



HAL
open science

Theoretical and numerical stability for the reduced MHD models in the JOREK code.

Emmanuel Franck, Matthias Hoelzl, Alexander Lessig, Eric Sonnendrücker

► **To cite this version:**

Emmanuel Franck, Matthias Hoelzl, Alexander Lessig, Eric Sonnendrücker. Theoretical and numerical stability for the reduced MHD models in the JOREK code.. 2014. hal-01053713v2

HAL Id: hal-01053713

<https://hal.science/hal-01053713v2>

Preprint submitted on 16 Sep 2014 (v2), last revised 30 Jan 2015 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Theoretical and numerical stability for the reduced MHD models in the JOREK code.

Emmanuel Franck*[†], Matthias Hölzl*, Alexander Lessig*, Eric Sonnendrücker*

September 16, 2014

Abstract

In this paper we present a rigorous derivation of the reduced MHD models with and without parallel velocity that are implemented in the non-linear MHD code JOREK. The model we obtain contains some terms that have been neglected in the implementation but might be relevant in the non-linear phase. These are necessary to guarantee exact conservation with respect to the full MHD energy.

For the second part of this work, we have replaced the linearized time stepping of JOREK by a non-linear solver based on the Inexact Newton method including adaptive time stepping. We demonstrate that this approach is more robust especially with respect to numerical errors in the saturation phase of an instability and allows to use larger time steps in the non-linear phase.

Contents

1	Physical context and resistive MHD	2
1.1	Physical context: ITER and ELM's simulations	2
1.2	Resistive MHD	2
1.3	Reduced MHD models	3
1.4	JOREK code	3
2	Derivation of the models	4
2.1	Notation	4
2.2	Derivation of the model	5
2.2.1	Magnetic poloidal flux equation	6
2.2.2	Poloidal momentum equation	7
2.2.3	Equation on ρ and T	11
2.2.4	Equation on the parallel velocity	12
2.3	Final Model	14
2.4	Energy estimate	15
3	Spatial and Time discretization	20
3.1	Spatial discretization	20
3.2	Original time discretization and preconditioning	21
3.3	Nonlinear time solvers	22
4	Numerical results	22
4.1	Model without parallel velocity	23
4.1.1	First test case	23
4.1.2	Second test case	25
4.2	Model with parallel velocity	26

*Max-Planck-Institut für Plasmaphysik, Boltzmannstraße 2 D-85748 Garching, Germany.

[†]Email adress: emmanuel.franck@ipp.mpg.de

1 Physical context and resistive MHD

1.1 Physical context: ITER and ELM's simulations

The aim of magnetic confinement fusion is to develop a power plant that gains energy from the fusion of deuterium and tritium in a magnetically confined plasma. ITER, a tokamak type fusion experiment currently being built in the South of France, is the next step towards this goal.

In order to achieve a reasonable lifetime of first wall materials in ITER and future fusion reactors, plasma instabilities like edge localized modes (ELMs) [17], [18] need to be well controlled. Numerical modelling can help to develop the necessary understanding of the relevant physical processes. A physical model well suited to describe those large scale instabilities is the set of magneto-hydrodynamic equations (MHD) or the simpler reduced MHD model.

1.2 Resistive MHD

We begin by introducing the resistive MagnetoHydrodynamic (MHD) fluid system in 3D. $\mathbf{x} \in \mathbb{R}^3$ is the spatial component. We note ρ the mass density of the plasma, \mathbf{v} the velocity, T the temperature, \mathbf{B} the magnetic field, \mathbf{J} the current and \mathbf{E} the electric field. The evolution of the plasma can be described by the following MHD model

$$\left\{ \begin{array}{l} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla(p) = \mathbf{J} \times \mathbf{B} + \nu \Delta \mathbf{v} \\ \partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \\ \partial_t \mathbf{B} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{J}) \\ \nabla \times \mathbf{B} = \mathbf{J} \\ \nabla \cdot \mathbf{B} = 0 \end{array} \right. \quad (1)$$

with ν the viscosity coefficient, η the resistivity coefficient. The pressure is given by $p = \rho T$. Now we consider the ideal MHD (without viscosity and resistivity terms). The total energy for the ideal MHD is given by

$$E = \rho \frac{|\mathbf{v}|^2}{2} + \frac{|\mathbf{B}|^2}{2} + \frac{1}{\gamma - 1} p.$$

with $p = \rho T$ and $\gamma = \frac{5}{3}$. The total energy is given by the sum of the kinetic energy, magnetic energy and internal energy. The conservation law for the total energy is given by

$$\partial_t E + \nabla \cdot \left[\mathbf{v} \left(\rho \frac{|\mathbf{v}|^2}{2} + \frac{\gamma}{\gamma - 1} p \right) + |\mathbf{B}|^2 \mathbf{v} - (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \right] = 0$$

If the unknowns are equal to zero at the boundary, using the flux-divergence theorem we obtain the total energy conservation

$$\frac{d}{dt} \int_{\Omega} E = 0.$$

This result comes from the flux divergence theorem and the assumptions on the boundary conditions. We consider the energy conservation important for the numerical stability.

1.3 Reduced MHD models

The reduced resistive MHD models are designed to reduce the CPU cost by making assumptions reasonable for the tokamak configuration. For the magnetic field we consider that it is much larger in the toroidal direction than the magnitude in the poloidal plane. Consequently we consider that the magnetic field is constant in time in the toroidal direction and depends on the poloidal flux function ψ . The magnetic field is given by

$$\mathbf{B} = \mathbf{B}_\phi + \mathbf{B}_{pol} = \frac{F_0}{R} \mathbf{e}_\phi + \frac{1}{R} \nabla \psi \times \mathbf{e}_\phi.$$

The velocity field depends on the electrical potential in the poloidal plane and the parallel velocity (parallel to the magnetic field). It is given by

$$\mathbf{v} = \mathbf{v}_{pol} + \mathbf{v}_{||} = -R \nabla u \times \mathbf{e}_\phi + v_{||} \mathbf{B}.$$

If the parallel velocity is equal to zero the term which depends on the electrical potential is nearly incompressible. This potential formulation allows to reduce the number of variables and filter the fast magnetosonic waves of the MHD for nearly incompressible flows. The full MHD system with all waves is a very stiff problem with restrictive CFL stability conditions or bad conditioning for the numerical methods. Consequently eliminating these waves allow to obtain a less stiff problem which is easier to solve. To obtain the final reduced models we plug the potential formulations in the full MHD model and use projections to obtain the equations on u and $v_{||}$. For the electrical potential equation we project by applying the operator $\mathbf{e}_\phi \cdot \nabla \times (R^2 \dots)$ to the momentum equation. To obtain the equation on $v_{||}$ we project by applying the operator $\mathbf{B} \cdot (\dots)$ to the momentum equation. In the following sections we derive exactly the reduced MHD model used in the JOREK code and prove that this model satisfies the energy conservation law. Indeed the energy conservation is a very important property to ensure the numerical stability of the time method for nonlinear models.

1.4 JOREK code

The non-linear JOREK code [7], [15] solves the reduced or full MHD equations in realistic three-dimensional tokamak geometry. The spatial discretization is given by isoparametric Bezier finite elements in the poloidal plane and a toroidal Fourier decomposition. As a first step in a simulation, the Grad Shafranov equation given by

$$\Delta^* \psi = -R^2 \frac{\partial p}{\partial \psi} - F \frac{\partial F}{\partial \psi}$$

with $\Delta^* \psi = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial Z^2}$, $F = R \mathbf{B}_\phi$, p the pressure and \mathbf{B}_ϕ the toroidal magnetic field, is solved on an initial grid (Fig. 1, on the left) to calculate the plasma equilibrium and again on a grid aligned to the equilibrium magnetic flux surfaces (Fig. 1, on the right in blue). This second grid is used during the following time integration as well, in which the (reduced) MHD equations are solved by a fully implicit method (Crank-Nicholson or Gear scheme). The resulting large sparse matrix system is solved using the iterative GMRES method with a physics-based preconditioning during which the direct sparse matrix solver Pastix is employed. JOREK is implemented in Fortran 90/95 and uses a hybrid MPI plus OpenMP parallelization suitable for large scale simulations on supercomputers. The realistic treatment of the tokamak geometry including the plasma region, separatrix and X-point, as well as scrape-off layer and divertor region makes the code suitable for simulations of many different types of plasma instabilities.

In this paper, we provide a rigorous algebraic derivation of the reduced MHD equations that are implemented in JOREK from the full MHD equations (Sections 2.1–2.3) and investigate the energy conservation properties of this reduced MHD model (Section 2.4). In Section 3, we describe an approach of introducing a non-linear time solver based on inexact Newton iterations into

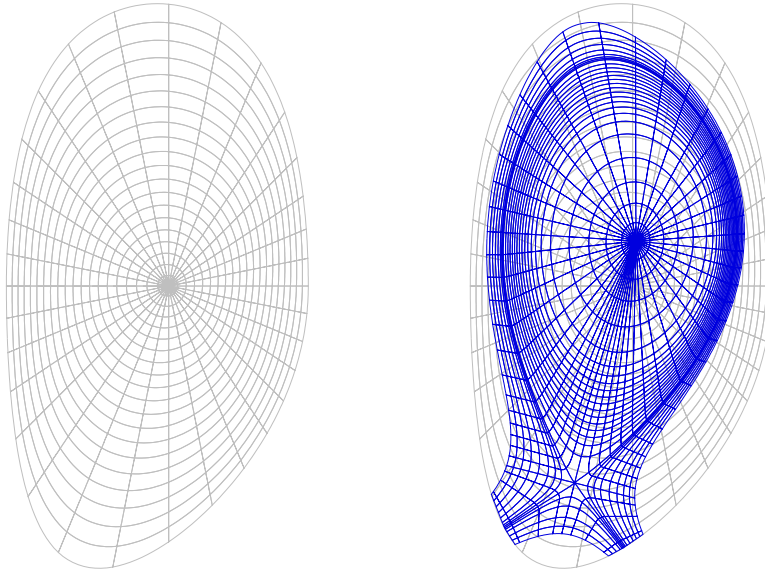


Figure 1: Initial grid (grey) and flux aligned grid (blue) used in JOREK X-point simulations (shown with reduced resolutions).

JOREK in order to increase the robustness and performance of the code in highly non-linear stages. Numerical tests of the non-linear time stepping are presented in Section 4 and brief conclusions of the work are provided in Section 5.

2 Derivation of the models

The derivation of the reduced MHD models is not a new topic of research. We can find the derivation of the model with parallel velocity for small curvature in the Tokamak in [13], [6]. This derivation is based on an asymptotic analysis with the small parameter ε which correspond to the curvature. In this type of derivation some terms are neglected. In our case we use an algebraic derivation. Using the same assumptions for the magnetic field and the velocity field as in [13]-[6]. We can find the same method and the same type of calculation in the works of R. Sart and B. Després in [1], [3]-[13]. In these papers the authors propose two methods to derive the reduced MHD in the low β case where β is the ratio between plasma and magnetic pressures (which correspond to $p \ll |\mathbf{B}_\phi|^2$) for general density profiles. In this work we use the same technique as the first paper, but we apply this method to obtain the more complicated models used in the JOREK code. So far, no rigorous derivation existed for the reduced MHD models implemented in JOREK. For this reason we give these proofs and identify previously neglected terms in the reduced MHD physics models.

2.1 Notation

The fundamental coordinate system used in JOREK is the cylindrical system (R, ϕ, Z) illustrated by the Fig. 2. The connection to Cartesian coordinates is given by

$$\begin{cases} X = R \cos \phi \\ Y = -R \sin \phi \\ Z = Z \end{cases} \quad (2)$$

We define $\mathbf{e}_R = \nabla R$, $\frac{1}{R}\mathbf{e}_\phi = \nabla \phi$ and $\mathbf{e}_Z = \nabla Z$ with R, ϕ, Z functions of (X, Y, Z) . By definition of the basis we have $\mathbf{e}_R \times \mathbf{e}_\phi = -\mathbf{e}_Z$, $\mathbf{e}_\phi \times \mathbf{e}_Z = -\mathbf{e}_R$ and $\mathbf{e}_Z \times \mathbf{e}_R = -\mathbf{e}_\phi$.

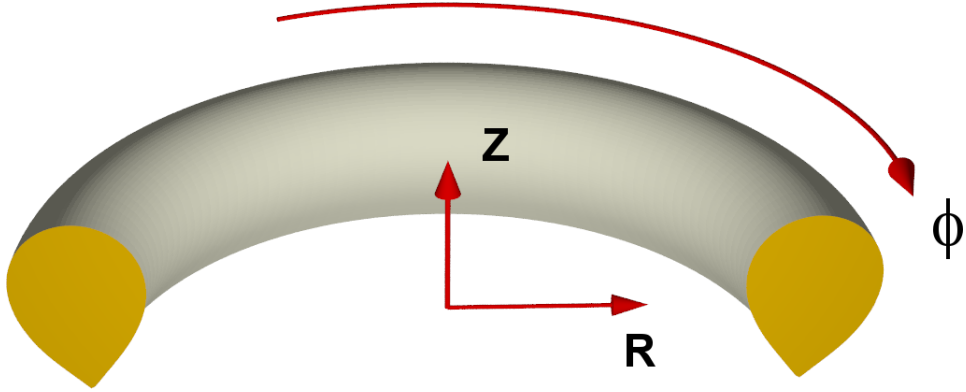


Figure 2: Illustration of the cylindrical coordinate system used in JOREK.

The domain is defined by $(R, Z, \phi) \in \Omega = D \times [0, 2\pi[$. To finish we define the different differential operators used for the calculation:

$$\begin{aligned}\nabla f &= \partial_R(f)\mathbf{e}_R + \frac{1}{R}\partial_\phi(f)\mathbf{e}_\phi + \partial_Z(f)\mathbf{e}_Z \\ \nabla_{pol} f &= \partial_R(f)\mathbf{e}_R + \partial_Z(f)\mathbf{e}_Z \\ \nabla \cdot \mathbf{f} &= \frac{1}{R}\partial_R(Rf_R) + \frac{1}{R}\partial_\phi(f_\phi) + \partial_Z(f_Z) \\ \nabla \times \mathbf{f} &= \left(\frac{1}{R}\partial_\phi f_Z - \partial_Z f_\phi\right)\mathbf{e}_R + (\partial_Z f_R - \partial_R f_Z)\mathbf{e}_\phi + \frac{1}{R}(\partial_R(Rf_\phi) - \partial_Z f_R)\mathbf{e}_Z \\ \Delta^* f &= R^2 \nabla \cdot \left(\frac{1}{R^2} \nabla^\perp f\right) = R \partial_R \left(\frac{1}{R} \partial_R f\right) + \partial_{ZZ} f \\ \Delta_{pol} &= \nabla \cdot (\nabla f) = \frac{1}{R} \partial_R (R \partial_R f) + \partial_{ZZ} f \\ [a, b] &= \mathbf{e}_\phi \cdot (\nabla a \times \nabla b) = \partial_R a \partial_Z b - \partial_Z a \partial_R b.\end{aligned}$$

The variables associated to the reduced MHD models are the poloidal magnetic flux ψ , the electrical potential u , ρ the density, T the temperature and $v_{||}$ the parallel velocity. To avoid high order differential operators we introduce two additional variables: the toroidal current j defined by $j = \Delta^* \psi$ and the vorticity w defined by $w = \Delta_{pol} u$. For the integration we note $dW = RdRdZ$ the cylindrical measure and $dV = dRdZ$. When no measure is given explicitly, $dRdZ$ is used.

2.2 Derivation of the model

To derive the model we begin with the following resistive MHD model

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{J} \times \mathbf{B} \\ \partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0 \\ \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{J}) \end{cases} \quad (3)$$

2.2.1 Magnetic poloidal flux equation

To begin we note that, since $\nabla \cdot \mathbf{B} = 0$ and $\partial_t F_0 = 0$ we get $\partial_t \mathbf{B} = \partial_t \left(\frac{F_0}{R} \mathbf{e}_\phi + \nabla \times \left(\frac{1}{R} \psi \mathbf{e}_\phi \right) \right) = \nabla \times \left(\partial_t \left(\frac{1}{R} \psi \mathbf{e}_\phi \right) \right)$. Consequently the equation on the magnetic field in (3) becomes

$$\nabla \times \left(\partial_t \left(\frac{1}{R} \psi \mathbf{e}_\phi \right) \right) = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \mathbf{J}) \quad (4)$$

with $\mathbf{J} = \nabla \times \mathbf{B}$. The equation becomes

$$\partial_t \left(\frac{1}{R} \psi \mathbf{e}_\phi \right) = \mathbf{v} \times \mathbf{B} - \eta \mathbf{J} + \nabla V \quad (5)$$

with V a potential. We begin by estimating the term $\mathbf{v} \times \mathbf{B}$. Since $\mathbf{B} \times \mathbf{B} = 0$ we obtain $\mathbf{v} \times \mathbf{B} = (-R^2 \nabla \times \left(\frac{1}{R} u \mathbf{e}_\phi \right)) \times \left(\frac{F_0}{R} \mathbf{e}_\phi + \nabla \times \left(\frac{1}{R} \psi \mathbf{e}_\phi \right) \right)$ which gives

$$\begin{aligned} \mathbf{v} \times \mathbf{B} &= \mathbf{v}_{pol} \times \mathbf{B} = (-R \partial_Z u \mathbf{e}_R + R \partial_R u \mathbf{e}_Z) \times \frac{F_0}{R} \mathbf{e}_\phi \\ &\quad + (-R \partial_Z u \mathbf{e}_R + R \partial_R u \mathbf{e}_Z) \times \left(\frac{1}{R} \partial_Z \psi \mathbf{e}_R - \frac{1}{R} \partial_R \psi \mathbf{e}_Z \right) \\ &= F_0 (\partial_Z u \mathbf{e}_Z + \partial_R u \mathbf{e}_R) + [\psi, u] \mathbf{e}_\phi. \end{aligned}$$

Now we study the term $\mathbf{J} = \nabla \times \mathbf{B}$.

$$\mathbf{J} = \nabla \times (F_0 \nabla \phi) + \nabla \times (\nabla \psi \times \nabla \phi).$$

Since F_0 is constant, using the properties of curl and gradient operators we have $\nabla \times \nabla \phi = 0$. So

$$\nabla \times (\nabla \psi \times \nabla \phi) = \nabla \times \left(\frac{1}{R} \partial_Z \psi \mathbf{e}_R - \frac{1}{R} \partial_R \psi \mathbf{e}_Z \right).$$

Since $\nabla \times \mathbf{e}_R = \nabla \times \nabla R = 0$ and $\nabla \times \mathbf{e}_Z = \nabla \times \nabla Z = 0$ we have

$$\nabla \times (\nabla \psi \times \nabla \phi) = \nabla \left(\frac{1}{R} \partial_Z \psi \right) \times \mathbf{e}_R - \nabla \left(\frac{1}{R} \partial_R \psi \right) \times \mathbf{e}_Z.$$

Therefore developing the gradient for each component we obtain

$$\nabla \times (\nabla \psi \times \nabla \phi) = -\frac{1}{R} \partial_{ZZ} \psi \mathbf{e}_\phi + \frac{1}{R^2} \partial_\phi (\partial_Z \psi) \mathbf{e}_Z - \partial_R \left(\frac{1}{R} \partial_R \psi \right) \mathbf{e}_\phi + \frac{1}{R^2} \partial_\phi (\partial_R \psi) \mathbf{e}_R,$$

and using the definition of Grad-Shafranov diffusion operator we have

$$\nabla \times (\nabla \psi \times \nabla \phi) = -\frac{1}{R} \Delta^* \psi \mathbf{e}_\phi + \frac{1}{R^2} \partial_\phi (\partial_Z \psi) \mathbf{e}_Z + \frac{1}{R^2} \partial_\phi (\partial_R \psi) \mathbf{e}_R.$$

We plug together all the terms to obtain

$$\begin{aligned} \partial_t \left(\frac{1}{R} \psi \mathbf{e}_\phi \right) &= +F_0 (\partial_Z u \mathbf{e}_Z + \partial_R u \mathbf{e}_R) + [\psi, u] \mathbf{e}_\phi \\ &\quad - \eta \left[-\frac{1}{R} \Delta^* \psi \mathbf{e}_\phi + \frac{1}{R^2} \partial_\phi (\partial_Z \psi) \mathbf{e}_Z + \frac{1}{R^2} \partial_\phi (\partial_R \psi) \mathbf{e}_R \right] + \nabla V \end{aligned}$$

Now we write the equation on V :

$$\begin{cases} \partial_R V = -F_0 \partial_R u + \frac{\eta}{R^2} \partial_\phi \psi \\ \partial_Z V = -F_0 \partial_Z u + \frac{\eta}{R^2} \partial_\phi \psi \end{cases} \quad (6)$$

Consequently $V = -F_0 u + \frac{\eta \partial_\phi \psi}{R^2}$ and $\nabla V \cdot \mathbf{e}_\phi = -\frac{F_0}{R} \partial_\phi u + \eta \frac{\partial_\phi \psi}{R^3}$. This definition of V gives the final result.

$$\partial_t \psi = R[\psi, u] + \eta \Delta^* \psi - F_0 \partial_\phi u + \eta \frac{\partial_\phi \psi}{R^2} \quad (7)$$

with $j = -R \mathbf{J} \cdot \mathbf{e}_\phi = \Delta^* \psi$ the toroidal current.

2.2.2 Poloidal momentum equation

To obtain an equation on the electric potential we apply the projection operator $\mathbf{e}_\phi \cdot \nabla \times (R^2 \dots)$ in the poloidal plane to the momentum equation. The equation obtained is

$$\mathbf{e}_\phi \cdot \nabla \times [R^2 (\rho \partial_t \mathbf{v} = -\rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p + \mathbf{J} \times \mathbf{B})]. \quad (8)$$

We begin by considering the first term of (8): $\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \partial_t \mathbf{v})$. Using the definition of \mathbf{v}_{pol} we obtain

$$\nabla \times (R^2 \rho \partial_t \mathbf{v}_{pol}) = \nabla \times [-\rho R^3 \partial_Z (\partial_t u) \mathbf{e}_R + \rho R^3 \partial_R (\partial_t u) \mathbf{e}_Z],$$

and

$$\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \partial_t \mathbf{v}_{pol}) = \partial_Z (\rho R^3 \partial_Z (\partial_t u)) + \partial_R (\rho R^3 \partial_R (\partial_t u)).$$

By definition of the gradient and the divergence we obtain

$$\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \partial_t \mathbf{v}_{pol}) = R \nabla \cdot (\rho R^2 \nabla_{pol} \partial_t u). \quad (9)$$

We consider the term associated to the time derivative of the parallel velocity $\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \partial_t \mathbf{v}_{||}) = \mathbf{e}_\phi \cdot \nabla \times [R^2 \rho \partial_t (v_{||} \mathbf{B})]$. Developing \mathbf{B} we obtain

$$\nabla \times (R^2 \rho \partial_t \mathbf{v}_{||}) = \nabla \times \left[R^2 \rho \left[\partial_t \left(v_{||} \frac{\partial_Z \psi}{R} \right) \mathbf{e}_R - \partial_t \left(v_{||} \frac{\partial_R \psi}{R} \right) \mathbf{e}_Z + \frac{F_0}{R} \partial_t v_{||} \mathbf{e}_\phi \right] \right].$$

Some computations give

$$\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \partial_t \mathbf{v}_{||}) = -\partial_Z [R \rho \partial_t (v_{||} \partial_Z \psi)] - \partial_R [R \rho \partial_t (v_{||} \partial_R \psi)] = -R \nabla \cdot [\rho \partial_t (v_{||} \nabla_{pol} \psi)]. \quad (10)$$

Secondly, we study the current term $\mathbf{e}_\phi \cdot \nabla \times (R^2 (\mathbf{J} \times \mathbf{B}))$. To begin we define the toroidal current $j = \Delta^* \psi$. By definition of the magnetic flux density $\mathbf{B} = \frac{1}{R} (F_0 \mathbf{e}_\phi + \partial_Z \psi \mathbf{e}_R - \partial_R \psi \mathbf{e}_Z)$ and using the definition of the curl operator we obtain

$$\mathbf{J} = \nabla \times \mathbf{B} = \frac{1}{R^2} (\partial_R \phi \psi \mathbf{e}_R - R j \mathbf{e}_\phi + \partial_Z \phi \psi \mathbf{e}_Z). \quad (11)$$

So, using (10) - (11) we have

$$\begin{aligned} R^2 \mathbf{J} \times \mathbf{B} &= + \frac{1}{R} [(-R j \partial_R \psi + F_0 \partial_Z \phi \psi) \mathbf{e}_R + (-R j \partial_Z \psi - F_0 \partial_R \phi \psi) \mathbf{e}_Z] \\ &+ \frac{1}{R} [-((\partial_R \psi)(\partial_R \phi \psi) + (\partial_Z \psi)(\partial_Z \phi \psi)) \mathbf{e}_\phi]. \end{aligned}$$

Applying the operator $\mathbf{e}_\phi \cdot \nabla \times (R^2 \dots)$ we obtain

$$\mathbf{e}_\phi \cdot \nabla \times (R^2 \mathbf{J} \times \mathbf{B}) = +\partial_Z (j \partial_R \psi) - \partial_Z \left(\frac{F_0}{R} \partial_Z \phi \psi \right) - \partial_R (j \partial_Z \psi) - \partial_R \left(\frac{F_0}{R} \partial_R \phi \psi \right).$$

The final calculation gives the following result

$$\mathbf{e}_\phi \cdot \nabla \times (R^2 \mathbf{J} \times \mathbf{B}) = [\psi, j] - \frac{F_0}{R} \partial_\phi \Delta^* \psi = [\psi, j] - \frac{F_0}{R} \partial_\phi j. \quad (12)$$

For the pressure term, trivial computations allow to obtain the following result

$$\nabla \times (R^2 \nabla p) \cdot \mathbf{e}_\phi = -\partial_R (R^2) \partial_Z p = -[p, R^2] = [R^2, p]. \quad (13)$$

The last term considered is $\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \mathbf{v} \cdot \nabla \mathbf{v})$. Firstly we study the part which depends only on the poloidal velocity: $\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol})$. To begin we note $\alpha = -R \partial_Z u$, $\beta = R \partial_R u$ and $\hat{\rho} = R^2 \rho$. So

$$\mathbf{v}_{pol} = \alpha \mathbf{e}_R + \beta \mathbf{e}_Z \text{ and } \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol} = (\alpha \partial_R \alpha + \beta \partial_Z \alpha) \mathbf{e}_R + (\alpha \partial_R \beta + \beta \partial_Z \beta) \mathbf{e}_Z.$$

To estimate this term we propose the following decomposition

$$\nabla \times (\widehat{\rho} \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol}) \cdot \mathbf{e}_\phi = (\widehat{\rho} \nabla \times (\mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol}) + \nabla \widehat{\rho} \times \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol}) \cdot \mathbf{e}_\phi = A + B. \quad (14)$$

One has the identities

$$A = -\widehat{\rho} [\partial_Z (\alpha \partial_R \alpha + \beta \partial_Z \alpha) - \partial_R (\alpha \partial_R \beta + \beta \partial_Z \beta)],$$

and

$$A = -\widehat{\rho} [\alpha \partial_R (\partial_Z \alpha - \partial_R \beta) + \beta \partial_Z (\partial_Z \alpha - \partial_R \beta) + (\partial_R \alpha + \partial_Z \beta) (\partial_Z \alpha - \partial_R \beta)].$$

Using $(\partial_Z \alpha - \partial_R \beta) = -R \Delta_{pol} u = -Rw$ and $\widehat{\rho} (\partial_R \alpha + \partial_Z \beta) = -\widehat{\rho} [R, u]$ we obtain $A = -R \widehat{\rho} [Rw, u] - R \widehat{\rho} w [R, u]$. In a first time we propose to estimate the term B . The definition of the vector product gives

$$B = -[\partial_Z (\widehat{\rho}) (\alpha \partial_R \alpha + \beta \partial_Z \alpha) - \partial_R (\widehat{\rho}) (\alpha \partial_R \beta + \beta \partial_Z \beta)]$$

which we can rewrite in the following form

$$B = -\left(\partial_Z (\widehat{\rho}) \left[\partial_R \left(\frac{1}{2} \alpha^2 + \frac{1}{2} \beta^2 \right) + \beta (\partial_Z \alpha - \partial_R \beta) \right] - \partial_R (\widehat{\rho}) \left[\partial_Z \left(\frac{1}{2} \alpha^2 + \frac{1}{2} \beta^2 \right) - \alpha (\partial_Z \alpha - \partial_R \beta) \right] \right)$$

Using $\partial_Z \alpha - \partial_R \beta = -Rw$, we obtain the final writing of the term B :

$$B = -\frac{1}{2} [R^2 |\nabla_{pol} u|^2, \widehat{\rho}] - \partial_Z (\widehat{\rho}) \beta (\partial_Z \alpha - \partial_R \beta) - \partial_R (\widehat{\rho}) \alpha (\partial_Z \alpha - \partial_R \beta) = -\frac{1}{2} [R^2 |\nabla_{pol} u|^2, \widehat{\rho}] - R^2 w [\widehat{\rho}, u].$$

To finish we sum A and B to obtain

$$\nabla \times (\widehat{\rho} \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol}) \mathbf{e}_\phi = -\frac{1}{2} [R^2 |\nabla_{pol} u|^2, \widehat{\rho}] - R [R \widehat{\rho} w, u] - R \widehat{\rho} w [R, u].$$

Therefore

$$\nabla \times (\widehat{\rho} \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol}) \mathbf{e}_\phi = -\frac{1}{2} [R^2 |\nabla_{pol} u|^2, \widehat{\rho}] - [R^2 \widehat{\rho} w, u]. \quad (15)$$

At this moment of the derivation we have obtained the equation on u implemented in the code. Now we propose to derive the terms neglected in the code which correspond to the following cross terms between the parallel and poloidal velocities

$$\mathbf{e}_\phi \cdot \nabla \times R^2 \rho (\partial_t \mathbf{v}_{||} + \mathbf{v}_{||} \cdot \nabla \mathbf{v}_{||} + \mathbf{v}_{||} \cdot \nabla \mathbf{v}_{pol} + \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{||}). \quad (16)$$

Firstly we consider the term $\mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \mathbf{v}_{||} \cdot \nabla \mathbf{v}_{||})$. We begin by splitting the term into two parts

$$A = \widehat{\rho} \nabla \times (v_{||} \mathbf{B} \cdot \nabla (v_{||} \mathbf{B})) \cdot \mathbf{e}_\phi \text{ and } B = \nabla \widehat{\rho} \times (v_{||} \mathbf{B} \cdot \nabla (v_{||} \mathbf{B})) \cdot \mathbf{e}_\phi$$

and we define $v_{||} \mathbf{B} = v_{||} a \mathbf{e}_R + v_{||} b \mathbf{e}_Z + v_{||} c \mathbf{e}_\phi$ with $a = \frac{\partial_Z \psi}{R}$, $b = -\frac{\partial_R \psi}{R}$ and $c = \frac{F_0}{R}$, consequently

$$\begin{aligned} \mathbf{B} \cdot \nabla (v_{||} \mathbf{B}) = &+ \left[a \partial_R (a v_{||}) + b \partial_Z (a v_{||}) + \frac{c}{R} \partial_\phi (a v_{||}) \right] \mathbf{e}_R + \left[a \partial_R (b v_{||}) + b \partial_Z (b v_{||}) + \frac{c}{R} \partial_\phi (b v_{||}) \right] \mathbf{e}_Z \\ &+ \left[a \partial_R (c v_{||}) + b \partial_Z (c v_{||}) + \frac{c}{R} \partial_\phi (c v_{||}) \right] \mathbf{e}_\phi - \left(v_{||}^2 \frac{c^2}{R} \right) \mathbf{e}_R + \left(v_{||}^2 \frac{c}{R} a \right) \mathbf{e}_\phi. \end{aligned}$$

The term B can be decomposed as $B = C + D$ with

$$C = (\partial_R \widehat{\rho}) [a v_{||} \partial_R (b v_{||}) + b v_{||} \partial_Z (b v_{||})] - (\partial_Z \widehat{\rho}) [a v_{||} \partial_R (a v_{||}) + b v_{||} \partial_Z (a v_{||})],$$

and

$$\begin{aligned} D = &+ (\partial_R \widehat{\rho}) \left[\frac{c v_{||}}{R} \partial_\phi (b v_{||}) \right] - (\partial_Z \widehat{\rho}) \left[\frac{c v_{||}}{R} \partial_\phi (a v_{||}) \right] + \partial_Z (\widehat{\rho}) (v_{||}^2 \frac{c^2}{R}) \\ = &-\frac{F_0}{R^3} v_{||} (\nabla_{pol} \widehat{\rho} \cdot \partial_\phi (v_{||} \nabla_{pol} \psi)) + \partial_Z (\widehat{\rho}) (v_{||}^2 \frac{F_0^2}{R^3}). \end{aligned}$$

We rewrite the term C to obtain

$$C = + \partial_R(\hat{\rho}) \left[\partial_Z \left(\frac{1}{2} v_{\parallel}^2 a^2 + \frac{1}{2} v_{\parallel}^2 b^2 \right) + av_{\parallel} \partial_R(bv_{\parallel}) - av_{\parallel} \partial_Z(av_{\parallel}) \right] \\ - \partial_Z(\hat{\rho}) \left[\partial_R \left(\frac{1}{2} v_{\parallel}^2 a^2 + \frac{1}{2} v_{\parallel}^2 b^2 \right) + bv_{\parallel} \partial_Z(av_{\parallel}) - bv_{\parallel} \partial_R(bv_{\parallel}) \right]$$

which is equal to

$$C = \frac{1}{2} [\hat{\rho}, v_{\parallel}^2 \frac{|\nabla_{pol}\psi|^2}{R^2}] - v_{\parallel} \partial_R(\hat{\rho}) a (\partial_Z(av_{\parallel}) - \partial_R(bv_{\parallel})) - v_{\parallel} \partial_Z(\hat{\rho}) b (\partial_Z(av_{\parallel}) - \partial_R(bv_{\parallel})). \quad (17)$$

We remark that $\partial_Z(av_{\parallel}) - \partial_R(bv_{\parallel}) = \frac{v_{\parallel}}{R} \Delta^* \psi + \frac{1}{R} (\nabla_{pol} v_{\parallel} \cdot \nabla_{pol} \psi)$. Using this result we obtain the following expression for B :

$$B = + \frac{1}{2} [\hat{\rho}, v_{\parallel}^2 |\mathbf{B}_{pol}|^2] - \frac{v_{\parallel}^2}{R^2} j[\hat{\rho}, \psi] - \frac{v_{\parallel}}{R^2} (\nabla_{pol} v_{\parallel} \cdot \nabla_{pol} \psi) [\hat{\rho}, \psi] \\ - \frac{F_0}{R^3} v_{\parallel} (\nabla_{pol} \hat{\rho} \cdot \partial_{\phi}(v_{\parallel} \nabla_{pol} \psi)) + \partial_Z(\hat{\rho}) (v_{\parallel}^2 \frac{F_0^2}{R^3}),$$

with $\mathbf{B}_{pol} = \frac{1}{R} \nabla \psi \times \mathbf{e}_{\phi}$.

Now we study the term $A = \hat{\rho} \nabla \times (v_{\parallel} \mathbf{B} \cdot \nabla(v_{\parallel} \mathbf{B})) \cdot \mathbf{e}_{\phi}$ which is equal to

$$A = - \hat{\rho} \partial_Z \left[av_{\parallel} \partial_R(av_{\parallel}) + bv_{\parallel} \partial_Z(av_{\parallel}) + \frac{cv_{\parallel}}{R} \partial_{\phi}(av_{\parallel}) - v_{\parallel}^2 \frac{c^2}{R} \right] \\ + \hat{\rho} \partial_R \left[av_{\parallel} \partial_R(bv_{\parallel}) + bv_{\parallel} \partial_Z(bv_{\parallel}) + \frac{cv_{\parallel}}{R} \partial_{\phi}(bv_{\parallel}) \right].$$

We split these terms into two terms $A = A_1 + A_2$ defined by $A_1 = -\hat{\rho} \partial_Z(av_{\parallel} \partial_R(av_{\parallel}) + bv_{\parallel} \partial_Z(av_{\parallel})) + \hat{\rho} \partial_R(av_{\parallel} \partial_R(bv_{\parallel}) + bv_{\parallel} \partial_Z(bv_{\parallel}))$ and $A_2 = -\hat{\rho} \partial_Z(\frac{cv_{\parallel}}{R} \partial_{\phi}(av_{\parallel})) + \hat{\rho} \partial_R(\frac{cv_{\parallel}}{R} \partial_{\phi}(bv_{\parallel})) + \hat{\rho} \partial_Z(v_{\parallel}^2 \frac{c^2}{R})$.

Factorizing the term A_1 we obtain

$$A_1 = - \hat{\rho} [av_{\parallel} \partial_R(\partial_Z(av_{\parallel}) - \partial_R(bv_{\parallel})) + bv_{\parallel} \partial_Z(\partial_Z(av_{\parallel}) - \partial_R(bv_{\parallel}))] \\ - \hat{\rho} [(\partial_R(av_{\parallel}) + \partial_Z(bv_{\parallel}))(\partial_Z(av_{\parallel}) - \partial_R(bv_{\parallel}))].$$

Using that $\partial_Z(av_{\parallel}) - \partial_R(bv_{\parallel}) = \frac{v_{\parallel}}{R} \Delta^* \psi + \frac{1}{R} (\nabla_{pol} v_{\parallel} \cdot \nabla_{pol} \psi)$ and $\partial_R(av_{\parallel}) + \partial_Z(bv_{\parallel}) = [\frac{v_{\parallel}}{R}, \psi]$ we obtain

$$A_1 = - \frac{\hat{\rho} v_{\parallel}}{R} [\frac{v_{\parallel}}{R} j, \psi] - \frac{\hat{\rho} v_{\parallel}}{R} [\frac{1}{R} (\nabla_{pol} v_{\parallel} \cdot \nabla_{pol} \psi), \psi] \\ - \frac{\hat{\rho} v_{\parallel}}{R} j [\frac{v_{\parallel}}{R}, \psi] - \frac{\hat{\rho}}{R} (\nabla_{pol} v_{\parallel} \cdot \nabla_{pol} \psi) [\frac{v_{\parallel}}{R}, \psi].$$

The properties of the Poisson bracket allow to conclude

$$A_1 = - \hat{\rho} [\frac{v_{\parallel}^2}{R^2} j, \psi] - \hat{\rho} [\frac{v_{\parallel}}{R^2} (\nabla_{pol} v_{\parallel} \cdot \nabla_{pol} \psi), \psi]. \quad (18)$$

For the term A_2 , some computations allow to obtain the result

$$A_2 = - \hat{\rho} \partial_Z \left(\frac{F_0}{R^3} v_{\parallel} \partial_{\phi}(v_{\parallel} \partial_Z \psi) \right) - \hat{\rho} \partial_R \left(\frac{F_0}{R^3} v_{\parallel} \partial_{\phi}(v_{\parallel} \partial_R \psi) \right) + \hat{\rho} \partial_Z \left(\frac{F_0^2}{R^3} v_{\parallel}^2 \right). \quad (19)$$

At the end using Poisson bracket properties and the derivative product we obtain

$$\begin{aligned} \mathbf{e}_\phi \cdot \nabla \times (R^2 \rho \mathbf{v}_\parallel \cdot \nabla \mathbf{v}_\parallel) &= -[\rho v_\parallel^2 j, \psi] - [\rho v_\parallel (\nabla_{pol} v_\parallel \cdot \nabla_{pol} \psi), \psi] + \frac{1}{2} [\hat{\rho}, v_\parallel^2 |\mathbf{B}_{pol}|^2] \\ &\quad - \partial_Z \left(\hat{\rho} \frac{F_0}{R^3} v_\parallel \partial_\phi (v_\parallel \partial_Z \psi) \right) - \partial_R \left(\hat{\rho} \frac{F_0}{R^3} v_\parallel \partial_\phi (v_\parallel \partial_R \psi) \right) + \partial_Z \left(\hat{\rho} \frac{F_0^2}{R^3} v_\parallel^2 \right). \end{aligned}$$

To finish the derivation associated with poloidal velocity, we study the last term $\mathbf{e}_\phi \cdot \nabla \times (\hat{\rho} \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_\parallel + \hat{\rho} \mathbf{v}_\parallel \cdot \nabla \mathbf{v}_{pol})$. Firstly we note

$$v_\parallel \mathbf{B} \cdot \nabla \mathbf{v}_{pol} = v_\parallel \left[a \partial_R \alpha + b \partial_Z \alpha + \frac{c}{R} \partial_\phi \alpha \right] \mathbf{e}_R + v_\parallel \left[a \partial_R \beta + b \partial_Z \beta + \frac{c}{R} \partial_\phi \beta \right] \mathbf{e}_Z + v_\parallel \frac{c}{R} \alpha \mathbf{e}_\phi$$

and

$$\mathbf{v}_{pol} \cdot \nabla (v_\parallel \mathbf{B}) = [\alpha \partial_R (a v_\parallel) + \beta \partial_Z (a v_\parallel)] \mathbf{e}_R + [\alpha \partial_R (b v_\parallel) + \beta \partial_Z (b v_\parallel)] \mathbf{e}_Z + [\alpha \partial_R (c v_\parallel) + \beta \partial_Z (c v_\parallel)] \mathbf{e}_\phi.$$

The term $\mathbf{e}_\phi \cdot \nabla \times (\hat{\rho} \mathbf{v}_{pol} \cdot \nabla (v_\parallel \mathbf{B}) + \hat{\rho} v_\parallel \mathbf{B} \cdot \nabla \mathbf{v}_{pol})$ can be split into two terms

$$\begin{aligned} (A) &= \nabla \hat{\rho} \times (\mathbf{v}_{pol} \cdot \nabla (v_\parallel \mathbf{B}) + v_\parallel \mathbf{B} \cdot \nabla \mathbf{v}_{pol}) \cdot \mathbf{e}_\phi \\ (B) &= \hat{\rho} \nabla \times (\mathbf{v}_{pol} \cdot \nabla (v_\parallel \mathbf{B}) + v_\parallel \mathbf{B} \cdot \nabla \mathbf{v}_{pol}) \cdot \mathbf{e}_\phi. \end{aligned}$$

Using our notation we obtain that $(A) = (A1) + (A2)$ with

$$\begin{aligned} (A1) &= -\partial_Z \hat{\rho} [v_\parallel a \partial_R \alpha + \alpha \partial_R (a v_\parallel) + v_\parallel b \partial_Z \alpha + \beta \partial_Z (a v_\parallel)] \\ &\quad + \partial_R \hat{\rho} [v_\parallel b \partial_Z \beta + \beta \partial_Z (b v_\parallel) + v_\parallel a \partial_R \beta + \alpha \partial_R (b v_\parallel)], \end{aligned}$$

and

$$(A2) = -\partial_Z \hat{\rho} \frac{c v_\parallel}{R} \partial_\phi \alpha + \partial_R \hat{\rho} \frac{c v_\parallel}{R} \partial_\phi \beta.$$

Straightforward calculations show that the term $(A2)$ is equal to $(A2) = \frac{F_0}{R} v_\parallel (\nabla_{pol} \hat{\rho} \cdot \nabla_{pol} (\partial_\phi u))$. Now we consider the term $(A1)$ which can be rewritten in the following form

$$\begin{aligned} (A1) &= -\partial_Z (\hat{\rho}) [\partial_R (a v_\parallel \alpha + b v_\parallel \beta) - v_\parallel b \partial_R \beta - \beta \partial_R (b v_\parallel) + v_\parallel b \partial_Z \alpha + \beta \partial_Z (a v_\parallel)] \\ &\quad + \partial_R (\hat{\rho}) [\partial_Z (a v_\parallel \alpha + b v_\parallel \beta) - v_\parallel a \partial_Z \alpha - \alpha \partial_Z (a v_\parallel) + \alpha \partial_R (b v_\parallel) + v_\parallel a \partial_R \beta]. \end{aligned}$$

Using the definition of the different coefficients we obtain that $(A1) = -[\hat{\rho}, v_\parallel (\nabla_{pol} \psi \cdot \nabla_{pol} u)] + (A3) + (A4)$ with

$$\begin{aligned} (A3) &= -\partial_Z (\hat{\rho}) v_\parallel (-b \partial_R \beta + b \partial_Z \alpha) + \partial_R (\hat{\rho}) v_\parallel (-a \partial_Z \alpha + a \partial_R \beta) \\ (A4) &= -\partial_Z (\hat{\rho}) (-\beta \partial_R (v_\parallel b) + \beta \partial_Z (a v_\parallel)) + \partial_R (\hat{\rho}) (-\alpha \partial_Z (v_\parallel a) + \alpha \partial_R (b v_\parallel)). \end{aligned}$$

Now we consider the term $(A3)$ which can be factorize in the following way

$$(A3) = (\partial_Z (\hat{\rho}) v_\parallel b + \partial_R (\hat{\rho}) v_\parallel a) (-\partial_Z \alpha + \partial_R \beta)$$

Using that $(\partial_Z \alpha - \partial_R \beta) = -R \Delta_{pol} u$ we obtain that $(A3) = v_\parallel w [\hat{\rho}, \psi]$ and $(A4) = -(\partial_Z \hat{\rho} \beta + \partial_R \hat{\rho} \alpha) (\partial_Z (a v_\parallel) - \partial_R (b v_\parallel))$.

We known that $\partial_Z (a v_\parallel) - \partial_R (b v_\parallel) = \frac{1}{R} (\nabla_{pol} \psi \cdot \nabla_{pol} v_\parallel) + \frac{1}{R} v_\parallel j$, consequently at the end we have $(A4) = [u, \hat{\rho}] (\nabla_{pol} \psi \cdot \nabla_{pol} v_\parallel + v_\parallel j)$ with $(\partial_Z (\hat{\rho}) \beta + \partial_R (\hat{\rho}) \alpha) = R [u, \hat{\rho}]$. Putting all the terms together we obtain that

$$\begin{aligned} (A) &= -[u, \hat{\rho}] (\nabla_{pol} \psi \cdot \nabla_{pol} v_\parallel) - [u, \hat{\rho}] v_\parallel j - [\hat{\rho}, v_\parallel (\nabla_{pol} \psi \cdot \nabla_{pol} u)] \\ &\quad + v_\parallel w [\hat{\rho}, u] + \frac{F_0}{R} v_\parallel (\nabla_{pol} \hat{\rho} \cdot \nabla_{pol} (\partial_\phi u)). \end{aligned}$$

Now we consider the term $(B) = \widehat{\rho} \nabla \times (\mathbf{v}_{pol} \cdot \nabla v_{||} \mathbf{B} + v_{||} \mathbf{B} \cdot \nabla \mathbf{v}_{pol}) \cdot \mathbf{e}_\phi$ decomposed into two terms $(B) = (B1) + (B2)$ with

$$(B1) = -\widehat{\rho} \partial_Z [v_{||} a \partial_R \alpha + v_{||} b \partial_Z \alpha + \alpha \partial_R (a v_{||}) + \beta \partial_Z (a v_{||})] \\ + \widehat{\rho} \partial_R [v_{||} a \partial_R \beta + v_{||} b \partial_Z \beta + \alpha \partial_R (b v_{||}) + \beta \partial_Z (b v_{||})],$$

and

$$(B2) = \widehat{\rho} \left(-\partial_Z \left(\frac{c v_{||}}{R} \partial_\phi \alpha \right) + \partial_R \left(\frac{c v_{||}}{R} \partial_\phi \beta \right) \right).$$

We consider the term $(B1)$. We begin by expanding $(B1)$ and after rearranging terms we obtain

$$(B1) = -\widehat{\rho} [v_{||} a \partial_R (\partial_Z \alpha - \partial_R \beta) + v_{||} b \partial_Z (\partial_Z \alpha - \partial_R \beta) \\ + \alpha \partial_R (\partial_Z (a v_{||}) - \partial_R (b v_{||})) + \beta \partial_Z (\partial_Z (a v_{||}) - \partial_R (b v_{||})) \\ + (\partial_R (v_{||} a) + \partial_Z (v_{||} b)) (\partial_Z \alpha - \partial_R \beta) + (\partial_R \alpha + \partial_Z \beta) (\partial_Z (a v_{||}) - \partial_R (b v_{||}))].$$

We use that $(\partial_Z \alpha - \partial_R \beta) = -R \Delta_{pol} u$ and $\partial_Z (a v_{||}) - \partial_R (b v_{||}) = \frac{v_{||}}{R} j + \frac{1}{R} (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||})$. Using $\partial_R (v_{||} a) + \partial_Z (v_{||} b) = [\frac{v_{||}}{R}, \psi]$ and $(\partial_R \alpha + \partial_Z \beta) = [u, R]$, we write the term $(B1)$ on the following form

$$(B1) = + \frac{1}{R} \widehat{\rho} v_{||} [R w, \psi] + \widehat{\rho} R w [\frac{v_{||}}{R}, \psi] - R \widehat{\rho} [u, \frac{v_{||}}{R} j] - \widehat{\rho} [u, R] \frac{v_{||}}{R} j \\ = -R \widehat{\rho} [u, \frac{1}{R} (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||})] - \frac{\widehat{\rho}}{R} [u, R] (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||}) \\ = + \widehat{\rho} [v_{||} w, \psi] - \widehat{\rho} [u, v_{||} j] - \widehat{\rho} [u, (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||})].$$

The term $(B2)$ is equal to $\widehat{\rho} R \nabla \cdot (\frac{F_0}{R^2} v_{||} \nabla_{pol} (\partial_\phi u))$, consequently we obtain

$$(B) = \widehat{\rho} [v_{||} w, \psi] - \widehat{\rho} [u, v_{||} j] - \widehat{\rho} [u, (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||})] + \widehat{\rho} R \nabla \cdot (\frac{F_0}{R^2} v_{||} \nabla_{pol} (\partial_\phi u)). \quad (20)$$

All together we have derived the following term

$$\mathbf{e}_\phi \cdot \nabla \times (\widehat{\rho} \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{||} + \widehat{\rho} \mathbf{v}_{||} \cot \nabla \mathbf{v}_{pol}) = -[\widehat{\rho}, v_{||} (\nabla_{pol} \psi \cdot \nabla_{pol} u)] + [\widehat{\rho} v_{||} w, \psi] - [u, \widehat{\rho} v_{||} j] \\ - [u, \widehat{\rho} (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||})] + R \nabla \cdot \left(\widehat{\rho} \frac{F_0}{R^2} v_{||} \nabla_{pol} (\partial_\phi u) \right).$$

In this work we don't derive the operator associated with the physical viscosity. We propose to consider the term used in the code: $R \nu_\perp \nabla^2 w$.

2.2.3 Equation on ρ and T

For the thermodynamic equations $\partial_t \rho = -\rho \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla \rho$ and $\partial_t p = -\gamma p \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \nabla p$, we propose to rewrite the equations in order to obtain a dependency on u and $v_{||}$. We begin with

$$\mathbf{v}_{pol} \cdot \nabla \rho = + (-R \nabla u \times \mathbf{e}_\phi) \cdot \nabla \rho, \\ = + (-R \partial_Z u \mathbf{e}_R + R \partial_R u \mathbf{e}_Z) \cdot \left[\partial_R \rho \mathbf{e}_R + \frac{1}{R} \partial_\phi \rho \mathbf{e}_\phi + \partial_Z \rho \mathbf{e}_Z \right], \\ = -R (\partial_Z u) (\partial_R \rho) + R (\partial_R u) (\partial_Z \rho) = -R [\rho, u].$$

Then we compute the second term $\rho \nabla \cdot \mathbf{v}_{pol}$.

$$\rho \nabla \cdot \mathbf{v}_{pol} = -\rho \frac{1}{R} \partial_R (R^2 \partial_Z u) + \rho \partial_Z (R \partial_R u) = -2\rho \partial_Z u$$

Now we derive the term associated to the parallel velocity $\mathbf{v}_{\parallel} = v_{\parallel} \mathbf{B}$.

$$\begin{aligned} v_{\parallel} \mathbf{B} \cdot \nabla \rho &= v_{\parallel} \left[\frac{F_0}{R} \mathbf{e}_{\phi} + \frac{1}{R} \partial_Z \psi \mathbf{e}_R - \frac{1}{R} \partial_R \psi \mathbf{e}_Z \right] \cdot \nabla \rho, \\ &= v_{\parallel} \frac{F_0}{R^2} \partial_{\phi} \rho + \frac{1}{R} v_{\parallel} [\rho, \psi]. \end{aligned}$$

The second term is

$$\begin{aligned} \rho \nabla \cdot (v_{\parallel} \mathbf{B}) &= \rho \nabla \cdot \left[v_{\parallel} \left(\frac{F_0}{R} \mathbf{e}_{\phi} + \frac{1}{R} \partial_Z \psi \mathbf{e}_R - \frac{1}{R} \partial_R \psi \mathbf{e}_Z \right) \right], \\ &= \frac{\rho}{R} [v_{\parallel}, \psi] + \frac{\rho F_0}{R^2} \partial_{\phi} v_{\parallel}. \end{aligned}$$

Consequently we obtain

$$\partial_t \rho = R[\rho, u] + 2\rho \partial_Z u - \frac{v_{\parallel} F_0}{R^2} \partial_{\phi} \rho - \frac{v_{\parallel}}{R} [\rho, \psi] - \frac{\rho}{R} [v_{\parallel}, \psi] - \frac{\rho F_0}{R^2} \partial_{\phi} v_{\parallel}, \quad (21)$$

and

$$\partial_t p = R[p, u] + 2\gamma p \partial_Z u - \frac{v_{\parallel} F_0}{R^2} \partial_{\phi} p - \frac{v_{\parallel}}{R} [p, \psi] - \frac{\gamma p}{R} [v_{\parallel}, \psi] - \frac{\gamma p F_0}{R^2} \partial_{\phi} v_{\parallel}. \quad (22)$$

2.2.4 Equation on the parallel velocity

We consider the equation $\rho \partial_t \mathbf{v} = -\rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla p + \mathbf{J} \times \mathbf{B}$. To obtain the equation on v_{\parallel} we project the momentum equation applying the operator $\mathbf{B} \cdot (\dots)$. Firstly we remark that $\mathbf{B} \cdot (\mathbf{J} \times \mathbf{B}) = \text{Det}(\mathbf{B}, \mathbf{J}, \mathbf{B}) = 0$. Secondly we consider $\mathbf{B} \cdot \rho \partial_t (\mathbf{v}_{\parallel} + \mathbf{v}_{pol})$. Using the definition of \mathbf{B} we prove that the term $\mathbf{B} \cdot \rho \partial_t (v_{\parallel} \mathbf{B})$ is equal to

$$\rho |\mathbf{B}|^2 \partial_t v_{\parallel} + \rho v_{\parallel} \frac{1}{R^2} \nabla_{pol} \psi \cdot \nabla_{pol} (\partial_t \psi).$$

For the poloidal term $\mathbf{B} \cdot \rho (\partial_t \mathbf{v}_{pol})$, quick computations show that this term is given by $\mathbf{B} \cdot \rho (\partial_t \mathbf{v}_{pol}) = -\rho \nabla_{pol} \psi \cdot \nabla_{pol} (\partial_t u)$.

For the pressure term $\mathbf{B} \cdot \nabla p$, we obtain

$$\mathbf{B} \cdot \nabla p = \left(\frac{F_0}{R} \mathbf{e}_{\phi} + \frac{1}{R} \partial_Z \psi \mathbf{e}_R - \frac{1}{R} \partial_R \psi \mathbf{e}_Z \right) (\partial_R p \mathbf{e}_R + \frac{1}{R} \partial_{\phi} p \mathbf{e}_{\phi} + \partial_Z p \mathbf{e}_Z) = \frac{F_0}{R^2} \partial_{\phi} p + \frac{1}{R} [p, \psi] \quad (23)$$

Now we consider the following terms (the four last terms which are neglected in the model implemented in the code JOREK):

$$\mathbf{B} \cdot \rho (\mathbf{v}_{\parallel} \cdot \nabla \mathbf{v}_{\parallel} + \mathbf{v}_{\parallel} \cdot \nabla \mathbf{v}_{pol} + \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{\parallel} + \mathbf{v}_{pol} \cdot \nabla \mathbf{v}_{pol}). \quad (24)$$

Firstly we study $\mathbf{B} \cdot (\rho \mathbf{v}_{\parallel} \cdot \nabla \mathbf{v}_{\parallel}) = \mathbf{B} \cdot (\rho v_{\parallel} \mathbf{B} \cdot \nabla (v_{\parallel} \mathbf{B}))$. For this we note $\mathbf{v}_{\parallel} = v_{\parallel} \mathbf{B} = v_{\parallel} (a \mathbf{e}_R + b \mathbf{e}_Z + c \mathbf{e}_{\phi})$ with $a = \frac{1}{R} \partial_Z \psi$, $b = -\frac{1}{R} \partial_R \psi$ and $c = \frac{F_0}{R}$. Using these notations we obtain

$$\begin{aligned} \rho \mathbf{v}_{\parallel} \cdot \nabla \mathbf{v}_{\parallel} &= \rho v_{\parallel} \left(a \partial_R (v_{\parallel} a) + b \partial_Z (v_{\parallel} a) + \frac{c}{R} \partial_{\phi} (v_{\parallel} a) - v_{\parallel} \frac{c^2}{R} \right) \mathbf{e}_R \\ &\quad + \rho v_{\parallel} \left(a \partial_R (v_{\parallel} b) + b \partial_Z (v_{\parallel} b) + \frac{c}{R} \partial_{\phi} (v_{\parallel} b) \right) \mathbf{e}_Z \\ &\quad + \rho v_{\parallel} \left(a \partial_R (v_{\parallel} c) + b \partial_Z (v_{\parallel} c) + \frac{c}{R} \partial_{\phi} (v_{\parallel} c) \right) \mathbf{e}_{\phi} + v_{\parallel} a \frac{c}{R} \mathbf{e}_{\phi}. \end{aligned}$$

Now we rewrite the term as $\mathbf{B} \cdot (\rho \mathbf{v}_{\parallel} \cdot \nabla \mathbf{v}_{\parallel}) = W_1 + W_2 + W_3$, where W_1 is given by

$$\begin{aligned} W_1 &= \rho \frac{c}{R} v_{\parallel} [a \partial_{\phi} (v_{\parallel} a) + b \partial_{\phi} (v_{\parallel} b) + c \partial_{\phi} (v_{\parallel} c)] + \\ &= \rho \frac{F_0}{2R^2} \partial_{\phi} (v_{\parallel}^2 a^2 + v_{\parallel}^2 b^2 + v_{\parallel}^2 c^2) = \rho \frac{F_0}{R^2} \partial_{\phi} \left(\frac{v_{\parallel}^2 |\mathbf{B}|^2}{2} \right). \end{aligned}$$

The term W_2 is given by

$$\begin{aligned} W_2 &= b\rho v_{||} [a\partial_Z(v_{||}a) + b\partial_Z(v_{||}b) + c\partial_Z(v_{||}c)] \\ &= b\rho\partial_Z\frac{1}{2} (v_{||}^2a^2 + v_{||}^2b^2 + v_{||}^2c^2) = -\frac{\rho}{R}\partial_R\psi\partial_Z\left(\frac{v_{||}^2|\mathbf{B}|^2}{2}\right). \end{aligned}$$

The term W_3 is given by

$$\begin{aligned} W_3 &= a\rho v_{||} [a\partial_R(v_{||}a) + b\partial_R(v_{||}b) + c\partial_R(v_{||}c)] \\ &= a\rho\partial_R\frac{1}{2} (v_{||}^2a^2 + v_{||}^2b^2 + v_{||}^2c^2) = \frac{\rho}{R}\partial_Z\psi\partial_R\left(\frac{v_{||}^2|\mathbf{B}|^2}{2}\right). \end{aligned}$$

At the end we obtain

$$\mathbf{B} \cdot (\rho\mathbf{v}_{||} \cdot \nabla\mathbf{v}_{||}) = -\frac{\rho}{R}[\psi, \frac{v_{||}^2|\mathbf{B}|^2}{2}] + \rho\frac{F_0}{R^2}\partial_\phi\left(\frac{v_{||}^2|\mathbf{B}|^2}{2}\right). \quad (25)$$

Now we propose to study the fourth term $\mathbf{B} \cdot (\rho\mathbf{v}_{pol} \cdot \nabla\mathbf{v}_{pol})$. To estimate this term we define $\alpha = -R\partial_Zu$ and $\beta = R\partial_Ru$. Using this notation we prove that

$$\mathbf{v}_{pol} = \alpha\mathbf{e}_R + \beta\mathbf{e}_Z \text{ and } \mathbf{v}_{pol} \cdot \nabla\mathbf{v}_{pol} = (\alpha\partial_R\alpha + \beta\partial_Z\alpha)\mathbf{e}_R + (\alpha\partial_R\beta + \beta\partial_Z\beta)\mathbf{e}_Z.$$

Using the definitions of the coefficients we obtain

$$\mathbf{B} \cdot (\rho\mathbf{v}_{pol} \cdot \nabla\mathbf{v}_{pol}) = \frac{\rho}{R}(\partial_Z(\psi)(\alpha\partial_R\alpha + \beta\partial_Z\alpha) - \partial_R(\psi)(\alpha\partial_R\beta + \beta\partial_Z\beta))$$

which is equal to

$$\mathbf{B} \cdot (\rho\mathbf{v}_{pol} \cdot \nabla\mathbf{v}_{pol}) = \frac{1}{R}\left(\partial_Z(\psi)(\partial_R(\frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2) + \beta(\partial_Z\alpha - \partial_R\beta)) - \partial_R(\psi)(\partial_Z(\frac{1}{2}\alpha^2 + \frac{1}{2}\beta^2) - \alpha(\partial_Z\alpha - \partial_R\beta))\right)$$

to obtain

$$\mathbf{B} \cdot (\rho\mathbf{v}_{pol} \cdot \nabla\mathbf{v}_{pol}) = \frac{1}{2R}\rho[R^2|\nabla_{pol}u|^2, \psi] + \frac{\rho}{R}[\partial_Z(\psi)\beta(\partial_Z\alpha - \partial_R\beta) + \partial_R(\psi)\alpha(\partial_Z\alpha - \partial_R\beta)].$$

After quick computations we obtain

$$\mathbf{B} \cdot (\rho\mathbf{v}_{pol} \cdot \nabla\mathbf{v}_{pol}) = \frac{1}{2R}\rho[R^2|\nabla_{pol}u|^2, \psi] + \rho R w[\psi, u]. \quad (26)$$

Now we consider the term $\mathbf{B} \cdot (\rho\mathbf{v}_{pol} \cdot \nabla(v_{||}\mathbf{B}))$. To estimate this term we define $\alpha = -R\partial_Zu$, $\beta = R\partial_Ru$, $a = \frac{1}{R}\partial_Z\psi$, $b = -\frac{1}{R}\partial_R\psi$ and $c = \frac{F_0}{R}$. Consequently we obtain

$$\mathbf{v}_{pol} = \alpha\mathbf{e}_R + \beta\mathbf{e}_Z, \text{ and } v_{||}\mathbf{B} = v_{||}(a\mathbf{e}_R + b\mathbf{e}_Z + c\mathbf{e}_\phi).$$

Using these notations we obtain

$$\begin{aligned} \rho\mathbf{v}_{pol} \cdot \nabla(v_{||}\mathbf{B}) &= +\rho[\alpha\partial_R(v_{||}a) + \beta\partial_Z(v_{||}a)]\mathbf{e}_R \\ &\quad + \rho[\alpha\partial_R(v_{||}b) + \beta\partial_Z(v_{||}b)]\mathbf{e}_Z + \rho[\alpha\partial_R(v_{||}c) + \beta\partial_Z(v_{||}c)]\mathbf{e}_\phi, \end{aligned}$$

consequently

$$\begin{aligned} \mathbf{B} \cdot (\rho\mathbf{v}_{pol} \cdot \nabla(v_{||}\mathbf{B})) &= +\rho a[\alpha\partial_R(v_{||}a) + \beta\partial_Z(v_{||}a)] \\ &\quad + \rho b[\alpha\partial_R(v_{||}b) + \beta\partial_Z(v_{||}b)] + \rho c[\alpha\partial_R(v_{||}c) + \beta\partial_Z(v_{||}c)]. \end{aligned}$$

Rearranging terms we obtain

$$\begin{aligned}\mathbf{B} \cdot (\rho \mathbf{v}_{pol} \cdot \nabla(v_{||}\mathbf{B})) &= +\rho(a^2 + b^2 + c^2)\alpha\partial_R(v_{||}) + \rho(a^2 + b^2 + c^2)\beta\partial_Z(v_{||}) \\ &+ \frac{1}{2}\rho\alpha v_{||}\partial_R(a^2 + b^2 + c^2) + \frac{1}{2}\rho\beta v_{||}\partial_Z(a^2 + b^2 + c^2).\end{aligned}$$

Using that $(a^2 + b^2 + c^2) = |\mathbf{B}|^2$ we obtain that

$$\mathbf{B} \cdot (\rho \mathbf{v}_{pol} \cdot \nabla(v_{||}\mathbf{B})) = R\rho|\mathbf{B}|^2[u, v_{||}] + R\rho v_{||}[u, \frac{|\mathbf{B}|^2}{2}]. \quad (27)$$

To finish we consider the term $\mathbf{B} \cdot (\rho v_{||}\mathbf{B} \cdot \nabla(\mathbf{v}_{pol})) = \rho v_{||}(\mathbf{B} \cdot (\mathbf{B} \cdot \nabla \mathbf{v}_{pol}))$. We define $\mathbf{v}_{pol} = \alpha\mathbf{e}_R + \beta\mathbf{e}_Z$ and $\mathbf{B} = a\mathbf{e}_R + b\mathbf{e}_Z + c\mathbf{e}_\phi$. Using these definitions we obtain

$$\begin{aligned}\mathbf{B} \cdot (\mathbf{B} \cdot \nabla \mathbf{v}_{pol}) &= a \left[a\partial_R\alpha + b\partial_Z\alpha + \frac{c}{R}\partial_\phi\alpha \right] + b \left[a\partial_R\beta + b\partial_Z\beta + \frac{c}{R}\partial_\phi\beta \right] + \frac{c^2}{R}\alpha, \\ &= a \left[a\partial_R\alpha + b\partial_Z\alpha \right] + b \left[a\partial_R\beta + b\partial_Z\beta \right] + \frac{ac}{R}\partial_\phi\alpha + \frac{bc}{R}\partial_\phi\beta + \frac{c^2}{R}\alpha.\end{aligned}$$

Now we consider the first term $A = a(a\partial_R\alpha + b\partial_Z\alpha) + b(a\partial_R\beta + b\partial_Z\beta)$. For this we rewrite the term in the following form

$$A = a\partial_R(a\alpha + b\beta) + b\partial_Z(a\alpha + b\beta) - \alpha\partial_R\left(\frac{a^2}{2}\right) - \beta\partial_Z\left(\frac{b^2}{2}\right) - \beta a\partial_R b - \alpha b\partial_Z a.$$

We define $C = a\partial_R(a\alpha + b\beta) + b\partial_Z(a\alpha + b\beta)$ and $D = -\alpha\partial_R\left(\frac{a^2}{2}\right) - \beta\partial_Z\left(\frac{b^2}{2}\right) - \beta a\partial_R b - \alpha b\partial_Z a$. We can rewrite the term D in the following form

$$D = -\left(\alpha\partial_R\left(\frac{a^2}{2} + \frac{b^2}{2}\right) + \beta\partial_Z\left(\frac{a^2}{2} + \frac{b^2}{2}\right) + \alpha b(\partial_Z a - \partial_R b) - \beta a(\partial_Z a - \partial_R b)\right).$$

We obtain

$$D = -R\left[u, \frac{|\mathbf{B}_{pol}|^2}{2}\right] + \frac{j}{R}[u, \psi]$$

Quick computations show that $C = \frac{1}{R}[\psi, (\nabla_{pol}\psi \cdot \nabla_{pol}u)]$. The term A is given by $A = C + D$.

Now we consider the term $B = \frac{ac}{R}\partial_\phi\alpha + \frac{bc}{R}\partial_\phi\beta + \frac{c^2}{R}\alpha$ and it is easy to prove that

$B = -\frac{F_0}{R^2}(\nabla_{pol}\psi \cdot \nabla_{pol}(\partial_\phi u)) - \frac{F_0^2}{R^2}\partial_Z u$. At the end we obtain

$$\begin{aligned}\mathbf{B} \cdot (\rho v_{||}\mathbf{B} \cdot \nabla(\mathbf{v}_{pol})) &= -R\rho v_{||}[u, \frac{|\mathbf{B}_{pol}|^2}{2}] + \rho v_{||}\frac{j}{R}[u, \psi] + \frac{\rho v_{||}}{R}[\psi, (\nabla_{pol}\psi \cdot \nabla_{pol}u)] \\ &- \frac{\rho v_{||}F_0}{R^2}(\nabla_{pol}\psi \cdot \nabla_{pol}(\partial_\phi u)) - \rho v_{||}\frac{F_0^2}{R^2}\partial_Z u.\end{aligned}$$

2.3 Final Model

We define the magnetic and velocity fields by $\mathbf{B} = \frac{F_0}{R}\mathbf{e}_\phi + \frac{1}{R}\nabla\psi \times \mathbf{e}_\phi$ and $\mathbf{v} = -R\nabla u \times \mathbf{e}_\phi + v_{||}\mathbf{B}$. Using all the previous results based on these definitions of the fields, we obtain the final reduced

MHD model with parallel velocity.

$$\left\{ \begin{array}{l}
\partial_t \frac{1}{R^2} \psi = \frac{1}{R} [\psi, u] + \frac{\eta}{R^2} \left(j + \frac{\partial_\phi \psi}{R^2} \right) - \frac{1}{R^2} F_0 \partial_\phi u \\
\\
\nabla \cdot (\widehat{\rho} \nabla_{pol} \partial_t u) - \nabla \cdot (\rho \partial_t (v_{||} \nabla_{pol} \psi)) = \frac{1}{2R} [R^2 |\nabla_{pol} u|^2, \widehat{\rho}] + \frac{1}{R} [\widehat{\rho} R^2 w, u] - \frac{1}{R} [R^2, p] + \frac{1}{R} [\psi, j] - \frac{F_0}{R^2} \partial_\phi j \\
+ \nabla \cdot (\nu_{perp} \nabla w) + \frac{1}{R} [\rho v_{||}^2 j, \psi] + \frac{1}{R} [\rho v_{||} (\nabla_{pol} v_{||} \cdot \nabla_{pol} \psi), \psi] - \nabla \cdot \left(\widehat{\rho} \frac{F_0}{R^2} v_{||} \nabla_{pol} (\partial_\phi u) \right) \\
+ \frac{1}{R} [\widehat{\rho}, v_{||} (\nabla_{pol} \psi \cdot \nabla_{pol} u)] - \frac{1}{2R} [\widehat{\rho}, v_{||}^2 |\mathbf{B}_{pol}|^2] - \frac{1}{R} [\widehat{\rho} v_{||} w, \psi] + \frac{1}{R} [u, \widehat{\rho} v_{||} j] + \frac{1}{R} [u, \widehat{\rho} (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||})] \\
+ \frac{1}{R} \partial_Z \left(\widehat{\rho} \frac{F_0}{R^3} v_{||} \partial_\phi (v_{||} \partial_Z \psi) \right) + \frac{1}{R} \partial_R \left(\widehat{\rho} \frac{F_0}{R^3} v_{||} \partial_\phi (v_{||} \partial_R \psi) \right) - \frac{1}{R} \partial_Z \left(\widehat{\rho} \frac{F_0^2}{R^3} v_{||}^2 \right) \\
\\
w = \Delta_{pol} u \\
\\
j = \Delta^* \psi \\
\\
\partial_t \rho = R[\rho, u] + 2\rho \partial_Z u - \frac{v_{||} F_0}{R^2} \partial_\phi \rho - \frac{v_{||}}{R} [\rho, \psi] - \frac{\rho}{R} [v_{||}, \psi] - \frac{\rho F_0}{R^2} \partial_\phi v_{||} \\
\\
\partial_t p = R[p, u] + 2\gamma p \partial_Z u - \frac{v_{||} F_0}{R^2} \partial_\phi p - \frac{v_{||}}{R} [p, \psi] - \frac{\gamma p}{R} [v_{||}, \psi] - \frac{\gamma p F_0}{R^2} \partial_\phi v_{||} \\
\\
\rho |\mathbf{B}^2| \partial_t v_{||} + \rho v_{||} \frac{1}{R^2} \nabla_{pol} \psi \cdot \nabla_{pol} (\partial_t \psi) - \rho \nabla_{pol} \psi \cdot \nabla_{pol} (\partial_t u) = -\frac{1}{R} [p, \psi] - \frac{F_0}{R^2} \partial_\phi p + \frac{\rho}{R} [\psi, \frac{v_{||}^2 |\mathbf{B}|^2}{2}] \\
- \frac{F_0}{R} \rho \partial_\phi \left(\frac{v_{||}^2 |\mathbf{B}|^2}{2R} \right) - \frac{1}{2R} \rho [R^2 |\nabla_{pol} u|^2, \psi] - \frac{\widehat{\rho}}{R} w [\psi, u] - R\rho |\mathbf{B}|^2 [u, v_{||}] - R\rho v_{||} [u, \frac{|\mathbf{B}|^2}{2}] + R\rho v_{||} [u, \frac{|\mathbf{B}_{pol}|^2}{2}] \\
- \rho v_{||} \frac{\Delta^* \psi}{R} [u, \psi] - \frac{\rho v_{||}}{R} [\psi, (\nabla_{pol} \psi \cdot \nabla_{pol} u)] + \frac{\rho v_{||} F_0}{R^2} (\nabla_{pol} \psi \cdot \nabla_{pol} (\partial_\phi u)) + \rho v_{||} \frac{F_0^2}{R^2} \partial_Z u
\end{array} \right.$$

2.4 Energy estimate

For the full MHD model the total energy is conserved in the ideal case and dissipated in the resistive case. To validate the derivation of the model, to validate the choice of the projection operators and to obtain results of stability important for the numerical method we prove that the reduced MHD model satisfies the same energy balance equation as the full MHD model [3]-[6].

Lemma 2.1. *We define the energy $E = \frac{|\mathbf{B}|^2}{2} + \rho \frac{v_{||}^2}{2} + \frac{1}{\gamma-1} p$. If $\eta = \nu = 0$ the total energy satisfies*

$$\frac{d}{dt} \int_{\Omega} E dW = \frac{d}{dt} \int_{\Omega} R E dV = 0$$

and if $\eta \neq 0, \nu \neq 0$

$$\frac{d}{dt} \int_{\Omega} E dW = -\nu \int_{\omega} w^2 dW - \eta \int_{\omega} j^2 dW - \eta \int_{\Omega} |\nabla_{pol} \left(\frac{\partial_\phi \psi}{R^2} \right)|^2 dW$$

Proof. To begin we compute $dE = \frac{d}{dt} \int_{\Omega} \frac{|\mathbf{B}|^2}{2} + \rho \frac{|\mathbf{v}|^2}{2} + \frac{1}{\gamma-1} p dW$. We obtain

$$\begin{aligned} dE &= \int_{\Omega} \partial_t \frac{|\nabla_{pol} \psi|^2}{2R^2} dW + \int_{\Omega} \rho \partial_t \frac{|\mathbf{v}_{pol}|^2}{2} dW + \int_{\Omega} \rho \partial_t \frac{|\mathbf{v}_{||}|^2}{2} dW + \int_{\Omega} \rho (\mathbf{v}_{||} \cdot \partial_t \mathbf{v}_{pol} + \mathbf{v}_{pol} \cdot \partial_t \mathbf{v}_{||}) dW \\ &\quad + \int_{\Omega} \frac{|\mathbf{v}_{pol}|^2}{2} \partial_t \rho dW + \int_{\Omega} \frac{|\mathbf{v}_{||}|^2}{2} \partial_t \rho dW + \int_{\Omega} (\mathbf{v}_{||} \cdot \mathbf{v}_{pol}) \partial_t \rho dW + \int_{\Omega} \frac{\partial_t p}{\gamma-1} dW. \end{aligned}$$

After quick computations we shows that the derivative of the energy is given by

$$\begin{aligned} dE &= \int_{\Omega} \partial_t \left(\frac{|\nabla_{pol} \psi|^2}{2R^2} \right) dW + \int_{\Omega} \widehat{\rho} \partial_t \left(\frac{|\nabla_{pol} u|^2}{2} \right) dW + \int_{\Omega} \rho |\mathbf{B}|^2 \partial_t \left(\frac{v_{||}^2}{2} \right) dW \\ &\quad + \int_{\Omega} \frac{\rho v_{||}^2}{R^2} (\nabla_{pol} \psi \cdot \nabla_{pol} (\partial_t \psi)) dW - \int_{\Omega} \rho (\nabla_{pol} u \cdot \partial_t (v_{||} \nabla_{pol} \psi)) dW - \int_{\Omega} \rho v_{||} (\nabla_{pol} \psi \cdot \partial_t (\nabla_{pol} u)) dW \\ &\quad + \int_{\Omega} \frac{|\nabla_{pol} u|^2}{2} \partial_t \widehat{\rho} dW + \int_{\Omega} \frac{v_{||}^2 |\mathbf{B}|^2}{2} \partial_t \rho dW - \int_{\Omega} v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) \partial_t \rho dW + \int_{\Omega} \frac{\partial_t p}{\gamma-1} dW. \end{aligned}$$

The term $\int_{\Omega} \partial_t \left(\frac{|\nabla_{pol} \psi|^2}{2R^2} \right) dW$ is equal to $\int_{\Omega} \left(\frac{\nabla_{pol} \psi}{R} \cdot \nabla_{pol} (\partial_t \psi) \right) dW$. Integrating by parts we obtain

$$\int_{\Omega} \partial_t \left(\frac{|\nabla_{pol} \psi|^2}{2R^2} \right) dW = \int_{\Omega} \partial_t \left(\frac{|\nabla_{pol} \psi|^2}{2R} \right) dV = - \int_{\Omega} \frac{\Delta^* \psi}{R} \partial_t \psi dV = - \int_{\Omega} \frac{j}{R^2} \partial_t \psi dW.$$

Using an integration by parts we obtain also

$$\int_{\Omega} \widehat{\rho} \partial_t \left(\frac{|\nabla_{pol} u|^2}{2} \right) = - \int_{\Omega} \nabla \cdot (\widehat{\rho} \nabla_{pol} (\partial_t u)) u dW.$$

Consequently we obtain

$$\begin{aligned} dE &= - \int_{\Omega} (\partial_t \psi) \frac{j}{R^2} dW - \int_{\Omega} (\nabla \cdot (\widehat{\rho} \nabla_{pol} (\partial_t u)) - \nabla \cdot (\rho \partial_t (v_{||} \nabla_{pol} \psi))) u dW \\ &\quad + \int_{\Omega} \rho \left(|\mathbf{B}|^2 \partial_t v_{||} + \frac{v_{||}}{R^2} (\nabla_{pol} \psi \cdot \nabla_{pol} (\partial_t \psi)) - (\nabla_{pol} \psi \cdot \partial_t (\nabla_{pol} u)) \right) v_{||} dW \\ &\quad + \int_{\Omega} \frac{|\nabla_{pol} u|^2}{2} \partial_t \widehat{\rho} dW + \int_{\Omega} \frac{v_{||}^2 |\mathbf{B}|^2}{2} \partial_t \rho dW - \int_{\Omega} v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) \partial_t \rho dW + \int_{\Omega} \frac{\partial_t p}{\gamma-1} dW. \end{aligned}$$

Using $2\partial_Z u = \frac{1}{R}[R^2, u]$, we obtain

$$\partial_t \rho = R[\rho, u] + \frac{\rho}{R}[R^2, u] - \frac{v_{||} F_0}{R^2} \partial_{\phi} \rho - \frac{v_{||}}{R}[\rho, \psi] - \frac{\rho}{R}[v_{||}, \psi] - \frac{\rho F_0}{R^2} \partial_{\phi} v_{||}.$$

Before computes the energy estimate we give an equation on $\widehat{\rho}$. For $\widehat{\rho}$ we multiply by R^2 the equation on ρ . We obtain

$$\begin{aligned} \partial_t \widehat{\rho} &= R^3[\rho, u] + 2\widehat{\rho} \partial_Z u + R[\psi, \rho v_{||}] - F_0 \partial_{\phi} (\rho v_{||}) \\ &= R[\widehat{\rho}, u] - \frac{\widehat{\rho}}{R}[R^2, u] + 2\widehat{\rho} \partial_Z u + R[\psi, \rho v_{||}] - F_0 \partial_{\phi} (\rho v_{||}). \end{aligned}$$

Using that $2\widehat{\rho} \partial_Z u = \frac{\widehat{\rho}}{R}[R^2, u]$, we obtain

$$\partial_t \widehat{\rho} = R[\widehat{\rho}, u] + R[\psi, \rho v_{||}] - F_0 \partial_{\phi} (\rho v_{||}),$$

which is equal to

$$\partial_t \widehat{\rho} = R[\widehat{\rho}, u] + R\rho[\psi, v_{||}] + Rv_{||}[\psi, \rho] - F_0 \rho \partial_{\phi} v_{||} - F_0 v_{||} \partial_{\phi} \rho.$$

To compute dE we add to the final model three equations on the density :

$$\left\{ \begin{array}{l} \frac{v_{||}^2 |\mathbf{B}|^2}{2} \partial_t \rho = \frac{v_{||}^2 |\mathbf{B}|^2}{2R} [\hat{\rho}, u] - \frac{v_{||}^2 |\mathbf{B}|^2}{2} \frac{F_0}{R^2} \partial_\phi (\rho v_{||}) - \frac{v_{||}^2 |\mathbf{B}|^2}{2} \frac{1}{R} [\rho v_{||}, \psi] \\ v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) \partial_t \rho = \frac{v_{||}}{R} (\nabla_{pol} u \cdot \nabla_{pol} \psi) [\hat{\rho}, u] - v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) \frac{F_0}{R^2} \partial_\phi (\rho v_{||}) - v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) \frac{1}{R} [\rho v_{||}, \psi] \\ \frac{|\nabla_{pol} u|^2}{2} \partial_t \hat{\rho} = \frac{|\nabla_{pol} u|^2}{2} R [\hat{\rho}, u] - \frac{|\nabla_{pol} u|^2}{2} F_0 \partial_\phi (\rho v_{||}) - \frac{|\nabla_{pol} u|^2}{2} R [\rho v_{||}, \psi] \end{array} \right.$$

Now we compute

$$\begin{aligned} dE = & - \int_{\Omega} (\partial_t \psi) \frac{j}{R^2} dW - \int_{\Omega} (\nabla \cdot (\hat{\rho} \nabla_{pol} (\partial_t u)) - \nabla \cdot (\rho \partial_t (v_{||} \nabla_{pol} \psi))) u dW \\ & + \int_{\Omega} \rho \left(|\mathbf{B}|^2 \partial_t v_{||} + \frac{v_{||}}{R^2} (\nabla_{pol} \psi \cdot \nabla_{pol} (\partial_t \psi)) - (\nabla_{pol} \psi \cdot \partial_t (\nabla_{pol} u)) \right) v_{||} dW \\ & + \int_{\Omega} \frac{|\nabla_{pol} u|^2}{2} \partial_t \hat{\rho} dW + \int_{\Omega} \frac{v_{||}^2 |\mathbf{B}|^2}{2} \partial_t \rho dW - \int_{\Omega} v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) \partial_t \rho dW + \int_{\Omega} \frac{\partial_t p}{\gamma - 1} dW. \end{aligned}$$

The derivative in time dE can be written as the sum of 17 groups of terms: $dE = (E1) + \dots + (E18)$.

Now we propose to prove that each group of terms is equal to zero or negative.

$$(E1) = - \int_{\Omega} \frac{1}{R} [\psi, u] j dW - \int_{\Omega} \frac{1}{R} [\psi, j] u dW = - \int_{\Omega} ([\psi, u] j + [\psi, j] u) dV = 0,$$

$$(E2) = - \int_{\Omega} \frac{F_0}{R^2} \partial_\phi (u) j dW + \int_{\Omega} \frac{F_0}{R^2} \partial_\phi (j) u dW = \int_{\Omega} \frac{F_0}{R} (\partial_\phi (u) j + \partial_\phi (j) u) dV = 0.$$

These results are obtained by integrating by part. Now we study the term (E3)

$$(E3) = - \int_{\Omega} \frac{1}{R} [\hat{\rho} R^2 w, u] u dW = - \int_{\Omega} [\hat{\rho} R^2 w, u] u dV = \int_{\Omega} [u, u] \hat{\rho} R^2 w dV = 0.$$

The term (E4) corresponds to the viscosity and resistivity terms:

$$\begin{aligned} (E4) &= -\nu \int_{\Omega} \Delta_{pol} w u dW - \int_{\Omega} \eta \frac{j^2}{R^2} dW - \eta \int_{\Omega} \frac{\partial_{\phi\phi} \psi}{R^4} j dW \\ &= -\nu \int_{\Omega} w^2 dW - \int_{\Omega} \eta \frac{j^2}{R^2} dW - \eta \int_{\Omega} \left| \nabla_{pol} \left(\frac{\partial_{\phi} \psi}{R^2} \right) \right|^2 dW. \end{aligned}$$

To obtain this result we use $w = \Delta_{pol} u$ and a double integrating by part. Now we define the term (E5) which depends on the pressure.

$$(E5) = \int_{\Omega} \frac{1}{R} [R^2, p] u dW + \frac{1}{\gamma - 1} \int_{\Omega} R [p, u] dW + \frac{2\gamma}{\gamma - 1} \int_{\Omega} p \partial_Z u dW.$$

Using $2p \partial_Z u = \frac{p}{R} [R^2, u]$ and integrating by parts we obtain

$$(E5) = \int_{\Omega} [R^2, p] u dV + \frac{1}{\gamma - 1} \int_{\Omega} R^2 [p, u] dV + \frac{\gamma}{\gamma - 1} \int_{\Omega} p [R^2, u] dV.$$

$$(E5) = - \int_{\Omega} [R^2, u] p dV - \frac{1}{\gamma - 1} \int_{\Omega} p [R^2, u] dV + \frac{\gamma}{\gamma - 1} \int_{\Omega} p [R^2, u] dV = 0.$$

Now we study the terms (E6) and (E7). In these two cases using integration by part and the anti-symmetric properties of bracket operator we conclude.

$$\begin{aligned} (E6) &= - \int_{\Omega} \frac{1}{2R} [R^2 |\nabla_{pol} u|^2, \hat{\rho}] u dW + \int_{\Omega} \frac{|\nabla_{pol} u|^2}{2} R [\hat{\rho}, u] dW, \\ &= - \int_{\Omega} \frac{1}{2} [R^2 |\nabla_{pol} u|^2, \hat{\rho}] u dV - \int_{\Omega} u [\hat{\rho}, \frac{|\nabla_{pol} u|^2}{2} R^2] dV = 0, \end{aligned}$$

$$\begin{aligned}
(E7) &= + \int_{\Omega} \frac{1}{R} [\widehat{\rho} v_{||} w, \psi] u dV - \int_{\Omega} \frac{\widehat{\rho}}{R} w [\psi, u] v_{||} dW \\
&= - \int_{\Omega} \widehat{\rho} v_{||} w [u, \psi] dV - \int_{\Omega} \widehat{\rho} w [\psi, u] v_{||} dV = 0.
\end{aligned}$$

The term (E8) correspond to the coupling between the pressure and the parallel velocity $v_{||}$. We obtain

$$\begin{aligned}
(E8) &= - \frac{1}{\gamma - 1} \int_{\Omega} \frac{F_0 v_{||}}{R^2} (\partial_{\phi} p) dW - \frac{1}{\gamma - 1} \int_{\Omega} \frac{v_{||}}{R} [p, \psi] dW - \frac{\gamma}{\gamma - 1} \int_{\Omega} \gamma \frac{p}{R} [v_{||}, R] dW \\
&\quad - \frac{\gamma}{\gamma - 1} \int_{\Omega} \frac{F_0}{R^2} p (\partial_{\phi} v_{||}) dW - \int_{\Omega} \frac{F_0 v_{||}}{R^2} (\partial_{\phi} p) dW - \int_{\Omega} \frac{v_{||}}{R} [p, \psi] dW.
\end{aligned}$$

Integrating by part the terms which depend on $\frac{\gamma}{\gamma-1}$ and factorizing we obtain that (E8) = 0. The term (E9) is defined by

$$\begin{aligned}
(E9) &= + \int_{\Omega} \frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2} \frac{1}{R} [\rho v_{||}, \psi] dW + \int_{\Omega} \frac{\rho v_{||}}{R} [\psi, \frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2}] dW \\
&\quad - \int_{\Omega} \frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2} \frac{F_0}{R^2} \partial_{\phi} (\rho v_{||}) dW - \int_{\Omega} \frac{F_0}{R^2} \rho v_{||} \partial_{\phi} \left(\frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2} \right) dW.
\end{aligned}$$

Integrating by parts we obtain

$$\begin{aligned}
(E9) &= + \int_{\Omega} \rho v_{||} \left([\psi, \frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2}] + [\frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2}, \psi] \right) dV \\
&\quad + \int_{\Omega} \frac{F_0}{R} \rho v_{||} \partial_{\phi} \left(\frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2} \right) dV - \int_{\Omega} \frac{F_0}{R} \rho v_{||} \partial_{\phi} \left(\frac{v_{||}^2 |\mathbf{B}_{pol}|^2}{2} \right) dV = 0.
\end{aligned}$$

The term (E10) is defined by

$$(E10) = - \int_{\Omega} \frac{|\nabla_{pol} u|^2}{2} R [\rho v_{||}, \psi] dW - \int_{\Omega} \frac{1}{2R} \rho v_{||} [R^2 |\nabla_{pol} u|^2, \psi] dW.$$

We apply the classical integration to conclude. Now we study the term (E11)

$$(E11) = - \int_{\Omega} \frac{1}{R} [u, \widehat{\rho} v_{||} j] u dW - \int_{\Omega} \frac{1}{R} [u, \widehat{\rho} (\nabla_{pol} \psi \cdot \nabla_{pol} v_{||})] u dW.$$

To conclude we use the integration by part and the fact that $[u, u] = 0$. The term (E12) depends of the toroidal direction

$$\begin{aligned}
(E12) &= + \int_{\Omega} \nabla \cdot \left(\widehat{\rho} \frac{F_0}{R^2} v_{||} \nabla_{pol} (\partial_{\phi} u) \right) dW - \int_{\Omega} \frac{|\nabla_{pol} u|^2}{2} F_0 \partial_{\phi} (\rho v_{||}) \\
&= - \int_{\Omega} \widehat{\rho} \frac{F_0}{R} v_{||} (\nabla_{pol} u \cdot \nabla_{pol} (\partial_{\phi} u)) dV - \int_{\Omega} \frac{|\nabla_{pol} u|^2}{2} F_0 \partial_{\phi} \left(\frac{\widehat{\rho}}{R} v_{||} \right).
\end{aligned}$$

Using that $(\nabla_{pol} u \cdot \nabla_{pol} (\partial_{\phi} u)) = \partial_{\phi} (\frac{|\nabla_{pol} u|^2}{2})$ and integrating by part the second term we conclude. The term (E13) also depends on the toroidal derivative. It is defined by

$$\begin{aligned}
(E13) &= - \int_{\Omega} \frac{1}{R} \partial_Z \left(\widehat{\rho} \frac{F_0}{R^3} v_{||} \partial_{\phi} (v_{||} \partial_Z \psi) \right) dW - \int_{\Omega} \frac{1}{R} \partial_R \left(\widehat{\rho} \frac{F_0}{R^3} v_{||} \partial_{\phi} (v_{||} \partial_R \psi) \right) dW \\
&\quad + \int_{\Omega} v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) \frac{F_0}{R^2} \partial_{\phi} (\rho v_{||}) dW + \int_{\Omega} \frac{\rho v_{||}^2}{R^2} F_0 (\nabla_{pol} \psi \cdot \nabla_{pol} (\partial_{\phi} u)) dW.
\end{aligned}$$

We integrate by part the first term and expend this term, we integrate by part the second one to obtain

$$(E13) = \int_{\Omega} \rho \frac{F_0}{R} v_{||} (\partial_{\phi}(\nabla_{pol}\psi) \cdot \nabla_{pol}u) dV + \int_{\Omega} \rho v_{||} \frac{F_0}{R} (\nabla_{pol}u \cdot \nabla_{pol}\psi) \partial_{\phi}(v_{||}) dV \\ - \int_{\Omega} \frac{F_0 \rho v_{||}}{R} \partial_{\phi}(v_{||}(\nabla_{pol}u \cdot \nabla_{pol}\psi)) dV + \int_{\Omega} \frac{\rho F_0}{R} v_{||}^2 (\nabla_{pol}\psi \cdot \nabla_{pol}(\partial_{\phi}u)) dV.$$

To conclude we expand the third term $\frac{F_0 \rho v_{||}}{R} \partial_{\phi}(v_{||}(\nabla_{pol}u \cdot \nabla_{pol}\psi))$ in two terms $\frac{F_0 \rho v_{||}^2}{R} \partial_{\phi}(\nabla_{pol}u \cdot \nabla_{pol}\psi)$ and $\frac{F_0 \rho v_{||}}{R} (\nabla_{pol}u \cdot \nabla_{pol}\psi) \partial_{\phi}(v_{||})$. The sum of the five terms obtained is equal to zero. Now we introduce the terms (E14) and (E15).

$$(E14) = - \int_{\Omega} \frac{1}{R} [\hat{\rho}, v_{||}(\nabla_{pol}\psi \cdot \nabla_{pol}u)] u dW - \int_{\Omega} \frac{1}{R} v_{||} (\nabla_{pol}u \cdot \nabla_{pol}\psi) [\hat{\rho}, u] dW,$$

$$(E15) = - \int_{\Omega} \frac{1}{R} [\rho v_{||}^2 \Delta^* \psi, \psi] u dW - \int_{\Omega} \rho v_{||} \frac{\Delta^* \psi}{R} [u, \psi] dW.$$

A integration by part of the first term is sufficient to prove that (E14) and (E15) are equal to zero. The term (E16) is defined by

$$(E16) = + \int_{\Omega} \frac{1}{2R} [\hat{\rho}, v_{||}^2 |\mathbf{B}_{pol}|^2] u dW + \int_{\Omega} \frac{v_{||}^2 |\mathbf{B}|^2}{2R} [\hat{\rho}, u] dW \\ - \int_{\Omega} R \rho v_{||}^2 [u, \frac{|\mathbf{B}|^2}{2}] dW - \int_{\Omega} R \rho v_{||} \frac{F_0}{R^2} [u, v_{||}] dW + \int_{\Omega} R \rho v_{||}^2 [u, \frac{|\mathbf{B}_{pol}|^2}{2}] dW.$$

The fourth term of (E16) is the toroidal part of the term $R\rho |\mathbf{B}|^2 [u, v_{||}]$ in the parallel velocity equation. Now we split (E16) between two terms (E16a) and (E16b). (E16a) is defined by

$$(E16a) = + \int_{\Omega} \frac{1}{2R} [\hat{\rho}, v_{||}^2 |\mathbf{B}_{pol}|^2] u dW + \int_{\Omega} v_{||}^2 \frac{|\mathbf{B}_{pol}|^2}{2R} [\hat{\rho}, u] dW \\ - \int_{\Omega} R \rho v_{||}^2 [u, \frac{|\mathbf{B}_{pol}|^2}{2}] dW + \int_{\Omega} R \rho v_{||}^2 [u, \frac{|\mathbf{B}_{pol}|^2}{2}] dW.$$

This term is equal to zero (integrating by part the first term is sufficient to prove this). The (E16b) is defined by

$$(E16b) = \int_{\Omega} \frac{v_{||}^2}{R} \frac{F_0^2}{2R^2} [\hat{\rho}, u] dW - \int_{\Omega} R \rho v_{||}^2 [u, \frac{F_0^2}{2R^2}] dW - \int_{\Omega} R \rho v_{||} \frac{F_0^2}{R^2} [u, v_{||}] dW$$

We rewrite the term (E16b) to obtain

$$(E16b) = + \int_{\Omega} v_{||}^2 \frac{F_0^2}{2R^2} [\hat{\rho}, u] dV - \int_{\Omega} \hat{\rho} v_{||}^2 [u, \frac{F_0^2}{2R^2}] dV \\ - \int_{\Omega} \hat{\rho} v_{||} \frac{F_0^2}{2R^2} [u, v_{||}] dV - \int_{\Omega} \hat{\rho} v_{||} \frac{F_0^2}{2R^2} [u, v_{||}] dV$$

We combine the second and third terms and use the anti-symmetry property of the bracket for the fourth term. We obtain

$$(E16b) = - \int_{\Omega} \hat{\rho} v_{||} [u, \frac{F_0^2}{2R^2} v_{||}] dV + \int_{\Omega} v_{||}^2 \frac{F_0^2}{2R^2} [\hat{\rho}, u] dV + \int_{\Omega} \hat{\rho} v_{||} \frac{F_0^2}{2R^2} [v_{||}, u] dV$$

Now we combine the two last terms and we use anti-symmetric property of the bracket in the first to obtain

$$(E16b) = \int_{\Omega} \hat{\rho} v_{||} [\frac{F_0^2}{2R^2} v_{||}, u] dV + \int_{\Omega} v_{||} \frac{F_0^2}{2R^2} [\hat{\rho} v_{||}, u] dV = 0$$

The result is obtained using an integration by part. The last (E17) is given by

$$(E17) = - \int_{\Omega} \frac{1}{R} [\rho v_{||} (\nabla_{pol} v_{||} \cdot \nabla_{pol} \psi), \psi] u dW + \int_{\Omega} \frac{1}{R} v_{||} (\nabla_{pol} u \cdot \nabla_{pol} v_{||}) [\rho v_{||}, \psi] dW \\ - \int_{\Omega} \frac{1}{R} \rho v_{||}^2 [\psi, (\nabla_{pol} \psi \cdot \nabla_{pol} u)] dW - \int_{\Omega} R \rho |\mathbf{B}_{pol}|^2 [u, v_{||}] v_{||} dW$$

Firstly $R \rho |\mathbf{B}_{pol}|^2 [u, v_{||}] v_{||} = \frac{\rho}{R} (\nabla_{pol} \psi \cdot \nabla_{pol} \psi) v_{||} [u, v_{||}]$, secondly we have the identity

$$(\nabla_{pol} v_{||} \cdot \nabla_{pol} \psi) [u, \psi] = (\nabla_{pol} \psi \cdot \nabla_{pol} \psi) [u, v_{||}] + (\nabla_{pol} u \cdot \nabla_{pol} \psi) [v_{||}, \psi].$$

Using these two identities we obtain

$$(E17) = + \int_{\Omega} \rho v_{||} (\nabla_{pol} v_{||} \cdot \nabla_{pol} \psi) [u, \psi] dV - \int_{\Omega} \rho v_{||}^2 [(\nabla_{pol} u \cdot \nabla_{pol} \psi), \psi] dV \\ - \int_{\Omega} \rho v_{||} (\nabla_{pol} u \cdot \nabla_{pol} \psi) [v_{||}, \psi] dV - \int_{\Omega} \rho v_{||}^2 [\psi, (\nabla_{pol} u \cdot \nabla_{pol} \psi)] dV \\ - \int_{\Omega} \rho v_{||} (\nabla_{pol} \psi \cdot \nabla_{pol} \psi) [u, v_{||}] dV$$

The sum of second and fourth terms is equal to zero (anti-symmetry property of the bracket). The sum of the other terms is equal to zero (second identity). To finish the proof we compute (E18) defined by

$$(E18) = \int_{\Omega} \frac{1}{R} \partial_Z \left(\widehat{\rho} \frac{F_0^2}{R^3} v_{||}^2 \right) u dW + \int_{\Omega} \rho v_{||}^2 \frac{F_0^2}{R^2} \partial_Z u dW$$

This term is equal to zero because the sum of the two terms is also equal to zero (using a integration by part). This last result finishes the proof. \square

This result proves that the physical energy associated with the MHD system is conserved in the ideal case ($\nu = \eta = 0$) and dissipated in the resistive case. Firstly this result is an argument to justify the derivation of the reduced MHD models as the conservation of the energy is an essential physical property. Secondly this result is useful to study at the mathematical level that the model is well-posed. For example in [1]-[3] the authors explain and detail the key role of the energy balance to prove the existence of weak solutions. Finally, this energy estimate is very important to ensure the numerical stability of the schemes. Indeed a way to ensure the stability is to design a numerical method which dissipates the energy at the discrete level and we cannot obtain this stability property without energy conservation on the continuous model.

A small remark: the term $\frac{\partial_{\phi\phi}\psi}{R^4}$ which is the poloidal current is neglected in the JOREK code. With or without this term we have a model which conserve the energy in the ideal case and dissipate the energy in the resistive case.

3 Spatial and Time discretization

3.1 Spatial discretization

In the JOREK code, different discretization methods are applied for the toroidal direction and the poloidal plane. For the toroidal direction we use a classical Fourier expansion. This discretization is easy to implement but generates a large matrix. Using a Fast Fourier transformation we obtain a faster algorithm to construct the matrix and the right hand side. For the Poloidal plane we use a classical finite element method with numerical viscosity to stabilize the method. The elements chosen are Cubic Bezier elements which allows to guarantee C^1 continuity useful to discretize the fourth order operators and preserve the free divergence constraints. However this C^1 reconstruction is not guaranteed for the grid center and for the X-Point. These elements use 4 degrees of freedom

by node contrary to the Lagrangian cubic element (9 degrees of freedom by node). Another advantage comes from the isoparametric formulation. Indeed we can discretize the geometrical quantities like R and Z with Bezier Splines. This property allows to construct the aligned grid with the magnetic surfaces easily. The details about the discretization using Bezier elements are given in [15].

3.2 Original time discretization and preconditioning

In this section we explain the time discretization used in JOREK and the preconditioning used for the linear solver. The different models implemented in the JOREK code (with or without parallel velocity) can be written in the following form

$$\partial_t A(\mathbf{U}) = B(\mathbf{U})$$

with A and B discrete nonlinear differential operators and $\mathbf{U} = (\psi, u, j, w, \rho, T, v_{||})$. For the time discretization we use the classical Crank Nicholson or a Gear second order scheme allowing to write the time scheme in the following form

$$(1 + \zeta)A(\mathbf{U}^{n+1}) - \theta\Delta t B(\mathbf{U}^{n+1}) = (1 + 2\zeta)A(\mathbf{U}^n) - \zeta A(\mathbf{U}^{n-1}) + (1 - \theta)\Delta t B(\mathbf{U}^n)$$

with ζ and θ the parameters of the scheme. If $\theta = 1$ and $\zeta = 0$ we obtain the implicit Euler scheme, if $\zeta = 0$ and $\theta = \frac{1}{2}$ we obtain the Crank-Nicholson scheme and if $\theta = 1$ and $\zeta = \frac{1}{2}$ we obtain the Gears scheme. These implicit schemes does not preserve the decay of the discrete time energy, because the system is too nonlinear. Find a accurate time with this property is an interesting problem for the future. Now we define two nonlinear vectors $G(\mathbf{U}) = (1 + \zeta)A(\mathbf{U}) - \theta\Delta t B(\mathbf{U})$ and $b(\mathbf{U}^n, \mathbf{U}^{n-1}) = (1 + 2\zeta)A(\mathbf{U}^n) - \zeta A(\mathbf{U}^{n-1}) + (1 - \theta)\Delta t B(\mathbf{U}^n)$. At the end we want to solve the following nonlinear system

$$G(\mathbf{U}^{n+1}) = b(\mathbf{U}^n, \mathbf{U}^{n-1}).$$

A first order linearization is applied in the original code to obtain the following linear system

$$\left(\frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}} \right) \delta \mathbf{U}^{n+1} = -G(\mathbf{U}^n) + b(\mathbf{U}^n, \mathbf{U}^{n-1}) = R(\mathbf{U}^n)$$

with $\delta \mathbf{U}^{n+1} = \mathbf{U}^{n+1} - \mathbf{U}^n$ and the Jacobian $J_n = \frac{\partial G(\mathbf{U}^n)}{\partial \mathbf{U}}$. To solve this system we use the classical GMRES method with left preconditioning. The principle of the left preconditioning is to replace the solver $J_n \delta \mathbf{U}^{n+1} = R(\mathbf{U}^n)$ by $M_n^{-1} J_n \delta \mathbf{U}^{n+1} = M_n^{-1} R(\mathbf{U}^n)$. The last system can be split between two steps. First we solve exactly

$$M_n \delta \mathbf{y} = R(\mathbf{U}^n)$$

and then we solve with the GMRES method

$$M_n^{-1} J_n \delta \mathbf{U}^{n+1} = \delta \mathbf{y}.$$

It is necessary to obtain the final algorithm that the preconditioning matrix M_n is invertible. The idea currently followed in the code is to write the Jacobian by block, each block corresponds to the coupling terms between two Fourier modes. Under the assumption of weak coupling it is possible to eliminate the non diagonal blocks. We obtain a diagonal block matrix where the blocks correspond to the equations for each Fourier mode. To compute the inverse we use a direct solver (LU method for example) to obtain the inverse of each block and consequently the inverse of M_n . To minimize the CPU cost we don't invert M_n at each time step, but only when the convergence for the previous linear step is too slow.

3.3 Nonlinear time solvers

A first order linearization may not be the optimal choice to solve the problem in the nonlinear phase of the run. Consequently we propose to replace this linearization by a Newton procedure. Since we use an iterative solver to compute the solution of the linear system, it will be interesting to use an inexact Newton procedure [4]-[5]. This variation of the Newton method means that the criterion of convergence of the GMRES method is adapted using the nonlinear residue and the convergence of the Newton procedure. The aim is to use the nonlinear convergence to minimize the number of GMRES iteration. Indeed it is not necessary to solve with a high accuracy the linear system but just enough to converge to the solution of the nonlinear system. In the following we detail the Inexact Newton algorithm to solve.

Algorithm

- At the time step n , we compute $b(\mathbf{U}^n, \mathbf{U}^{n-1})$, $G(\mathbf{U}^n)$
- We choose ϵ_0 and the initial guess $\delta\mathbf{U}_0$
- At each iteration k of the Newton method we have the solution \mathbf{U}_k .
- We compute $G(\mathbf{U}_k)$ and the Jacobian J_k .
- We solve the linear system with GMRES $J_k\delta\mathbf{U}_k = -G(\mathbf{U}_k) + b(\mathbf{U}^n, \mathbf{U}^{n-1}) = R(\mathbf{U}^k, \mathbf{U}^n)$ and the following convergence criterion

$$\frac{\|J_k\delta\mathbf{U}_k - R(\mathbf{U}^k, \mathbf{U}^n)\|}{\|R(\mathbf{U}^k, \mathbf{U}^n)\|} \leq \epsilon_{gmres}^k$$

with

$$\epsilon_{gmres}^k = \gamma \left(\frac{\|R(\mathbf{U}^k, \mathbf{U}^n)\|}{\|R(\mathbf{U}^{k-1}, \mathbf{U}^n)\|} \right)^\alpha$$

- We iterate with $\mathbf{U}_{k+1} = \mathbf{U}_k + \delta\mathbf{U}_k$.
- We apply a convergence test (for example $\|R(\mathbf{U}_k, \mathbf{U}^n)\| < \epsilon_a + \epsilon_r\|R(\mathbf{U}^n)\|$).
- When the Newton method has converged we define $\mathbf{U}^{n+1} = \mathbf{U}_{k+1}$.

We couple this algorithm with an adaptive time stepping which allows to use large time steps in the linear phase and smaller time steps in the nonlinear phase. Actually the principle is simple if the Newton process converges very quickly we increase the time step and if the convergence is slow we decrease the time step for the following iteration. If the Newton process does not converge or if $\|R(\mathbf{U}^{k+1}, \mathbf{U}^n)\| > \|R(\mathbf{U}^k, \mathbf{U}^n)\|$ during two or three consecutive linear steps we decrease the time step and restart the Newton iterations. To have a smooth evolution of the time step it is necessary to avoid a large increasing or decreasing of the time step.

4 Numerical results

In general the different test cases used in this paper have the same structure. First we compute the equilibrium on the poloidal grid (Fig 1., left), compute the aligned grid (Fig 1., right) and begin the time loop. At the beginning of the time loop peeling-ballooning modes set in which are responsible for the appearance of edge localized modes (ELMs). These linear instabilities are driven by large pressure gradients (steep pressure pedestal) and large current densities in the edge. During these instabilities the energy associated with the non principal modes grow exponentially. The background profiles are modified. When the energies associated with the non principal modes are sufficiently large, the pressure gradients get smaller which stabilizes the instability. This is the

nonlinear saturation phase. The implicit time methods are known to be stable without restriction on the time step (classical property), however this type of result is valid for stable physical dynamic and stable models. In our cases we have physical instabilities consequently the numerical stability is not ensured. Typically we will show that if the numerical error (time error, linearization error) becomes too large the numerical simulation does not capture correctly the beginning of the salutation phase and generates critical numerical instabilities.

In this section we present some numerical results for the models with and without parallel velocity. We add to the reduced MHD models, numerical diffusion operators for each equation and two anisotropic diffusion operators on the density and the temperature (density and pressure equations). For example, for the pressure equation we add the following diffusion operator

$$\nabla \cdot (k_{\parallel} \nabla_{\parallel} T + k_{\perp} \nabla_{\perp} T) = \nabla \cdot ((k_{\parallel} - k_{\perp}) \nabla_{\parallel} T + k_{\perp} \nabla T)$$

with $\nabla_{\parallel} T = \frac{\mathbf{B}}{\|\mathbf{B}\|} \cdot (\frac{\mathbf{B}}{\|\mathbf{B}\|} \cdot \nabla T)$ and $\nabla_{\perp} = \nabla - \nabla_{\parallel}$.

We propose to compare the different methods (Exact and Inexact Newton methods and classical linearization) mainly in the nonlinear phase. Indeed in the linear phase the classical method is clearly more efficient. In this phase the preconditioning is very efficient and the GMRES solver converges quickly (between 1 and 5 iterations). The Newton procedure converges with 3 iterations in general. Consequently using the Newton method the cost is clearly higher for each time step in the linear phase. In the nonlinear phase the situation is more complicated. The nonlinear phase begins when the quantities associated with the non principal modes have the same order of magnitude as the quantities associated to the principal mode. To compare the numerical results, we define the beginning of the nonlinear phase as the time where the kinetic and magnetic energies for $n \neq 0$ are at the level of the energies associated to $n = 0$. To compare the classical linearization and the Newton procedure we use the adaptive time stepping. If the algorithm for one time step does not converge we recompute it with a smaller time step (typically $\Delta t_{new} = 0.8 \Delta t_{old}$. For the Newton and the linearization methods the factorization is recomputed for each time step and during a Newton step the factorization is recomputed if the number of GMRES iterations associated with the two last Newton steps is superior to 50.

4.1 Model without parallel velocity

4.1.1 First test case

This first case corresponds to a simplified equilibrium configuration associated to the JET reactor. We solve the model without parallel velocity. In this case the numerical viscosity is zero and the numerical resistivity is 10^{-10} . The physical viscosity and resistivity, dependent on the temperature are given by $\eta(T) = 2 \times 10^{-6} T^{-\frac{5}{2}}$ and $\nu(T) = 4 \times 10^{-6} T^{-\frac{5}{2}}$. We consider a geometry with X-point. For the Linearization procedure the criterion of convergence for the GMRES procedure is $\varepsilon = 10^{-8}$. For the Newton procedure the maximum number of Newton iteration is 10 and the criterion of convergence for the Newton procedure $\varepsilon_a = 10^{-5}$, the ε_{gmres}^0 of the GMRES convergence criterion is 0.0005. Using $\Delta t = 30$ we compare the results for the linearization method, the exact Newton method and the inexact Newton method. These results are given between the time 1400 and 3500 corresponding to the nonlinear saturation phase. The code is executed with 2 MPI and 16 OpenMP threads per MPI process. In the tables (Tab 1) - (Tab 2) - (Tab 3) we give the average of different quantities associated to the solver during one time step.

Linearization method			
$\Delta t =$	GMRES Iter.	LU fact.	time
30	19	1	53.25

Table 1: For the linearization method the average number of gmres iterations and LU factorizations per time step are given as well as the wall clock time

Exact Newton					
$\Delta t =$	GMRES Iter.	LU fact.	Newton iter.	Total Gmres iter.	time
30	19.8	1.05	3	59	79.6
40	26.6	1.28	3.2	85.5	102

Table 2: For the exact Newton method the average number of total GMRES iterations, LU factorizations, number of Newton iteration and number to GMRES iteration per Newton step per time step are given as well as the wall clock time.

Inexact Newton					
$\Delta t =$	GMRES Iter.	LU fact.	Newton iter.	Total Gmres iter.	time
30	3.3	1	5.7	18.7	76.25
40	5.4	1	5.8	31	82.9

Table 3: For the inexact Newton method the average number of total GMRES iterations, LU factorizations, number of Newton iteration and number to GMRES iteration per Newton step per time step are given as well as the wall clock time.

Some remarks about these results. It is clear that the CPU cost associated with the Newton procedure is higher compared to the classical linearization for the same time stepping. This result is expected, indeed by definition of the Newton method, the number of linear problems solved is larger with the Newton procedure. First in the nonlinear phase we remark that the Newton procedure is also less performing, but using an inexact Newton method we can reduce the CPU cost. In the (Tab 2) - (Tab 3). We remark that for the time step $\Delta t = 30$, the main difference between the Inexact and exact Newton method is small, but for $\Delta t = 40$ the difference is larger. The main difference between exact and inexact Newton method can be explained by the fact that the number of GMRES iterations is larger using exact Newton method consequently the factorization for the preconditioning is called more often. At the end we remark that the inexact Newton method is clearly more efficient when the problem is more nonlinear and similar when the problem is not too nonlinear. This result verify the usefulness of the inexact Newton method.

Now we propose to compare the linearization method and the inexact Newton method. In the nonlinear phase the difference is less important. Indeed in the nonlinear phase the number of GMRES iterations for each linear problem is larger. Using an inexact Newton procedure we have more linear problems to solve but each linear system is solved with a small accuracy. Consequently the cost associated with each linear system is smaller when we use the inexact Newton procedure. The table (Tab 1) - (Tab 3) for $\Delta t = 30$ show that the total GMRES iteration for one time step are similar between an inexact Newton procedure and the linearization procedure. Consequently the additional cost associated with the inexact Newton method come from the computation of the matrix and in this case is around 1.5 which is an acceptable additional cost. Additionally, the parallel scaling is better for the construction of the Matrix than the iterative solver and the preconditioning. Consequently with more MPI process the difference between the CPU cost associated with the inexact Newton method and the linearization method can be reduced.

Secondly we compare the two methods with $\Delta t = 40, 50, 60$. For the Newton procedure the maximum number of Newton iterations is 20 and the criterion of convergence for the Newton procedure $\varepsilon = 10^{-7}$, the ε_0 of the GMRES convergence criterion is 0.0005. We plot the kinetic and magnetic energies associated with the different modes for the two procedures and the different time steps (Fig 3.) and (Fig 4.).

These results show that the Newton procedure with adaptive time stepping is more robust than the classical linearization and allows to use a larger time step. When we use the classical linearization with a very large time step, the numerical error linked to the time discretization and the linearization becomes too large such that consequently the numerical scheme does not capture correctly the beginning of the saturation phase. In this case, numerical instabilities appear and

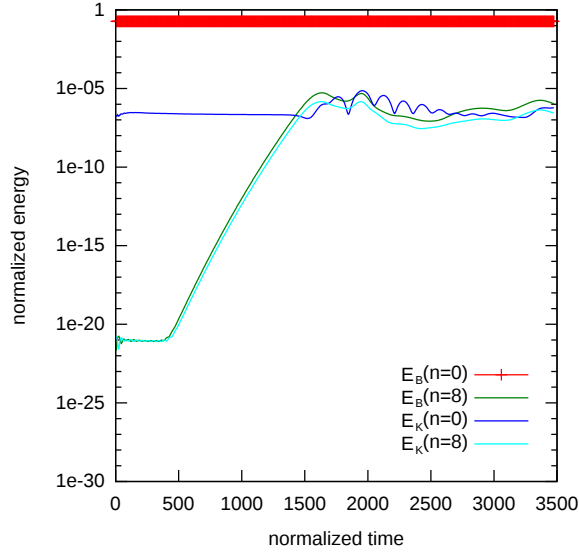


Figure 3: Reference solution: Kinetic and Magnetic energies for $\Delta t = 5$ gives by the Newton method.

the iterative solver does not converge after the beginning of the numerical instability. If we use an adaptive time stepping the situation is the same because in general the scheme computes the beginning of the numerical instabilities and at this moment is too late to adapt and decrease the time step.

With the Newton procedure the situation is different. First the error of linearization and consequently the global numerical error is smaller consequently we can use larger time step and capture correctly the beginning of the saturation phase. Secondly we don't have the problem associated with the numerical instabilities with the Newton procedure as the Newton method does not converge in case of the numerical instability such that the time step is recalculated with smaller Δt . To conclude the adaptive time stepping works with the Newton method because this procedure detects the beginning of the numerical instabilities by non convergence of the method contrary to the linearization and to continue the computation it is necessary to adapt the Δt before the beginning of the numerical instability. Consequently the Newton procedure is more robust because allowing an efficient adaptive time stepping which avoids numerical instabilities for large time steps and non convergence issues. The figure (Fig 4.) shows that the code with the linearization method does not converge with $\Delta t = 40$ contrary to the Newton method which converges even with $\Delta t = 60$.

This test case is not too nonlinear and consequently not too stiff for the numerical method. For more nonlinear test cases the Newton procedure gives better results when the problems get stiffer.

4.1.2 Second test case

This second test case corresponds to a realistic ASDEX Upgrade equilibrium configuration with unrealistically large resistivity which makes the instability especially violent. We solve the model without parallel velocity. In this case the numerical viscosity and the numerical resistivity are close to 10^{-11} . The physical viscosity and resistivity are dependent of the temperature : $\eta(T) = 2 \times 10^{-5} T^{-\frac{5}{2}}$ and $\nu(T) = 3 \times 10^{-5} T^{-\frac{5}{2}}$. We consider a geometry with X-point. For the Linearization procedure the maximum number of GMRES iteration is 500 and the the criterion of convergence for the GMRES procedure $\varepsilon = 10^{-8}$. Firstly we propose to compare the two methods $\Delta t = 5, 10, 20$. For the Newton procedure the maximum number of Newton iterations is 10

and the the criterion of convergence for the Newton procedure $\varepsilon = 10^{-5}$, the ε_0 of the GMRES convergence criterion is 0.0005.

This test case with violent physical instabilities allows to confirm the previous remarks about adaptive time stepping and numerical instabilities. First if we choose a too big time step with the linearization method, we have a numerical instability which appears and the adaptive time stepping is not efficient.

Gmres $\Delta t = 1$	Newton + adaptive time method
18800	7600

Table 4: Total time of the simulation for the GMRES method with $\Delta t = 1$ and for Newton method with adaptive time method (initial time step $\Delta t = 10$). Final time $T = 450$

Due to the violent physical instabilities the problem is strongly nonlinear in the saturation phase. Contrary to the previous test case, using the Newton method allows to reduce significantly the CPU cost for the total run (Tab 5.).

The inexact Newton method with adaptive time stepping is more robust than the linearization method and allows to reduce the CPU costs for highly nonlinear cases due to the larger time steps possible.

4.2 Model with parallel velocity

This test case is the same as the one used in section 4.1.1 but we solve the model with parallel velocity. First we compare the two methods in the nonlinear phase with $\Delta t = 20$. For the Newton procedure the maximum number of Newton iteration is 10 and the the criterion of convergence for the Newton procedure $\varepsilon=10^{-5}$, the ε_0 of the GMRES convergence criterion is 0.0005. These results are given between the time 1250 and 3500 which correspond to the saturation phase (stiff part of the computation).

Linearization method			
models	Gmres Iter.	Precon . called .	time
with neglected terms	25.4	1	75.7
without neglected terms	28	1	83.6

Table 5: Average of number of Gmres iteration and preconditioning called during a time step. Linearization method. $\Delta t = 20$

Inexact Newton method					
models	Gmres Iter.	Precon . called	Newton iter.	Total Gmres iter.	time
with neglected terms	5.1	1	6.4	32.7	119.3
without neglected terms	5.2	1	6.4	33.4	122.5

Table 6: Average of number of Gmres iteration and preconditioning called during a time step. Inexacte Newton method. $\Delta t = 20$

The conclusions on the comparison between the Newton procedure and the Linearization procedure are similar to the conclusion to the first test case. Indeed in the Nonlinear phase we have an additional cost around 1.4 - 1.5. But this additional cost can be reduced using a larger time step. Indeed with the Newton procedure (as previously) we can use larger time steps with the Newton procedure without numerical instabilities. For example in this case the Linearization method is unstable with $\Delta t = 25$ and the Newton method is stable with $\Delta t = 40$.

For all these results we recompute the factorization for the preconditioning at each time step. For the Newton method we have added an additional rule. The factorization is recomputed if the

Inexact Newton method					
models	Gmres Iter.	Precon . called	Newton iter.	Total Gmres iter.	time
with neglected terms	10.9	1.1	6.95	75.6	152
without neglected terms	8.7	1	6.7	58	142

Table 7: Average of number of Gmres iteration and preconditioning called during a time step. Inexacte Newton method. $\Delta t = 20$

convergence is too slow for the previous linear step. To reduce the CPU time we can use only the second rule for the Newton procedure and the Linearization method. In this case, it is not necessary to compute the factorization for each time step. The different test cases show that for the Newton procedure it will be important to use a smaller ε_0 (initial ε for the GMRES method in the Inexact Newton procedure) to compute correctly the first Newton iteration.

The last remark about this result is on the difference between the model with and without neglected terms. These terms in the potential and parallel velocities equations comes from to the fact the poloidal and parallel velocity are not perpendicular, this is the cross terms between the poloidal velocity and the poloidal part of the parallel velocity. In the (Fig 7.) we remark that we have small differences in the dynamics of kinetic and magnetic energies between the models with and without neglected terms. We observe these differences for the linearization method with $\Delta = 20$ and for the Newton methods with $\Delta = 40$. With the Newton procedure and $\Delta t = 20$ the difference is smaller. In theory these terms are small consequently it is expected that the impact of these terms is small when the numerical error (Time and linearization errors) is small. When the error is larger (Linearization method with $\Delta = 20$, Newton method with $\Delta = 40$) the impact of these terms is more important. However the impact of these additional terms on the stability, conditioning and convergence issues is not clear and require additional studies for exemple when the resistivity and viscosity are small.

5 Conclusion

In this paper, we have presented a rigorous analytical derivation of the reduced MHD models implemented in the non-linear MHD code JOREK. Starting from the potential formulation of the magnetic field vector and fluid velocity used in JOREK we obtain a few additional terms that have been neglected in the code but might be relevant in the non-linear phase. We have also given a proof of the conservation of total energy for this reduced MHD model if the additional terms are taken into account. This is an important validation for the choices of the projections and the assumptions of the derivation and an important basis for numerical stability analysis.

The second part of the paper is focused on the time solver of JOREK. The original method used in JOREK for the time-stepping of the nonlinear system is a linearization solved iteratively by GMRES with physics-based preconditioning. We have replaced this by the nonlinear inexact Newton method in which the linear convergence accuracy of GMRES depends on the non-linear convergence. Especially at the onset of non-linear saturation, large numerical errors can cause numerical instabilities and prevent convergence. The non-linear time stepping reduces those errors and consequently allows to use larger time steps as confirmed by numerical tests. We have also implemented and tested an adaptive time stepping that works very efficiently with the Newton method and allows to reduce computational costs. The Newton method is more robust than the linearization method as it avoids certain numerical instabilities, is well suited for adaptive time stepping, and allows to reduce computational costs in highly non-linear cases. The Newton method is currently implemented for the single fluid reduced MHD equations in JOREK, and will be extended to two-fluid terms and further extended models in the future.

References

- [1] B. Després, R. Sart *Reduced resistive MHD in Tokamaks with general density*, Mathematical Modelling and numerical analysis, 46, 1081-1106, (2012)
- [2] E. Deriaz, B. Després, G. Faccanoni, K. P. Gostaf, L. M. Imbert, G. Sadaka, R. Sart *Magnetic equations with FreeFem ++: the Grad-Shafranov equation and the Current Hole*, ESAIM : Proc. 32, 149–162, (2011)
- [3] B. Després, R. Sart *Derivation of hierarchies of reduced MHD models in Tokamaks*, submitted to Archive of Rational Mechanics and Analysis, preprint: http://hal.archives-ouvertes.fr/docs/00/79/64/25/PDF/mhd_tout_12.pdf
- [4] R. S. Dembo, S. C. Eisenstat, T. Steihaug *Inexact newton methods*, SIAM J. Numer. Anal 19 (1982)
- [5] S. C. Eisenstat, H. F. Walker *Globally convergent Inexact newton methods*, SIAM J. Sci. Stat. Comput., 6 (1985), pp 793-832
- [6] S. E. Kruger, C. C. Hegna, J. D. Callen *Generalized reduced magnetohydrodynamic equations*, physics of plasmas volume 5, number 12, 1998
- [7] G. T. A. Huysmans, O. Czarny *MHD stability in X-point geometry: simulation of ELMs*, Nucl. Fusion 47 (2007) 659-666
- [8] L. Chacón, D. A. Knoll, J. M. Finn *An implicit, nonlinear reduced resistive MHD solver*, JCP volume 178 issue 1, 2002
- [9] B. Phillip, L. Chacón, M. Pernice *Implicit adaptive mesh refinement for 2D reduced resistive Magnetohydrodynamics*, JCP volume 227 issue 20, 2008
- [10] L. Chacón, D. A. Knoll *A 2D high β hall MHD implicit nonlinear solver*, JCP volume 188 issue 2, 2003
- [11] L. Chacón, *An optimal, parallel, fully implicit Newton-Krylov solver for three-dimensional viscoresistive magnetohydrodynamics*, Physics of Plasmas vol 15 issue 5, 2008
- [12] L. Chacón, *Scalable parallel implicit solver for 3D magnetohydrodynamics*, Physics of Physics, conference series, volume 125
- [13] S. K. Malapaka, B. Després, R. Sart *Unconditionally stable numerical simulations of a new generalized reduced resistive magnetohydrodynamics model*, International Journal for Numerical Methods in Fluids Volume 74, Issue 4, pages 231-249, 10 February 2014.
- [13] H. R. Strauss *Reduced MHD in nearly potential magnetic fields*, Journal of Plasma Physics / Volume 57 / Issue 01 / January 1997, pp 83-87
- [14] M Hölzl, S Günter, RP Wenninger, W-C Müller, GTA Huysmans, K Lackner, I Krebs, ASDEX Upgrade Team *Reduced-magnetohydrodynamic simulations of toroidally and poloidally localized edge localized modes*, Physics of Plasmas, volume 19, 2007
- [15] G. T. A. Huysmans, O. Czarny, *Bézier surfaces and finite elements for MHD simulations*, Journal of Computational Physics archive Volume 227 Issue 16, August, 2008, pages 7423-7445
- [16] Marie Martin, *Modélisations fluides pour les plasmas de fusion : approximation par éléments finis C1 de Bell*, 2013.
- [17] H. Zohm, *Edge localized modes (ELMs)*, 1996 Plasma Phys. Control. Fusion 38.

- [18] PB Snyder, HR Wilson, JR Ferron, LL Lao, AW Leonard, TH Osborne, AD Turnbull, D Mossessian, M Murakami, XQ Xu, *Edge localized modes and the pedestal: A model based on coupled peeling ballooning modes*, Physics of Plasmas 9 (5), 2037-2043, 2002

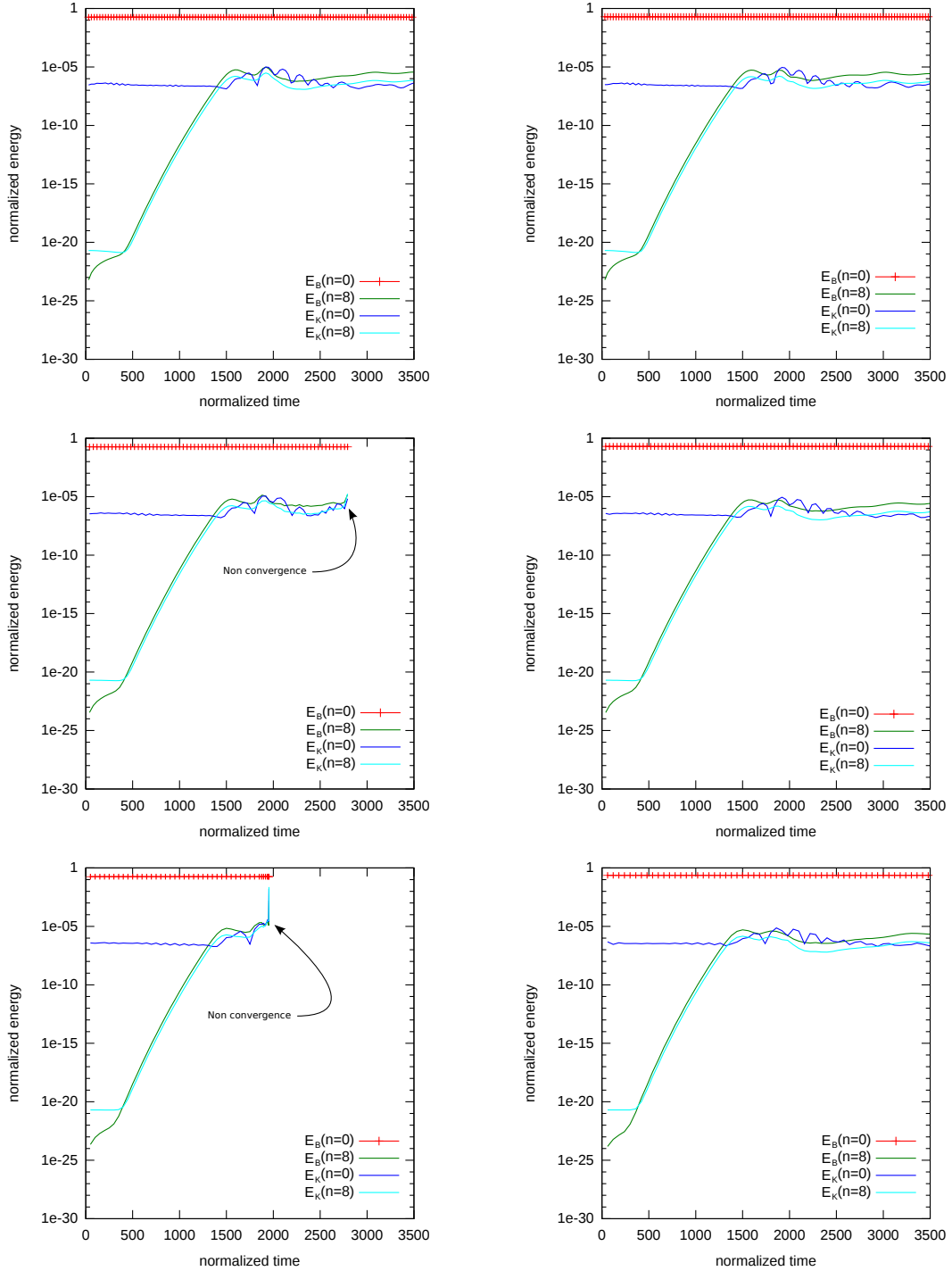


Figure 4: In the left Kinetic and Magnetic energies for Linearization method for $\Delta t = 30, 40, 50$. In the right Kinetic and Magnetic energies for Newton method for $\Delta t = 30, 40, 60$

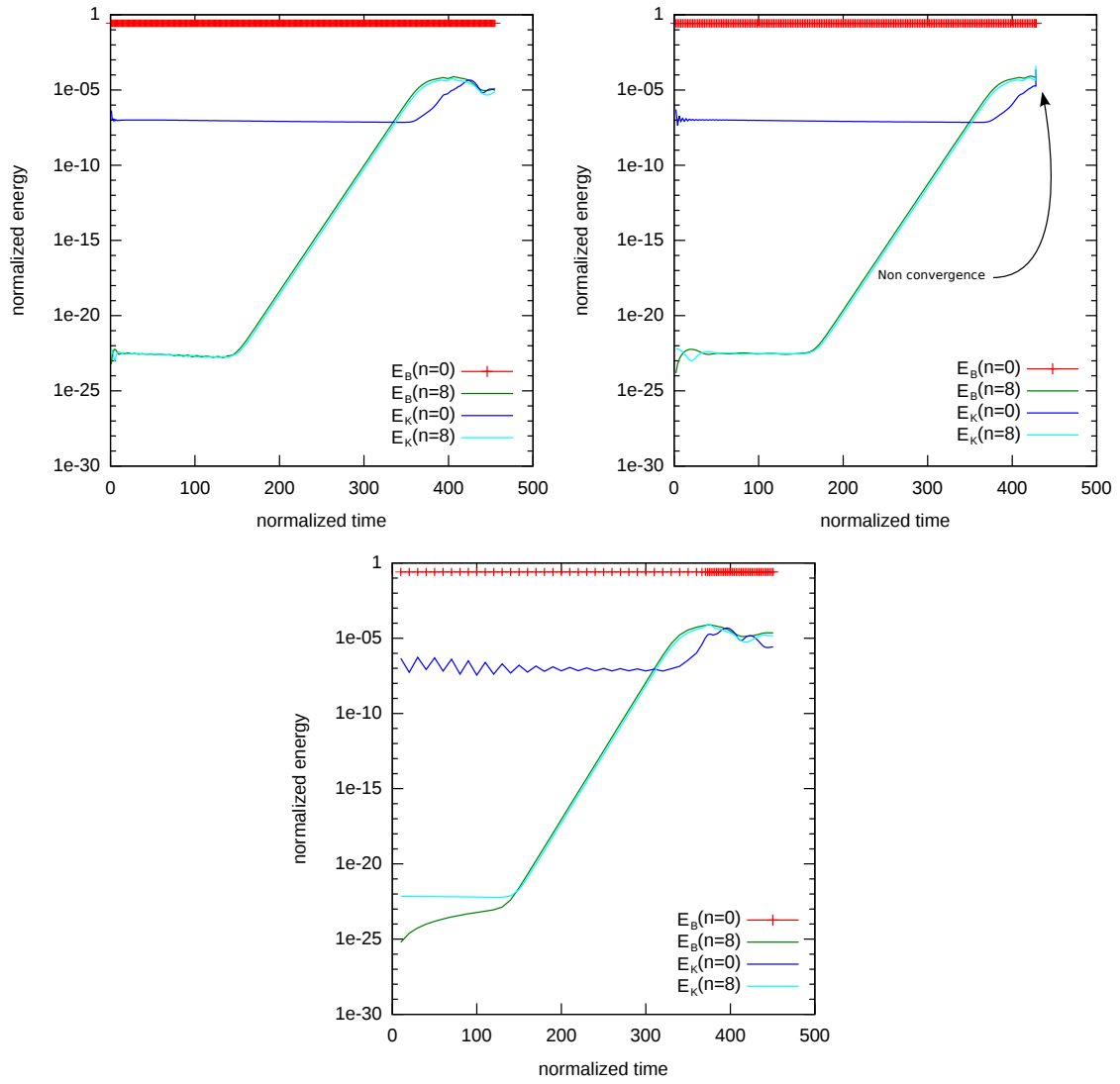


Figure 5: In the left Kinetic and Magnetic energies for Linearization method for $\Delta t = 1$. In the middle Kinetic and Magnetic energies for Linearization method for $\Delta t = 2$. In the right Kinetic and Magnetic energies for the Newton method for $\Delta t = 10$ with adaptive time stepping.

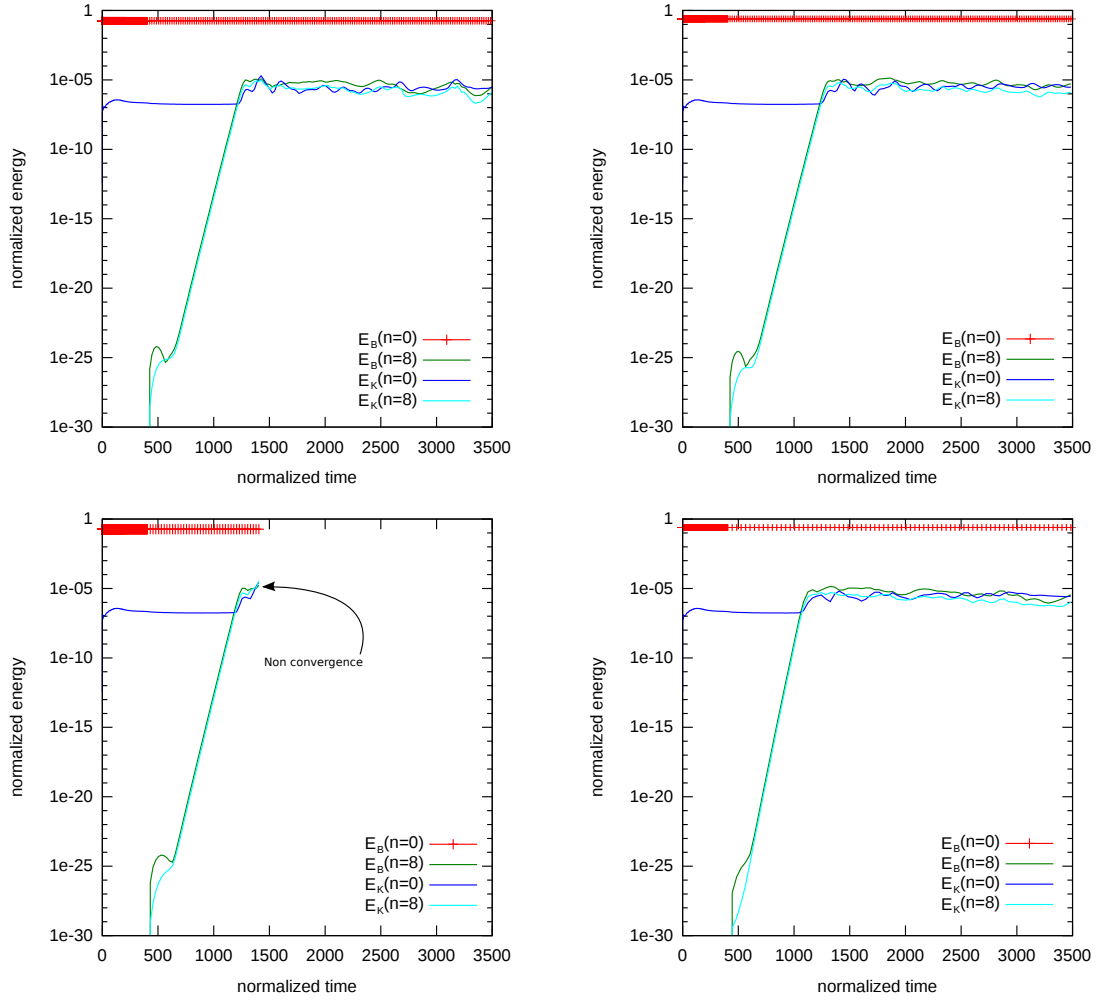


Figure 6: In the left Kinetic and Magnetic energies for Linearization method for $\Delta t = 20, 25$. In the left Kinetic and Magnetic energies for Newton method for $\Delta t = 20, 40$

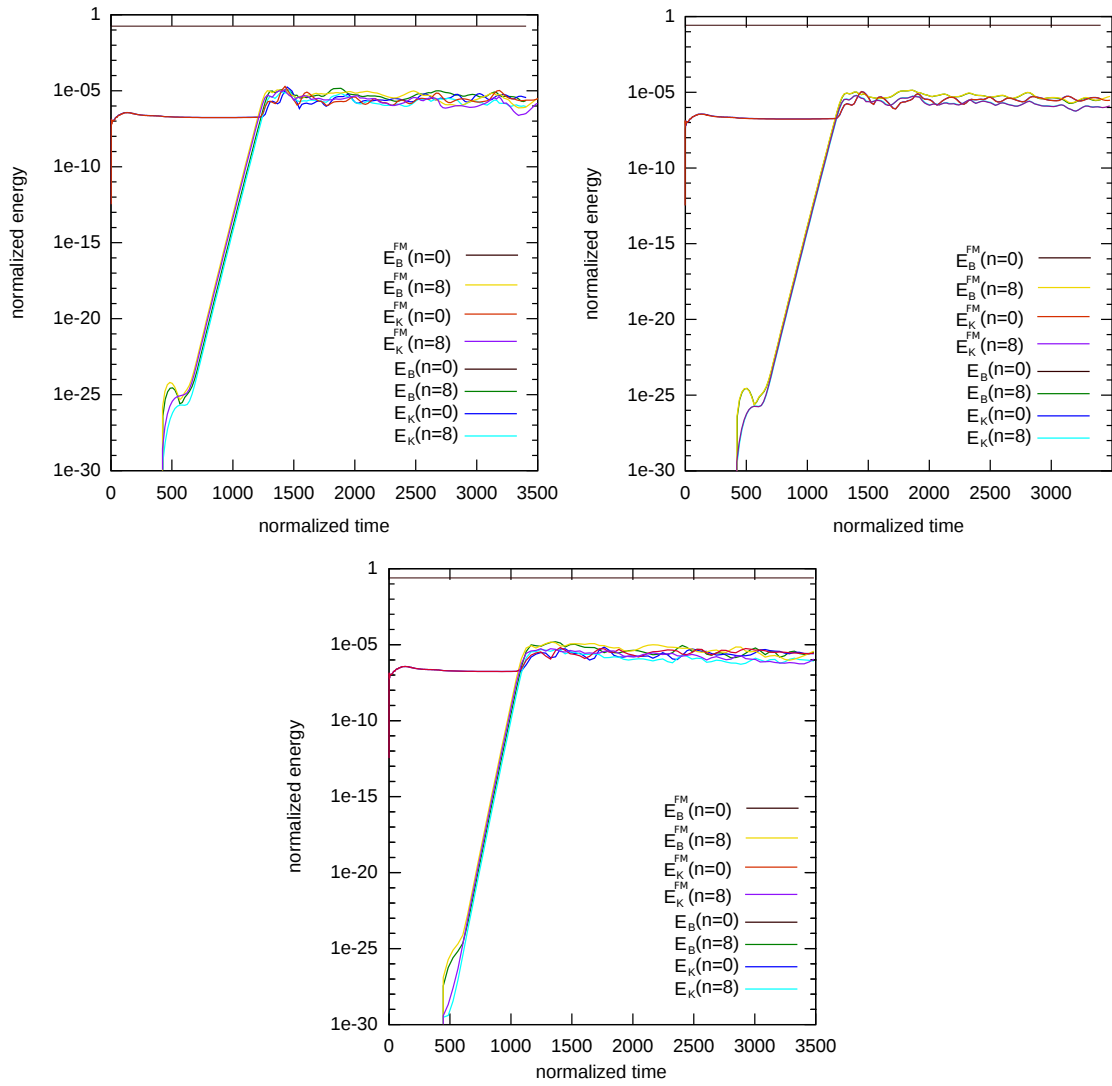


Figure 7: Comparison between the full model (model with neglected terms) and the model without neglected terms. In the top and left results given by the Linearization method with $\Delta t = 20$, in the top and right, results given by the Newton method with $\Delta t = 20$ and in bottom results given by the Newton method with $\Delta t = 40$.