Families of affine ruled surfaces: existence of cylinders
Adrien Dubouloz, Takashi Kishimoto

To cite this version:
hal-01053674

HAL Id: hal-01053674
https://hal.science/hal-01053674
Preprint submitted on 31 Jul 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
FAMILIES OF AFFINE RULED SURFACES: EXISTENCE OF CYLINDERS

ADRIEN DUBOULOZ AND TAKASHI KISHIMOTO

Abstract. We show that the generic fiber of a family \( f : X \to S \) of smooth \( \mathbb{A}^1 \)-ruled affine surfaces always carries an \( \mathbb{A}^1 \)-fibration, possibly after a finite extension of the base \( S \). In the particular case where the general fibers of the family are irrational surfaces, we establish that up to shrinking \( S \), such a family actually factors through an \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) over a certain \( S \)-scheme \( Y \to S \) induced by the MRC-fibration of a relative smooth projective model of \( X \) over \( S \). For affine threefolds \( X \) equipped with a fibration \( f : X \to B \) by irrational \( \mathbb{A}^1 \)-ruled surfaces over a smooth curve \( B \), the induced \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) can also be obtained from a relative Minimal Model Program applied to a smooth projective model of \( X \) over \( B \).

Introduction

The general structure of smooth non complete surfaces \( X \) with negative (logarithmic) Kodaira dimension is not fully understood yet. For say smooth quasi-projective surfaces over an algebraically closed field of characteristic zero, it was established by Keel and McKernan [10] that the negativity of the Kodaira dimension is equivalent to the fact that \( X \) is generically covered by images of the affine line \( \mathbb{A}^1 \) in the sense that the set of points \( x \in X \) with the property that there exists a non constant morphism \( f : \mathbb{A}^1 \to X \) such that \( x \in f(\mathbb{A}^1) \) is dense in \( X \) with respect to the Zariski topology. This property, called \( \mathbb{A}^1 \)-uniruledness is equivalent to the existence of an open embedding \( X \to (\bar{X},B) \) into a complete variety \( \bar{X} \) covered by proper rational curves meeting the boundary \( B = \bar{X} \setminus X \) in at most one point. In the case where \( X \) is smooth and affine, an earlier deep result of Miyanishi-Sugie [14] asserts the stronger property that \( X \) is \( \mathbb{A}^1 \)-ruled: there exists a Zariski dense open subset \( U \subset X \) of the form \( U \cong Z \times \mathbb{A}^1 \) for a suitable smooth curve \( Z \). Equivalently, \( X \) admits a surjective flat morphism \( \rho : X \to C \) to an open subset \( C \) of a smooth projective model \( \bar{Z} \) of \( Z \), whose generic fiber is isomorphic to the affine line over the function field of \( C \). Such a morphism \( \rho : X \to C \) is called an \( \mathbb{A}^1 \)-fibration, and we say that \( \rho \) is of affine type or complete type when the base curve \( C \) is affine or complete, respectively.

Smooth \( \mathbb{A}^1 \)-uniruled but not \( \mathbb{A}^1 \)-ruled affine varieties are known to exist in every dimension \( \geq 3 \) [1]. Many examples of \( \mathbb{A}^1 \)-uniruled affine threefolds can be constructed in the form of flat families \( f : X \to B \) of smooth \( \mathbb{A}^1 \)-ruled affine surfaces parametrized by a smooth base curve \( B \). For instance, the complement \( X \) of a smooth cubic surface \( S \subset \mathbb{P}^3 \) is the total space of a family \( f : X \to \mathbb{A}^1 = \text{Spec}(\mathbb{C}[t]) \) of \( \mathbb{A}^1 \)-ruled surfaces induced by the restriction of a pencil \( T : \mathbb{P}^3 \to \mathbb{P}^1 \) on \( \mathbb{P}^3 \) generated by \( S \) and three times a tangent hyperplane \( H \) to \( S \) whose intersection with \( S \) consists of a cuspidal cubic curve. The general fibers of \( f \) have negative Kodaira dimension, carrying \( \mathbb{A}^1 \)-fibrations of complete type only, and the failure of \( \mathbb{A}^1 \)-ruledness is intimately related to the fact that the generic fiber \( X_\eta \) of \( f \), which is a surface defined over the field \( K = \mathbb{C}(t) \), does not admit any \( \mathbb{A}^1 \)-fibration defined over \( \mathbb{C}(t) \). Nevertheless, it was noticed in [3, Theorem 6.1] that one can infer straight from the construction of \( f : X \to \mathbb{A}^1 \) the existence a finite base extension \( \text{Spec}(L) \to \text{Spec}(K) \) for which the surface \( X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \) carries an \( \mathbb{A}^1 \)-fibration \( \rho : X_\eta \times_{\text{Spec}(K)} \text{Spec}(L) \to \mathbb{P}^1_L \) defined over the field \( L \).

A natural question is then to decide whether this phenomenon holds in general for families \( f : X \to B \) of \( \mathbb{A}^1 \)-ruled affine surfaces parameterized by a smooth base curve \( B \), namely, does the existence of \( \mathbb{A}^1 \)-fibrations on the general fibers of \( f \) imply the existence of one on the generic fiber of \( f \), possibly after a finite extension of the base \( B \)? A partial positive answer is given by Gurjar-Masuda-Miyanishi in [3, Theorem 3.8] under the additional assumption that the general fibers of \( f \) carry \( \mathbb{A}^1 \)-fibrations of affine type. The main result in loc. cit. is derived from the study of log-deformations of suitable relative normal projective models.

\begin{itemize}
\item 2000 Mathematics Subject Classification. 14R25; 14D06; 14M20; 14E30.
\item This project was partially funded by ANR Grant “BirPol” ANR-11-BS01-004-01, Grant-in-Aid for Scientific Research of JSPS no. 24740003, and Polish Ministry of Science and Higher Education, "Inventus Plus", Grant no. IP/2012/038/272. The research was done during a visit of the first author at the University of Saitama. The authors thank this institution for its generous support and the excellent working conditions.
\end{itemize}
Let \( f : (X, D) \to B \) of \( X \) over \( B \) with appropriate boundaries \( D \). It is established in particular that the structure of the boundary divisor of a well chosen smooth projective completion of a general closed fiber \( X_s \) is stable under small deformations, a property which implies in turn, possibly after a finite extension of the base \( B \), the existence of an \( \mathbb{A}^1 \)-fibration of affine type on the generic fiber of \( f \). This log-deformation theoretic approach is also central in the related recent work of Flennor-Kaliman-Zaidenberg [2] on the classification of normal affine surfaces with \( \mathbb{A}^1 \)-fibrations of affine type up to a certain notion of deformation equivalence, defined for families which admit suitable relative projective models satisfying Kamawata’s axioms of logarithmic deformations of pairs [8]. The fact that the \( \mathbb{A}^1 \)-fibrations under consideration are of affine type plays again a crucial role and, in contrast with the situation considered in [3], the restrictions imposed on the families imply the existence of \( \mathbb{A}^1 \)-fibrations of affine type on their generic fibers.

Our main result (Theorem 6) consists of a generalization of the results in [3] to families \( f : X \to S \) of \( \mathbb{A}^1 \)-ruled surfaces over an arbitrary normal base \( S \), which also includes the case where a general closed fiber \( X_s \) of \( f \) admits \( \mathbb{A}^1 \)-fibrations of complete type only. In particular, we obtain the following positive answer to Conjecture 6.2 in [3]:

**Theorem.** Let \( f : X \to S \) be dominant morphism between normal complex algebraic varieties whose general fibers are smooth \( \mathbb{A}^1 \)-ruled affine surfaces. Then there exists a dense open subset \( S_* \subset S \), a finite étale morphism \( T \to S_* \) and a normal \( T \)-scheme \( h : Y \to T \) such that the induced morphism \( f_T = p \circ f : X_T = X \times_{S_*} T \to T \) factors as

\[
f_T = h \circ \rho : X_T \to Y \to T,
\]

where \( \rho : X_T \to Y \) is an \( \mathbb{A}^1 \)-fibration.

In contrast with the log-deformation theoretic strategy used in [3], which involves the study of certain Hilbert schemes of rational curves on well-chosen relative normal projective models \( \overline{f} : (\overline{X}, B) \to S \) of \( X \) over \( S \), our approach is more elementary, based on the notion of Kodaira dimension [7] adapted to the case of geometrically connected varieties defined over arbitrary base fields of characteristic zero. Indeed, the hypothesis means equivalently that the general fibers of \( f \) have negative Kodaira dimension. This property is in turn inherited by the generic fiber of \( f \), which is a smooth affine surface defined over the function field of \( S \), thanks to a standard Lefschetz principle argument. Then we are left with checking that a smooth affine surface \( X \) defined over an arbitrary base field \( k \) of characteristic zero and with negative Kodaira dimension admits an \( \mathbb{A}^1 \)-fibration, possibly after a suitable finite base extension \( \text{Spec}(k_0) \to \text{Spec}(k) \), a fact which follows immediately from finite type hypotheses and the aforementioned characterization of Miyanishi-Sugie [14].

The article is organized as follows. The first section contains a review of the structure of smooth affine surfaces of negative Kodaira dimension over arbitrary base fields \( k \) of characteristic zero. We show in particular that every such surface \( X \) admits an \( \mathbb{A}^1 \)-fibration after a finite extension of the base field \( k \), and we give criteria for the existence of \( \mathbb{A}^1 \)-fibrations defined over \( k \). These results are then applied in the second section to the study of deformations \( f : X \to S \) of smooth \( \mathbb{A}^1 \)-ruled affine surfaces: after giving the proof of the main result, Theorem 6, we consider in more detail the particular situation where the general fibers of \( f : X \to S \) are irrational. In this case, after shrinking \( S \) if necessary, we show that the morphism \( f \) actually factors through an \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) over an \( S \)-scheme \( h : Y \to S \) which coincides, up to birational equivalence, with the Maximally Rationally Connected quotient of a relative smooth projective model \( \overline{f} : \overline{X} \to S \) of \( X \) over \( S \). The last section is devoted to the case of affine threefolds equipped with a fibration \( f : X \to Y \) by irrational \( \mathbb{A}^1 \)-ruled surfaces over a smooth base curve \( B \): we explain in particular how to construct an \( \mathbb{A}^1 \)-fibration \( \rho : X \to Y \) factoring \( f \) by means of a relative Minimal Model Program applied to a smooth projective model \( \overline{f} : \overline{X} \to B \) of \( X \) over \( B \).

1. \( \mathbb{A}^1 \)-ruledness of affine surfaces over non closed field

1.1. Logarithmic Kodaira dimension.

1.1.1. Let \( X \) be a smooth geometrically connected algebraic variety defined over a field \( k \) of characteristic zero. By virtue of Nagata compactification [15] and Hironaka desingularization [5] theorems, there exists an open immersion \( X \hookrightarrow (\overline{X}, B) \) into a smooth complete algebraic variety \( \overline{X} \) with reduced SNC boundary divisor \( B = \overline{X} \setminus X \). The (logarithmic) Kodaira dimension \( \kappa(X) \) of \( X \) is then defined as the Iitaka dimension [6] of the pair
\( (\mathcal{X}; \omega_{\mathcal{X}}(\log B)) \) where \( \omega_{\mathcal{X}}(\log B) = (\det \Omega_{\mathcal{X}/k}) \otimes \mathcal{O}_{\mathcal{X}}(B) \). So letting \( \mathcal{R}(\mathcal{X}, B) = \bigoplus_{m \geq 0} H^0(\mathcal{X}, \omega_{\mathcal{X}}(\log B)^{\otimes m}) \), we have \( \kappa(X) = \text{tr. deg}_k \mathcal{R}(\mathcal{X}, B) - 1 \) if \( H^0(\mathcal{X}, \omega_{\mathcal{X}}(\log B)^{\otimes m}) \neq 0 \) for sufficiently large \( m \). Otherwise, if \( H^0(\mathcal{X}, \omega_{\mathcal{X}}(\log B)^{\otimes m}) = 0 \) for every \( m \geq 1 \), we set by convention \( \kappa(X) = -\infty \) and we say that \( \kappa(X) \) is negative. The so-defined element \( \kappa(X) \in \{0, \ldots, \dim_k \mathcal{X}\} \cup \{-\infty\} \) is independent of the choice of a smooth complete model \((\mathcal{X}, B)\) [7].

Furthermore, the Kodaira dimension of \( \mathcal{X} \) is invariant under arbitrary extensions of the base field \( k \). Indeed, given an extension \( k \subset k' \), the pair \((\mathcal{X}_{k'}, B_{k'})\) obtained by the base change \( \text{Spec}(k') \rightarrow \text{Spec}(k) \) is a smooth complete model of \( \mathcal{X}_{k'} = \mathcal{X} \times_{\text{Spec}(k)} \text{Spec}(k') \) with reduced SNC boundary \( B_{k'} \). Furthermore letting \( \pi : \mathcal{X}_{k'} \rightarrow \mathcal{X} \) be the corresponding faithfully flat morphism, we have \( \omega_{\mathcal{X}_{k'}}(\log B_{k'}) \simeq \pi^* \omega_{\mathcal{X}}(\log B) \) and so \( \mathcal{R}(\mathcal{X}_{k'}) \simeq \mathcal{R}(\mathcal{X}) \otimes_k k' \) by the flat base change theorem [4, Proposition III.9.3]. Thus \( \kappa(X) = \kappa(X_{k'}) \).

**Example 1.** The affine line \( \mathbb{A}^1_k \) is the only smooth geometrically connected non complete curve \( C \) with negative Kodaira dimension. Indeed, let \( \mathcal{C} \) be a smooth projective model of \( C \) and let \( \mathcal{C}_\mathcal{F} \) be the curve obtained by the base change to an algebraic closure \( \overline{k} \) of \( k \). Since \( C \) is non complete, \( B = \mathcal{C}_\mathcal{F} \setminus \mathcal{C} \) consists of a finite collection of closed points \( p_1, \ldots, p_s \), \( s \geq 1 \), on which the Galois group \( \text{Gal}(\overline{k}/k) \) acts by \( k \)-automorphisms of \( \mathcal{C}_\mathcal{F} \). Clearly, \( H^0(\mathcal{C}_\mathcal{F}, \omega_{\mathcal{C}_\mathcal{F}}(\log B)^{\otimes m}) \neq 0 \) unless \( \mathcal{C}_\mathcal{F} \simeq \mathbb{P}^1_k \) and \( s = 1 \). Since \( p_1 \) is then necessarily \( \text{Gal}(\overline{k}/k) \)-invariant, \( \mathcal{C} \setminus B \) consists of unique \( k \)-rational point, showing that \( \mathcal{C} \simeq \mathbb{P}^1_k \) and \( C \simeq \mathbb{A}^1_k \).

**1.2. Smooth affine surfaces with negative Kodaira dimension.**

Recall that by virtue of [14], a smooth affine surface \( X \) defined over an algebraically closed field of characteristic zero has negative Kodaira dimension if and only if it is \( \mathbb{A}^1 \)-ruled: there exists a Zariski dense open subset \( U \subset X \) of the form \( U \approx Z \times \mathbb{A}^1 \) for a suitable smooth curve \( Z \). In fact, the projection \( \text{pr}_2 : U \rightarrow \mathbb{A}^1 \rightarrow Z \) always extends to an \( \mathbb{A}^1 \)-fibration \( \rho : X \rightarrow C \) over an open subset \( C \) of a smooth projective model \( \mathcal{Z} \) of \( Z \). This characterization admits the following straightforward generalization to arbitrary base fields of characteristic zero:

**Theorem 2.** Let \( X \) be a smooth geometrically connected affine surface defined over a field \( k \) of characteristic zero. Then the following are equivalent:

a) The Kodaira dimension \( \kappa(X) \) of \( X \) is negative.

b) For some finite extension \( k_0 \) of \( k \), the surface \( X_{k_0} \) contains an open subset \( U \approx Z \times \mathbb{A}^1_{k_0} \) for some smooth curve \( Z \) defined over \( k_0 \).

c) There exists a finite extension \( k_0 \) of \( k \) and an \( \mathbb{A}^1 \)-fibration \( \rho : X_{k_0} \rightarrow C_0 \) over a smooth curve \( C_0 \) defined over \( k_0 \).

*Proof.\* Clearly c) implies b) and b) implies a). To show that a) implies c), we observe that letting \( \overline{k} \) be an algebraic closure of \( k \), we have \( \kappa(X_{\overline{k}}) = \kappa(X) < 0 \). It then follows from the aforementioned result of Miyanishi-Sugie [14] that \( X_{\overline{k}} \) admits an \( \mathbb{A}^1 \)-fibration \( q : X_{\overline{k}} \rightarrow C \) over a smooth curve \( C \), with smooth projective model \( \mathcal{C}_{\overline{k}} \). Since \( X_{\overline{k}} \) and \( \mathcal{C} \) are of finite type over \( \overline{k} \), there exists a finite extension \( k \subset k_0 \) such that \( q : X_{k_0} \rightarrow \mathcal{C}_{k_0} \) is obtained from a morphism \( \rho : X_{k_0} \rightarrow \mathcal{C}_{k_0} \) to a smooth projective curve \( \mathcal{C}_{k_0} \) defined over \( k_0 \) by the base extension \( \text{Spec}(\overline{k}) \rightarrow \text{Spec}(k_0) \). By virtue of Example 1, \( \rho : X_{k_0} \rightarrow \mathcal{C}_{k_0} \) is an \( \mathbb{A}^1 \)-fibration. \( \square \)

Examples of smooth affine surfaces \( X \) of negative Kodaira dimension without any \( \mathbb{A}^1 \)-fibration defined over the base field but admitting \( \mathbb{A}^1 \)-fibrations of complete type after a finite base extension were already constructed in [1]. The following example illustrates the fact that a similar phenomenon occurs for \( \mathbb{A}^1 \)-fibrations of affine type, providing in particular a negative answer to Problem 3.13 in [3].

**Example 3.** Let \( B \subset \mathbb{P}^2_k = \text{Proj}(k[x, y, z]) \) be a smooth conic without \( k \)-rational points defined by a quadratic form \( q = a x^2 + a y^2 + b z^2 \), where \( a, b \in k^* \), and let \( \mathfrak{X} \subset \mathbb{P}^3_k = \text{Proj}(k[x, y, z, t]) \) be the smooth quadric surface defined by the equation \( q(x, y, z) - t^2 = 0 \). The complement \( X \subset \mathfrak{X} \) of the hyperplane section \( \{t = 0\} \) is a \( k \)-rational smooth affine surface with \( \kappa(X) < 0 \), which does not admit any \( \mathbb{A}^1 \)-fibration \( \rho : X \rightarrow C \) over a smooth, affine or projective curve \( C \). Indeed, if such a fibration existed then a smooth projective model of \( C \) would be isomorphic to \( \mathbb{P}^1_k \); since the fiber of \( \rho \) over a general \( k \)-rational point of \( C \) is isomorphic to \( \mathbb{A}^1_k \), its closure in \( \mathfrak{X} \) would intersect the boundary \( \mathfrak{X} \setminus X \approx B \) in a unique point, necessarily \( k \)-rational, in contradiction with the choice of \( B \).

In contrast, for a suitable finite extension \( k \subset k' \), the surface \( X_{k'} \) becomes isomorphic to the complement of the diagonal in \( \mathfrak{X}_{k'} \approx \mathbb{P}^1_{k'} \times \mathbb{P}^1_{k'} \) and hence, it admits at least two distinct \( \mathbb{A}^1 \)-fibrations over \( \mathbb{P}^1_{k'} \), induced
by the restriction of the first and second projections from $\overline{X}_{k'}$. Furthermore, since $X_{k'}$ is isomorphic to the smooth affine quadric in $A^3_{k'} = \text{Spec}(k'[u,v,w])$ with equation $uv - w^2 = 1$, it also admits two distinct $\mathbb{A}^1$-fibrations over $A^1_{k'}$, induced by the restrictions of the projections $\text{pr}_u$ and $\text{pr}_v$.

### 1.3. Existence of $\mathbb{A}^1$-fibrations defined over the base field.

#### 1.3.1. The previous example illustrates the general fact that if $X$ is a smooth geometrically connected affine surface with $\kappa(X) < 0$ which does not admit any $\mathbb{A}^1$-fibration, then there exists a finite extension $k'$ of $k$ such that $X_{k'}$ admits at least two $\mathbb{A}^1$-fibrations of the same type, either affine or complete, with distinct general fibers. Indeed, by virtue of Theorem 2, there exists a finite extension $k_0$ of $k$ such that $X_{k_0}$ admits an $\mathbb{A}^1$-fibration $\rho : X_{k_0} \to C$. Let $k'$ be the Galois closure of $k_0$ in an algebraic closure of $k$ and let $\rho_{k'} : X_{k'} \to C_{k'}$ be the $\mathbb{A}^1$-fibration deduced from $\rho$. If $\rho_{k'} : X_{k'} \to C_{k'}$ is globally invariant under the action of the Galois group $\text{Gal}(k'/k)$ on $X_{k'}$, in the sense that for every $\Phi \in \text{Gal}(k'/k)$ considered as a Galois automorphism of $X_{k'}$ there exists a commutative diagram

$$
\begin{array}{ccc}
X_{k'} & \xrightarrow{\rho} & X_{k'} \\
\rho_{k'} \downarrow & & \downarrow \rho_{k'} \\
C_{k'} & \xrightarrow{\Phi} & C_{k'}
\end{array}
$$

for a certain $k'$-automorphism $\phi$ of $C_{k'}$, then we would obtain a Galois action of $\text{Gal}(k'/k)$ on $C_{k'}$ for which $\rho_{k'} : X_{k'} \to C_{k'}$ becomes an equivariant morphism. Since $C_{k'}$ is quasi-projective and $\rho'_k$ is affine, it would follow from Galois descent that there exists a curve $\tilde{C}$ defined over $k$ and a morphism $q : X \to \tilde{C}$ defined over $k$ such that $\rho_{k'} : X_{k'} \to C_{k'}$ is obtained from $q$ by the base change $\text{Spec}(k') \to \text{Spec}(k)$. Since by virtue of Example 1 the affine line does not have any nontrivial form, the generic fiber of $q$ would be isomorphic to the affine line over the field of rational functions of $\tilde{C}$ and so, $q : X \to \tilde{C}$ would be an $\mathbb{A}^1$-fibration defined over $k$, in contradiction with our hypothesis. So there exists at least an element $\Phi \in \text{Gal}(k'/k)$ considered as a $k$-automorphism of $X_{k'}$ such that the $\mathbb{A}^1$-fibrations $\rho_{k'} : X_{k'} \to C_{k'}$ and $\rho_{k'} \circ \phi : X_{k'} \to C_{k'}$ have distinct general fibers.

Arguing backward, we obtain the following useful criterion:

**Proposition 4.** Let $X$ be a smooth geometrically connected affine surface with $\kappa(X) < 0$. If there exists a finite Galois extension $k'$ of $k$ such that $X_{k'}$ admits a unique $\mathbb{A}^1$-fibration $\rho' : X_{k'} \to C_{k'}$ up to composition by automorphisms of $C_{k'}$, then $\rho' : X_{k'} \to C_{k'}$ is obtained by base extension from an $\mathbb{A}^1$-fibration $\rho : X \to C$ defined over $k$.

**Corollary 5.** A smooth geometrically connected irrational affine surface $X$ has negative Kodaira dimension if and only if it admits an $\mathbb{A}^1$-fibration $\rho : X \to C$ over a smooth irrational curve $C$ defined over the base field $k$. Furthermore for every extension $k'$ of $k$, $\rho_{k'} : X_{k'} \to C_{k'}$ is the unique $\mathbb{A}^1$-fibration on $X_{k'}$ up to composition by automorphisms of $C_{k'}$.

**Proof.** Uniqueness is clear since otherwise $C_{k'}$ would be dominated by a general fiber of another $\mathbb{A}^1$-fibration on $X_{k'}$, and hence would be rational, implying in turn the rationality of $X$. By virtue of Theorem 2, there exists a finite Galois extension $k'$ of $k$ and an $\mathbb{A}^1$-fibration $\rho' : X_{k'} \to C'$ over a smooth curve $C'$. The latter is irrational as $X$ is irrational, which implies that $\rho' : X_{k'} \to C'$ is the unique $\mathbb{A}^1$-fibration on $X_{k'}$. So $\rho'$ descend to an $\mathbb{A}^1$-fibration $\rho : X \to C$ over a smooth irrational curve $C$ defined over $k$. □

### 2. Families of $\mathbb{A}^1$-ruled affine surfaces

#### 2.1. Existence of étale $\mathbb{A}^1$-cylinders. This subsection is devoted to the proof of the following:

**Theorem 6.** Let $X$ and $S$ be normal algebraic varieties defined over a field $k$ of infinite transcendence degree over $\mathbb{Q}$, and let $f : X \to S$ be a dominant affine morphism with the property that for a general closed point $s \in S$, the fiber $X_s$ is a smooth geometrically connected affine surface with negative Kodaira dimension. Then there exists an open subset $S_x \subset S$, a finite étale morphism $T \to S_x$ and a normal $T$-scheme $h : Y \to T$ such that $f_T = \text{pr}_T : X_T = X \times S_x \to T$ factors as

$$
f_T = h \circ \rho : X_T \xrightarrow{\rho} Y \xrightarrow{h} T
$$

where $\rho : X_T \to Y$ is an $\mathbb{A}^1$-fibration.
Proof. Shrinking $S$ if necessary, we may assume that $S$ is affine, that $f : X \to S$ is smooth and that $\kappa(X) < 0$ for every closed point $s \in S$. It is enough to show that the fiber $X_\eta$ of $f$ over the generic point $\eta$ of $S$ is geometrically connected, with negative Kodaira dimension. Indeed, if so, then by Theorem 2 above, there exists a finite extension $L$ of $K = \text{Frac}(\Gamma(S, O_S))$ and an $\mathbb{A}^1$-fibration $\rho : X_\eta \times \text{Spec}(K) \text{Spec}(L) \to C$ onto a smooth curve $C$ defined over $L$. Letting $T$ be the normalization of $S$ in $L$ and shrinking $T$ again if necessary, we obtain a finite étale morphism $T \to S$ such that the generic fiber of $\text{pr}_T : X_T \to T$ is isomorphic to the $\mathbb{A}^1$-fibered surface $\rho : X_\eta \times \text{Spec}(K) \text{Spec}(L) \to C$ and then the assertion follows from Lemma 7 below.

Since $X$ and $S$ are affine and of finite type over $k$, there exists a subfield $k_0$ of $k$ of finite transcendence degree over $\mathbb{Q}$, and a smooth morphism $f_0 : X_0 \to S_0$ of $k_0$-varieties such that $f : X \to S$ is obtained from $f_0 : X_0 \to S_0$ by the base extension $\text{Spec}(k) \to \text{Spec}(k_0)$. The field $K_0 = \text{Frac}(\Gamma(S_0, O_{S_0}))$ has finite transcendence degree over $\mathbb{Q}$ and hence, it admits a $k_0$-embedding $\xi : K_0 \hookrightarrow k$. Letting $(X_0)_\eta_0$ be the fiber of $f_0$ over the generic point $\eta_0 : \text{Spec}(K_0) \to S_0$ of $S_0$, the composition $\Gamma(S_0, O_{S_0}) \hookrightarrow K_0 \hookrightarrow k$ induces a $k$-homomorphism $\Gamma(S_0, O_{S_0}) \otimes_{k_0} k \to k$ defining a closed point $s : \text{Spec}(k) \to \text{Spec}(\Gamma(S_0, O_{S_0}) \otimes_{k_0} k) = S$ of $S$ for which obtain the following commutative diagram:

![Diagram](image)

The bottom square of the cube being cartesian by construction, we deduce that

$$(X_0)_\eta_0 \times \text{Spec}(K) \simeq \text{Spec}(k) \simeq X_0 \times \text{Spec}(k) \simeq X \times S \text{Spec}(k) = X_s.$$

Since by assumption, $X_s$ is geometrically connected with $\kappa(X_s) < 0$, we conclude that $(X_0)_\eta_0$ is geometrically connected and has negative Kodaira dimension. This implies in turn that $X_\eta$ is geometrically connected and that $\kappa(X_\eta) < 0$ as desired.

In the proof of the above theorem, we used the following lemma:

**Lemma 7.** Let $f : X \to S$ be a dominant affine morphism between normal varieties defined over a field $k$ of characteristic zero. Then the following are equivalent:

a) The generic fiber $X_\eta$ of $f$ admits an $\mathbb{A}^1$-fibration $q : X_\eta \to C$ over a smooth curve $C$ defined over the fraction field $K$ of $S$.

b) There exists an open subset $S_*$ of $S$ and a normal $S_*$-scheme $h : Y \to S_*$ of relative dimension 1 such that the restriction of $f$ to $V = f^{-1}(S_*)$ factors as $f|_V = h \circ \rho : V \to Y \to S_*$ where $\rho : V \to Y$ is an $\mathbb{A}^1$-fibration.

**Proof.** If b) holds then letting $L$ be the fraction field of $Y$, we have a commutative diagram

$$
\begin{array}{c c c c c c c c c}
V_\xi & \xrightarrow{h_\xi} & V_\eta & \xrightarrow{h_\eta} & V \\
\downarrow \rho_\xi & & \downarrow \rho_\eta & & \downarrow \rho \\
\text{Spec}(L) & \xrightarrow{\xi} & C & \xrightarrow{h} & \text{Spec}(K) \\
& & \downarrow h_\eta & & \downarrow h \\
& & S_0 & & \eta \\
\end{array}
$$

in which each square is cartesian. It follows that $h_\eta : C \to \text{Spec}(K)$ is a normal whence smooth curve defined over $K$ and that $\rho_\eta : X_\eta \to C$ is an $\mathbb{A}^1$-fibration. Conversely, suppose that $X_\eta$ admits an $\mathbb{A}^1$-fibration $q : X_\eta \to C$ and let $\overline{C}$ be a smooth projective model of $C$ over $K$. Then there exists an open subset $S_0$ of $S$ and a projective $S_0$-scheme $h : Y \to S_0$ whose generic fiber is isomorphic to $\overline{C}$. After shrinking $S_0$ if necessary, the rational map $\rho : V \dashrightarrow Y$ of $S_0$-schemes induced by $q$ becomes a morphism and
we obtain a factorization $f |_V = h \circ \rho$. By construction, the generic fiber $V_\xi$ of $\rho : V \to Y$ is isomorphic to $V \times_Y \text{Spec}(L) \simeq (V \times_Y C) \times_C \text{Spec}(L) \simeq X_\eta \times \text{Spec}(L) \simeq A^1_L$ since $V \times_Y C \simeq V_\eta \simeq X_\eta$ and $\rho : X_\eta \to C \to \overline{C}$ is an $A^1$-fibration. So $\rho : V \to Y$ is an $A^1$-fibration.

Example 8. Let $R = \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$, $S = \text{Spec}(R)$ and let $D$ be the relatively ample divisor in $\mathbb{P}^2_R = \text{Proj}(R[x, y, z])$ defined by the equation $x^2 + sy^2 + tz^2 = 0$. The restriction $h : X = \mathbb{P}^2_R \setminus D \to S$ of the structure morphism defines a family of smooth affine surfaces with the property that for every closed point $s \in S$, $X_s$ is isomorphic to the complement of the smooth conic $D_s$. In particular $X_s$ admits a continuum of pairwise distinct $A^1$-fibrations $X_s \to A^1_s$, induced by the restrictions to $X_s$ of the rational pencils on $\mathbb{P}^2_s$ generated by $D_s$ and twice its tangent line at an arbitrary closed point $p_s \in D_s$. On the other hand the fiber of $D$ over the generic point $\eta$ of $S$ is a conic without $\mathbb{C}(s, t)$-rational point in $\mathbb{P}^2_{\mathbb{C}(s, t)}$ and hence, we conclude by a similar argument as in Example 3 that $X_\eta$ does not admit any $A^1$-fibration defined over $\mathbb{C}(s, t)$. Therefore there is no open subset $S_\eta$ of $S$ over which $h$ can be factored through an $A^1$-fibration.

2.2. Deformations of irrational $A^1$-ruled affine surfaces.

In this subsection, we consider the particular situation of a flat family $f : X \to S$ over a normal variety $S$ whose general fibers are irrational $A^1$-ruled affine surfaces. A combination of Corollary 5 and Theorem 6 above implies that if $f : X \to S$ is smooth and defined over a field of infinite transcendence degree over $\mathbb{Q}$, then the generic fiber $X_\eta$ of $f$ is $A^1$-ruled. Equivalently, there exists an open subset $S_\eta \subset S$ and a normal $S_\eta$-scheme $h : Y \to S_\eta$ such that the restriction of $f$ to $X_\eta = X \times_S S_\eta$ factors through an $A^1$-fibration $\rho : X_\eta \to Y$ (see Lemma 7). The restriction of $\rho$ to the fiber of $f$ over a general closed point $s \in S_\eta$ is an $A^1$-fibration $\rho_s : X_s \to Y_s$ over the normal, whence smooth, curve $Y_s$. Since $X_s$ is irrational, $Y_s$ is irrational, and so $\rho_s : X_s \to Y_s$ is the unique $A^1$-fibration on $X_s$ up to composition by automorphisms of $Y_s$. So in this case, we can identify $\rho_s : X_s \to Y_s$ with the Maximal Rationally Connected fibration (MRC-fibration) $\varphi : \overline{X}_s \dashrightarrow Y_s$ of a smooth projective model $\overline{X}_s$ of $X_s$ in the sense of [11, IV.5]: recall that $\varphi$ is unique, characterized by the property that its general fibers are rationally connected and that for a very general point $y \in Y_s$ any rational curve in $\overline{X}_s$ which meets $\overline{X}_s$ is actually contained in $\overline{X}_y$. The $A^1$-fibration $\rho : X_s \to Y_s$ can therefore be re-interpreted as being the MRC-fibration of a relative smooth projective model $\overline{X}_s$ of $X_s$ over $S$.

Reversing the argument, general existence and uniqueness results for MRC-fibrations allow actually to get rid of the smoothness hypothesis of a general fiber of $f : X \to S$ and to extend the conclusion of Theorem 6 to arbitrary base fields of characteristic zero. Namely, we obtain the following characterization:

Theorem 9. Let $X$ and $S$ be normal varieties defined over a field $k$ of characteristic zero and let $f : X \to S$ be a dominant affine morphism with the property that for a general closed point $s \in S$, the fiber $X_s$ is irrational and $A^1$-ruled. Then there exists an open subset $S_\eta$ and a normal $S_\eta$-scheme $h : Y \to S_\eta$ such that the restriction of $f$ to $X_\eta = X \times_S S_\eta$ factors as

$$f |_{X_\eta} = h \circ \rho : X_\eta \dashrightarrow Y \dashrightarrow h : S_\eta$$

where $\rho : X_\eta \to Y$ is an $A^1$-fibration.

Proof. Shrinking $S$ if necessary, we may assume that for every closed point $s \in S$, $X_s$ is irrational and $A^1$-ruled, hence carrying a unique $A^1$-fibration $\pi_s : X_s \to C_s$ over an irrational normal curve $C_s$. Since $f : X \to S$ is affine, there exists a normal projective $S$-scheme $\overline{X} \to S$ and an open embedding $X \hookrightarrow \overline{X}$ of schemes over $S$. Letting $W \to \overline{X}$ be a resolution of the singularities of $\overline{X}$, we may assume up to shrinking $S$ again if necessary that $W \to S$ is a smooth morphism. We let $j : X \dashrightarrow W$ be the birational map of $S$-schemes induced by the embedding $X \hookrightarrow \overline{X}$. By virtue of [11, Theorem 5.9], there exists an open subset $W'$ of $W$, an $S$-scheme $h : Z \to S$ and a proper morphism $\overline{h} : W' \to Z$ such that for every $s \in S$, the induced rational map $\overline{\eta}_s : W_s' \dashrightarrow Z_s$ is the MRC-fibration for $W_s$. On the other hand, since $W_s$ is a smooth projective model of $X_s$, the induced rational map $\pi_s : \overline{X}_s \dashrightarrow C_s$ is the MRC-fibration for $W_s$. Consequently, for a general closed point $z \in Z$ with $h(z) = s$, the fiber $W_s$ of $\overline{\eta}_s$, which is an irreducible proper rational curve contained in $W_s$, must coincide with the closure of the image by $j$ of a general closed fiber of $\pi_s$. The latter being isomorphic to the affine line $A^1_k$ over the residue field $k$ of the corresponding point of $C_s$, we conclude that there exists an affine open subset $U$ of $X$ on which the composition $\overline{\eta} \circ j : U \to Z$ is a well defined morphism with general closed fibers isomorphic to affine lines over the corresponding residue fields.
So $\varphi \circ j : U \to Z$ is an $\mathcal{A}^1$-fibration by virtue of [9]. The generic fiber of $f : X \to S$ is thus $\mathcal{A}^1$-ruled and the assertion follows from Lemma 7 above.

Example 10. Let $h : Y \to S$ be smooth family of complex projective curves of genus $g \geq 2$ over a normal affine base $S$ and let $\mathcal{T}_{Y/S}$ be the relative tangent sheaf of $h$. Since by Riemann-Roch $H^i(Y_s, \mathcal{T}_{Y/S}) = 0$ and $\dim H^1(Y_s, \mathcal{T}_{Y/S}) = g - 1$ for every point $s \in S$, $h^*_s \mathcal{T}_{Y/S} = 0$, $R^1 h_* \mathcal{T}_{Y/S}$ is locally free of rank $g - 1$ [4, Corollary III.12.9] and so, $H^1(Y, \mathcal{T}_{Y/S}) \cong H^0(S, R^1 h_* \mathcal{T}_{Y/S})$ by the Leray spectral sequence. Replacing $S$ by an open subset, we may assume that $R^1 h_* \mathcal{T}_{Y/S}$ admits a nowhere vanishing global section $\sigma$. Via the isomorphism $H^1(Y, \mathcal{T}_{Y/S}) \cong \text{Ext}^1_0(\mathcal{O}_Y, \mathcal{T}_{Y/S})$, we may interpret this section as the class of a non trivial extension $0 \to \mathcal{T}_{Y/S} \to E \to \mathcal{O}_Y \to 0$ of locally free sheaves over $Y$. The inclusion $\mathcal{T}_{Y/S} \to E$ defines a section $D$ of the locally trivial $\mathbb{P}^1$-bundle $\overline{\mathbb{P}} : \mathbb{X} = \text{Proj}(\mathbb{S}(\mathcal{O}_Y, E^\vee)) \to Y$ and the non vanishing of $\sigma$ guarantees that $D$ is the support of an $S$-ample divisor. Indeed the $S$-ampleness of $D$ is equivalent to the property that for every $s \in S$ the induced section $D_s$ of the $\mathbb{P}^1$-bundle $\overline{\mathbb{P}}_s : \mathbb{X}_s \to Y_s$ over the smooth projective curve $Y_s$ is ample. Since by construction, $\overline{\mathbb{P}}_s : \mathbb{X}_s \to Y_s$ is a nontrivial torsor under the line bundle $\text{Spec}(\mathbb{S}(\mathcal{T}_Y^\vee)) \to Y_s$, it follows that $D_s$ intersects positively every section $D$ of $\overline{\mathbb{P}}_s$ except maybe $D_s$ itself. On the other hand, we have $\langle D_s^2 \rangle = -\deg \mathcal{T}_{Y/s} = 2g(Y_s) - 2 > 0$, and so the ampleness of $D_s$ follows from the Nakai-Moishezon criterion and the description of the cone effective cycles on an irrational projective ruled surface given in [4, Proposition 2.20-2.21].

Letting $X = \mathbb{X} \setminus D$, we obtain a smooth family

$$f = g \circ \overline{\mathbb{P}} \mid X : X \xrightarrow{\mathbb{P}} Y \xrightarrow{\mathcal{A}} S$$

where $\overline{\mathbb{P}} \mid X : X \to Y$ is nontrivial, locally trivial, $\mathcal{A}^1$-bundle such that for every $s \in S$, $X_s$ is an affine surface with an $\mathcal{A}^1$-fibration $\rho_s : X_s \to Y_s$ of complete type.

In contrast with the previous example, the following proposition shows in particular that if the total space of a family of irrational $\mathcal{A}^1$-ruled affine surfaces $f : X \to S$ has finite divisor class group, then the induced $\mathcal{A}^1$-fibration on a general fiber of $f : X \to S$ is of affine type.

Proposition 11. Let $X$ be a geometrically integral normal variety with finite divisor class group $\text{Cl}(X)$ and let $f : X \to S$ be a dominant morphism to a normal variety $S$ with the property that for a general closed point $s \in S$, the fiber $X_s$ is irrational and $\mathcal{A}^1$-ruled, say with unique $\mathcal{A}^1$-fibration $\pi_s : X_s \to C_s$. Then there exists an effective $\mathbb{G}_a, S$-action on $X$ such that for a general closed point $s \in S$, the $\mathcal{A}^1$-fibration $\pi_s : X_s \to C_s$ factors through the algebraic quotient $\rho_s : X_s \to X_s / \mathbb{G}_a, s = \text{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^\mathbb{G}_a)$. 

Proof. Let $f \mid X_s = h \circ \rho : X_s \xrightarrow{\rho} Y \xrightarrow{h} S_s$ be as in Theorem 9. Since $\rho$ is an $\mathcal{A}^1$-fibration, there exists an affine open subset $U \subset Y$ such that $\rho^{-1}(U) \cong X \times \mathcal{A}^1$ as schemes over $U$. Since $\rho^{-1}(U)$ is affine, its complement in $X$ is of pure codimension 1, and the finiteness of $\text{Cl}(X)$ implies that it is actually the support of an effective principal divisor $\text{div}_X(a)$ for some $a \in \Gamma(X, \mathcal{O}_X)$. Letting $\partial_h$ be the locally nilpotent derivation of $\Gamma(\rho^{-1}(U), \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)_{\rho}$ corresponding to the $\mathbb{G}_a, U$-action by translations on the second factor, the finite generation of $\Gamma(X, \mathcal{O}_X)$ guarantees that for a suitably chosen $n \geq 0$, $a^n \partial_h$ lifts to a locally nilpotent derivation $\partial$ of $\Gamma(X, \mathcal{O}_X)$. By construction, the restriction of $f$ to the dense open subset $\rho^{-1}(U)$ of $X$ is invariant under the corresponding $\mathbb{G}_a$-action, and so $f : X \to S$ is $\mathbb{G}_a$-invariant. For a general closed point $s \in S$, the induced $\mathbb{G}_a$-action on $X_s$ is nontrivial, and its algebraic quotient $\rho_s : X_s \to X_s / \mathbb{G}_a = \text{Spec}(\Gamma(X_s, \mathcal{O}_{X_s})^\mathbb{G}_a)$ is a surjective $\mathcal{A}^1$-fibration over a normal affine curve $X_s / \mathbb{G}_a$. Since $C_s$ is irrational, the general fibers of $\rho_s$ and $\pi_s$ must coincide. It follows that $\pi_s$ is $\mathbb{G}_a$-invariant, whence factors through $\rho_s$. 

3. Affine threefolds fibered in irrational $\mathcal{A}^1$-ruled surfaces

In this section we consider in more detail the case of normal complex affine threefolds $X$ admitting a fibration $f : X \to B$ by irrational $\mathcal{A}^1$-ruled surfaces, over a smooth curve $B$. We explain how to derive the variety $h : Y \to B$ for which $f$ factors through an $\mathcal{A}^1$-fibration $\rho : X \to Y$ from a relative minimal model program applied to a suitable projective model of $X$ over $B$. In the case where the divisor class group of $X$ is finite, we provide a complete classification of such fibrations in terms of additive group actions on $X$. 
3.1. \(A^1\)-cylinders via relative Minimal Model Program.

Let \(X\) be a normal complex affine threefold and let \(f : X \to B\) be a flat morphism onto a smooth curve \(B\) with the property that a general closed fiber \(X_b\) of \(f\) is an irreducible irrational \(A^1\)-ruled surface. We let \(\overline{f} : W \to B\) be a smooth projective model of \(X\) over \(B\) obtained from an arbitrary normal relative projective completion \(X \to \overline{X}\) of \(X\) over \(B\) by resolving the singularities. We let \(j : X \to W\) be the birational map induced by the open immersion \(X \to \overline{X}\).

By applying a minimal model program for \(W\) over \(B\), we obtain a sequence of birational \(B\)-maps
\[ W = W_0 \rightarrow W_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_{\ell-1} \rightarrow W_\ell = W', \]
between \(B\)-schemes \(\overline{f}_i : W_i \to B\), where \(\varphi_i : W_i \to W_{i+1}\) is either a divisorial contraction or a flip, and the rightmost variety \(W'\) is the output of a minimal model program over \(B\). The hypotheses imply that \(W'\) has the structure of a Mori conic bundle \(\overline{p} : W' \to Y\) over a \(B\)-scheme \(h : Y \to B\) corresponding to the contraction of an extremal ray \(\text{NE}(W'/B)\). Indeed, a general fiber of \(\overline{f}\) being a birationaly ruled projective surface, the output \(W'\) is not a minimal model of \(W\) over \(B\). So \(W'\) is either a Moric conic bundle over a \(B\)-scheme \(Y\) of dimension 2 or a del Pezzo fibration over \(B\), the second case being excluded by the fact that the general fibers of \(\overline{f}\) are irrational.

**Proposition 12.** The induced map \(\rho = \overline{f}|_X : X \to Y\) is a rational \(A^1\)-fibration.

**Proof.** Since a general closed fiber \(X_b\) is a normal affine surface with an \(A^1\)-fibration \(\pi_b : X_b \to C_b\) over a certain irrational smooth curve \(C_b\), it follows that there exists a unique maximal affine open subset \(U_b\) of \(C_b\) such that \(\pi_b^{-1}(U_b) \simeq U_b \times A^1\) and such that the rational map \(j_b : \pi_b^{-1}(U_b) \to W_b\) induced by \(j\) is proper, inducing an isomorphism between \(\pi_b^{-1}(U_b)\) and its image. Each step \(\varphi_i : W_i \to W_{i+1}\) consists of either a flip whose flipping and flipped curves are contained in fibers of \(\overline{f}_i : W_i \to B\) and \(\overline{f}_{i+1} : W_{i+1} \to B\) respectively, or a divisorial contraction whose exceptional divisor is contained in a fiber of \(\overline{f}_i : W_i \to B\), or a divisorial contraction whose exceptional divisor intersects a general fiber of \(\overline{f}_i : W_i \to B\). Clearly, a general closed fiber of \(\overline{f}_i : W_i \to B\) is not affected by the first two types of birational maps. On the other hand, if \(\varphi_i : W_i \to W_{i+1}\) is the contraction of a divisor \(E_i \subset W_i\) which dominates \(B\), then a general fiber of \(\varphi_i |_{E_i}\) is a smooth proper rational curve. The intersection of \(E_i\) with a general closed fiber \(W_i\) of \(\overline{f}_i\) thus consists of proper rational curves, and its intersection with the image of the maximal affine cylinder like open subset \(\pi_b^{-1}(U_b)\) of \(X_b\) is either empty or composed of affine rational curves. Since \(U_b\) is an irrational curve, it follows that each irreducible component of \(E_i \cap (\pi_b^{-1}(U_b))\) is contained in a fiber of \(\pi_b\). This implies that there exists an open subset \(U_{b,0}\) of \(U_b\) with the property that for every \(i = 1, \ldots, \ell\), the restriction of \(\varphi_i \circ \cdots \circ \varphi_1 \circ j\) to \(\pi_b^{-1}(U_{b,0}) \subset X_b\) is an isomorphism onto its image in \(W_{i,b}\). A general fiber of \(\overline{p} : W' \to Y\) over a closed point \(y \in Y\) being a smooth proper rational curve, its intersection with \(\pi_{b(y)}^{-1}(U_{b(y),0})\) viewed as an open subset of \(W_{b(y)}\) is thus either empty or equal to a fiber of \(\pi_{b(y)}\). So by virtue of [9], there exists an open subset \(V\) of \(X\) on which \(\overline{p}\) restricts to an \(A^1\)-fibration \(\overline{p}|_V : V \to Y\).

**Corollary 13.** Let \(X\) be a normal complex affine threefold \(X\) equipped with a morphism \(f : X \to B\) onto a smooth curve \(B\) whose general closed fibers are irrational \(A^1\)-ruled surfaces. Then \(X\) is birationally equivalent to the product of \(\mathbb{P}^1\) with a family \(h_0 : C_0 \to B_0\) of smooth projective curves of genus \(g \geq 1\) over an open subset \(B_0 \subset B\).

**Proof.** By the previous Proposition, \(X\) has the structure of a rational \(A^1\)-fibration \(\rho : X \to Y\) over a 2-dimensional normal proper \(B\)-scheme \(h : Y \to B\). In particular, \(X\) is birational to \(Y \times \mathbb{P}^1\). On the other hand, for a general closed point \(b \in B\), the curve \(Y_b\) is birational to the base \(C_b\) of the unique \(A^1\)-fibration \(\pi_b : X_b \to C_b\) on the irrational affine surface \(X_b\). Letting \(\sigma : Y \to Y\) be a desingularization of \(Y\), there exists an open subset \(B_0\) of \(B\) over which the composition \(h \circ \sigma : Y \to Y\) restricts to a smooth family \(h_0 : C_0 \to B_0\) of projective curves of a certain genus \(g \geq 1\). By construction, \(X\) is birational to \(C_0 \times \mathbb{P}^1\).

**Remark 14.** Example 10 above shows conversely that for every smooth family \(h : C \to B\) of projective curves of genus \(g \geq 2\), there exists a smooth \(A^1\)-ruled affine threefold \(X\) birationally equivalent to \(C \times \mathbb{P}^1\). Actually, in the setting of the previous Corollary 13, if we assume further that a general fiber of \(f : X \to B\) carries an \(A^1\)-fibration \(\pi_0 : X_0 \to C_0\) over a smooth curve \(C_0\) whose smooth projective model has genus \(g \geq 2\), then there exists a uniquely determined family \(h : C \to B\) of proper stable curves of genus \(g\) such that \(X\) is birationally equivalent to \(C \times \mathbb{P}^1\); indeed, the moduli stack \(\overline{M}_g\) of stable curves of genus \(g \geq 2\) being proper
and separated, the smooth family \( h_0 : C_0 \to B_0 \) extends in a unique way to a family \( h : C \to B \) of stable curves of genus \( g \).

### 3.2. Factorial threefolds.

**Proposition 15.** Let \( X \) be a normal affine threefold with finite divisor class group \( \text{Cl}(X) \) and let \( f : X \to B \) be a morphism onto a smooth curve \( B \) whose general closed fibers are irrational \( \mathbb{A}^1 \)-ruled surfaces. Then there exists a factorization \( f = h \circ \rho : X \to Y \to B \) where \( \rho : X \to Y \) is the algebraic quotient morphism of an effective \( \mathbb{G}_{a,B} \)-action on \( X \). In particular, a general fiber of \( f \) admits an \( \mathbb{A}^1 \)-fibration of affine type.

**Proof.** By virtue of Proposition 11, there exist an effective \( \mathbb{G}_{a,B} \)-action on \( X \) such that for a general closed point \( b \in B \), the \( \mathbb{A}^1 \)-fibration \( \pi_b : X_b \to C_b \) on \( X_b \) factors through the algebraic quotient \( \rho_b : X_b \to X_b/\mathbb{G}_{a,b} = \text{Spec}(\Gamma(X_b, \mathcal{O}_{X_b})^{\mathbb{G}_{a,b}}) \). Since \( X \) is a threefold, the ring of invariants \( \Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,B}} \) is finitely generated [16]. The quotient morphism \( \rho : X \to Y = \text{Spec}(\Gamma(X, \mathcal{O}_X)^{\mathbb{G}_{a,B}}) \) is an \( \mathbb{A}^1 \)-fibration, and since \( Y \) is a categorical quotient in the category of algebraic varieties, the invariant morphism \( f : X \to B \) factors through \( \rho \). \( \square \)

**Corollary 16.** Let \( f : \mathbb{A}^3 \to B \) be a morphism onto a smooth curve \( B \) with irrational \( \mathbb{A}^1 \)-ruled general fibers. Then \( B \) is isomorphic to either \( \mathbb{P}^1 \) or \( \mathbb{A}^1 \) and there exists a factorization \( f = h \circ \rho : \mathbb{A}^3 \to \mathbb{A}^2 \to B \), where \( \rho : \mathbb{A}^3 \to \mathbb{A}^2 \) is the quotient morphism of an effective \( \mathbb{G}_{a,B} \)-action on \( \mathbb{A}^3 \).

**Proof.** Since \( B \) is dominated by a general line in \( \mathbb{A}^3 \), it is necessarily isomorphic to \( \mathbb{P}^1 \) or \( \mathbb{A}^1 \). The second assertion follows from Proposition 15 and the fact that the algebraic quotient of every nontrivial \( \mathbb{G}_{a} \)-action on \( \mathbb{A}^3 \) is isomorphic to \( \mathbb{A}^2 \) [13]. \( \square \)

**Example 17.** In Corollary 16 above, the base curve \( B \) need not be affine. For instance, the morphism

\[
\pi : \mathbb{A}^3 \to \mathbb{P}^1, (x, y, z) \mapsto [(xz - y^2)x^2 + 1 : (xz - y^2)^3]
\]

defines a family whose general member is isomorphic to the product \( \mathbb{C}_\lambda \times \mathbb{A}^1 \), where \( \mathbb{C}_\lambda \subset \mathbb{A}^2 = \text{Spec}(\mathbb{C}[xz - y^2, x]) \) is the affine elliptic curve with equation \( (xz - y^2)^3 + \lambda((xz - y^2)x^2 + 1) = 0 \). The subring \( \mathbb{C}[xz - y^2, x] \) of \( \mathbb{C}[x, y, z] \) coincides with the ring of invariants of the \( \mathbb{G}_{a,B} \)-action associated with the locally nilpotent \( \mathbb{C}[x]- \)derivation \( x \partial_x + 2y \partial_z \) and \( f \) is the composition of the quotient morphism \( \rho : \mathbb{A}^3 \to \mathbb{A}^2 = \mathbb{A}^3/\mathbb{G}_a = \text{Spec}(\mathbb{C}[u, v]), (x, y, z) \mapsto (xz - y^2, x) \) and the morphism \( h : \mathbb{A}^2 \to \mathbb{P}^1, (u, v) \mapsto [uv^2 + 1 : v^3] \).

Corollary 16 above implies in particular that a general fiber of a regular function \( f : \mathbb{A}^3 \to \mathbb{A}^1 \) cannot be an irrational surface equipped with an \( \mathbb{A}^1 \)-fibration of complete type only. In contrast, regular functions \( f : \mathbb{A}^3 \to \mathbb{A}^1 \) whose general fibers are rational and equipped with \( \mathbb{A}^1 \)-fibractions of complete type only do exist, as illustrated by the following example.

**Example 18.** Let \( f = x^3 - y^3 + z(z + 1) \in \mathbb{C}[x, y, z] \) and let \( \pi : \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z]) \to \mathbb{A}^1 = \text{Spec}(\mathbb{C}[\lambda]) \) be the corresponding morphism. The closure \( S_\lambda \) in \( \mathbb{P}^3 = \text{Proj}(\mathbb{C}[x, y, z, t]) \) of a general fiber \( S_\lambda = f^*(\lambda) \) of \( f \) is a smooth cubic surface which intersects the hyperplane \( H_\infty = \{ t = 0 \} \) along the union \( B_3 \) of three lines meeting at the Eckardt point \( p = [0:0:1:0] \). Thus \( S_\lambda \) is rational and a direct computation reveals that \( \kappa(S_\lambda) = -\infty \). So by virtue [14], \( S_\lambda \) admits an \( \mathbb{A}^1 \)-fibration \( \pi_\lambda : S_\lambda \to C_\lambda \) over a smooth rational curve \( C_\lambda \). If \( C_\lambda \) was affine, then there would exist a non trivial \( \mathbb{G}_{a,B} \)-action on \( S_\lambda \) having the general fibers of \( \pi_\lambda \) as general orbits. But it is straightforward to check that every automorphism of \( S_\lambda \) considered as a birational self-map of \( \overline{S}_\lambda \) is in fact a biregular automorphism of \( \overline{S}_\lambda \) preserving the boundary \( B_\lambda \). So the automorphism group of \( S_\lambda \) injects into the group \( \text{Aut}(\overline{S}_\lambda, B_\lambda) \) of automorphisms of the pair \( (\overline{S}_\lambda, B_\lambda) \). The latter being a finite group, we conclude that no such \( \mathbb{G}_{a,B} \)-action exists, and hence that \( S_\lambda \) only admits \( \mathbb{A}^1 \)-fibrations of complete type. An \( \mathbb{A}^1 \)-fibration \( \pi_\lambda : S_\lambda \to \mathbb{P}^1 \) can be obtained as follows: letting \( B_\lambda = L_1 \cup L_2 \cup L_3 \), \( L_1 \) is a member of a 6-tuple of pairwise disjoint lines whose simultaneous contraction realizes \( S_\lambda \) as a blow-up \( \sigma : \overline{S}_\lambda \to \mathbb{P}^2 \) of \( \mathbb{P}^2 \) in two such a way that \( \sigma(L_2) \) and \( \sigma(L_3) \) are respectively a smooth conic and its tangent line at the point \( p = \sigma(L_1) \). The birational transform \( \overline{\pi}_\lambda : \overline{S}_\lambda \to \mathbb{P}^1 \) on \( \overline{S}_\lambda \) of the pencil generated by \( \sigma(L_2) \) and \( 2\sigma(L_1) \) restricts to an \( \mathbb{A}^1 \)-fibration \( \pi_\lambda : S_\lambda \to \mathbb{P}^1 \) with two degenerate fibers: an irreducible one, of multiplicity two, consisting of the intersection with \( S_\lambda \) of the unique exceptional divisor of \( \sigma \) whose center is supported on \( \sigma(L_3) \setminus \{ p \} \), and a smooth one consisting of the intersection with \( S_\lambda \) of the four exceptional divisors of \( \sigma \) with centers supported on \( \sigma(L_2) \setminus \{ p \} \).
Families of Affine Ruled Surfaces: Existence of Cylinders

References


Institut de Mathématiques de Bourgogne, 9 avenue Alain Savary, 21 078 Dijon, France
E-mail address: adrien.dubouloz@u-bourgogne.fr

Department of Mathematics, Faculty of Science, Saitama University, Saitama 338-8570, Japan
E-mail address: tkishimo@rimath.saitama-u.ac.jp