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FAMILIES OF AFFINE RULED SURFACES: EXISTENCE OF CYLINDERS

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

ABSTRACT. We show that the generic fiber of a family $f : X \rightarrow S$ of smooth \mathbb{A}^1 -ruled affine surfaces always carries an \mathbb{A}^1 -fibration, possibly after a finite extension of the base S . In the particular case where the general fibers of the family are irrational surfaces, we establish that up to shrinking S , such a family actually factors through an \mathbb{A}^1 -fibration $\rho : X \rightarrow Y$ over a certain S -scheme $Y \rightarrow S$ induced by the MRC-fibration of a relative smooth projective model of X over S . For affine threefolds X equipped with a fibration $f : X \rightarrow B$ by irrational \mathbb{A}^1 -ruled surfaces over a smooth curve B , the induced \mathbb{A}^1 -fibration $\rho : X \rightarrow Y$ can also be obtained from a relative Minimal Model Program applied to a smooth projective model of X over B .

INTRODUCTION

The general structure of smooth non complete surfaces X with negative (logarithmic) Kodaira dimension is not fully understood yet. For say smooth quasi-projective surfaces over an algebraically closed field of characteristic zero, it was established by Keel and McKernan [10] that the negativity of the Kodaira dimension is equivalent to the fact that X is generically covered by images of the affine line \mathbb{A}^1 in the sense that the set of points $x \in X$ with the property that there exists a non constant morphism $f : \mathbb{A}^1 \rightarrow X$ such that $x \in f(\mathbb{A}^1)$ is dense in X with respect to the Zariski topology. This property, called \mathbb{A}^1 -uniruledness is equivalent to the existence of an open embedding $X \hookrightarrow (\overline{X}, B)$ into a complete variety \overline{X} covered by proper rational curves meeting the boundary $B = \overline{X} \setminus X$ in at most one point. In the case where X is smooth and affine, an earlier deep result of Miyanishi-Sugie [14] asserts the stronger property that X is \mathbb{A}^1 -ruled: there exists a Zariski dense open subset $U \subset X$ of the form $U \simeq Z \times \mathbb{A}^1$ for a suitable smooth curve Z . Equivalently, X admits a surjective flat morphism $\rho : X \rightarrow C$ to an open subset C of a smooth projective model \overline{Z} of Z , whose generic fiber is isomorphic to the affine line over the function field of C . Such a morphism $\rho : X \rightarrow C$ is called an \mathbb{A}^1 -fibration, and we say that ρ is of affine type or complete type when the base curve C is affine or complete, respectively.

Smooth \mathbb{A}^1 -uniruled but not \mathbb{A}^1 -ruled affine varieties are known to exist in every dimension ≥ 3 [1]. Many examples of \mathbb{A}^1 -uniruled affine threefolds can be constructed in the form of flat families $f : X \rightarrow B$ of smooth \mathbb{A}^1 -ruled affine surfaces parametrized by a smooth base curve B . For instance, the complement X of a smooth cubic surface $S \subset \mathbb{P}_{\mathbb{C}}^3$ is the total space of a family $f : X \rightarrow \mathbb{A}^1 = \text{Spec}(\mathbb{C}[t])$ of \mathbb{A}^1 -ruled surfaces induced by the restriction of a pencil $\overline{f} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ on \mathbb{P}^3 generated by S and three times a tangent hyperplane H to S whose intersection with S consists of a cuspidal cubic curve. The general fibers of f have negative Kodaira dimension, carrying \mathbb{A}^1 -fibrations of complete type only, and the failure of \mathbb{A}^1 -ruledness is intimately related to the fact that the generic fiber X_η of f , which is a surface defined over the field $K = \mathbb{C}(t)$, does not admit any \mathbb{A}^1 -fibration defined over $\mathbb{C}(t)$. Nevertheless, it was noticed in [3, Theorem 6.1] that one can infer straight from the construction of $f : X \rightarrow \mathbb{A}^1$ the existence a finite base extension $\text{Spec}(L) \rightarrow \text{Spec}(K)$ for which the surface $X_\eta \times_{\text{Spec}(K)} \text{Spec}(L)$ carries an \mathbb{A}^1 -fibration $\rho : X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \rightarrow \mathbb{P}_L^1$ defined over the field L .

A natural question is then to decide whether this phenomenon holds in general for families $f : X \rightarrow B$ of \mathbb{A}^1 -ruled affine surfaces parameterized by a smooth base curve B , namely, does the existence of \mathbb{A}^1 -fibrations on the general fibers of f imply the existence of one on the generic fiber of f , possibly after a finite extension of the base B ? A partial positive answer is given by Gurjar-Masuda-Miyanishi in [3, Theorem 3.8] under the additional assumption that the general fibers of f carry \mathbb{A}^1 -fibrations of affine type. The main result in *loc. cit.* is derived from the study of log-deformations of suitable relative normal projective models

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$\bar{f} : (\bar{X}, D) \rightarrow B$ of X over B with appropriate boundaries D . It is established in particular that the structure of the boundary divisor of a well chosen smooth projective completion of a general closed fiber X_s is stable under small deformations, a property which implies in turn, possibly after a finite extension of the base B , the existence of an \mathbb{A}^1 -fibration of affine type on the generic fiber of f . This log-deformation theoretic approach is also central in the related recent work of Flenner-Kaliman-Zaidenberg [2] on the classification of normal affine surfaces with \mathbb{A}^1 -fibrations of affine type up to a certain notion of deformation equivalence, defined for families which admit suitable relative projective models satisfying Kamawata's axioms of logarithmic deformations of pairs [8]. The fact that the \mathbb{A}^1 -fibrations under consideration are of affine type plays again a crucial role and, in contrast with the situation considered in [3], the restrictions imposed on the families imply the existence of \mathbb{A}^1 -fibrations of affine type on their generic fibers.

Our main result (Theorem 6) consists of a generalization of the results in [3] to families $f : X \rightarrow S$ of \mathbb{A}^1 -ruled surfaces over an arbitrary normal base S , which also includes the case where a general closed fiber X_s of f admits \mathbb{A}^1 -fibrations of complete type only. In particular, we obtain the following positive answer to Conjecture 6.2 in [3]:

Theorem. *Let $f : X \rightarrow S$ be dominant morphism between normal complex algebraic varieties whose general fibers are smooth \mathbb{A}^1 -ruled affine surfaces. Then there exists a dense open subset $S_* \subset S$, a finite étale morphism $T \rightarrow S_*$ and a normal T -scheme $h : Y \rightarrow T$ such that the induced morphism $f_T = \text{pr}_T : X_T = X \times_{S_*} T \rightarrow T$ factors as*

$$f_T = h \circ \rho : X_T \xrightarrow{\rho} Y \xrightarrow{h} T,$$

where $\rho : X_T \rightarrow Y$ is an \mathbb{A}^1 -fibration.

In contrast with the log-deformation theoretic strategy used in [3], which involves the study of certain Hilbert schemes of rational curves on well-chosen relative normal projective models $\bar{f} : (\bar{X}, B) \rightarrow S$ of X over S , our approach is more elementary, based on the notion of Kodaira dimension [7] adapted to the case of geometrically connected varieties defined over arbitrary base fields of characteristic zero. Indeed, the hypothesis means equivalently that the general fibers of f have negative Kodaira dimension. This property is in turn inherited by the generic fiber of f , which is a smooth affine surface defined over the function field of S , thanks to a standard Lefschetz principle argument. Then we are left with checking that a smooth affine surface X defined over an arbitrary base field k of characteristic zero and with negative Kodaira dimension admits an \mathbb{A}^1 -fibration, possibly after a suitable finite base extension $\text{Spec}(k_0) \rightarrow \text{Spec}(k)$, a fact which follows immediately from finite type hypotheses and the aforementioned characterization of Miyanishi-Sugie [14].

The article is organized as follows. The first section contains a review of the structure of smooth affine surfaces of negative Kodaira dimension over arbitrary base fields k of characteristic zero. We show in particular that every such surface X admits an \mathbb{A}^1 -fibration after a finite extension of the base field k , and we give criteria for the existence of \mathbb{A}^1 -fibrations defined over k . These results are then applied in the second section to the study of deformations $f : X \rightarrow S$ of smooth \mathbb{A}^1 -ruled affine surfaces: after giving the proof of the main result, Theorem 6, we consider in more detail the particular situation where the general fibers of $f : X \rightarrow S$ are irrational. In this case, after shrinking S if necessary, we show that the morphism f actually factors through an \mathbb{A}^1 -fibration $\rho : X \rightarrow Y$ over an S -scheme $h : Y \rightarrow S$ which coincides, up to birational equivalence, with the Maximally Rationally Connected quotient of a relative smooth projective model $\bar{f} : \bar{X} \rightarrow S$ of X over S . The last section is devoted to the case of affine threefolds equipped with a fibration $f : X \rightarrow B$ by irrational \mathbb{A}^1 -ruled surfaces over a smooth base curve B : we explain in particular how to construct an \mathbb{A}^1 -fibration $\rho : X \rightarrow Y$ factoring f by means of a relative Minimal Model Program applied to a smooth projective model $\bar{f} : \bar{X} \rightarrow B$ of X over B .

1. \mathbb{A}^1 -RULEDNESS OF AFFINE SURFACES OVER NON CLOSED FIELD

1.1. Logarithmic Kodaira dimension.

1.1.1. Let X be a smooth geometrically connected algebraic variety defined over a field k of characteristic zero. By virtue of Nagata compactification [15] and Hironaka desingularization [5] theorems, there exists an open immersion $X \hookrightarrow (\bar{X}, B)$ into a smooth complete algebraic variety \bar{X} with reduced SNC boundary divisor $B = \bar{X} \setminus X$. The (logarithmic) Kodaira dimension $\kappa(X)$ of X is then defined as the Iitaka dimension [6] of the pair

$(\overline{X}; \omega_{\overline{X}}(\log B))$ where $\omega_{\overline{X}}(\log B) = (\det \Omega_{\overline{X}/k}^1) \otimes \mathcal{O}_{\overline{X}}(B)$. So letting $\mathcal{R}(\overline{X}, B) = \bigoplus_{m \geq 0} H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m})$, we have $\kappa(X) = \text{tr. deg}_k \mathcal{R}(\overline{X}, B) - 1$ if $H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m}) \neq 0$ for sufficiently large m . Otherwise, if $H^0(\overline{X}, \omega_{\overline{X}}(\log B)^{\otimes m}) = 0$ for every $m \geq 1$, we set by convention $\kappa(X) = -\infty$ and we say that $\kappa(X)$ is negative. The so-defined element $\kappa(X) \in \{0, \dots, \dim_k X\} \cup \{-\infty\}$ is independent of the choice of a smooth complete model (\overline{X}, B) [7].

Furthermore, the Kodaira dimension of X is invariant under arbitrary extensions of the base field k . Indeed, given an extension $k \subset k'$, the pair $(\overline{X}_{k'}, B_{k'})$ obtained by the base change $\text{Spec}(k') \rightarrow \text{Spec}(k)$ is a smooth complete model of $X_{k'} = X \times_{\text{Spec}(k)} \text{Spec}(k')$ with reduced SNC boundary $B_{k'}$. Furthermore letting $\pi : \overline{X}_{k'} \rightarrow \overline{X}$ be the corresponding faithfully flat morphism, we have $\omega_{\overline{X}_{k'}}(\log B_{k'}) \simeq \pi^* \omega_X(\log B)$ and so $\mathcal{R}(X_{k'}) \simeq \mathcal{R}(X) \otimes_k k'$ by the flat base change theorem [4, Proposition III.9.3]. Thus $\kappa(X) = \kappa(X_{k'})$.

Example 1. The affine line \mathbb{A}_k^1 is the only smooth geometrically connected non complete curve C with negative Kodaira dimension. Indeed, let \overline{C} be a smooth projective model of C and let $\overline{C}_{\overline{k}}$ be the curve obtained by the base change to an algebraic closure \overline{k} of k . Since C is non complete, $B = \overline{C}_{\overline{k}} \setminus C_{\overline{k}}$ consists of a finite collection of closed points p_1, \dots, p_s , $s \geq 1$, on which the Galois group $\text{Gal}(\overline{k}/k)$ acts by k -automorphisms of $\overline{C}_{\overline{k}}$. Clearly, $H^0(\overline{C}_{\overline{k}}, \omega_{\overline{C}_{\overline{k}}}(\log B)^{\otimes m}) \neq 0$ unless $\overline{C}_{\overline{k}} \simeq \mathbb{P}_{\overline{k}}^1$ and $s = 1$. Since p_1 is then necessarily $\text{Gal}(\overline{k}/k)$ -invariant, $\overline{C} \setminus C$ consists of unique k -rational point, showing that $\overline{C} \simeq \mathbb{P}_k^1$ and $C \simeq \mathbb{A}_k^1$.

1.2. Smooth affine surfaces with negative Kodaira dimension.

Recall that by virtue of [14], a smooth affine surface X defined over an algebraically closed field of characteristic zero has negative Kodaira dimension if and only if it is \mathbb{A}^1 -ruled: there exists a Zariski dense open subset $U \subset X$ of the form $U \simeq Z \times \mathbb{A}^1$ for a suitable smooth curve Z . In fact, the projection $\text{pr}_Z : U \simeq Z \times \mathbb{A}^1 \rightarrow Z$ always extends to an \mathbb{A}^1 -fibration $\rho : X \rightarrow C$ over an open subset C of a smooth projective model \overline{Z} of Z . This characterization admits the following straightforward generalization to arbitrary base fields of characteristic zero:

Theorem 2. *Let X be a smooth geometrically connected affine surface defined over a field k of characteristic zero. Then the following are equivalent:*

- a) *The Kodaira dimension $\kappa(X)$ of X is negative.*
- b) *For some finite extension k_0 of k , the surface X_{k_0} contains an open subset $U \simeq Z \times \mathbb{A}_{k_0}^1$ for some smooth curve Z defined over k_0 .*
- c) *There exists a finite extension k_0 of k and an \mathbb{A}^1 -fibration $\rho : X_{k_0} \rightarrow C_0$ over a smooth curve C_0 defined over k_0 .*

Proof. Clearly c) implies b) and b) implies a). To show that a) implies c), we observe that letting \overline{k} be an algebraic closure of k , we have $\kappa(X_{\overline{k}}) = \kappa(X) < 0$. It then follows from the aforementioned result of Miyanishi-Sugie [14] that $X_{\overline{k}}$ admits an \mathbb{A}^1 -fibration $q : X_{\overline{k}} \rightarrow C$ over a smooth curve C , with smooth projective model \overline{C} . Since $X_{\overline{k}}$ and \overline{C} are of finite type over \overline{k} , there exists a finite extension $k \subset k_0$ such that $q : X_{\overline{k}} \rightarrow \overline{C}$ is obtained from a morphism $\rho : X_{k_0} \rightarrow \overline{C}_0$ to a smooth projective curve \overline{C}_0 defined over k_0 by the base extension $\text{Spec}(\overline{k}) \rightarrow \text{Spec}(k_0)$. By virtue of Example 1, $\rho : X_{k_0} \rightarrow \overline{C}_0$ is an \mathbb{A}^1 -fibration. \square

Examples of smooth affine surfaces X of negative Kodaira dimension without any \mathbb{A}^1 -fibration defined over the base field but admitting \mathbb{A}^1 -fibrations of complete type after a finite base extension were already constructed in [1]. The following example illustrates the fact that a similar phenomenon occurs for \mathbb{A}^1 -fibrations of affine type, providing in particular a negative answer to Problem 3.13 in [3].

Example 3. Let $B \subset \mathbb{P}_k^2 = \text{Proj}(k[x, y, z])$ be a smooth conic without k -rational points defined by a quadratic form $q = x^2 + ay^2 + bz^2$, where $a, b \in k^*$, and let $\overline{X} \subset \mathbb{P}_k^3 = \text{Proj}(k[x, y, z, t])$ be the smooth quadric surface defined by the equation $q(x, y, z) - t^2 = 0$. The complement $X \subset \overline{X}$ of the hyperplane section $\{t = 0\}$ is a k -rational smooth affine surface with $\kappa(X) < 0$, which does not admit any \mathbb{A}^1 -fibration $\rho : X \rightarrow C$ over a smooth, affine or projective curve C . Indeed, if such a fibration existed then a smooth projective model of C would be isomorphic to \mathbb{P}_k^1 ; since the fiber of ρ over a general k -rational point of C is isomorphic to \mathbb{A}_k^1 , its closure in \overline{X} would intersect the boundary $\overline{X} \setminus X \simeq B$ in a unique point, necessarily k -rational, in contradiction with the choice of B .

In contrast, for a suitable finite extension $k \subset k'$, the surface $X_{k'}$ becomes isomorphic to the complement of the diagonal in $\overline{X}_{k'} \simeq \mathbb{P}_{k'}^1 \times \mathbb{P}_{k'}^1$ and hence, it admits at least two distinct \mathbb{A}^1 -fibrations over $\mathbb{P}_{k'}^1$, induced

by the restriction of the first and second projections from $\overline{X}_{k'}$. Furthermore, since $X_{k'}$ is isomorphic to the smooth affine quadric in $\mathbb{A}_{k'}^3 = \text{Spec}(k'[u, v, w])$ with equation $uv - w^2 = 1$, it also admits two distinct \mathbb{A}^1 -fibrations over $\mathbb{A}_{k'}^1$, induced by the restrictions of the projections pr_u and pr_v .

1.3. Existence of \mathbb{A}^1 -fibrations defined over the base field.

1.3.1. The previous example illustrates the general fact that if X is a smooth geometrically connected affine surface with $\kappa(X) < 0$ which does not admit any \mathbb{A}^1 -fibration, then there exists a finite extension k' of k such that $X_{k'}$ admits at least two \mathbb{A}^1 -fibrations of the same type, either affine or complete, with distinct general fibers. Indeed, by virtue of Theorem 2, there exists a finite extension k_0 of k such that X_{k_0} admits an \mathbb{A}^1 -fibration $\rho : X_{k_0} \rightarrow C$. Let k' be the Galois closure of k_0 in an algebraic closure of k and let $\rho_{k'} : X_{k'} \rightarrow C_{k'}$ be the \mathbb{A}^1 -fibration deduced from ρ . If $\rho_{k'} : X_{k'} \rightarrow C_{k'}$ is globally invariant under the action of the Galois group $\text{Gal}(k'/k)$ on $X_{k'}$, in the sense that for every $\Phi \in \text{Gal}(k'/k)$ considered as a Galois automorphism of $X_{k'}$ there exists a commutative diagram

$$\begin{array}{ccc} X_{k'} & \xrightarrow{\Phi} & X_{k'} \\ \rho_{k'} \downarrow & & \downarrow \rho_{k'} \\ C_{k'} & \xrightarrow{\varphi} & C_{k'} \end{array}$$

for a certain k' -automorphism φ of $C_{k'}$, then we would obtain a Galois action of $\text{Gal}(k'/k)$ on $C_{k'}$ for which $\rho_{k'} : X_{k'} \rightarrow C_{k'}$ becomes an equivariant morphism. Since $C_{k'}$ is quasi-projective and $\rho_{k'}$ is affine, it would follow from Galois descent that there exists a curve \tilde{C} defined over k and a morphism $q : X \rightarrow \tilde{C}$ defined over k such that $\rho_{k'} : X_{k'} \rightarrow C_{k'}$ is obtained from q by the base change $\text{Spec}(k') \rightarrow \text{Spec}(k)$. Since by virtue of Example 1 the affine line does not have any nontrivial form, the generic fiber of q would be isomorphic to the affine line over the field of rational functions of \tilde{C} and so, $q : X \rightarrow \tilde{C}$ would be an \mathbb{A}^1 -fibration defined over k , in contradiction with our hypothesis. So there exists at least an element $\Phi \in \text{Gal}(k'/k)$ considered as a k -automorphism of $X_{k'}$ such that the \mathbb{A}^1 -fibrations $\rho_{k'} : X_{k'} \rightarrow C_{k'}$ and $\rho_{k'} \circ \varphi : X_{k'} \rightarrow C_{k'}$ have distinct general fibers.

Arguing backward, we obtain the following useful criterion:

Proposition 4. *Let X be a smooth geometrically connected affine surface with $\kappa(X) < 0$. If there exists a finite Galois extension k' of k such that $X_{k'}$ admits a unique \mathbb{A}^1 -fibration $\rho' : X_{k'} \rightarrow C_{k'}$ up to composition by automorphisms of $C_{k'}$, then $\rho' : X_{k'} \rightarrow C_{k'}$ is obtained by base extension from an \mathbb{A}^1 -fibration $\rho : X \rightarrow C$ defined over k .*

Corollary 5. *A smooth geometrically connected irrational affine surface X has negative Kodaira dimension if and only if it admits an \mathbb{A}^1 -fibration $\rho : X \rightarrow C$ over a smooth irrational curve C defined over the base field k . Furthermore for every extension k' of k , $\rho_{k'} : X_{k'} \rightarrow C_{k'}$ is the unique \mathbb{A}^1 -fibration on $X_{k'}$ up to composition by automorphisms of $C_{k'}$.*

Proof. Uniqueness is clear since otherwise $C_{k'}$ would be dominated by a general fiber of another \mathbb{A}^1 -fibration on $X_{k'}$, and hence would be rational, implying in turn the rationality of X . By virtue of Theorem 2, there exists a finite Galois extension k' of k and an \mathbb{A}^1 -fibration $\rho' : X_{k'} \rightarrow C'$ over a smooth curve C' . The latter is irrational as X is irrational, which implies that $\rho' : X_{k'} \rightarrow C'$ is the unique \mathbb{A}^1 -fibration on $X_{k'}$. So ρ' descend to an \mathbb{A}^1 -fibration $\rho : X \rightarrow C$ over a smooth irrational curve C defined over k . \square

2. FAMILIES OF \mathbb{A}^1 -RULED AFFINE SURFACES

2.1. **Existence of étale \mathbb{A}^1 -cylinders.** This subsection is devoted to the proof of the following:

Theorem 6. *Let X and S be normal algebraic varieties defined over a field k of infinite transcendence degree over \mathbb{Q} , and let $f : X \rightarrow S$ be a dominant affine morphism with the property that for a general closed point $s \in S$, the fiber X_s is a smooth geometrically connected affine surface with negative Kodaira dimension. Then there exists an open subset $S_* \subset S$, a finite étale morphism $T \rightarrow S_*$ and a normal T -scheme $h : Y \rightarrow T$ such that $f_T = \text{pr}_T : X_T = X \times_{S_*} T \rightarrow T$ factors as*

$$f_T = h \circ \rho : X_T \xrightarrow{\rho} Y \xrightarrow{h} T$$

where $\rho : X_T \rightarrow Y$ is an \mathbb{A}^1 -fibration.

Proof. Shrinking S if necessary, we may assume that S is affine, that $f : X \rightarrow S$ is smooth and that $\kappa(X_s) < 0$ for every closed point $s \in S$. It is enough to show that the fiber X_η of f over the generic point η of S is geometrically connected, with negative Kodaira dimension. Indeed, if so, then by Theorem 2 above, there exists a finite extension L of $K = \text{Frac}(\Gamma(S, \mathcal{O}_S))$ and an \mathbb{A}^1 -fibration $\rho : X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \rightarrow C$ onto a smooth curve C defined over L . Letting T be the normalization of S in L and shrinking T again if necessary, we obtain a finite étale morphism $T \rightarrow S$ such that the generic fiber of $\text{pr}_T : X_T \rightarrow T$ is isomorphic to the \mathbb{A}^1 -fibred surface $\rho : X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \rightarrow C$ and then the assertion follows from Lemma 7 below.

Since X and S are affine and of finite type over k , there exists a subfield k_0 of k of finite transcendence degree over \mathbb{Q} , and a smooth morphism $f_0 : X_0 \rightarrow S_0$ of k_0 -varieties such that $f : X \rightarrow S$ is obtained from $f_0 : X_0 \rightarrow S_0$ by the base extension $\text{Spec}(k) \rightarrow \text{Spec}(k_0)$. The field $K_0 = \text{Frac}(\Gamma(S_0, \mathcal{O}_{S_0}))$ has finite transcendence degree over \mathbb{Q} and hence, it admits a k_0 -embedding $\xi : K_0 \hookrightarrow k$. Letting $(X_0)_{\eta_0}$ be the fiber of f_0 over the generic point $\eta_0 : \text{Spec}(K_0) \rightarrow S_0$ of S_0 , the composition $\Gamma(S_0, \mathcal{O}_{S_0}) \hookrightarrow K_0 \hookrightarrow k$ induces a k -homomorphism $\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k \rightarrow k$ defining a closed point $s : \text{Spec}(k) \rightarrow \text{Spec}(\Gamma(S_0, \mathcal{O}_{S_0}) \otimes_{k_0} k) = S$ of S for which obtain the following commutative diagram

$$\begin{array}{ccccccc}
 & & X_s & \xrightarrow{\quad} & X & & \\
 & \swarrow & \downarrow & & \downarrow & \searrow & \\
 (X_0)_{\eta_0} & \xrightarrow{\quad} & X_0 & & X & & \\
 \downarrow & & \downarrow & \xrightarrow{f_0} & \downarrow & \xrightarrow{f} & \\
 \text{Spec}(k) & \xrightarrow{\quad} & \text{Spec}(k) & \xrightarrow{s} & S & \xrightarrow{\quad} & \text{Spec}(k) \\
 \downarrow \xi^* & & \downarrow & & \downarrow & & \downarrow \\
 \text{Spec}(K_0) & \xrightarrow{\eta_0} & S_0 & \xrightarrow{\quad} & \text{Spec}(k_0) & &
 \end{array}$$

The bottom square of the cube being cartesian by construction, we deduce that

$$(X_0)_{\eta_0} \times_{\text{Spec}(K_0)} \text{Spec}(k) \simeq X_0 \times_{S_0} \text{Spec}(k) \simeq X \times_S \text{Spec}(k) = X_s.$$

Since by assumption, X_s is geometrically connected with $\kappa(X_s) < 0$, we conclude that $(X_0)_{\eta_0}$ is geometrically connected and has negative Kodaira dimension. This implies in turn that X_η is geometrically connected and that $\kappa(X_\eta) < 0$ as desired. \square

In the proof of the above theorem, we used the following lemma:

Lemma 7. *Let $f : X \rightarrow S$ be a dominant affine morphism between normal varieties defined over a field k of characteristic zero. Then the following are equivalent:*

a) *The generic fiber X_η of f admits an \mathbb{A}^1 -fibration $q : X_\eta \rightarrow C$ over a smooth curve C defined over the fraction field K of S .*

b) *There exists an open subset S_* of S and a normal S_* -scheme $h : Y \rightarrow S_*$ of relative dimension 1 such that the restriction of f to $V = f^{-1}(S_*)$ factors as $f|_V = h \circ \rho : V \rightarrow Y \rightarrow S_*$ where $\rho : V \rightarrow Y$ is an \mathbb{A}^1 -fibration.*

Proof. If b) holds then letting L be the fraction field of Y , we have a commutative diagram

$$\begin{array}{ccccc}
 V_\xi = X_\xi & \rightarrow & V_\eta = X_\eta & \rightarrow & V \\
 \downarrow \rho_\xi & & \downarrow \rho_\eta & & \downarrow \rho \\
 \text{Spec}(L) & \xrightarrow{\xi} & C = Y_\eta & \rightarrow & Y \\
 & & \downarrow h_\eta & & \downarrow h \\
 & & \text{Spec}(K) & \xrightarrow{\eta} & S_*
 \end{array}$$

in which each square is cartesian. It follows that $h_\eta : C \rightarrow \text{Spec}(K)$ is a normal whence smooth curve defined over K and that $\rho_\eta : X_\eta \rightarrow C$ is an \mathbb{A}^1 -fibration. Conversely, suppose that X_η admits an \mathbb{A}^1 -fibration $q : X_\eta \rightarrow C$ and let \overline{C} be a smooth projective model of C over K . Then there exists an open subset S_0 of S and a projective S_0 -scheme $h : Y \rightarrow S_0$ whose generic fiber is isomorphic to \overline{C} . After shrinking S_0 if necessary, the rational map $\rho : V \dashrightarrow Y$ of S_0 -schemes induced by q becomes a morphism and

we obtain a factorization $f|_V = h \circ \rho$. By construction, the generic fiber V_ξ of $\rho : V \rightarrow Y$ is isomorphic to $V \times_Y \text{Spec}(L) \simeq (V \times_Y C) \times_C \text{Spec}(L) \simeq X_\eta \times_C \text{Spec}(L) \simeq \mathbb{A}_L^1$ since $V \times_Y C \simeq V_\eta \simeq X_\eta$ and $\rho : X_\eta \rightarrow C \hookrightarrow \overline{C}$ is an \mathbb{A}^1 -fibration. So $\rho : V \rightarrow Y$ is an \mathbb{A}^1 -fibration. \square

Example 8. Let $R = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$, $S = \text{Spec}(R)$ and let D be the relatively ample divisor in $\mathbb{P}_S^2 = \text{Proj}_R(R[x, y, z])$ defined by the equation $x^2 + sy^2 + tz^2 = 0$. The restriction $h : X = \mathbb{P}_S^2 \setminus D \rightarrow S$ of the structure morphism defines a family of smooth affine surfaces with the property that for every closed point $s \in S$, X_s is isomorphic to the complement in $\mathbb{P}_{\mathbb{C}}^2$ of the smooth conic D_s . In particular X_s admits a continuum of pairwise distinct \mathbb{A}^1 -fibrations $X_s \rightarrow \mathbb{A}_{\mathbb{C}}^1$, induced by the restrictions to X_s of the rational pencils on $\mathbb{P}_{\mathbb{C}}^2$ generated by D_s and twice its tangent line at an arbitrary closed point $p_s \in D_s$. On the other hand the fiber of D over the generic point η of S is a conic without $\mathbb{C}(s, t)$ -rational point in $\mathbb{P}_{\mathbb{C}(s, t)}^2$ and hence, we conclude by a similar argument as in Example 3 that X_η does not admit any \mathbb{A}^1 -fibration defined over $\mathbb{C}(s, t)$. Therefore there is no open subset S_* of S over which h can be factored through an \mathbb{A}^1 -fibration.

2.2. Deformations of irrational \mathbb{A}^1 -ruled affine surfaces.

In this subsection, we consider the particular situation of a flat family $f : X \rightarrow S$ over a normal variety S whose general fibers are irrational \mathbb{A}^1 -ruled affine surfaces. A combination of Corollary 5 and Theorem 6 above implies that if $f : X \rightarrow S$ is smooth and defined over a field of infinite transcendence degree over \mathbb{Q} , then the generic fiber X_η of f is \mathbb{A}^1 -ruled. Equivalently, there exists an open subset $S_* \subset S$ and a normal S_* -scheme $h : Y \rightarrow S_*$ such that the restriction of f to $X_* = X \times_S S_*$ factors through an \mathbb{A}^1 -fibration $\rho : X_* \rightarrow Y$ (see Lemma 7). The restriction of ρ to the fiber of f over a general closed point $s \in S_0$ is an \mathbb{A}^1 -fibration $\rho_s : X_s \rightarrow Y_s$ over the normal, whence smooth, curve Y_s . Since X_s is irrational, Y_s is irrational, and so $\rho_s : X_s \rightarrow Y_s$ is the unique \mathbb{A}^1 -fibration on X_s up to composition by automorphisms of Y_s . So in this case, we can identify $\rho_s : X_s \rightarrow Y_s$ with the Maximally Rationally Connected fibration (MRC-fibration) $\varphi : \overline{X}_s \dashrightarrow Y_s$ of a smooth projective model \overline{X}_s of X_s in the sense of [11, IV.5]: recall that φ is unique, characterized by the property that its general fibers are rationally connected and that for a very general point $y \in Y_s$ any rational curve in \overline{X}_s which meets \overline{X}_y is actually contained in \overline{X}_y . The \mathbb{A}^1 -fibration $\rho : X_* \rightarrow Y$ can therefore be re-interpreted as being the MRC-fibration of a relative smooth projective model \overline{X} of X over S .

Reversing the argument, general existence and uniqueness results for MRC-fibrations allow actually to get rid of the smoothness hypothesis of a general fiber of $f : X \rightarrow S$ and to extend the conclusion of Theorem 6 to arbitrary base fields of characteristic zero. Namely, we obtain the following characterization:

Theorem 9. *Let X and S be normal varieties defined over a field k of characteristic zero and let $f : X \rightarrow S$ be a dominant affine morphism with the property that for a general closed point $s \in S$, the fiber X_s is irrational and \mathbb{A}^1 -ruled. Then there exists an open subset S_* and a normal S_* -scheme $h : Y \rightarrow S_*$ such that the restriction of f to $X_* = X \times_S S_*$ factors as*

$$f|_{X_*} = h \circ \rho : X_* \xrightarrow{\rho} Y \xrightarrow{h} S_*$$

where $\rho : X_* \rightarrow Y$ is an \mathbb{A}^1 -fibration.

Proof. Shrinking S if necessary, we may assume that for every closed point $s \in S$, X_s is irrational and \mathbb{A}^1 -ruled, hence carrying a unique \mathbb{A}^1 -fibration $\pi_s : X_s \rightarrow C_s$ over an irrational normal curve C_s . Since $f : X \rightarrow S$ is affine, there exists a normal projective S -scheme $\overline{X} \rightarrow S$ and an open embedding $X \hookrightarrow \overline{X}$ of schemes over S . Letting $W \rightarrow \overline{X}$ be a resolution of the singularities of \overline{X} , we may assume up to shrinking S again if necessary that $W \rightarrow S$ is a smooth morphism. We let $j : X \dashrightarrow W$ be the birational map of S -schemes induced by the embedding $X \hookrightarrow \overline{X}$. By virtue of [11, Theorem 5.9], there exists an open subset W' of W , an S -scheme $h : Z \rightarrow S$ and a proper morphism $\overline{q} : W' \rightarrow Z$ such that for every $s \in S$, the induced rational map $\overline{q}_s : W_s \dashrightarrow Z_s$ is the MRC-fibration for W_s . On the other hand, since W_s is a smooth projective model of X_s , the induced rational map $\pi_s : \overline{X}_s \dashrightarrow C_s$ is the MRC-fibration for W_s . Consequently, for a general closed point $z \in Z$ with $h(z) = s$, the fiber W_z of \overline{q}_s , which is an irreducible proper rational curve contained in W_s , must coincide with the closure of the image by j of a general closed fiber of π_s . The latter being isomorphic to the affine line \mathbb{A}_κ^1 over the residue field κ of the corresponding point of C_s , we conclude that there exists an affine open subset U of X on which the composition $\overline{q} \circ j : U \rightarrow Z$ is a well defined morphism with general closed fibers isomorphic to affine lines over the corresponding residue fields.

So $\bar{q} \circ j : U \rightarrow Z$ is an \mathbb{A}^1 -fibration by virtue of [9]. The generic fiber of $f : X \rightarrow S$ is thus \mathbb{A}^1 -ruled and the assertion follows from Lemma 7 above. \square

Example 10. Let $h : Y \rightarrow S$ be smooth family of complex projective curves of genus $g \geq 2$ over a normal affine base S et let $\mathcal{T}_{Y/S}$ be the relative tangent sheaf of h . Since by Riemann-Roch $H^0(Y_s, \mathcal{T}_{Y/S,s}) = 0$ and $\dim H^1(Y_s, \mathcal{T}_{Y/S,s}) = g - 1$ for every point $s \in S$, $h_* \mathcal{T}_{Y/S,s} = 0$, $R^1 h_* \mathcal{T}_{Y/S}$ is locally free of rank $g - 1$ [4, Corollary III.12.9] and so, $H^1(Y, \mathcal{T}_{Y/S}) \simeq H^0(S, R^1 h_* \mathcal{T}_{Y/S})$ by the Leray spectral sequence. Replacing S by an open subset, we may assume that $R^1 h_* \mathcal{T}_{Y/S}$ admits a nowhere vanishing global section σ . Via the isomorphism $H^1(Y, \mathcal{T}_{Y/S}) \simeq \text{Ext}_Y^1(\mathcal{O}_Y, \mathcal{T}_{Y/S})$, we may interpret this section as the class of a non trivial extension $0 \rightarrow \mathcal{T}_{Y/S} \rightarrow \mathcal{E} \rightarrow \mathcal{O}_Y \rightarrow 0$ of locally free sheaves over Y . The inclusion $\mathcal{T}_{Y/S} \rightarrow \mathcal{E}$ defines a section D of the locally trivial \mathbb{P}^1 -bundle $\bar{\rho} : \bar{X} = \text{Proj}(\text{Sym}_{\mathcal{O}_Y} \mathcal{E}^\vee) \rightarrow Y$ and the non vanishing of σ guarantees that D is the support of an S -ample divisor. Indeed the S -ampleness of D is equivalent to the property that for every $s \in S$ the induced section D_s of the \mathbb{P}^1 -bundle $\bar{\rho}_s : \bar{X}_s \rightarrow Y_s$ over the smooth projective curve Y_s is ample. Since by construction, $\bar{\rho}_s|_{\bar{X}_s \setminus D_s} : \bar{X}_s \setminus D_s \rightarrow Y_s$ is a nontrivial torsor under the line bundle $\text{Spec}(\text{Sym} \mathcal{T}_{Y_s}^\vee) \rightarrow Y_s$, it follows that D_s intersects positively every section D of $\bar{\rho}_s$ except maybe D_s itself. On the other hand, we have $(D_s^2) = -\deg \mathcal{T}_{Y_s} = 2g(Y_s) - 2 > 0$, and so the ampleness of D_s follows from the Nakai-Moishezon criterion and the description of the cone effective cycles on an irrational projective ruled surface given in [4, Proposition 2.20-2.21].

Letting $X = \bar{X} \setminus D$, we obtain a smooth family

$$f = g \circ \bar{\rho}|_X : X \xrightarrow{\bar{\rho}|_X} Y \xrightarrow{h} S$$

where $\bar{\rho}|_X : X \rightarrow Y$ is nontrivial, locally trivial, \mathbb{A}^1 -bundle such that for every $s \in S$, X_s is an affine surface with an \mathbb{A}^1 -fibration $\rho_s : X_s \rightarrow Y_s$ of complete type.

In contrast with the previous example, the following proposition shows in particular that if the total space of a family of irrational \mathbb{A}^1 -ruled affine surfaces $f : X \rightarrow S$ has finite divisor class group, then the induced \mathbb{A}^1 -fibration on a general fiber of $f : X \rightarrow S$ is of affine type.

Proposition 11. *Let X be a geometrically integral normal variety with finite divisor class group $\text{Cl}(X)$ and let $f : X \rightarrow S$ be a dominant affine morphism to a normal variety S with the property that for a general closed point $s \in S$, the fiber X_s is irrational and \mathbb{A}^1 -ruled, say with unique \mathbb{A}^1 -fibration $\pi_s : X_s \rightarrow C_s$. Then there exists an effective $\mathbb{G}_{a,s}$ -action on X such that for a general closed point $s \in S$, the \mathbb{A}^1 -fibration $\pi_s : X_s \rightarrow C_s$ factors through the algebraic quotient $\rho_s : X_s \rightarrow X_s // \mathbb{G}_{a,s} = \text{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^{\mathbb{G}_{a,s}})$.*

Proof. Let $f|_{X_*} = h \circ \rho : X_* \xrightarrow{\rho} Y \xrightarrow{h} S_*$ be as in Theorem 9. Since ρ is an \mathbb{A}^1 -fibration, there exists an affine open subset $U \subset Y$ such that $\rho^{-1}(U) \simeq U \times \mathbb{A}^1$ as schemes over U . Since $\rho^{-1}(U)$ is affine, its complement in X is of pure codimension 1, and the finiteness of $\text{Cl}(X)$ implies that it is actually the support of an effective principal divisor $\text{div}_X(a)$ for some $a \in \Gamma(X, \mathcal{O}_X)$. Letting ∂_0 be the locally nilpotent derivation of $\Gamma(\rho^{-1}(U), \mathcal{O}_X) \simeq \Gamma(X, \mathcal{O}_X)_a$ corresponding to the $\mathbb{G}_{a,U}$ -action by translations on the second factor, the finite generation of $\Gamma(X, \mathcal{O}_X)$ guarantees that for a suitably chosen $n \geq 0$, $a^n \partial_0$ lifts to a locally nilpotent derivation ∂ of $\Gamma(X, \mathcal{O}_X)$. By construction, the restriction of f to the dense open subset $\rho^{-1}(U)$ of X is invariant under the corresponding \mathbb{G}_a -action, and so $f : X \rightarrow S$ is \mathbb{G}_a -invariant. For a general closed point $s \in S$, the induced \mathbb{G}_a -action on X_s is nontrivial, and its algebraic quotient $\rho_s : X_s \rightarrow X_s // \mathbb{G}_a = \text{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^{\mathbb{G}_a})$ is a surjective \mathbb{A}^1 -fibration over a normal affine curve $X_s // \mathbb{G}_a$. Since C_s is irrational, the general fibers of ρ_s and π_s must coincide. It follows that π_s is \mathbb{G}_a -invariant, whence factors through ρ_s . \square

3. AFFINE THREEFOLDS FIBERED IN IRRATIONAL \mathbb{A}^1 -RULED SURFACES

In this section we consider in more detail the case of normal complex affine threefolds X admitting a fibration $f : X \rightarrow B$ by irrational \mathbb{A}^1 -ruled surfaces, over a smooth curve B . We explain how to derive the variety $h : Y \rightarrow B$ for which f factors through an \mathbb{A}^1 -fibration $\rho : X \rightarrow Y$ from a relative minimal model program applied to a suitable projective model of X over B . In the case where the divisor class group of X is finite, we provide a complete classification of such fibrations in terms of additive group actions on X .

3.1. \mathbb{A}^1 -cylinders via relative Minimal Model Program.

Let X be a normal complex affine threefold and let $f : X \rightarrow B$ be a flat morphism onto a smooth curve B with the property that a general closed fiber X_b of f is an irreducible irrational \mathbb{A}^1 -ruled surface. We let $\bar{f} : W \rightarrow B$ be a smooth projective model of X over B obtained from an arbitrary normal relative projective completion $X \hookrightarrow \bar{X}$ of X over B by resolving the singularities. We let $j : X \dashrightarrow W$ be the birational map induced by the open immersion $X \hookrightarrow \bar{X}$.

By applying a minimal model program for W over B , we obtain a sequence of birational B -maps

$$W = W_0 \xrightarrow{\varphi_1} W_1 \xrightarrow{\varphi_2} W_2 \dashrightarrow \cdots \dashrightarrow W_{\ell-1} \xrightarrow{\varphi_\ell} W_\ell = W',$$

between B -schemes $\bar{f}_i : W_i \rightarrow B$, where $\varphi_i : W_i \dashrightarrow W_{i+1}$ is either a divisorial contraction or a flip, and the rightmost variety W' is the output of a minimal model program over B . The hypotheses imply that W' has the structure of a Mori conic bundle $\bar{\rho} : W' \rightarrow Y$ over a B -scheme $h : Y \rightarrow B$ corresponding to the contraction of an extremal ray of $\overline{\text{NE}}(W'/B)$. Indeed, a general fiber of \bar{f} being a birationally ruled projective surface, the output W' is not a minimal model of W over B . So W' is either a Mori conic bundle over a B -scheme Y of dimension 2 or a del Pezzo fibration over B , the second case being excluded by the fact that the general fibers of \bar{f} are irrational.

Proposition 12. *The induced map $\rho = \bar{\rho}|_X : X \dashrightarrow Y$ is a rational \mathbb{A}^1 -fibration.*

Proof. Since a general closed fiber X_b is a normal affine surface with an \mathbb{A}^1 -fibration $\pi_b : X_b \rightarrow C_b$ over a certain irrational smooth curve C_b , it follows that there exists a unique maximal affine open subset U_b of C_b such that $\pi_b^{-1}(U_b) \simeq U_b \times \mathbb{A}^1$ and such that the rational map $j_b : \pi_b^{-1}(U_b) \dashrightarrow W_b$ induced by j is regular, inducing an isomorphism between $\pi_b^{-1}(U_b)$ and its image. Each step $\varphi_i : W_i \dashrightarrow W_{i+1}$ consists of either a flip whose flipping and flipped curves are contained in fibers of $\bar{f}_i : W_i \rightarrow B$ and $\bar{f}_{i+1} : W_{i+1} \rightarrow B$ respectively, or a divisorial contraction whose exceptional divisor is contained in a fiber of $\bar{f}_i : W_i \rightarrow B$, or a divisorial contraction whose exceptional divisor intersects a general fiber of $\bar{f}_i : W_i \rightarrow B$. Clearly, a general closed fiber of $\bar{f}_i : W_i \rightarrow B$ is not affected by the first two types of birational maps. On the other hand, if $\varphi_i : W_i \dashrightarrow W_{i+1}$ is the contraction of a divisor $E_i \subset W_i$ which dominates B , then a general fiber of $\varphi_i|_{E_i}$ is a smooth proper rational curve. The intersection of E_i with a general closed fiber $W_{i,b}$ of \bar{f}_i thus consists of proper rational curves, and its intersection with the image of the maximal affine cylinder like open subset $\pi_b^{-1}(U_b)$ of X_b is either empty or composed of affine rational curves. Since U_b is an irrational curve, it follows that each irreducible component of $E_i \cap (\pi_b^{-1}(U_b))$ is contained in a fiber of π_b . This implies that there exists an open subset $U_{b,0}$ of U_b with the property that for every $i = 1, \dots, \ell$, the restriction of $\varphi_i \circ \cdots \circ \varphi_1 \circ j$ to $\pi_b^{-1}(U_{b,0}) \subset X_b$ is an isomorphism onto its image in $W_{i,b}$. A general fiber of $\bar{\rho} : W' \rightarrow Y$ over a closed point $y \in Y$ being a smooth proper rational curve, its intersection with $\pi_{h(y)}^{-1}(U_{h(y),0})$ viewed as an open subset of $W'_{h(y)}$, is thus either empty or equal to a fiber of $\pi_{h(y)}$. So by virtue of [9], there exists an open subset V of X on which $\bar{\rho}$ restricts to an \mathbb{A}^1 -fibration $\bar{\rho}|_V : V \rightarrow Y$. \square

Corollary 13. *Let X be a normal complex affine threefold X equipped with a morphism $f : X \rightarrow B$ onto a smooth curve B whose general closed fibers are irrational \mathbb{A}^1 -ruled surfaces. Then X is birationally equivalent to the product of \mathbb{P}^1 with a family $h_0 : \mathcal{C}_0 \rightarrow B_0$ of smooth projective curves of genus $g \geq 1$ over an open subset $B_0 \subset B$.*

Proof. By the previous Proposition, X has the structure of a rational \mathbb{A}^1 -fibration $\rho : X \dashrightarrow Y$ over a 2-dimensional normal proper B -scheme $h : Y \rightarrow B$. In particular, X is birational to $Y \times \mathbb{P}^1$. On the other hand, for a general closed point $b \in B$, the curve Y_b is birational to the base C_b of the unique \mathbb{A}^1 -fibration $\pi_b : X_b \rightarrow C_b$ on the irrational affine surface X_b . Letting $\sigma : \tilde{Y} \rightarrow Y$ be a desingularization of Y , there exists an open subset B_0 of B over which the composition $h \circ \sigma : \tilde{Y} \rightarrow Y$ restricts to a smooth family $h_0 : \mathcal{C}_0 \rightarrow B_0$ of projective curves of a certain genus $g \geq 1$. By construction, X is birational to $\mathcal{C}_0 \times \mathbb{P}^1$. \square

Remark 14. Example 10 above shows conversely that for every smooth family $h : \mathcal{C} \rightarrow B$ of projective curves of genus $g \geq 2$, there exists a smooth \mathbb{A}^1 -ruled affine threefold X birationally equivalent to $\mathcal{C} \times \mathbb{P}^1$. Actually, in the setting of the previous Corollary 13, if we assume further that a general fiber of $f : X \rightarrow B$ carries an \mathbb{A}^1 -fibration $\pi_b : X_b \rightarrow C_b$ over a smooth curve C_b whose smooth projective model has genus $g \geq 2$, then there exists a uniquely determined family $h : \mathcal{C} \rightarrow B$ of proper stable curves of genus g such that X is birationally equivalent to $\mathcal{C} \times \mathbb{P}^1$: indeed, the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ being proper

and separated, the smooth family $h_0 : \mathcal{C}_0 \rightarrow B_0$ extends in a unique way to a family $h : \mathcal{C} \rightarrow B$ of stable curves of genus g .

3.2. Factorial threefolds.

Proposition 15. *Let X be a normal affine threefold with finite divisor class group $\text{Cl}(X)$ and let $f : X \rightarrow B$ be a morphism onto a smooth curve B whose general closed fibers are irrational \mathbb{A}^1 -ruled surfaces. Then there exists a factorisation $f = h \circ \rho : X \rightarrow Y \rightarrow B$ where $\rho : X \rightarrow Y$ is the algebraic quotient morphism of an effective $\mathbb{G}_{a,B}$ -action on X . In particular, a general fiber of f admits an \mathbb{A}^1 -fibration of affine type.*

Proof. By virtue of Proposition 11, there exist an effective $\mathbb{G}_{a,B}$ -action on X such that for a general closed point $b \in B$, the \mathbb{A}^1 -fibration $\pi_b : X_b \rightarrow C_b$ on X_b factors through the algebraic quotient $\rho_b : X_b \rightarrow X_b // \mathbb{G}_{a,b} = \text{Spec}(\Gamma(X_b, \mathcal{O}_{X_b})^{\mathbb{G}_{a,b}})$. Since X is a threefold, the ring of invariants $\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,B}}$ is finitely generated [16]. The quotient morphism $\rho : X \rightarrow Y = \text{Spec}(\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,B}})$ is an \mathbb{A}^1 -fibration, and since Y is a categorical quotient in the category of algebraic varieties, the invariant morphism $f : X \rightarrow B$ factors through ρ . \square

Corollary 16. *Let $f : \mathbb{A}^3 \rightarrow B$ be a morphism onto a smooth curve B with irrational \mathbb{A}^1 -ruled general fibers. Then B is isomorphic to either \mathbb{P}^1 or \mathbb{A}^1 and there exists a factorization $f = h \circ \rho : \mathbb{A}^3 \rightarrow \mathbb{A}^2 \rightarrow B$, where $\rho : \mathbb{A}^3 \rightarrow \mathbb{A}^2$ is the quotient morphism of an effective $\mathbb{G}_{a,B}$ -action on \mathbb{A}^3 .*

Proof. Since B is dominated by a general line in \mathbb{A}^3 , it is necessarily isomorphic to \mathbb{P}^1 or \mathbb{A}^1 . The second assertion follows from Proposition 15 and the fact that the algebraic quotient of every nontrivial \mathbb{G}_a -action on \mathbb{A}^3 is isomorphic to \mathbb{A}^2 [13]. \square

Example 17. In Corollary 16 above, the base curve B need not be affine. For instance, the morphism

$$f : \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \longrightarrow \mathbb{P}^1, (x, y, z) \mapsto [(xz - y^2)x^2 + 1 : (xz - y^2)^3]$$

defines a family whose general member is isomorphic to the product $C_\lambda \times \mathbb{A}^1$ where $C_\lambda \subset \mathbb{A}^2 = \text{Spec}(\mathbb{C}[xz - y^2, x])$ is the affine elliptic curve with equation $(xz - y^2)^3 + \lambda((xz - y^2)x^2 + 1) = 0$. The subring $\mathbb{C}[xz - y^2, x]$ of $\mathbb{C}[x, y, z]$ coincides with the ring of invariants of the \mathbb{G}_a -action associated with the locally nilpotent $\mathbb{C}[x]$ -derivation $x\partial_y + 2y\partial_z$ and f is the composition of the quotient morphism $\rho : \mathbb{A}^3 \rightarrow \mathbb{A}^2 = \mathbb{A}^3 // \mathbb{G}_a = \text{Spec}(\mathbb{C}[u, v])$, $(x, y, z) \mapsto (xz - y^2, x)$ and the morphism $h : \mathbb{A}^2 = \text{Spec}(\mathbb{C}[u, v]) \rightarrow \mathbb{P}^1$, $(u, v) \mapsto [uv^2 + 1 : u^3]$.

Corollary 16 above implies in particular that a general fiber of a regular function $f : \mathbb{A}^3 \rightarrow \mathbb{A}^1$ cannot be an irrational surface equipped with an \mathbb{A}^1 -fibration of complete type only. In contrast, regular functions $f : \mathbb{A}^3 \rightarrow \mathbb{A}^1$ whose general fibers are rational and equipped with \mathbb{A}^1 -fibrations of complete type only do exist, as illustrated by the following example.

Example 18. Let $f = x^3 - y^3 + z(z + 1) \in \mathbb{C}[x, y, z]$ and let $f : \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \rightarrow \mathbb{A}^1 = \text{Spec}(\mathbb{C}[\lambda])$ be the corresponding morphism. The closure \overline{S}_λ in $\mathbb{P}^3 = \text{Proj}(\mathbb{C}[x, y, z, t])$ of a general fiber $S_\lambda = f^*(\lambda)$ of f is a smooth cubic surface which intersects the hyperplane $H_\infty = \{t = 0\}$ along the union B_λ of three lines meeting at the Eckardt point $p = [0 : 0 : 1 : 0]$. Thus S_λ is rational and a direct computation reveals that $\kappa(S_\lambda) = -\infty$. So by virtue [14], S_λ admits an \mathbb{A}^1 -fibration $\pi_\lambda : S_\lambda \rightarrow C_\lambda$ over a smooth rational curve C_λ . If C_λ was affine, then there would exist a non trivial \mathbb{G}_a -action on S_λ having the general fibers of π_λ as general orbits. But it is straightforward to check that every automorphism of S_λ considered as a birational self-map of \overline{S}_λ is in fact a biregular automorphism of \overline{S}_λ preserving the boundary B_λ . So the automorphism group of S_λ injects into the group $\text{Aut}(\overline{S}_\lambda, B_\lambda)$ of automorphisms of the pair $(\overline{S}_\lambda, B_\lambda)$. The latter being a finite group, we conclude that no such \mathbb{G}_a -action exists, and hence that S_λ only admits \mathbb{A}^1 -fibrations of complete type. An \mathbb{A}^1 -fibration $\pi_\lambda : S_\lambda \rightarrow \mathbb{P}^1$ can be obtained as follows: letting $B_\lambda = L_1 \cup L_2 \cup L_3$, L_1 is a member of a 6-tuple of pairwise disjoint lines whose simultaneous contraction realizes \overline{S}_λ as a blow-up $\sigma : \overline{S}_\lambda \rightarrow \mathbb{P}^2$ of \mathbb{P}^2 in such a way that $\sigma(L_2)$ and $\sigma(L_3)$ are respectively a smooth conic and its tangent line at the point $p = \sigma(L_1)$. The birational transform $\overline{\pi}_\lambda : \overline{S}_\lambda \dashrightarrow \mathbb{P}^1$ on \overline{S}_λ of the pencil generated by $\sigma(L_2)$ and $2\sigma(L_3)$ restricts to an \mathbb{A}^1 -fibration $\pi_\lambda : S_\lambda \rightarrow \mathbb{P}^1$ with two degenerate fibers: an irreducible one, of multiplicity two, consisting of the intersection with S_λ of the unique exceptional divisor of σ whose center is supported on $\sigma(L_3) \setminus \{p\}$, and a smooth one consisting of the intersection with S_λ of the four exceptional divisors of σ with centers supported on $\sigma(L_2) \setminus \{p\}$.

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