})$, implies that the word $\sigma_u(i)$ starts with 1, and the substitution σ_u is left proper.

The propositions 5 and 8 imply the existence of a left proper primitive substitution ξ' such that (Ω_σ, S) is conjugate to $(\Omega_{\xi'}, S)$, moreover by Proposition 7 there exists an integer $l > 0$ such that the incidence matrices M_σ^l and $M_{\xi'}$ share the same eigenvalues, except perhaps 0 and 1.

To obtain a proper substitution we need to modify ξ'. Let a be the letter such that for all letter b, $\xi'(b) = aw(b)$ for some word $w(b)$. Now consider the substitution ξ'' defined by $\xi'' \colon b \mapsto w(b)a$. Then, ξ' and ξ'' define the same language, so we have $\Omega_{\xi'} = \Omega_{\xi''} = \Omega_\xi$ where ξ is the composition of substitutions $\xi' \circ \xi''$ and is proper. We conclude observing that $M_\xi = M_{\xi'} M_{\xi''} = M_{\xi'}^2$. $\qquad \square$

In terms of Pisot substitutions, Corollary 9 becomes:

Corollary 10. *Let σ be an aperiodic substitution of Pisot type, then the substitutive subshift associated with σ is conjugate to a substitutive subshift (Ω_ξ, S) where ξ is a proper primitive substitution of weakly irreducible Pisot type.*

The example after Proposition 7 shows that the use of return substitutions seems to force to deal with W. I. Pisot substitutions. In fact, it is unavoidable to consider W. I. Pisot substitution to represent a substitutive subshift by a proper substitution. For instance, consider the non-proper substitution $\sigma : 0 \mapsto 001$, $1 \mapsto 10$. The dimension group of the associated subshift, computed in [Dur96], is of rank 3. As a consequence, any proper substitution ξ representing the subshift Ω_σ should be, at least, on 3 letters (see [DHS99] for the details). Moreover Cobham's theorem (see Theorem 14 in [Dur98c]) for minimal substitutive subshifts implies that, taking

powers if needed, ξ and σ share the same dominant eigenvalue. So, the substitution ξ can not be irreducible.

4. Conjugacy with a domain exchange

In this section we give sufficient conditions on a primitive proper substitution so that the associated substitutive system is measurably conjugate to a domain exchange in an Euclidean space.

4.1. **Using Kakutani-Rohlin partitions.**

In this subsection, we will assume that ξ is a primitive proper substitution on a finite alphabet A equipped with a fixed order.

First let us recall a structure property of the system (Ω_ξ, S) in terms of Kakutani-Rohlin towers.

Proposition 11 ([DHS99]). *Let ξ be a primitive proper substitution on a finite alphabet A. Then for every $n > 0$,*

$$\mathcal{P}_n = \{S^{-k}\xi^{n-1}([a]); \ a \in A, \ 0 \le k \le |\xi^{n-1}(a)| - 1\}$$

is a clopen partition of Ω_ξ defining a nested sequence of Kakutani-Rohlin partition of Ω_ξ, more precisely:

- *The sequence of bases $(\xi^n(\Omega_\xi))_{n \ge 0}$ is decreasing and the intersection is only one point;*
- *For every $n > 0$, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$;*
- *The sequence $(\mathcal{P}_n)_{n>0}$ spans the topology of Ω_ξ.*

To be coherent with the notations in [BDM05], we take the conventions $\mathcal{P}_0 = \{\Omega_\xi\}$ and for an integer $n \ge 1$, $r_n(x)$ denotes the *entrance time* of a point $x \in \Omega_\xi$ in the base $\xi^{n-1}(\Omega_\xi)$, that is

$$r_n(x) = \min\{k \ge 0; \ S^k x \in \xi^{n-1}(\Omega_\xi)\}.$$

By minimality, this value is finite for any $x \in \Omega_\xi$ and the function r_n is continuous. The homeomorphism $S_{\xi(\Omega_\xi)} \colon \xi(\Omega_\xi) \ni x \mapsto S^{r_2(Sx)}(Sx) \in \xi(\Omega_\xi)$ is then the induced map of the system (X, S) on the clopen set $\xi(\Omega_\xi)$. Since we have the relation

$$(4.1) \qquad \xi \circ S = S_{\xi(\Omega_\xi)} \circ \xi,$$

the induced system $(\xi(\Omega_\xi), S_{\xi(\Omega_\xi)})$ is a factor of (Ω_ξ, S) via the map ξ (and in fact a conjugacy).

Note that for any integer $n > 0$,

$$(4.2) \qquad r_n(Sx) - r_n(x) = \begin{cases} -1 & \text{if } x \notin \xi^{n-1}(\Omega_\xi) \\ |\xi^{n-1}(a)| - 1 & \text{if } x \in \xi^{n-1}([a]), a \in A. \end{cases}$$

More precisely, we can relate the entrance time and the incidence matrix by the following equality (see Lemma in [BDM05]): For a primitive proper substitution ξ, we have for any $x \in \Omega_\xi$ and $n \ge 2$

$$(4.3) \qquad r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^k H(1) \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product, M_ξ^t is the transpose of the incidence matrix, $H(1) = (1, \cdots, 1)^t$ and $s_k : \Omega_\xi \to \mathbb{Z}^{\#A}$ is a continuous function defined by

$$s_k(x)_a = \#\{r_k(x) < i \le r_{k+1}(x); \ S^i x \in \xi^{k-1}([a])\}, \qquad \text{for } a \in A.$$

In other words, the vector $s_k(x)$ counts, in each coordinate $a \in A$, the number of time that the positive iterates of x meet the clopen set $\xi^{k-1}([a])$ before meeting for the first time the clopen set $\xi^k(\Omega_\xi)$ and after meeting the clopen set $\xi^{k-1}(\Omega_\xi)$.

The proof of the following lemma is direct from the definition and Proposition 11.

Lemma 12. *For ξ a primitive proper substitution, we have, for any $x \in \Omega_\xi$,*

$$s_1(\xi x) = 0 \qquad and \qquad \forall k > 1, \ s_k(\xi x) = s_{k-1}(x).$$

For any letter $a \in A$, $k \in \mathbb{N}^*$, we also have $s_k(x)_a \le \sup_{b \in A} |\xi(b)|$.

From the ergodic point of view, it is well-known (see [Que87]) that subshifts generated by primitive substitutions are uniquely ergodic. We call μ the unique probability shift-invariant measure of (Ω_ξ, S). We have the following relations, for any positive integer n,

$$(4.4) \qquad \vec{\mu}(n) = M_\xi \vec{\mu}(n+1), \qquad \text{and} \qquad \langle H(1), \vec{\mu}(1) \rangle = 1,$$

where $\vec{\mu}(n) \in \mathbb{R}^{\sharp A}$ is the vector defined by

$$\vec{\mu}(n)_a = \mu(\xi^{n-1}([a])), \qquad \text{for any letter } a \in A.$$

4.2. **On the spectrum of a substitutive subshift.** From this subsection, we assume that ξ is a primitive proper substitution on a finite alphabet A.

Taking a power of ξ if needed, from classical results of linear algebra, there are M_ξ^t-invariant $\mathbb{R}$-vectorial subspaces E^0, E^u, E^b and E^s such that

(1) $\mathbb{R}^{\#A} = E^0 \oplus E^s \oplus E^u \oplus E^b$,
(2) $M_\xi^t v = 0$ for all $v \in E^0$,
(3) $\lim_{k \to +\infty} (M_\xi^t)^k v = 0$, $(M_\xi^t)^n v \ne 0$ for all $v \in E^s \setminus \{0\}$ and any $n \in \mathbb{N}$,
(4) $\lim_{k \to +\infty} \|(M_\xi^t)^k v\| = +\infty$ for all $v \in E^u \setminus \{0\}$ and
(5) $((M_\xi^t)^k v)_{k \in \mathbb{Z}}$ is bounded and $(M_\xi^t)^n v \ne 0$ for all $v \in E^b \setminus \{0\}$ and $n \in \mathbb{N}$.

Let us apply some well-know facts to our context (see [Hos86] or [FMN96] for substitutions and [BDM05] for a wider context). Let r_n and s_n be as defined in Section 4.1.

Proposition 13. *Let ξ be a primitive proper substitution on an alphabet A. If $\lambda \in \mathbb{S}$ is an eigenvalue of the system (Ω_ξ, S), then $(\lambda^{-r_n})_{n \ge 1}$ converges uniformly to a continuous eigenfunction associated with λ. Moreover, $\sum_{n \ge 1} \max_{a \in A} |\lambda^{|\xi^n(a)|} - 1|$ converges.*

So if $\exp(2i\pi\alpha)$ is an eigenvalue of the substitutive system (Ω_ξ, S), for any letter a of the alphabet $|\xi^n(a)|\alpha$ converges to 0 mod $\mathbb{Z}$ as n goes to infinity. In an equivalent way the vector $(M_\xi^t)^n \alpha(1, \cdots, 1)^t$ tends to 0 mod $\mathbb{Z}^{\#A}$. The next lemma precises this for the usual convergence.

Lemma 14. *Let $\lambda = \exp(2i\pi\alpha)$ be an eigenvalue of a substitutive system (Ω_ξ, S) for a primitive proper substitution ξ on a finite alphabet A. Then, there exist $m \in \mathbb{N}$, $v \in \mathbb{R}^{\#A}$ and $w \in \mathbb{Z}^{\#A}$ such that*

$$\alpha H(1) = v + w, \qquad (M_\xi^t)^m w \in \mathbb{Z}^{\#A} \ and \ (M_\xi^t)^n v \to_{n \to \infty} 0,$$

where all entries of $H(1)$ are equal to 1. Moreover

i) *The convergence is geometric: there exist $0 \le \rho < 1$ and a constant C such that*

$$||(M_\xi^t)^n v|| \le C\rho^n, \ \text{for any } n \in \mathbb{N}.$$

ii) *For any positive integer n,*

$$\langle v, \vec{\mu}(n)\rangle = 0 \qquad and \qquad \alpha = \langle (M_\xi^t)^{n-1}w, \vec{\mu}(n)\rangle.$$

Proof. The first claim and item $i)$ comes from [Hos86]. We have just to show the item $ii)$. Notice that the relations (4.4) give us for any positive integer

$$\langle v, \vec{\mu}(n)\rangle = \langle v, M_\xi^p \vec{\mu}(n+p)\rangle = \langle (M_\xi^t)^p v, \vec{\mu}(n+p)\rangle \to_{p\to+\infty} 0.$$

We deduce then

$$\alpha = \alpha\langle H(1), \vec{\mu}(1)\rangle = \langle v, \vec{\mu}(1)\rangle + \langle w, \vec{\mu}(1)\rangle = \langle w, \vec{\mu}(1)\rangle = \langle (M_\xi^t)^{n-1}w, \vec{\mu}(n)\rangle.$$

$\square$

Remark. We get by Item $ii)$ of Lemma 14, that if $\exp(2i\pi\alpha)$ is an eigenvalue of a substitutive system, then α is in the subgroup of $\mathbb{R}$ generated by the component of the vector $\vec{\mu}(n)$, that is, in the subgroup generated by the frequency of occurrences of the words. This provides a converse to Proposition 3.

If $\exp(2i\pi\alpha_1), \ldots, \exp(2i\pi\alpha_{d-1})$ are $d-1$ eigenvalues of the substitutive system (Ω_ξ, S), from Proposition 13 and Lemma 14 there exist $m \in \mathbb{N}$, $v(1), \ldots v(d-1) \in \mathbb{R}^{\sharp A}$ and $w(1), \ldots, w(d-1) \in \mathbb{Z}^{\sharp A}$ such that for all $i \in \{1, \ldots, d-1\}$:

(4.5) $\alpha_i H(1) = v(i) + w(i), \quad (M_\xi^t)^m w(i) \in \mathbb{Z}^{\#A}$ and $\sum_{n \ge 1}(M_\xi^t)^n v(i)$ converges.

Notice that up to take a power of ξ, if needed, we can assume that the constant $m = 1$ and that any $v(i)$ has no component in E^0.

Let us recall Proposition 3: a unimodular Pisot substitutive subshift on d letters admits $d-1$ non trivial eigenvalues $\exp(2i\pi\alpha_1), \ldots, \exp(2i\pi\alpha_{d-1})$ that are *multiplicatively independent*, i.e., $1, \alpha_1, \ldots, \alpha_{d-1}$ are rationally independent. This motivates the next proposition that interprets the arithmetical properties of the eigenvalues in terms of the vectors $v(i)$ and $w(i)$.

Proposition 15. *If $\exp(2i\pi\alpha_1), \ldots, \exp(2i\pi\alpha_{d-1})$ are $d-1$ multiplicatively independent eigenvalues of the substitutive system (Ω_ξ, S) for a proper primitive substitution ξ. Then, both families of vectors $\{M_\xi^t v(1), \ldots, M_\xi^t v(d-1)\}$ and $\{M_\xi^t H(1), M_\xi^t w(1), \ldots, M_\xi^t w(d-1)\}$ are linearly independent.*

Notice it implies also that both family of vectors $\{v(1), \ldots, v(d-1)\}$ and $\{H(1), w(1), \ldots, w(d-1)\}$ are linearly independent.

Proof. The proof is similar to Proposition 10 in [BDM05]. We adapt it to our case. Assume there exist reals $\delta_0, \delta_1, \ldots, \delta_{d-1}$, one being different from 0, such that $\delta_0 M_\xi^t H(1) + \sum_{i=1}^{d-1}\delta_i M_\xi^t w(i) = 0$. Since all the vectors are in $\mathbb{Z}^{\sharp A}$, by an algebraic classical result, we can assume that any δ_i is an integer. Taking the inner product of this sum with the vector $\vec{\mu(2)}$, the normalization and recurrence relations of this vector (Relation (4.4)) together with the normalization with respect to each $w(i)$ in item $ii)$ of Lemma 14, give us $\delta_0 + \sum_{i=1}^{d-1}\delta_i\alpha_i = 0$. The rational

independence of the numbers $1, \alpha_1, \ldots, \alpha_{d-1}$ implies any $\delta_i = 0$. So the vectors $M_\xi^t H(1), M_\xi^t w(1), \ldots, M_\xi^t w(d-1)$ are independent.

Now, assume that there exist real numbers λ_i such that $\sum_{i=1}^{d-1} \lambda_i M_\xi^t v(i) = 0$. We obtain $(\sum_{i=1}^{d-1} \lambda_i \alpha_i) M_\xi^t H(1) - \sum_{i=1}^{d-1} \lambda_i M_\xi^t w(i) = 0$. The independence of the vectors $M_\xi^t H(1), M_\xi^t w(1), \ldots, M_\xi^t w(d-1)$ implies that $\lambda_i = 0$ for any i. So the vectors $M_\xi^t v(1), \ldots, M_\xi^t v(d-1)$ are independent. $\qquad\square$

The following property gives a bound on the number of multiplicatively independent eigenvalues for a substitutive subshift.

Proposition 16. *Let ξ be a proper primitive substitution. If the substitutive system (Ω_ξ, S) admits $d-1$ eigenvalues $\exp(2i\pi\alpha_1)$, ..., $\exp(2i\pi\alpha_{d-1})$, then the vectorial space spanned by the vectors $v(i)$, $E_\xi = \mathrm{Vect}(v(1), \ldots, v(d-1))$, is a subspace of E^s.*
Moreover if the eigenvalues are muliplicatively independent, then $d-1 \leq \dim E^s$.

Proof. For $i \in \{1, \ldots, d-1\}$, the vector $v(i)$ can be decomposed using the $\mathbb{R}$-vectorial subspaces E^0, E^u, E^b and E^s. From Lemma 14 it has no component in E^u and E^b. From the choice we made in (4.5), it has no component in E^0. Thus $v(i)$ belongs to E^s. So we get $E_\xi \subset E^s$. The bound by the dimension is obtained with Proposition 15. $\qquad\square$

To construct the domain exchange of a Pisot substitution we will need the following direct corollary.

Corollary 17. *Let ξ be a proper primitive substitution. If the substitutive system (Ω_ξ, S) admits $\dim E^s$ multiplicatively independent eigenvalues, then $E_\xi := \mathrm{Vect}(v(1), \ldots, v(\dim E^s)) = E^s$. In particular, we have $M_\xi^t(E_\xi) = E_\xi$.*

Notice that for a unimodular Pisot substitution σ, $\dim E^s + 1$ equals the degree of the associated Pisot number, or the number of letters in the alphabet. Thus, by Proposition 3, the proper W. I. Pisot substitution ξ associated to σ in Corollary 10, fulfills the conditions of Corollary 17.

4.3. **Semi-conjugacy with the domain exchange.** We prove the main result, Theorem 1, in this section. For this, we start recalling the very hypotheses we need to get the result.

Hypotheses P. *Let ξ be a primitive proper substitution on a finite alphabet A such that:*

   i) The characteristic polynomial P_ξ admits a unique root greater than one in modulus.
   ii) The substitutive subshift (Ω_ξ, S) admits $\dim E^s = d-1$ eigenvalues $\exp(2i\pi\alpha_1)$, ..., $\exp(2i\pi\alpha_{d-1})$ such that $1, \alpha_1, \ldots, \alpha_{d-1}$ are rationally independent.
   iii) Its Perron number β satisfies $\beta|\det M_{\xi|E^s}^t| = 1$.

For instance, all these hypotheses apply to the proper substitution ξ of Corollary 10 associated with a unimodular Pisot substitution on d letters: The statement $i)$ is obvious, the others come from the fact that the space E^s is spanned by the eigenspaces associated with the algebraic conjugates $\beta_1, \ldots, \beta_{d-1}$ of the Pisot number leading eigenvalue β of M_ξ. The unimodular hypothesis implies $|\beta\beta_1 \cdots \beta_{d-1}| = 1$.

From Hypotheses P ii) and by a byproduct of the formula (4.3) on the entrance time r_n, with Formula (4.5) on the vectors $v(i)$, up to consider a power of ξ, we get for any $i \in \{1, \dots, d-1\}$ and $x \in \Omega_\xi$

$$\alpha_i r_n(x) = \sum_{k=1}^{n-1} \langle s_k(x), (M_\xi^t)^k v(i) \rangle \mod \mathbb{Z}.$$

Let $F_n = \left(\sum_{k=0}^{n-1} \langle s_k, (M_\xi^t)^k v(i) \rangle \right)_{1 \leq i \leq d-1}^t$. The Proposition 13 and Lemma 14 ensure the sequence $(F_n)_n$ uniformly converges to a continuous function $F \colon \Omega_\xi \to \mathbb{R}^{d-1}$, explicitly defined for $x \in \Omega_\xi$ by

$$F(x) = \left(\sum_{k=1}^{+\infty} \langle s_k(x), (M_\xi^t)^k v(i) \rangle \right)_{1 \leq i \leq d-1}^t .$$

Let V be the matrix with rows $v(1)^t, \dots, v(d-1)^t$. Then, the map F may be written as

$$F(x) = V \sum_{k=1}^{+\infty} M_\xi^k s_k(x).$$

Lemma 18. *Assume Hypotheses P i), ii). There exist a continuous map $\Delta \colon \Omega_\xi \to \mathbb{R}^{\#A}$ and a bijective linear map $N \colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ such that for $\alpha = (\alpha_1, \dots, \alpha_{d-1})^t$ and for any $x \in \Omega_\xi$,*

  (1) $F \circ S(x) = F(x) + \alpha \mod \mathbb{Z}^{d-1}$;
  (2) $F(x) = V \Delta(x)$;
  (3) $M_\xi^t V^t = V^t N$;
  (4) *the matrix N is conjugated to the matrix $M_{\xi|E^s}^t$ restricted to the space E^s;*
  (5) $F \circ \xi(x) = N^t(F(x))$.

Proof. By the approximation property of the eigenfunctions in Proposition 13 (see also Relation (4.2)), we get $F \circ S(x) = F(x) + \alpha \mod \mathbb{Z}^{d-1}$.
Let us prove Statement (2). We have

(4.6) $$F_n(x) = \quad V \left(\sum_{k=1}^{n-1} M_\xi^k s_k(x) \right)$$

(4.7) $$= \quad V Proj \left(\sum_{k=1}^{n-1} M_\xi^k s_k(x) \right),$$

where $Proj \colon \mathbb{R}^{\#A} \to E_\xi = \text{Vect} (v(1), \dots, v(d-1))$ denotes the orthogonal projection onto E_ξ. Recall that by Corollary 17, E_ξ has dimension $d-1$. Since $(F_n)_n$ uniformly converges (see Proposition 13 and Lemma 14), the projection $Proj(\sum_{k=1}^{n-1} M_\xi^k s_k(x))$ converges when n goes to infinity to the vector $\Delta(x)$ belonging to E_ξ for any $x \in \Omega_\xi$. Therefore, we obtain Statement (2).
Let us prove the other statements. The basic properties of $s_n \circ \xi$ (Lemma 12) give for any $x \in \Omega_\xi$ and $n > 2$,

(4.8) $$F_n \circ \xi = V M_\xi \left(\sum_{k=1}^{n-2} M_\xi^k s_k \right).$$

By the $\mathbb{R}$-independence of the vectors $v(i)$ (Proposition 15), the linear map $V^t \colon \mathbb{R}^{d-1} \to E_\xi$ is bijective and since $M_\xi^t(E_\xi) = E_\xi$ (Corollary 17), there exists a bijective linear map $N \colon \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ such that

(4.9) $$M_\xi^t V^t = V^t N.$$

This shows Statement (4). Therefore, using (4.6) and (4.8), we obtain for $n > 2$,

$$F_n \circ \xi = V M_\xi \sum_{k=1}^{n-2} M_\xi^k s_k = N^t F_{n-1}.$$

Passing through the limit in n, we get (5) and this achieves the proof. $\qquad\square$

From Lemma 19 to Proposition 21, we use the strategy developed in [CS01] to tackle the Pisot conjecture. Recall that μ denotes the unique probability shift-invariant measure of the system (Ω_ξ, S), and λ denotes the Lebesgue measure on $F(\Omega_\xi)$.

Lemma 19. *Assume Hypotheses P $i) - iii)$. There exists a constant C such that for any letter $a \in A$ we have:*

(1) $\lambda(F([a])) = C\mu([a])$,

(2) *for any integer n large enough, $F([a])$ is the union of the measure theoretically disjoint sets*

$$F(S^{-k}\xi^n([b])), \;\; with \; 0 \le k < |\xi^n(b)|, [a] \cap S^{-k}\xi^n([b]) \ne \emptyset,$$

(3) *for any Borel set $B \subset [a]$,*

$$\lambda(F(B)) = C\mu(B).$$

Proof. Let $G = F \circ S - F - (\alpha_1, \ldots, \alpha_{d-1})^t$. From the basic properties of the map F (Lemma 5), it takes integer values. Being continuous, it is locally constant. Hence, there exists some integer $n_0 \ge 0$ such that G is constant on each sets $S^{-k}\xi^n([b])$, with $n > n_0$, $b \in A$ and $0 \le k < |\xi^n(b)|$ (see Proposition 11).
Therefore, from Item (5) of Lemma 18, for any such b and k, there exists a vector $\delta(k, b) \in \mathbb{R}^{d-1}$ such that

$$F(S^{-k}\xi^n([b])) = \delta(k, b) + F(\xi^n([b])) = \delta(k, b) + (N^t)^n F([b]).$$

By the very hypothesis P $iii)$, we have $|\det N^t| = 1/\beta$, so we get

$$\lambda(F(S^{-k}\xi^n([b]))) = \lambda((N^t)^n F([b])) = |\det(N^t)^n|\lambda(F([b])) = \frac{1}{\beta^n}\lambda(F([b])).$$

Let $a \in A$, the partitions of Ω_ξ in Proposition 11 provide

$$[a] = \bigcup_{\substack{0 \le k < |\xi(j)|, b \in A \\ [a] \cap S^{-k}\xi^n([b]) \ne \emptyset}} S^{-k}\xi^n([b]).$$

Consequently,

(4.10) $$\lambda(F([a])) \le \sum_{\substack{k,b; 0 \le k < |\xi^n(b)|, \\ [a] \cap S^{-k}\xi^n([b]) \ne \emptyset}} \frac{1}{\beta^n}\lambda(F([b])) = \frac{1}{\beta^n}(M_\xi^n(\lambda(F([b])))_{b \in A}^t)_a.$$

From the Perron's Theorem, the above inequality is an equality and $(\lambda(F([b])))_{b \in A}^t$ is a multiple of the eigenvector $(\mu([a]))_{a \in A}^t = \vec{\mu}(1)$ of the dominant eigenvalue β^n of M_ξ^n. This shows Item (1). Notice that the equality in (4.10) also implies Item (2).

To prove Item (3), it is enough to use the partitions of Ω_ξ given in Proposition 11 and the ideas in the beginning of this proof. This part is similar to the proof of Proposition 4.3 in [CS01] and we left it to the reader. $\square$

With the next proposition, we continue to follow the approach (and the proofs) in [CS01].

Proposition 20. *Assume Hypotheses P $i) - iii$). There exists a μ-negligeable measurable subset $\mathcal{N} \subset \Omega_\xi$ such that F is one-to-one on each cylinder set $[a]$: for any x and y in $[a] \setminus \mathcal{N}$ satisfying $F(x) = F(y)$, we have $x = y$.*

Proof. Let $a \in A$. From Lemma 19, the sets

$$\mathcal{N}_a^{(\ell)} = \bigcup_{\substack{(k_1, j_1) \neq (k_2, j_2); \\ 0 \leq k_1 < |\xi^\ell(b_1)|, [a] \cap S^{-k_1}\xi^\ell([b_1]) \neq \emptyset \\ 0 \leq k_2 < |\xi^\ell(b_2)|, [a] \cap S^{-k_2}\xi^\ell([b_2]) \neq \emptyset}} F(S^{-k_1}\xi^\ell([b_1])) \cap F(S^{-k_2}\xi^\ell([b_2]))$$

have zero λ-measure, for any $\ell \in \mathbb{N}$ big enough. Item (3) of Lemma 19, gives furthermore, the sets $\mathcal{M}_a^{(\ell)} = F^{-1}(\mathcal{N}_a^{(\ell)})$ have zero measure with respect to μ.

Let x_1 and x_2 be two distinct elements of $[a]$ such that $F(x_1) = F(x_2)$. It suffices to show that they belong to some $\mathcal{M}_a^{(\ell)}$. Considering the partitions $\{\mathcal{P}_\ell\}_{l \geq 0}$ of Proposition 11, there exist infinitely many $\ell \in \mathbb{N}$ with two distinct couples (k_1, b_1) and (k_2, b_2), such that $0 \leq k_1 < |\xi^\ell(b_1)|$, $0 \leq k_2 < |\xi^\ell(b_2)|$, $x_1 \in S^{-k_1}\xi^\ell([b_1])$ and $x_2 \in S^{-k_2}\xi^\ell([b_2])$. Then, x_1 and x_2 belong to $\mathcal{M}_a^{(\ell)}$ for infinitely many ℓ, which achieves the proof. $\square$

Proposition 21. *Assume Hypotheses P $i) - iii$). The map F is one-to-one except on a set of measure zero.*

Proof. As ξ is proper, there exists a letter a such that $\xi(\Omega_\xi)$ is included in $[a]$. Therefore, from Proposition 20, F is one-to-one on $\xi(\Omega_\xi)$ except on a set $\mathcal{N}$ of zero measure. By the basic properties of the map F (precisely Item (5) of Lemma 18), if two points $x, y \in \Omega_\xi$ have the same image through F, then $F(\xi(x)) = F(\xi(y))$, and hence $x, y \in \xi^{-1}(\mathcal{N})$.

Recall that the induced system on $\xi(\Omega_\xi)$ is a factor of (Ω_ξ, S) via the map ξ (see Relation (4.1)). This implies that the measure $\mu(\xi^{-1}(\cdot))$ is invariant for the induced system $(\xi(\Omega_\xi), S_{\xi(\Omega_\xi)})$. Since it is uniquely ergodic with respect to the induced probability measure, $\mu(\xi^{-1}(\mathcal{N}))$ is proportional to $\mu(\mathcal{N})$, so it is null. This achieves the proof. $\square$

The following proposition is a modification of the arguments in [Kul95] Lemma 2.1.

Proposition 22. *Assume Hypotheses P $i), ii$). For any clopen set c in Ω_ξ, the set $F(c)$ is regular,* i.e.,

$$\overline{\mathrm{int}\ F(c)} = F(c),$$

where int *A denotes the interior of the set A for the usual Euclidean topology.*

Proof. First we show that int $F(\Omega_\xi) \neq \emptyset$. Since $1, \alpha_1, \ldots, \alpha_{d-1}$ are rationally independant, by Lemma 18, denoting by π the canonical projection $\mathbb{R}^{d-1} \to \mathbb{R}^{d-1}/\mathbb{Z}^{d-1} = \mathbb{T}^{d-1}$, the map $\pi \circ F \colon \Omega_\xi \to \mathbb{T}^{d-1}$ has a dense image hence is onto. It follows that for any small ϵ, there exist a finite family $\mathcal{V}$ of integer vectors such that

$$B_\epsilon(0) \subset \bigcup_{p \in \mathcal{V}} F(\Omega_\xi) + p.$$

By the Baire Category Theorem, the set $F(\Omega_\xi)$ has a non empty interior.

Now let $\Omega^* = \Omega_\xi \setminus \bigcup \{O; O$ is open and int $F(O) = \emptyset\}$. From the previous remark it is a non empty compact set. Notice that $\Omega_\xi \setminus \Omega^*$ is the union of countably many open (and then σ-compact) subsets. The image $F(\Omega_\xi \setminus \Omega^*)$ is then a countable union of compact sets each of those with an empty interior. Again by the Baire Category Theorem, $F(\Omega^*)$ is dense in $F(\Omega_\xi)$ and since Ω^* is compact, $F(\Omega^*) = F(\Omega_\xi)$.

Let us show that Ω^* is S invariant. Let O be an open set in Ω_ξ such that int $F(O)$ is empty. By Lemma 18, the function $F \circ S - F - (\alpha_1, \ldots, \alpha_{d-1}) : \Omega_\xi \to \mathbb{Z}$ is constant on a partition by clopen sets $\mathcal{P}$ of Ω_ξ. For any atom c of $\mathcal{P}$, int $F(c \cap O) = \emptyset$ and then int $F(S(c \cap O))$ is empty. We have $F(SO) = \cup_{c \in \mathcal{P}} F(S(c \cap O))$ is then a countable union of compact sets with empty interiors. Again by the Baire Category Theorem, $F(SO)$ has empty interior, and Ω^* is S-invariant.

By minimality, we get that $\Omega^* = \Omega_\xi$, so the image by F of any open set has a non empty interior.

Finally, let C be a clopen set, and assume that $A := F(C) \setminus \overline{\text{int } F(C)}$ is not empty. From the previous assertion, $F(F^{-1}(A) \cap C) = A$ contains a ball and then A intersects int $F(C)$: a contradiction. This shows the statement of the proposition. $\qquad\square$

Let $\pi : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}/\mathbb{Z}^{d-1} = \mathbb{T}^{d-1}$ be the canonical projection.

Proposition 23. *Assume Hypotheses P $i) - iii)$. The map $Z : \Omega_\xi \to \mathbb{Z} \cup \{\infty\}$ defined by $Z(x) = \#(\pi \circ F)^{-1}(\{\pi \circ F(x)\})$ is finite and constant μ-a.e..*

Proof. We claim Z is measurable. For any $z \in \mathbb{Z}^{d-1}$, let A_z be the set $A_z = \{x \in \Omega_\xi; \exists y \in \Omega_\xi, F(x) = F(y) + z\}$. We have $A_z = F^{-1}(F(\Omega_\xi) + z)$, so it is a Borel set. Notice that for any integer n, $Z^{-1}(\{n\})$ is a finite intersection of such sets, so the claim is proved. By Proposition 21, the map F is a.e. one-to-one, and by compacity of the set $F(\Omega_\xi)$, the projection $\pi : F(\Omega_\xi) \to \mathbb{T}^{d-1}$ is finite-to-one, so the map Z is a.e. finite. It suffices to notice that Z is T-invariant, to conclude by ergodicity. $\qquad\square$

Proof of Theorem 1. Let ξ be a unimodular Pisot substitution. By Corollary 10 and Proposition 3, we can assume that ξ satisfies the hypotheses P $i) - iii)$ (Subsection 4.3). Let E be the compact set $F(\Omega_\xi)$. Proposition 11 and Lemma 18 on the properties of the map F both ensure the existence of an integer n such that the map $F \circ S - F$ is constant on any set $E_{n,a,k} := F(S^{-k}\xi^n([a]))$ with $a \in A$ and $0 \leq k < |\xi^n(a)|$. Let T be the transformation defined on $E_{n,a,k}$ by the translation of the vector $(F \circ S - F)_{|E_{n,a,k}}$. It follows from Lemma 19 and Proposition 22 that E and T define a domain exchange transformation on regular sets. Moreover, Item (5) of Lemma 18 provides it is self-affine with respect to the sets $E_{n,a,k}$ and the linear part $(N^t)^n$. Finally, Proposition 21 shows this domain exchange is measurably conjugate to the subshift (Ω_ξ, S) and Proposition 23 gives the map $\pi \circ F : \Omega_\xi \to \mathbb{T}^{d-1}$ is a.e. Z-to-one for some constant Z. $\qquad\square$

In the sequel, we denote by Z the constant of Proposition 23. We give here a characterization of this constant in term of the volume of the set $F(\Omega_\xi)$.

Proposition 24. *Assume Hypotheses P $i) - iii)$. We have $\lambda(F(\Omega_\xi)) = Z$.*

Proof. The canonical projection $\pi : \mathbb{R}^{d-1} \to \mathbb{T}^{d-1}$ defines a factor map from the domain exchange to a minimal translation on the torus. So the image measure of

the normalized measure $\frac{\lambda}{\lambda(F(\Omega_\xi))}$ is the Lebesgue measure on the torus. For any integrable function $f : F(\Omega_\xi) \to \mathbb{R}$, the conditional expectation $E(f|\pi^{-1}(\mathcal{B}_\mathbb{T}))$, with respect to the Borel σ-algebra of the torus $\mathcal{B}_\mathbb{T}$, is constant over any π-fiber. So it follows for a.e. points $y \in F(\Omega_\xi)$,

$$E(f|\pi^{-1}(\mathcal{B}_\mathbb{T}))(y) = \sum_{x \in F(\Omega_\xi);\ \pi(x)=\pi(y)} \gamma_{x,\pi(y)} f(x),$$

for some non negative measurable function $x \mapsto \gamma_{x,\pi(x)}$ such that

$$(4.11) \qquad\qquad \sum_{x;\pi(x)=\pi(y)} \gamma_{x,\pi(y)} = 1 \quad \text{for a.e. } y.$$

Since for any integrable function $f : F(\Omega_\xi) \to \mathbb{R}$ with support in a unit square U, we have

$$
\begin{aligned}
\frac{1}{\lambda F(\Omega_\xi)} \int_U f d\lambda &= \int_U E(f|\pi^{-1}(\mathcal{B}_\mathbb{T})) d\lambda \\
&= \int_{U \cap F(\Omega_\xi)} \gamma_{x,\pi(x)} f(x) d\lambda(x).
\end{aligned}
$$

We obtain that $\gamma_{x,\pi(x)} = \frac{1}{\lambda F(\Omega_\xi)}$. We get the conclusion by the equation (4.11) $\quad\square$

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