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Matthieu Arfeux

► **To cite this version:**

Matthieu Arfeux. Compactification and trees of spheres covers. *Conformal Geometry and Dynamics*, 2017, 21, pp.225-246. 10.1090/ecgd/309 . hal-01053433v1

HAL Id: hal-01053433

<https://hal.science/hal-01053433v1>

Submitted on 31 Jul 2014 (v1), last revised 31 Oct 2015 (v2)

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Compactification and trees of spheres covers

Matthieu Arfeux

Abstract

We already saw in [A1] that the space of dynamically marked rational maps can be identified to a subspace of the space of covers between trees of spheres on which there is a notion of convergence that makes it sequentially compact. In the following we describe a topology on this space quotiented by the natural action of its group of isomorphisms. This topology corresponds to the previous convergence notion and makes this space compact.

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1 Introduction

Motivations.

Define $\mathbb{S} := \mathbb{P}^1(\mathbb{C})$ the Riemann sphere. According to the uniformisation Theorem, every compact surface of genus 0 with a projective structure is isomorphic to \mathbb{S} . For $d \geq 1$, we denote by Rat_d the set of rational maps $f : \mathbb{S} \rightarrow \mathbb{S}$

of degree d . In particular, $\text{Aut}(\mathbb{S}) := \text{Rat}_1$ is the set of Moebius transformations. This set acts on Rat_d by conjugacy :

$$\text{Aut}(\mathbb{S}) \times \text{Rat}_d \ni (\phi, f) \mapsto \phi \circ f \circ \phi^{-1} \in \text{Rat}_d.$$

We are interested in quotient rat_d of Rat_d by this action which is not a compact set.

We propose in this paper a compactification that allows to understand such behaviors. However we will not study the compactification of the set rat_d but the one of a subset consisting of conjugacy classes of rational maps marked by a given portrait. We define this notion in the following.

Let X be a finite set with at least 3 elements.

Definition (Marked sphere). *A sphere marked (by X) is an injection $x : X \rightarrow \mathbb{S}$.*

A portrait \mathbf{F} of degree $d \geq 2$ is a couple (F, deg) where

- $F : Y \rightarrow Z$ is a map between two finite sets Y and Z and
- $\text{deg} : Y \rightarrow \mathbb{N} - \{0\}$ is a function that satisfies

$$\sum_{a \in Y} (\text{deg}(a) - 1) = 2d - 2 \quad \text{and} \quad \sum_{a \in F^{-1}(b)} \text{deg}(a) = d \quad \text{for all } b \in Z.$$

Typically, $Z \subset \mathbb{S}$ is a finite set, $F : Y \rightarrow Z$ is the restriction of a rational map $F : \mathbb{S} \rightarrow \mathbb{S}$ to $Y := F^{-1}(Z)$ and $\text{deg}(a)$ is the local degree of F at a . In this case, the Riemann-Hurwitz formula and the conditions on the function deg implies that Z contains the set V_F of the critical values of F in order to let $F : \mathbb{S} - Y \rightarrow \mathbb{S} - Z$ be a cover.

Definition (Marked rational maps). *A rational map marked by \mathbf{F} is a triple (f, y, z) where*

- $f \in \text{Rat}_d$
- $y : Y \rightarrow \mathbb{S}$ and $z : Z \rightarrow \mathbb{S}$ are marked spheres,
- $f \circ y = z \circ F$ on Y and
- $\text{deg}_{y(a)} f = \text{deg}(a)$ for $a \in Y$.

If (f, y, z) is marked by \mathbf{F} , we have the following commutative diagram :

$$\begin{array}{ccc} Y & \xrightarrow{y} & \mathbb{S} \\ F \downarrow & & \downarrow f \\ Z & \xrightarrow{z} & \mathbb{S} \end{array}$$

Moreover suppose that $X \subseteq Y \cap Z$.

Definition (Dynamically marked rational map). *A rational map dynamically marked by (\mathbf{F}, X) is a rational map (f, y, z) marked by \mathbf{F} such that $y|_X = z|_X$.*

We denote by $\text{Rat}_{\mathbf{F}}$ the set of rational maps marked by \mathbf{F} and $\text{Rat}_{\mathbf{F},X}$ the set of rational maps dynamically marked by (\mathbf{F}, X) .

The group $\text{Aut}(\mathbb{S})$ acts on $\text{Rat}_{\mathbf{F}}$ by pre-composition and post-composition: a couple of Moebius transformations $(\phi, \psi) \in \text{Aut}(\mathbb{S}) \times \text{Aut}(\mathbb{S})$ maps the marked rational map $(f, y, z) \in \text{Rat}_{\mathbf{F}}$ on

$$(\phi \circ f \circ \psi^{-1}, \psi \circ y, \phi \circ z) \in \text{Rat}_{\mathbf{F}}$$

as on the following diagram:

$$\begin{array}{ccccc} Y & \xrightarrow{y} & \mathbb{S} & \xrightarrow{\psi} & \mathbb{S} \\ \downarrow F & & \downarrow f & & \downarrow \phi \circ f \circ \psi^{-1} \\ Z & \xrightarrow{z} & \mathbb{S} & \xrightarrow{\phi} & \mathbb{S} \end{array}$$

We denote by $\text{rat}_{\mathbf{F}}$ the quotient of $\text{Rat}_{\mathbf{F}}$ by the action of $\text{Aut}(\mathbb{S}) \times \text{Aut}(\mathbb{S})$.

Likewise, the group $\text{Aut}(\mathbb{S})$ acts on $\text{Rat}_{\mathbf{F},X}$ by conjugacy : a Moebius transformation $\phi \in \text{Aut}(\mathbb{S})$ maps the dynamically marked rational map $(f, y, z) \in \text{Rat}_{\mathbf{F},X}$ on

$$(\phi \circ f \circ \phi^{-1}, \phi \circ y, \phi \circ z) \in \text{Rat}_{\mathbf{F},X}.$$

We denote by $\text{rat}_{\mathbf{F},X}$ the quotient of $\text{Rat}_{\mathbf{F},X}$ by the action of $\text{Aut}(\mathbb{S})$.

According to the work of Adam Epstein and Xavier Buff, $\text{rat}_{\mathbf{F}}$ and $\text{rat}_{\mathbf{F},X}$ are smooth varieties. If $\text{card}X \geq 3$ and if $(f, y, z) \in \text{Rat}_{\mathbf{F}}$, then f is determined by the pair (y, z) . Indeed, a rational map is totally determined if we know the preimages, with multiplicities, of any triple of points. Thus $[F] \in \text{rat}_{\mathbf{F},X}$ lies naturally in the product of the moduli space of spheres marked by Y and by Z . Recall the definition of these spaces.

Definition (Moduli space). *The moduli space Mod_X is the space of spheres marked by X modulo post-composition by Moebius transformations.*

There exists a natural compactification of Mod_X introduced by Deligne and Mumford in [DM]. In the following, I explicit a compactification which is known to be equivalent to this one.

The point of view is to consider $\text{rat}_{\mathbf{F}}$ as a subspace of Mod_Y and to compactify it using this compactification. We will see that elements of this compactification can be identified to isomorphism classes of trees of spheres covers where the covers between two trees having a unique internal vertex can be identified to rational maps. We give on this compactified space an analytic structure through a totally different approach than the one exposed in [HK] for example.

Outline. In section 2 we define the set $\overline{\text{Mod}}_X$ of trees of spheres marked by X modulo a certain notion of isomorphism on trees of spheres. Considering

Quad_X , the set of quadruples of distinct elements of X we recall the embedding :

$$\mathfrak{B} : \text{Mod}_X \rightarrow \mathbb{S}^{\text{Quad}_X}.$$

This defines a natural compactification of Mod_X . We identify Mod_X with \mathbf{Mod}_X the set of trees of spheres with only one internal vertex modulo isomorphism on trees of spheres. We prove that the convergence notion that we defined in [A1] on trees of spheres agrees with this topology on Mod_X and prove the following theorem :

Theorem 1. *The space $\overline{\mathbf{Mod}}_X$ is compact as the adherence of \mathbf{Mod}_X (in $\mathbb{S}^{\text{Quad}_X}$),*

$$ie: \mathfrak{B}(\overline{\mathbf{Mod}}_X) = \text{Ad}(\mathfrak{B}(\mathbf{Mod}_X)).$$

In section 3 we define $\overline{\mathbf{rev}}_{\mathbf{F}}$ the set of covers between trees of spheres modulo a certain notion of isomorphism. We define on it a topology through the natural projection map

$$\mathbf{I} : \overline{\mathbf{rev}}_{\mathbf{F}} \rightarrow \mathbf{Mod}_Y,$$

and we prove that this topology agrees with the convergence notion defined in [A1].

We identify the set $\text{rat}_{\mathbf{F}}$ of to the set $\mathbf{rev}_{\mathbf{F}}$ of covers between elements of \mathbf{Mod}_Y and \mathbf{Mod}_Z modulo a natural notion of isomorphism to the set of marked rational maps modulo their natural isomorphism. Then we prove the following theorem :

Theorem 2. *The topological space $\overline{\mathbf{rev}}_{\mathbf{F}}$ is compact as the adherence of $\mathbf{rev}_{\mathbf{F}}$,*

$$ie: \mathbf{I}(\overline{\mathbf{rev}}_{\mathbf{F}}) = \text{Ad}(\mathbf{I}(\mathbf{rev}_{\mathbf{F}})).$$

In section 4 we define $\mathbf{dyn}_{\mathbf{F},X}$ the set of dynamical systems between trees of spheres modulo a certain notion of isomorphism and we identify $\text{rat}_{\mathbf{F},X}$ as a subset of this space. We prove that $\mathbf{dyn}_{\mathbf{F},X}$ can be identified to a subspace of the topological space $\overline{\mathbf{rev}}_{\mathbf{F}}$. With this topology we prove the following result.

Theorem 3. *The space $\mathbf{dyn}_{\mathbf{F},X}$ is compact.*

We conclude this section by looking at questions of the choice of representatives and the relation between this topology and the dynamical convergence defined in [A1]. From this study we prove the following proposition.

Proposition. *We have the following inclusions:*

$$\text{Ad}(\text{rat}_{\mathbf{F},X}) \subsetneq \mathbf{dyn}_{\mathbf{F},X} \subsetneq \overline{\mathbf{rev}}_{\mathbf{F}}.$$

Acknowledgments. I would want to thanks my advisor Xavier Buff for all the time he spent to teach me how to write and make clear my ideas.

2 Isomorphism classes of trees of spheres

2.1 Background

In this subsection we recall notions and notations introduced in [A1].

Let X be a finite set with at least 3 elements. A (projective) tree of spheres \mathcal{T} marked by X is the following data :

- a combinatorial tree T whose leaves are the elements of X (marking) and every internal vertex has at least valence 3 (stability),
- for each internal vertex v of T , an injection $i_v : E_v \rightarrow \mathcal{S}_v$ of the set of edges E_v adjacent to v into a topological sphere \mathcal{S}_v , and
- for every $v \in IV$ (internal vertex) of a projective structure on \mathcal{S}_v .

We use the notation $X_v := i_v(E_v)$ and define the map $a_v : X \rightarrow \mathcal{S}_v$ such that $a_v(x) := i_v(e)$ if x and e lie in the same connected component of $T - \{v\}$. We denote by $[v, v']$ the path between v and v' including these vertices.

A particular case is the notion of spheres marked by X defined below.

Definition 2.1 (Marked sphere). *A sphere marked (by X) is an injection*

$$x : X \rightarrow \mathbb{S}.$$

We identify trees with only one internal vertex with the marked spheres. We define the notion of convergence of a sequence of marked spheres to a marked tree of spheres as follows.

Definition 2.2 (Convergence of marked spheres). *A sequence of marked spheres $x_n : X \rightarrow \mathbb{S}_n$ converges to a tree of spheres \mathcal{T}^X if for all internal vertex v of \mathcal{T}^X , there exists a (projective) isomorphism $\phi_{n,v} : \mathbb{S}_n \rightarrow \mathbb{S}_v$ such that $\phi_{n,v} \circ x_n$ converges to a_v .*

We will use the notation $x_n \rightarrow \mathcal{T}^X$ or $x_n \xrightarrow[\phi_n]{} \mathcal{T}^X$ and we have the following property.

Lemma 2.3. *Let v and v' be two distinct internal vertices of \mathcal{T}^X (having each one at least three edges) and a sequence of marked spheres $(\mathcal{T}_n)_n$ such that $\mathcal{T}_n \xrightarrow[\phi_n]{} \mathcal{T}^X$. Then the sequence of isomorphisms $(\phi_{n,v'} \circ \phi_{n,v}^{-1})_n$ converges locally uniformly outside $a_v(v')$ to the constant $a_{v'}(v)$.*

2.2 Isomorphism of combinatorial trees and partitions

Definition 2.4 (Isomorphism of marked trees). *An isomorphism between two trees marked by X is a tree map which is bijective and restricts to the identity on X .*

Define $P_v^X := \{B_v(e) \cap X \mid e \in E_v\}$. Denote by \mathfrak{P}_X the set of partitions of X . Recall that a partition does not contain the element \emptyset .

Lemma 2.5. *For all $v \in IV^X$, P_v^X is a partition of X .*

Proof. Indeed, v is connected to every element of X by a unique path. These paths begin by an edge of E_v so we can associate to every point of X a unique element of E_v . Every branch is not empty so all the $B_v(e)$ are not empty. \square

Let ψ be the map between the set of trees marked by X and the set of partitions of \mathfrak{P}_X that maps T to

$$\psi(T) = \{P_v^X | v \in IV^X\}.$$

The goal of this section is to give a characterization of the image of this map and of the isomorphism classes of combinatorial trees marked by X .

Definition 2.6 (Admissible set of partitions). *A set \mathcal{P} of partitions is admissible if it satisfies the following properties :*

1. every partition $P \in \mathcal{P}$ contains at least three distinct elements,
2. for all partition $P \in \mathcal{P}$ and all subset $B \in P$, either there exists a partition $P' \in \mathcal{P}$ containing $X - B$, or $B = \{x\}$ with $x \in X$,
3. if $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ are two distinct partitions, then $P_1 \cap P_2 = \emptyset$.

We prove the following theorem:

Theorem 2.7. *If T is a combinatorial tree, then $\psi(T)$ is a set of admissible partitions. The trees T and T' are isomorphic if and only if $\psi(T) = \psi(T')$. Every admissible set of partition is the image of a (stable) tree.*

Corollary 2.8. *The map ψ induces a bijection between the set of isomorphism classes of trees and the set of admissible sets of partitions.*

The end of this section is the proof of this theorem.

Lemma 2.9. *Let T^X be a tree. Let $[v, v', v'']$ be a path in T^X . Then*

$$B_{v'}(v'') \cap X \subsetneq B_v(v') \cap X.$$

Proof. Define $e := \{v, v'\}$. If $x \in B_{v'}(v'') \cap X$ then $e \notin [v', x]$ so $\{v, \{v, v'\}\} \cup [v', x]$ is a path between v and x so $x \in B_v(v') \cap X$. As a tree is stable, there exists a third edge $\{v', v'''\}$ on v' . So, in the same way, $B_{v'}(v''') \cap X \subset B_v(v') \cap X$. But by definition $B_{v'}(v'') \cap X$ and $B_{v'}(v''') \cap X$ are disjoint and from the previous lemma they are not empty. Then $B_{v'}(v'') \cap X \subsetneq B_v(v') \cap X$. \square

We deduce the following properties:

Lemma 2.10. *Let T^X be a tree. Let v and v' be two vertices of T^X . Take $e \in E_v \cap [v, v']$ and $e' \in E_{v'} \cap [v, v']$. Then, for every edge $e'' \in E_{v'} - \{e'\}$, we have $B_{v'}(e'') \cap X \subsetneq B_v(e) \cap X$.*

Proof. We prove that if $[v_0, v_1, v_2]$ is a path of T^X , then

$$B_{v_1}(v_2) \cap X \subsetneq B_{v_0}(v_1) \cap X.$$

Set $e := \{v_0, v_1\}$. If $x \in B_{v_1}(v_2) \cap X$ then $e \notin [v_1, x]$ so $\{v_0, \{v_0, v_1\}\} \cup [v_1, x]$ is a path between v_0 and x so $x \in B_{v_0}(v_1) \cap X$. By stability we have a third edge $\{v_1, v_3\}$ on v_1 . Then, by the same way, $B_{v_1}(v_3) \cap X \subset B_{v_0}(v_1) \cap X$. But by definition $B_{v_1}(v_2) \cap X$ and $B_{v_1}(v_3) \cap X$ are disjoint and from the previous lemma, these are not empty sets. So we have $B_{v_1}(v_2) \cap X \subsetneq B_{v_0}(v_1) \cap X$.

The lemma follows by using this result a finite number of time on every part of the path $[v, v']$ that contains three vertices. \square

Lemma 2.11. *The set $\psi(T)$ is an admissible set of partitions.*

Proof. The property 1 is true because trees are stable.

For property 2, take such $P \in \psi(T)$ and $B \in P$. Then B is associated to an edge $e = \{v, v'\}$ on some tree internal vertex v . Either v' is a leave x , then every element of $X - \{x\}$ is connected to x by a path containing the edge e so we are in the second following case. Or v' is an internal vertex, so we are in the first case according to the inequality of lemma 2.10.

For property 3, from lemma 2.10, if two vertices are distinct, then we can find a path connecting them. Let B be an element common to P_1 and P_2 . We have three cases: either the edges associated to B are in this path, or none of them, or only one of them. In the first case we can take $B' \in P_1$ distinct to B (because T is stable) and lemma 2.10 assures that $B' \subset B$, which is absurd because P_1 is a partition. In the second case, lemma 2.10 gives $B \subsetneq (X - B)$, absurd. In the third case, lemma 2.10 gives a contradiction. \square

Take an admissible set of partitions \mathcal{P} . Define the set of vertices $V_T = \mathcal{P} \cup X$. Define the set of edges E_T as the set of $\{P_1, P_2\}$ for all $P_1 \in \mathcal{P}$ and $P_2 \in \mathcal{P}$ such that we have $B_1 \in P_1$ and $B_2 \in P_2$ satisfying $B_1 \cup B_2 = X$ with $B_1 \cap B_2 = \emptyset$ and the $\{P_0, x\}$ satisfying $P_0 \in \mathcal{P}$ and $\{x\} \in P_0$.

Lemma 2.12. *The graph T is a tree and $\psi(T) = \mathcal{P}$.*

Proof. We first prove that T is a tree.

Claim: Let $x \in X$. Every vertex $v_1 \in V_T - \{x\}$ can be connected to the vertex x by a unique path. Moreover, if the first edge of this path is $\{P_1, P_2\}$, then $x \in P_1$. (We will prove later this claim in lemma 2.13).

Then we have:

-connectivity: to connect two distinct vertices v and v' , we take $x \in X$ and the paths $[v, v_1, \dots, v_k, x]$ and $[v', v'_1, \dots, v'_{k'}, x]$. We have $v_k = v_{k'}$ because there is only one edge connecting x to a vertex of T . Consider the first common element of these paths, $v_i = v'_{i'}$. The path we were looking for is

$$[v, v_1, \dots, v_{i-1}, v_i, v'_{i'}, v'_{i'-1}, \dots, v'_{2'}, v'_{1'}].$$

-no cycles: suppose that we have a cycle $C = [v_1, v_2, \dots, v_k] \cup \{\{v_k, v_1\}\}$. The claim assures that we can find a path $[v_1, v'_2, \dots, v'_{k'}, x]$ for some $x \in X$ with

$x \neq v_1$. Let i be the biggest index such that C crosses v'_i . Define j such that $v'_i = v_j$. Thus the existences of $[v_1, v_2, \dots, v_{j-1}, v'_i, \dots, v'_{k'}, x]$ and of $[v_1, v_k, v_{k-1}, \dots, v_{j-1}, v'_i, \dots, v'_{k'}, x]$ contradicts the unicity in the claim.
 -stability: from the first property and by construction the leaves of our tree are the elements of X .

Now we prove that $\psi(T) = \mathcal{P}$. Let $v_1 \in V_T$. Denote by $P = \{p_1, \dots, p_k\}$ the associated partition at the edges $\{p_i, \star\}$ of v_1 . The last part of the claim assures that $B_{v_1}^{\{p_i, \star\}} \subseteq p_1$. But P is a partition so it is an equality. \square

Now we prove the claim.

Lemma 2.13. *Let $x \in X$. Every vertex $v_1 \in V_T - \{x\}$ can be connected to the vertex x by a unique path. Moreover, if the first edge of this path is $\{P_1, P_2\}$, then $x \in P_1$.*

Proof. We are looking for a path $[v_1, v_2, v_3, \dots, v_k, x]$.

If $v_1 \in X$ then the third property assures the existence of a vertex v_2 such that $\{v_1\}$ lies in the partition. Then we are in the case $v_1 \notin X$. We find the v_i recursively.

Recurrence hypothesis: we have find v_2, \dots, v_i such that $[v_1, v_2, \dots, v_i]$ is a path and the subset B_i of X of v_i containing x is included in the one of v_{i-1} containing x . Suppose that it is true for some $i \in \mathbb{N}$. Let B_i be this subset. If $B_i = \{x\}$ then by construction $\{v_i, \{x\}\} \in E_T$ and $[v_1, v_2, \dots, v_i, \{x\}]$ is the desired path. If not, we find v_{i+1} containing $X - B_i \in V_T$. Thus $\{v_i, v_{i+1}\} \in E_T$. If B_{i+1} is the subset of v_{i+1} containing x then B_{i+1} and $X - B_i$ are two elements of the partition v_{i+1} so we have $B_{i+1} \subset B_i$ as desired. The property is true for $i + 1$.

This construction stops because the inclusions of B_i are strict. In addition we always have $x \in B_i$. It follows that if v_k is the last vertex of the constructed path then $v_k = x$.

This path is unique because the hypothesis $B_{i+1} \subset B_i$ is necessary and induces the unicity of the vertices choices at every step.

By construction, we proved the end of the lemma. \square

We just proved that the map ψ is surjective onto the set of admissible sets of partitions.

Proof. (Theorem 2.7) It remains to prove that the quotiented map is bijective. It is sufficient to prove that the map that associate to an edge its corresponding branch behaves well to the quotient.

Take T and T' two trees in the same class, and F the bijection on the set of vertices respecting the edges. Let $v \in V_T$, $e \in E_v$ and $x \in X \cap B_v(e)$. Then if $[v, v_1, \dots, v_k, x]$ is a path, $[F(v), F(v_1), \dots, F(v_k), F(x) = x]$ is a path too so x lies in $B_{F(v)}(F(e))$.

In addition, if two marked trees have same image, the number of internal vertices and the number of their edges is the same because there is one and only one partition associated to each internal vertex. The vertices adjacent to the

vertices that are elements of X are determined by the third property. The one that are adjacent to these ones are determined by the second property and by the same way we prove that the structure connecting the vertices to the others is rigid and then that these two trees are in the same class. \square

2.3 Isomorphism of trees of spheres and topology

Definition 2.14 (Isomorphism of trees of spheres). *An isomorphism of trees of spheres marked by X is a cover between trees of spheres with degree 1 which restricts to the identity on X .*

Note that the associated map on the combinatorial trees is an isomorphism of combinatorial trees.

We define on the set \mathfrak{T}_X of trees of spheres marked by X an equivalence relation given by $\mathcal{T} \sim \mathcal{T}'$ if and only if there exists an isomorphism $\mathcal{M} : \mathcal{T} \rightarrow \mathcal{T}'$ of trees of spheres marked by X . Note that it follows that for all internal vertex v of T , $m_v : \mathbb{S}_v \rightarrow \mathbb{S}_{M(v)}$ is an isomorphism and $a_{M(v)} = m_v \circ a_v$. We will sometime use the notation $T \sim_{\mathcal{M}} T'$.

We denote by \mathfrak{T}_X the set of trees of spheres marked by X . We call moduli space of trees of spheres marked by X and denote by $\overline{\mathbf{Mod}}_X$ the quotient of the set \mathfrak{T}_X by this equivalence relation. Remark that \mathbf{Mod}_X is the set of isomorphism classes of marked spheres.

Remark 2.15. The isomorphism class of a tree of spheres with a unique internal vertex v marked by X is determined by the element $[a_v] \in \mathbf{Mod}_X$. We will do the confusion between \mathbf{Mod}_X and $\overline{\mathbf{Mod}}_X$.

Recall. The moduli space \mathbf{Mod}_X of spheres marked by X is the set of injections of X in \mathbb{S} modulo post-composition by a Moebius transformation. It is equipped with a quasi projective variety structure. Indeed, if we choose three distinct points of X , we can associate to every element of \mathbf{Mod}_X the set of their cross ratios with the other elements of X and this does not depend on the choice representatives.

For this method, the three points that we chose plays a particular role. A way to don't have this problem is to consider \mathbf{Quad}_X , the set of quadruples of distinct elements of X and to consider the embedding :

$$\mathfrak{B} : \mathbf{Mod}_X \rightarrow \mathbb{S}^{\mathbf{Quad}_X}$$

that associates to $[i] \in \mathbf{Mod}_X$ the collection of the cross ratios

$$[i(x_1), i(x_2), i(x_3), i(x_4)]_{(x_1, x_2, x_3, x_4) \in \mathbf{Quad}_X}.$$

We are going to use this approach to give to $\overline{\mathbf{Mod}}_X$ a projective variety structure.

Denote by \mathbf{Trip}_X the set of triples of distinct elements of X . Consider a combinatorial tree T marked by X . Take $t := (x_0, x_1, x_\infty) \in \mathbf{Trip}_X$. The

vertices x_0, x_1 and x_∞ are separated by a unique vertex v . We say that this vertex separates the triple t .

If T is the combinatorial tree of a tree of spheres \mathcal{T} , the map a_v maps the three elements of t to distinct images. So there exists a unique projective chart $\sigma_t : \mathbb{S}_v \rightarrow \mathbb{S}$ satisfying $\sigma_t \circ a_v(x_0) = 0, \sigma_t \circ a_v(x_1) = 1$ and $\sigma_t \circ a_v(x_\infty) = \infty$.

Definition 2.16 (t -charts). *The map σ_t is called the t -chart of T . The map*

$$\alpha_t := \sigma_t \circ a_v : X \rightarrow \mathbb{S}$$

is called the marking of the t -chart of T .

The following lemma justifies that we can talk about the t -chart of an isomorphism class of tree of spheres. We will denote it by α_t .

Lemma 2.17. *If $T \sim T'$ then for all $t \in \text{Trip}_X$ we have $\alpha_t = \alpha'_t$.*

Proof. Suppose that $T \sim_{\mathcal{M}} T'$. Let $v \in V$ and $v' \in V'$ be the vertices associated to the triple t . As it has degree 1, \mathcal{M} maps the branches on v to branches of $M(v)$ (see [A2]) but is the identity on X so $v' := M(v)$ separates the elements of t .

Let $\sigma_t : \mathbb{S}_v \rightarrow \hat{\mathbb{C}}$ satisfying $\sigma_t \circ a_v(x_\star) = \star$ and identically $\sigma'_t : \mathbb{S}_{v'} \rightarrow \hat{\mathbb{C}}$ for v' . As $\sigma'_t \circ m_v \circ \sigma_t^{-1}$ fixes three points it is the identity. For all $x \in X$, we have

$$\sigma_t \circ a_v(x) = \sigma'_t \circ m_v \circ \sigma_t^{-1} \circ \sigma_t \circ a_v(x) = \sigma'_t \circ a_v(x).$$

□

Recall that Quad_X is the set of quadruples of distinct elements of X .

Definition 2.18 (Topology). *We define the following map:*

$$\mathfrak{B}_X : \overline{\text{Mod}}_X \rightarrow \mathbb{S}^{\text{Quad}_X}$$

that maps every $[T] \in \overline{\text{Mod}}_X$ to the collection of the $(\alpha_t(x))_{(t,x) \in \text{Quad}_X}$.

The map \mathfrak{B}_X defines a topology on $\overline{\text{Mod}}_X$. We will sometime simply write \mathfrak{B} when there is no possible confusion. The following lemma implies that this topology is Hausdorff.

Lemma 2.19. *The map \mathfrak{B} is injective.*

Proof. Let T be a tree of spheres marked by X . For a fixed $t \in \text{Trip}_X$, the data of $\alpha_t(x)$ is sufficient to build the map a_v when t is separated by the vertex v of T . As trees are stables, for all vertex $v \in V_X$ we have $\text{card}(E_v) \geq 3$ and we can always find an element of Trip_X separated by v . Thus, Theorem 2.7 assures that the class of T is uniquely determined. □

Corollary 2.20. *The map \mathfrak{B} is an homeomorphism onto its image that equips Mod_X with a smooth quasi projective variety structure which is the same as the one of Mod_X (via the identification).*

Proof. Indeed Mod_X and $\mathbb{S}^{\text{Quad}_X}$ are smooth spaces and the restriction of \mathfrak{B} to Mod_X is algebraic. \square

First we show that this topology is compatible with the convergence notion.

Lemma 2.21. *Let $(\mathcal{T}_n)_n$ and $(\mathcal{T}'_n)_n$ be two sequences of spheres marked by X and let \mathcal{T} and \mathcal{T}' be two trees of spheres marked by X .*

1. (quotient)

- if $\mathcal{T} \sim \mathcal{T}'$, then $\mathcal{T}_n \rightarrow \mathcal{T} \iff \mathcal{T}_n \rightarrow \mathcal{T}'$.
- if $\mathcal{T}_n \sim \mathcal{T}'_n$, then $\mathcal{T}_n \rightarrow \mathcal{T} \iff \mathcal{T}'_n \rightarrow \mathcal{T}$.

2. (uniquity of the limit) if $\mathcal{T}_n \rightarrow \mathcal{T}$ and $\mathcal{T}_n \rightarrow \mathcal{T}'$, then $\mathcal{T} \sim \mathcal{T}'$.

Proof. If $\mathcal{T}_n \xrightarrow{\phi_n} \mathcal{T}$ and $\mathcal{T}' \sim_{\mathcal{M}} \mathcal{T}$ then $\mathcal{T}_n \xrightarrow{\phi'_n} \mathcal{T}'$ with $\phi_{n,v} = m_v \circ \phi'_{n,v}$. In addition, if $\mathcal{T}_n \sim_{\mathcal{M}} \mathcal{T}'_n \xrightarrow{\phi'_n} \mathcal{T}$ then $\mathcal{T}_n \xrightarrow{\phi'_n \circ \mathcal{M}} \mathcal{T}$ which concludes the proof of point 1.

For point 2, suppose that $\mathcal{T}_n \xrightarrow{\phi_n} \mathcal{T}$ and $\mathcal{T}_n \xrightarrow{\phi'_n} \mathcal{T}'$. For every internal vertex v of \mathcal{T} , we have $\phi'^{-1}_{n,v} \circ \phi_{n,v} \rightarrow m_v$ an isomorphism. Indeed, if we take a $t \in \text{Trip}_X$ separated by v , then $\sigma'_t \circ \phi'^{-1}_{n,v} \circ \phi_{n,v} \circ \sigma_t^{-1}$ is a Moebius transformation that fixes 0, 1 and ∞ so it is the identity. Thus $\phi'^{-1}_{n,v} \circ \phi_{n,v} \rightarrow \sigma'^{-1}_t \circ \sigma_t$ is an isomorphism. \square

Lemma 2.22. *The map \mathfrak{B} defines the same convergence notion as the one on trees of spheres on \mathbf{Mod}_X , ie :*

$$\mathcal{T}_n \rightarrow \mathcal{T} \text{ if and only if } \mathfrak{B}([\mathcal{T}_n]) \rightarrow \mathfrak{B}([\mathcal{T}]).$$

Proof. Lemma 2.21 assures that these two formulations are equivalent. Suppose that $\mathcal{T}_n \xrightarrow{\phi_n} \mathcal{T}$. Let $t \in \text{Trip}_X$. Let $x \in X$ which does not lie in t . Let $\sigma_{n,t}$ be the t -chart of \mathcal{T}_n . Let ϕ_t be the t -chart of \mathcal{T} . Let v be the vertex of \mathcal{T} defined by t . Then $m_n := \sigma_t \circ \phi'^{-1}_{n,v} \circ \sigma_{n,t}$ (cf the following diagram) is a Moebius transformation that fixes 0, 1 and ∞ so m_n is the identity.

$$\begin{array}{ccccc} X & \xrightarrow{a_n} & \mathbb{S}_n & \xrightarrow{\phi_{n,t}} & \mathbb{S} \\ & \searrow a_v & \downarrow \phi_{n,v} & & \downarrow m_n \\ & & \mathbb{S}_v & \xrightarrow{\phi_t} & \mathbb{S} \end{array}$$

Then we have

$$\sigma_{n,t} \circ a_n(x) = m_n \circ \sigma_{n,t} \circ a_n(x) = \sigma_v \circ \phi_{n,v} \circ a_n(x) \rightarrow \sigma_t \circ a_v(x).$$

Thus $\alpha_{n,t} \rightarrow \alpha_t$ so $\mathfrak{B}([\mathcal{T}_n]) \rightarrow \mathfrak{B}([\mathcal{T}])$.

If in addition $\mathfrak{B}([\mathcal{T}_n]) \rightarrow \mathfrak{B}([\mathcal{T}])$, for all internal vertex v of \mathcal{T} denote by t_v a triple that defines v and σ_{n,t_v} the t_v -chart of \mathcal{T}_n . Define $\phi_{n,v} := \sigma_{n,t_v}^{-1} \circ \sigma_{t_v}$. Then we have $\phi_{n,v} \circ a_n \rightarrow a_v$. \square

Remark 2.23 (Convergence of trees). Let $(\mathcal{T}_n)_n$ be a sequence of trees marked by X . For all $t \in \text{Trip}_Y$ we denote by $v_{n,t}$ the vertex of \mathcal{T}_n^Y separating t .

Let $\mathcal{T} \in \overline{\mathbf{Mod}}_X$. By the definition of \mathfrak{B} we know that $(\mathcal{T}_n)_n$ converges to \mathcal{T} if

$$\forall t \in \text{Trip}_X, \exists \phi_{n,v_{n,t}}^X \in \text{Aut}(\mathbb{S}_{v_{n,t}}, \mathbb{S}_v), \phi_{n,v_{n,t}}^X \circ a_{n,v_{n,t}} \rightarrow a_v.$$

Notation. We will use the notation $\phi_{n,t}^X := \phi_{n,v_{n,t}}^X$.

2.4 Compactness, projective variety

In this section we prove the following theorem:

Theorem 2.24. *The space $\overline{\mathbf{Mod}}_X$ is the adherence of \mathbf{Mod}_X (in $\mathbb{S}^{\text{Quad}_X}$), ie:*

$$\mathfrak{B}(\overline{\mathbf{Mod}}_X) = \text{Ad}(\mathfrak{B}(\mathbf{Mod}_X)).$$

The proof of this result will be divided in two inclusions (lemmas 2.26 and 2.29). We deduce the following corollary :

Corollary 2.25. *The topological space $\overline{\mathbf{Mod}}_X$ is compact, it is the adherence of \mathbf{Mod}_X .*

Proof. Indeed, $\mathfrak{B}(\overline{\mathbf{Mod}}_X)$ is closed in a compact set so it is compact. \square

This compactification corresponds to the one of Deligne-Mumford (in [DM]), it is exposed in a closer way in [B] for example. For the following we will do the confusion by calling it the Deligne-Mumford compactification. It is known that $\mathfrak{B}(\overline{\mathbf{Mod}}_X)$ is a smooth projective sub variety so it is equipped with a smooth projective variety structure (which is not described in this paper).

Lemma 2.26. *The set \mathbf{Mod}_X is dense in $\overline{\mathbf{Mod}}_X$. In particular we have*

$$\mathfrak{B}(\overline{\mathbf{Mod}}_X) \subseteq \text{Ad}(\mathfrak{B}(\mathbf{Mod}_X)).$$

In order to prove this, we use the notion of convex hull :

Definition 2.27 (Convex hull). *For every combinatorial tree T and every set of vertices $V' \subset T$, the convex hull of V' is the sub tree consisting in the paths connecting the elements of V' .*

Note that it is the smallest subtree of T containing V' (connected hull).

Proof. By lemma 2.22, the two formulations are equivalents: it is sufficient to show that every tree of spheres marked by X is the limit of spheres marked by X . Define $X = \{x_1, x_2, \dots, x_{n_0}\}$. For $3 \leq k \leq n_0$, define by $X_k := \{x_1, \dots, x_k\}$ and denote by Conv_k the set of vertices of valence greater then 3 of the convex hull of X_k in \mathcal{T} . We prove by recurrence on k that we can find a sequence of spheres $(\mathcal{T}_n)_n$ marked by X_k and for all internal vertex $v \in \text{Conv}_k$ a sequence

of isomorphisms $\phi_{n,v} : \mathbb{S}_n \rightarrow \mathbb{S}_v$ such that $\phi_{n,v} \circ a_n \rightarrow a_v$ where a_n will always denote the marking of \mathbb{S}_n which is the sphere of the internal vertex of \mathcal{T}_n .

If $k = 3$, Conv_k has a unique vertex v . Take for all $n \in \mathbb{N}$ a sphere equipped with a complex structure \mathbb{S}_n and some injection $a_n : X_k \rightarrow \mathbb{S}_n$. As X_k has only three elements, there exists a unique isomorphism $\phi_{n,v} : \mathbb{S}_n \rightarrow \mathbb{S}_v$ such that $\phi_{n,v} \circ a_n$ and a_v are equal on X_k . Thus we have $\phi_{n,v} \circ a_n \rightarrow a_v$.

Suppose that the property is true for a given k with $3 \leq k < n_0$. Denote by $(\mathbb{S}_n)_n$ and $(\phi_{n,v})_{n \in \mathbb{N}, v \in \text{Conv}_k}$ the sequences given by the recursive property. Let v_0 be the vertex of Conv_{k+1} which is the closest to x_{k+1} (counting the number of vertices in $[v_0, x_{k+1}]$).

If $v_0 \in \text{Conv}_k$ then $\text{Conv}_k = \text{Conv}_{k+1}$. Define

$$U := \bigcup_{n \in \mathbb{N}} \phi_{n,v_0} \circ a_n(X_k).$$

As $\phi_{n,v} \circ a_n \rightarrow a_v$ and $B_{v_0}(x_{k+1}) \cap X_k = \emptyset$, U has a finite number of elements in a small enough neighborhood of $a_{v_0}(x_{k+1})$. Then we can find a sequence $(\zeta_n)_n$ of elements of $\mathbb{S}_{v_0} - U$ such that $\zeta_n \rightarrow a_{v_0}(x_{k+1})$. We define $a'_n : X_{k+1} \rightarrow \mathbb{S}_n$ equal to a_n on X_k and such that $a'_n(x_{k+1}) := \zeta_n$. As $\zeta_n \notin U$, the map a_n is an injection and we have $\phi_{n,v} \circ a'_n(x_{k+1}) \rightarrow a_v(x_{k+1})$. In addition lemma 2.3 assures that for every other vertex of Conv_k we have $\phi_{n,v} \circ a'_n(x_{k+1}) \rightarrow a_v(x_{k+1})$.

If $v_0 \notin \text{Conv}_k$, then either v_0 lies on a path between two spheres or there exists a leaf $x \in X_k$ such that x and v_0 are adjacent.

In the first case, take these two spheres $v_1 \in B_1, v_2 \in B_2$ of Conv_k where B_1 and B_2 are two branches on v_0 (cf Figure 1). Define $X^i = B_i \cap X_k$. We know that v_1 lies in a path $[z_1, z'_1]$ with $z_1, z'_1 \in X^1$ and that v_2 lies in a path $[z_2, z'_2]$. We define the triples $t_1 := (z_1, z'_1, z_2)$ separated by the sphere v_1 and $t_2 := (z_2, z'_2, z_1)$ separated by the sphere v_2 . Recall that σ_{t_\star} is the t_\star -chart of \mathcal{T} . If we define $M_n := \sigma_{t_2} \circ \phi_{n,v_2} \circ \phi_{n,v_1}^{-1} \circ \sigma_{t_1}^{-1}$, from the choices of t_\star we have $\forall \xi \in \hat{\mathbb{C}}, M_n(\xi) = \lambda_n/\xi$ with $\lambda_n \rightarrow \infty$.

Define

$$U_1 := \bigcup_{n \in \mathbb{N}} \sigma_{t_1} \circ \phi_{n,v_1} \circ a_n(X_k),$$

and $\xi_n := \sqrt{\lambda_n} + \varepsilon$ with $\varepsilon \in \mathbb{C}$ independent of n and chosen such that $(\xi_n)_n$ avoids U_1 . We define $a_n(x_{k+1}) := \phi_{n,v_1}^{-1} \circ \sigma_{t_1}^{-1}(\xi_n)$. (By definition, we have $\phi_{n,v_1} \circ a_n(x_{k+1}) \rightarrow a_{v_2}(x_{k+1})$ and $M_n(\xi_n) = \lambda_n/(\sqrt{\lambda_n} + \varepsilon) \rightarrow \infty$, we have $\phi_{n,v_2} \circ a_n(x_{k+1}) \rightarrow a_{v_2}(x_{k+1})$ too.)

Let $\phi_{n,v_0} : \mathbb{S}_n \rightarrow \mathbb{S}_{v_0}$ be the unique isomorphism such that

$$\phi_{n,v_0}(a_n(z_1)) = a_{v_0}(z_1), \quad \phi_{n,v_0}(a_n(z_2)) = a_{v_0}(z_2) \quad \text{and} \quad \phi_{n,v_0}(a_n(x)) = a_{v_0}(x).$$

Soit $t := (z_1, x, z_2)$ and σ_t the t -chart of \mathcal{T} . Define $N_n := \sigma_{t_0} \circ \phi_{n,v_0} \circ \phi_{n,v_1}^{-1} \circ \sigma_{t_1}^{-1}$. We note that $\forall \xi \in \hat{\mathbb{C}}, N_n(\xi) = \xi/(\xi_n)$. As for all $x \in X^1, \sigma_{t_1} \circ \phi_{n,v_1} \circ a_n(x)$ converges to a finite limit, we have

$$\sigma_t = \phi_{n,v_0}(a_n(x)) = N_n(\sigma_{t_1} \circ \phi_{n,v_1} \circ a_n(x)) \rightarrow 0 = \sigma_t \circ a_{v_0}(x).$$

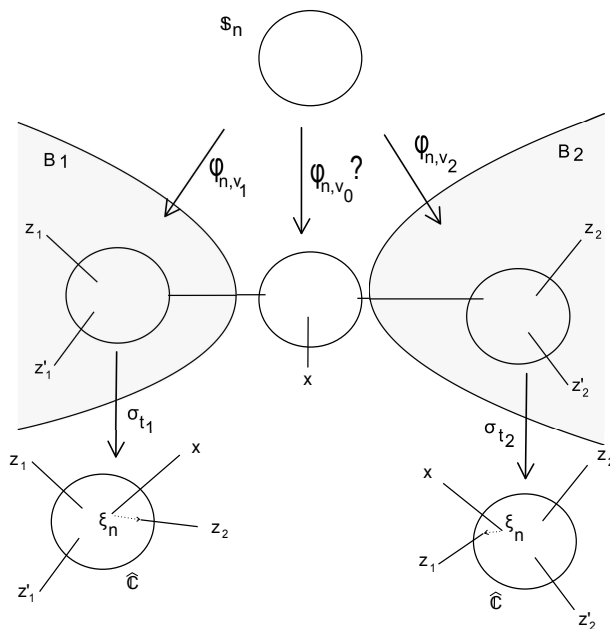


Figure 1:

By the same kind of considerations on v_2 , we prove that for every $v \in \text{Conv}_k$, from lemma 2.3, we have $\phi_{n,v} \circ a'_n(x_{k+1}) \rightarrow a_v(x_{k+1})$.

If there exists a leaf $x \in X_k$ such that x and v_0 are adjacent, then v_0 is adjacent to a unique internal vertex v_1 of Conv_k and separates the vertices x, v_0 and v_1 . We define $a_n(x_{k+1})$ as a sequence such that $\phi_{n,v_1} \circ a_n(x_{k+1}) \rightarrow \phi_{n,v_1}(x_k)$ and such that $a_n|_{X_{k+1}}$ is injective. We conclude as before by taking ϕ_{n,v_0} the unique isomorphism mapping the attaching points on \mathbb{S}_n of the branches containing x, x_k and x' to the one of \mathbb{S}_v . \square

Remark 2.28. This lemma can be proven by gluing spheres minus a finite number of points. This other method is called a "plumbing". We will use it for example in the proof of proposition 3.18.

Lemma 2.29. *The set $\mathfrak{B}(\overline{\text{Mod}}_X)$ is closed and*

$$\text{Ad}(\mathfrak{B}(\text{Mod}_X)) \subseteq \mathfrak{B}(\overline{\text{Mod}}_X).$$

Proof.

Let $(\mathcal{T}_n)_n$ be some sequence of spheres marked by X . For every $t \in \text{Trip}_X$, we denote by $\sigma_{n,t}$ the t -chart of \mathcal{T}_n , then we have $\sigma_{t,n} \circ a_{n,t}$ converges to a map that we will denote by $a_t : X \rightarrow \hat{\mathcal{C}}$.

Every a_t defines a partition P_t of X which are classes of the following equivalence relation: $x \sim x'$ if and only if $a_t(x) = a_t(x')$. We prove that the set \mathcal{P} of the P_t for $t \in \text{Trip}_X$ is an admissible set of partitions.

-Property 1. Elements of t have distinct images so P_t contains at least three elements.

-Property 2. Take $P_t \in \mathcal{P}$ and $B \in P_t$. By definition, for all element $x \notin B$, we have $a_t(x) \notin a_t(B) = \{\star\}$. Let t_0 be a triple of points with at least two elements of $X - B$. From lemma 2.3, the sphere P_{t_0} have a set B_0 containing $X - B$. If $B_0 = X - B$ then we are done. If not, $B_0 \cap B \neq \emptyset$. Then we consider an other triple $t_1 \in (X - B) \times (B_0 \cap B) \times B$ that contains an edge B_1 containing $X - B$ but such that $\text{card}(B_1) < \text{card}(B_0)$. We continue until that $\text{card}(B_i) = \text{card}(X - B)$.

-Property 3. We first note that if t is a triple of elements of X in distinct subsets of $P_{t'}$, then we have $P_t = P_{t'}$. Take t_1 and t_2 such that $B \in P_{t_1} \cap P_{t_2}$ is non empty. Suppose by contradiction that $P_{t_1} \neq P_{t_2}$ but $B \in P_{t_1} \cap P_{t_2}$. Then we can find $x_1, x_2 \in B_2 \in P_{t_2}$ such that x_1 and x_2 are in distinct elements of P_{t_1} . As P_{t_2} has at least three elements we take $x_3 \notin B_2$. Take $x_B \in B$. Define $t'_2 := (x_1, x_3, x_B)$ and $t'_1 = (x_1, x_2, x_3)$. According to the preceding remark, we have $P_{t_1} = P_{t'_1}$ and $P_{t_2} = P_{t'_2}$. From lemma 2.3, as $a_{n,t'_2}(x_1)$ and $a_{n,t'_2}(x_2)$ tend to the same limit, $a_{n,t'_1}(x_3)$ and $a_{n,t'_1}(x_4)$ too. As $x_4 \in B$ then we have $x_3 \in B$ which is a contradiction.

According to corollary 2.8, the set \mathcal{P} determines a unique combinatorial tree (up to isomorphism) and, at each of its vertices, the associated partition corresponds to the associated partition at an a_t . Fix a combinatorial tree T in this isomorphism class and for each of its internal vertices v a triple v_t such that the partition of a_{t_v} corresponds to the partition of v . Define $\phi_{n,v} = a_{n,t_v}$ and $\mathbb{S}_v = \mathbb{S}$ for every $v \in IV$. The tree T equipped to the spheres \mathbb{S}_v and the $a_v := a_{t_v}$ is a tree of spheres \mathcal{T} and by construction we have $\mathcal{T}_n \rightarrow_{\phi_n} \mathcal{T}$. \square

3 Isomorphism classes of covers

3.1 Background

In this subsection we recall notions and notations introduced in [A1].

In the same spirit we generalized the notion of rational maps marked by a portrait defined below :

Definition 3.1 (Marked rational maps). *A rational map marked by \mathbf{F} is a triple (f, y, z) where*

- $f \in \text{Rat}_d$
- $y : Y \rightarrow \mathbb{S}$ and $z : Z \rightarrow \mathbb{S}$ are marked spheres,
- $f \circ y = z \circ F$ on Y and
- $\deg_{y(a)} f = \deg(a)$ for $a \in Y$.

Where a portrait \mathbf{F} of degree $d \geq 2$ is a pair (F, deg) such that

- $F : Y \rightarrow Z$ is a map between two finite sets Y and Z and

- $\deg : Y \rightarrow \mathbb{N} - \{0\}$ is a function that satisfies

$$\sum_{a \in Y} (\deg(a) - 1) = 2d - 2 \quad \text{and} \quad \sum_{a \in F^{-1}(b)} \deg(a) = d \quad \text{for all } b \in Z.$$

If (f, y, z) is marked by \mathbf{F} , we have the following commutative diagram :

$$\begin{array}{ccc} Y & \xrightarrow{y} & \mathbb{S} \\ F \downarrow & & \downarrow f \\ Z & \xrightarrow{z} & \mathbb{S} \end{array}$$

Typically, $Z \subset \mathbb{S}$ is a finite set, $F : Y \rightarrow Z$ is the restriction of a rational map $F : \mathbb{S} \rightarrow \mathbb{S}$ to $Y := F^{-1}(Z)$ and $\deg(a)$ is the local degree of F at a . In this case, the Riemann-Hurwitz formula and the conditions on the function \deg implies that Z contains the set V_F of the critical values of F so that $F : \mathbb{S} - Y \rightarrow \mathbb{S} - Z$ is a cover.

The generalization of marked rational maps is the notion of (holomorphic) cover between trees of spheres. A cover $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ between two trees of spheres marked by Y and Z is the following data

- a map $F : T^Y \rightarrow T^Z$ mapping leaves to leaves, internal vertices to internal vertices, and edges to edges,
- for each internal vertex v of T^Y and $w := F(v)$ of T^Z , an holomorphic ramified cover $f_v : \mathcal{S}_v \rightarrow \mathcal{S}_w$ that satisfies the following properties:
 - the restriction $f_v : \mathcal{S}_v - Y_v \rightarrow \mathcal{S}_w - Z_w$ is a cover,
 - $f_v \circ i_v = i_w \circ F$,
 - if e is an edge between v and v' , then the local degree of f_v at $i_v(e)$ is the same as the local degree of $f_{v'}$ at $i_{v'}(e)$.

We saw that a cover between trees of spheres \mathcal{F} has a global degree, denoted by $\deg(\mathcal{F})$.

We define the notion of convergence of a sequence of marked spheres covers to marked cover between trees of spheres as follows.

Definition 3.2 (Non dynamical convergence). *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres of portrait \mathbf{F} . A sequence $\mathcal{F}_n := (f_n, a_n^Y, a_n^Z)$ of marked spheres covers converges to \mathcal{F} if their portrait is \mathbf{F} and if for all pair of internal vertices v and $w := F(v)$, there exists sequences of isomorphisms $\phi_{n,v}^Y : \mathbb{S}_n^Y \rightarrow \mathbb{S}_v$ and $\phi_{n,w}^Z : \mathbb{S}_n^Z \rightarrow \mathbb{S}_w$ such that*

- $\phi_{n,v}^Y \circ a_n^Y : Y \rightarrow \mathbb{S}_v$ converges to $a_v^Y : Y \rightarrow \mathbb{S}_v$,
- $\phi_{n,w}^Z \circ a_n^Z : Z \rightarrow \mathbb{S}_w$ converges to $a_w^Z : Z \rightarrow \mathbb{S}_w$ and
- $\phi_{n,w}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1} : \mathbb{S}_v \rightarrow \mathbb{S}_w$ converges locally uniformly outside Y_v to $f_v : \mathbb{S}_v \rightarrow \mathbb{S}_w$.

We use the notation $\mathcal{F}_n \rightarrow \mathcal{F}$ or $\mathcal{F}_n \xrightarrow{(\phi_n^Y, \phi_n^Z)} \mathcal{F}$.

Recall some properties of these convergences.

Lemma 3.3. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres with portrait \mathbf{F} and of degree D . Let $v \in IV^Y$ with $\deg(v) = D$ and let $\mathcal{F}_n := (f_n, a_n^Y, a_n^Z)$ be a sequence of covers between trees of spheres that satisfies $\mathcal{F}_n \xrightarrow{\phi_n^Y, \phi_n^Z} \mathcal{F}$. Then*

the sequence $\phi_{n, F(v)}^Z \circ f_n \circ (\phi_{n, v}^Y)^{-1} : \mathbb{S}_v \rightarrow \mathbb{S}_{F(v)}$ converges uniformly to $f_v : \mathbb{S}_v \rightarrow \mathbb{S}_{F(v)}$.

3.2 Isomorphisms of covers between trees

Definition 3.4 (Isomorphism between covers). *An isomorphism between two covers between trees of spheres $\mathcal{F}^1 : \mathcal{T}_1^Y \rightarrow \mathcal{T}_1^Z$ and $\mathcal{F}^2 : \mathcal{T}_2^Y \rightarrow \mathcal{T}_2^Z$ is a couple of isomorphisms between trees of spheres $(\mathcal{M}^Y, \mathcal{M}^Z)$ such that:*

- $\mathcal{T}_1^Y \sim_{\mathcal{M}^Y} \mathcal{T}_2^Y$ and $\mathcal{T}_1^Z \sim_{\mathcal{M}^Z} \mathcal{T}_2^Z$;
- for all the vertices $v_1 \in \mathcal{T}_1^Y$, $v_2 := \mathcal{M}^Y(v_1) \in \mathcal{T}_2^Y$, $w_1 := \mathcal{F}^1(v_1) \in \mathcal{T}_1^Z$ and $w_2 := \mathcal{F}^2(v_2) \in \mathcal{T}_2^Z$, the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{S}_{v_1} & \xrightarrow{m_{v_1}^Y} & \mathbb{S}_{v_2} \\
 f_{v_1} \downarrow & & \downarrow f_{v_2} \\
 \mathbb{S}_{w_1} & \xrightarrow{m_{w_1}^Z} & \mathbb{S}_{w_2}
 \end{array}$$

Thus we write $\mathcal{F}^1 \sim \mathcal{F}^2$ or $\mathcal{F}^1 \sim_{(\mathcal{M}^Y, \mathcal{M}^Z)} \mathcal{F}^2$. As \mathcal{M}^Y and \mathcal{M}^Z are invertible, it is an equivalence relation. Equivalence classes of this relation are called Isomorphism classes of covers between tree of spheres.

Note that two covers between trees of spheres which are isomorphic have same degree. Thus we can talk about the degree of an isomorphism class of covers between trees of spheres. On the same way, all the covers in a same class have same portrait, thus we can talk about the portrait of an isomorphism class of covers between trees of spheres.

Notation. We will denote by $\overline{\mathbf{Rev}}_{\mathbf{F}}$ the set of covers between trees of spheres \mathcal{F} with portrait $\mathbf{F} = (F|_Y, \deg|_Y)$ and $\mathbf{Rev}_{\mathbf{F}}$ the set of covers between two trees that have a unique internal vertex (we respectively talk about covers between trees of spheres marked by \mathbf{F} and of covers between spheres marked by \mathbf{F}). We denote by $\overline{\mathbf{rev}}_{\mathbf{F}}$ the quotient of $\overline{\mathbf{Rev}}_{\mathbf{F}}$ by this equivalence relation and $\mathbf{rev}_{\mathbf{F}}$ the one of $\mathbf{Rev}_{\mathbf{F}}$.

3.3 Marked covers, projections and topology

Recall that \mathfrak{T}_X denote the set of trees of spheres marked by X . Define

$$\mathcal{I} : \overline{\mathbf{Rev}}_{\mathbf{F}} \rightarrow \mathfrak{T}_Y \times \mathfrak{T}_Z$$

that associate $(\mathcal{T}^Y, \mathcal{T}^Z)$ to $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$. We prove the following proposition by recurrence on the cardinal of Y .

Proposition 3.5. *The map $\mathcal{I} : \overline{\mathbf{Rev}}_{\mathbf{F}} \rightarrow \mathfrak{T}_Y \times \mathfrak{T}_Z$ is an injection. It can be naturally quotiented to an injective map*

$$[I] : \overline{\mathbf{rev}}_{\mathbf{F}} \rightarrow \mathbf{Mod}_Y \times \mathbf{Mod}_Z.$$

This proposition proof follows essentially from the fact that two maps from the Riemann sphere to itself such that preimages of three distinct points coincide (with multiplicity) are equals. First we prove the following lemma:

Lemma 3.6. *Every tree of sphere is either a marked sphere or it has an internal vertex which is adjacent to exactly one other one.*

Proof. Indeed, consider a leaf and a path from this leaf which has a maximal number of edges. If this path is empty, this means that there is only one vertex and we don't have to consider this case. It is the same for the case where the tree has only two vertices. Suppose that we are not in these cases.

Then the path has the form

$$C = [v_1, v_2, \dots, v_{k-1}, v_k].$$

with $v_{k-1} \neq v_1$. Note that v_k is necessarily a leaf because, if not, it will have an edge connecting it to an other vertex that allows to extend the path. If v_{k-1} does not satisfies the property then v_{k-1} is adjacent to an other internal vertex v'_k that doesn't lie in the path. As this one is an internal vertex, it is adjacent to a vertex v'_{k+1} too that doesn't lie in the path. Then $C' = [v_1, v_2, \dots, v_{k-1}, v'_k, v'_{k+1}]$ would be a path longer than C . Thus v_{k-1} satisfies the desired property. \square

For $\mathcal{T}' \subset \mathcal{T}$ non empty, the tree of sphere $\overline{\mathcal{T}'}$ design the natural smallest subtree of \mathcal{T} containing \mathcal{T}' (cf [A1] for more details). The main ingredient for the proof of Proposition 3.5 will be the following lemma proved in [A1].

Lemma 3.7. *Let $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ be a cover between trees of spheres. Let T'' be an open, non empty and connected subset of T^Z and let T' be a connected component of $F^{-1}(T'')$. Then the map $\overline{\mathcal{F}} : \overline{\mathcal{T}'} \rightarrow \overline{\mathcal{T}''}$ defined by*

- $\overline{F} := F : \overline{\mathcal{T}'} \rightarrow \overline{\mathcal{T}''}$ and
- $\overline{f}_v := f_v$ if $v \in V' - Y'$

is a cover between trees of spheres.

Proof. (Proposition 3.5) As we said before, we prove this result by induction on the cardinal of Y .

We begin with the case $\text{card}(Y) = 3$. Take $\mathcal{F} \in \overline{\mathbf{Rev}}_{\mathbf{F}}$ with $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$, we prove that \mathcal{F} is uniquely determined by $\mathcal{I}(\mathcal{F})$. If Y has only three elements then T^Y has a unique internal vertex v . Then T^Z has only one internal vertex v' which is the image of v . The combinatorial tree map is well uniquely determined. Moreover, as Z has three elements and as we know all their preimages, we know the preimages of three attaching points of three edges on $\mathbb{S}_{v'}$ by f_v . So f_v is uniquely determined too.

Let Y' be a set of cardinal $n > 3$. Suppose that \mathcal{I} is injective for every set Y satisfying $\text{card}(Y) < n$. Now we prove it for the case $Y = Y'$. Take $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ in $\overline{\mathbf{Rev}}_{\mathbf{F}}$. Suppose that we know $(\mathcal{T}^Y, \mathcal{T}^Z)$ and we prove that \mathcal{F} is uniquely determined.

If T^Y has only one internal vertex then we do the same proof as before. We suppose that it is not the case. According to lemma 3.6, T^Y has an internal vertex w_0 adjacent to a unique internal vertex. Let y be a leaf adjacent to w (it exists because T^Y is stable). The image of w_0 is necessarily adjacent to $z := F(y)$ which is a leaf; $v := F(w_0)$ is uniquely determined. As there are more than one internal vertices, v is adjacent to an internal vertex v' . The preimages of v are the vertices adjacent to the preimages of the z . Identically the one of v' are all the internal vertices (if not v' would be a leaf) adjacent to v . Thus the preimages of v and v' are uniquely determined.

Now suppose that w is a preimage of v . Given that T^Z is stable, v is adjacent to a $z'' \in Z - \{z\}$. So we know the preimages of two of its points by f_w . Define $e := \{v, v'\}$. As we know the preimages of v' and of v , the preimages by f_w of the attaching point of e on v are the attaching points on w of the edges of w connecting w to some internal vertices. As we know the preimages of three distinct points of v by f_w , the map f_w is uniquely determined.

Thus the preimage by F of $B := B_{v'}(e)$ is uniquely determined. Define $T'' := V^Z - B$ and $T' := T^Y - F^{-1}(B) = F^{-1}(T'')$. Now we prove that $F|_{T'}$ is uniquely determined. By lemma 3.7, we construct a cover between trees of spheres $\overline{\mathcal{F}} : \overline{\mathcal{T}}' \rightarrow \overline{\mathcal{T}}''$. The leaves of $\overline{\mathcal{T}}'$ which are not leaves of T^Y are elements of $F^{-1}(v)$ and on the internal vertices of $v \in \overline{\mathcal{T}}'$ we have $\overline{F}(v) = F(v)$ and $\overline{f}_v = f_v$. However $\overline{\mathcal{T}}'$ is a tree marked by some elements of Y and some preimages of v so we know the portrait of $F|_{\overline{\mathcal{T}}'}$. Moreover $\text{card}(B \cap Z) \geq 2$ and the elements of this set are not leaves of $\overline{\mathcal{T}}''$ so $\overline{\mathcal{T}}''$ has at most $\text{card}(Z) - 1$ leaves and $\overline{\mathcal{T}}'$ has at most $n - 1$ leaves. So the induction property assures that $F|_{\overline{\mathcal{T}}'}$ (and then $F|_{T'}$) is uniquely determined.

Thus \mathcal{F} is uniquely determined by $\mathcal{I}(\mathcal{F})$ on $F^{-1}(B)$ and all the connected components of its complementary set. \square

Denote by π_1 the projection on the first coordinate.

Definition 3.8 (Topology). *We define the map*

$$\mathbf{I} : \overline{\mathbf{rev}}_{\mathbf{F}} \rightarrow \mathbf{Mod}_Y \text{ by setting } \mathbf{I} := \pi_1 \circ [I].$$

Proposition 3.9. *The map $\mathbf{I} : \overline{\text{rev}}_{\mathbf{F}} \rightarrow \mathbf{Mod}_Y$ is injective.*

Proof. Take $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ a cover between tree of sphere. Let v_0 be a vertex given by lemma 3.6. Let v'_0 be its image. Let V_0 be the set of the leaves adjacent to v_0 . The portrait (\mathbf{F}, deg) determines the images of the elements of V_0 that have to be adjacent to v'_0 . The other preimages of v_0 are adjacent to the elements of $\mathbf{F}^{-1} \circ \mathbf{F}(V_0)$. As we did in the last proof, lemma 3.7 allows to determine the vertices of the tree T^Z and the map F from the data of $T^Y - F^{-1}(v'_0)$ by induction on the number of vertices of T^Y .

Thus, it is possible to reconstruct the combinatorial tree T^Z and the combinatorial tree map from \mathcal{T}^Y . We prove that the attaching points of the edges of T^Z on the vertices of \mathcal{T}^Z are well determined up to post-composition by automorphisms. For this, it is sufficient to show that for each internal vertex of T^Z , the attaching points of all the edges on this vertex are determined by the data of three of them.

For every internal vertex v of T^Z , we suppose that we know the attaching points z_0, z_1 and z_∞ of three distinct edges e_0, e_1, e_∞ on v . For every preimage w of v , there exists a unique holomorphic cover $f_w : \mathbb{S}_w \rightarrow \mathbb{S}_v$ mapping the preimages of the edge e_0 (resp. e_1, e_∞) on z_0 (resp. z_1, z_∞). If e is an edge on v then e has a preimage e' on w so its attaching point has to be $f_w(e'_w)$. \square

We define a topology on the set of isomorphism classes of the covers between trees of spheres with the map \mathbf{I} . Below we show that this topology is compatible with the previous one.

Lemma 3.10. *Let (f_n) and (f'_n) be two sequences of spheres marked by \mathbf{F} and let \mathcal{F} and \mathcal{F}' be two trees of spheres marked by X .*

1. (quotient)

- if $\mathcal{F} \sim \mathcal{F}'$, then $f_n \rightarrow \mathcal{F} \iff f_n \rightarrow \mathcal{F}'$.
- if $f_n \sim f'_n$, then $f_n \rightarrow \mathcal{F} \iff f'_n \rightarrow \mathcal{F}$.

2. (uniquity of the limit) if $f_n \rightarrow \mathcal{F}$ and $f_n \rightarrow \mathcal{F}'$, then $\mathcal{F} \sim \mathcal{F}'$.

Proof. If $f_n \xrightarrow{(\phi_n^Y, \phi_n^Z)} \mathcal{F}$ and $\mathcal{F} \sim_{(\mathcal{M}^Y, \mathcal{M}^Z)} \mathcal{F}'$ then

$$f_n \xrightarrow{(\psi_n^Y, \psi_n^Z)} \mathcal{F}' \text{ with } \psi_{n,v}^* := M_v^* \circ \phi_{n,v}^*.$$

Moreover, suppose that $f'_n \sim_{(\mathcal{M}_n^Y, \mathcal{M}_n^Z)} f_n$,

$$\text{if } f_n \xrightarrow{(\phi_n^Y, \phi_n^Z)} \mathcal{F} \text{ then } f'_n \xrightarrow{(\mathcal{M}^Y \circ \phi_n^Y, \mathcal{M}^Y \circ \phi_n^Z)} \mathcal{F}.$$

So this convergence notion well behave in the quotient.

For point 2, we suppose that $f_n \xrightarrow{(\phi_n^Y, \phi_n^Z)} \mathcal{F}$ and $f_n \xrightarrow{(\psi_n^Y, \psi_n^Z)} \mathcal{F}'$ then

$$\mathcal{F} \sim_{(\mathcal{M}^Y, \mathcal{M}^Z)} \mathcal{F}' \text{ with } m_v^* := \lim_{n \rightarrow \infty} \psi_{n,v}^* \circ (\phi_{n,v}^*)^{-1}.$$

Indeed, $a_{n,v}^*$ tends to a_v^* and $(\psi_{n,v}^* \circ (\phi_{n,v}^*)^{-1})^* \circ a_{n,v}^*$ tends to a_v^{t*} on Y which contains at least three points so m_v is an isomorphism. \square

Corollary 3.11. *The convergence notion defined on $\overline{\mathbf{Rev}}_{\mathbf{F}}$ implies the one given by the topology given by \mathbf{I} :*

$$\text{if } f_n \rightarrow \mathcal{F} \text{ then } \mathbf{I}([f_n]) \rightarrow \mathbf{I}([\mathcal{F}]).$$

Proof. Indeed, if $(f_n : \mathcal{T}_n^Y \rightarrow \mathcal{T}_n^Z) \rightarrow (\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z)$, then by definition we have $\mathcal{T}_n^Y \rightarrow \mathcal{T}^Y$, ie $I(f_n) \rightarrow I(\mathcal{F})$ so in the quotient $\mathbf{I}([f_n]) \rightarrow \mathbf{I}([\mathcal{F}])$. \square

We will prove the reciprocal property in the following section.

3.4 Compactness

The aim of this section is to prove the following result:

Theorem 3.12. *The topological space $\overline{\mathbf{rev}}_{\mathbf{F}}$ is the adherence of $\mathbf{rev}_{\mathbf{F}}$:*

$$\mathbf{I}(\overline{\mathbf{rev}}_{\mathbf{F}}) = \text{Ad}(\mathbf{I}(\mathbf{rev}_{\mathbf{F}})).$$

Thus the space $\overline{\mathbf{rev}}_{\mathbf{F}}$ is a compact space that is injected in a product of \mathbb{P}^1 . To be more precise, we will prove that the map \mathbf{I} is an homeomorphism onto its image and that $\mathbf{rev}_{\mathbf{F}}$ is a dense open set of $\overline{\mathbf{rev}}_{\mathbf{F}}$. We prove the result by proving the two inclusions (propositions 3.18 and 3.17).

First note the fundamental result:

Lemma 3.13. *Let $(f_n : \mathbb{S} \rightarrow \mathbb{S})_n$ be a sequence of rational maps of same degree. Then, there exists a subsequence $(f_{n_k})_{n_k}$ and a sequence of Moebius transformations $(M_{n_k})_{n_k}$ such that $(M_{n_k} \circ f_{n_k})_{n_k}$ converges to a non constant rational map f uniformly outside a finite number of points.*

Proof. Define $x_0 = \infty$. We extract a subsequence in order to have $X_n := f_n^{-1}(f_n(x_0)) \rightarrow X$ with multiplicity. Define $y_0 \in \mathbb{C} - X$. We extract a subsequence in order to have $Y_n := f_n^{-1}(f_n(y_0)) \rightarrow Y$ with multiplicity. Define $z_0 \in \mathbb{C} - X \cap Y$. Again, we extract a subsequence in order to have $Z_n := f_n^{-1}(f_n(z_0)) \rightarrow Z$.

By construction, for all n we can find a Moebius transformation satisfying:

$$M_n \circ f_n(x_0) = \infty, \quad M_n \circ f_n(y_0) = 0, \quad M_n \circ f_n(z_0) = 1.$$

Thus we have

$$\forall w \in \mathbb{C}, M_n \circ f_n(w) = \frac{\prod_{x \in X_n} (w - x)}{\prod_{y \in Y_n} (w - y)} \cdot \frac{\prod_{y \in Y_n} (z_0 - y)}{\prod_{x \in X_n} (z_0 - x)}.$$

This sequence of rational maps converges uniformly to a non constant rational map outside a finite number of points which corresponds to common zeros of $\prod_{x \in X} (w - x)^{m_x}$ and $\prod_{y \in Y} (w - y)^{m_y}$. \square

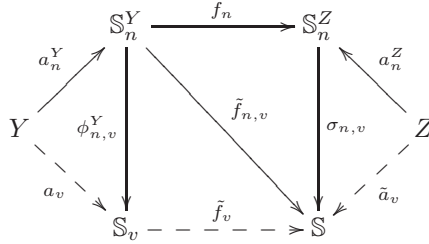
Proposition 3.14. *Let $(\mathcal{F}_n)_n$ be a sequence in $\mathbf{Rev}_{\mathbf{F},X}$. If $(\mathbf{I}([\mathcal{F}_n]))_n$ converges in $\mathbb{S}^{\text{Quadry}}$ then $(\mathcal{F}_n)_n$ converges to a cover between trees of spheres \mathcal{F} .*

Proof. Let $(\mathcal{F}_n : \mathcal{T}_n^Y \rightarrow \mathcal{T}_n^Z)_n$ be a sequence of element of $\mathbf{Rev}_{\mathbf{F}}$ such that $([I](\mathcal{F}_n))_n$ converges in $\mathfrak{T}_{\mathbf{Y}}$. Suppose that $\mathcal{T}_n^Y \rightarrow_{\phi_n^Y} \mathcal{T}^Y$.

Define the isomorphism $\sigma_n^Z : \mathbb{S}_n^Z \rightarrow \mathbb{S}$. For every internal vertex v of \mathcal{T}^Y , we set $\tilde{f}_{n,v} := (\sigma_n^Z)^{-1} \circ f_n \circ \phi_{n,v}^Y$. According to lemma 3.13, up to consider a subsequence, we can find a sequence of isomorphisms $(M_{n,v} : \mathbb{S} \rightarrow \mathbb{S})_n$ such that $(M_{n,v} \circ \tilde{f}_{n,v})_n$ converges uniformly outside a finite number of points to a non constant holomorphic morphism $\tilde{f}_v : \mathbb{S}_v \rightarrow \mathbb{S}$. We set

- $\sigma_{n,v}^Z := M_{n,v} \circ \sigma_n^Z$;
- $\tilde{a}_v = \lim \sigma_{n,v}^Z \circ a_n^Z$;
- $Y_v = a_v(Y)$ and $\tilde{Z}_v = \tilde{a}_v(Z)$.

Note that $\tilde{f}_v(Y_v) = \tilde{a}_v(Z)$.

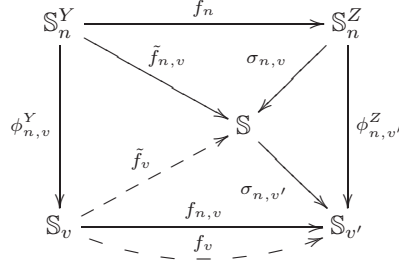


Lemma. *Let γ_z be the boundary of a small disk around $z \in \tilde{Z}_v$. Let $y \in Y_v$ be such that $\tilde{f}_v(y) = z$. Then there exists γ_y surrounding y such that $\tilde{f}_v(\gamma_y) = \gamma_z$ and $\tilde{f}_{n,v}(\gamma_y) \rightarrow \gamma_z$.*

Proof. Indeed, if γ_z is small enough, $\tilde{f}_v^{-1}(\gamma_z)$ is a loop γ_y which is the boundary of a disk containing y and avoiding the other elements of Y_v . As on $\mathbb{S}_v - Y_v$, the convergence is uniform, so $\tilde{f}_{n,v}(\gamma_y) \rightarrow \gamma_z$. \square

Lemma. *For all $v \in IV^Y$, we have $\text{card} \tilde{Z}_v \geq 3$.*

Proof. Consider small disks around the \tilde{Z}_v . Suppose n large enough such that the $\tilde{f}_{n,v}(Y)$ are in these disks. Denote by D_Z the set \mathbb{S} minus these disks and $D_Y := f_v^{-1}(D_Z)$. The Riemann-Hurwitz formula gives $-3 \geq \chi(D_Y) = \deg(f_v)\chi(D_Z)$ because \mathbb{S}_v has at least three edges and D_Y has no critical points. As $\deg(f_{n,v}) \geq 1$, then $\chi(D_Z) \leq -1$ so $\text{card} \tilde{Z}_v \geq 3$. \square



Thus $\text{card} \tilde{a}_v(Z) \geq 3$. Let t_v be a triple of points of Z which have pairwise distinct images by \tilde{a}_v . Let v' be the unique vertex of T^Z separating t_v . As on the previous diagram we use the notation $\sigma_{n,v'} := \phi_{n,v'}^Z \circ \sigma_{n,v}^{-1}$. From the choice of t_v , we know that $\sigma_{n,v'}$ converges to an isomorphism $\sigma_{v'}$. Thus $\sigma_{n,v'} \circ \tilde{f}_{n,v} \rightarrow \sigma_{v'} \circ \tilde{f}_v := f_v$ locally uniformly outside a finite number of points and $\deg(f_v) \geq 1$.

So we have $f_{n,v} := \phi_{n,v'}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1} \rightarrow f_v$ locally uniformly outside a finite number of points and $\deg(f_v) \geq 1$.

Lemma. *The map $F : V^Y \rightarrow V^Z$ that maps v defined by t to the vertex defined by t_v extends to a map between trees.*

Proof. Let v_1 and v_2 be two adjacent vertices in T^Y connected by an edge e and let v'_1 and v'_2 be their respective images. Let D_1 (reps. D_2) be a topological disk neighborhood of e_{v_1} (reps. e_{v_2}) and containing only this attaching point of edge and let C_1 (resp. C_2) be its boundary. Denote by $A_n := (\phi_{n,v_1}^Y)^{-1}(D_1) \cap (\phi_{n,v_2}^Y)^{-1}(D_2) \subset \mathbb{S}_n$, Denote by $C'_\star := f_{v_\star}(C_\star)$ and $A'_n := f_n(A_n)$. We now suppose that n is large enough such that A_n is an annulus and does not contain any attaching point of edges. Thus A'_n does not contain any attaching point of edges neither. As the critical points of f_n are attaching points of edges, A_n does not contain critical points and A'_n is an annulus.

Suppose that there is a vertex v' between v'_1 and v'_2 . As $\phi_{n,v_\star}^Z(C'_\star) \rightarrow C'_\star$, lemma 2.3 implies that $\phi_{n,v'}^Z(A'_n)$ tends to $\mathbb{S}_{v'}$ minus the attaching points of the branches containing respectively v'_1 and v'_2 . As A'_n does not contain attaching points of edge, $\mathbb{S}_{v'}$ has only two attaching points of edges which contradicts the stability of T^Z . Thus F maps two adjacent vertices to two adjacent vertices. \square

In particular we proved that the image of the attaching point of e on v_\star is the attaching point of $F(e)$ on v'_\star ie $f_{v_\star}(e_{v_\star})$.

Lemma. *The map $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ defined by F and the f_v is a cover between trees of spheres.*

Proof. Take $v'_1 := F(v_1)$ with $v_1 \in IV^Y$. Let $e' := \{v'_1, v'_2\}$ be an edge of T^Z . Let C'_1 (resp. C'_2) be a topological circle surrounding a disk D'_1 (resp. D'_2) containing a unique attaching point on v'_1 (resp. on v'_2), the one of e' . Define $A'_n = \phi_{n,v'_1}^Z(D'_1) \cap \phi_{n,v'_2}^Z(D'_2)$ and suppose n big enough such that A'_n is an annulus. Let A_n be a connected component of $f_n^{-1}(A'_n)$. From the Riemann-Hurwitz Formula, we deduce that A_n is an annulus. Denote by $C_{1,n}$ and $C_{2,n}$

the preimages of C'_1 and C'_2 surrounding A_n and by $D_{1,n}$ the disks bounded by $C_{1,n}$ containing A_n . We suppose n large enough such that the partition of $a_n(Y)$ (resp. $a_n(Z)$) given by the two connected components of $\mathbb{S}_n - A_n$ (resp. $\mathbb{S}_n - A'_n$) is constant.

Take $z_2 \in Z \cap B_{v'_1}(F(e))$ and $z_1 \in Z \cap B_{v'_2}(F(e))$. Then $a_n(z_1)$ and $a_n(z_2)$ are respectively in each of the two connected components of $\mathbb{S}_n - A'_n$. After choosing a projective chart σ_n such that $\sigma_n \circ a_n(z_1) = 0$ and $\sigma_n \circ a_n(z_2) = \infty$, we suppose that $\mathbb{S}_n = \mathbb{S}$, $a_n(z_1) = 0$ and $a_n(z_2) = \infty$.

Denote by

$$n_0 := \text{card}\{y \in Y \cap D_{1,n} \mid f_n(y) = 0\}$$

and

$$n_\infty := \text{card}\{y \in Y \cap D_{1,n} \mid f_n(y) = \infty\}.$$

The local degree of f_{v_1} at $i_{v_1}(e)$ is the same as the one of f_{v_1} on $C_{1,n}$ which is the one of f_n on $(\phi_{n,v_1}^Y)^{-1}(C_{1,n})$, ie

$$\text{deg}_{f_{v_1}}(e) = n_0 - n_\infty.$$

Note that these two cardinals don't depend on the choice of the pair (z_1, z_2) in the connected components of $\mathbb{S}_n - A'_n$. Again these cardinals are the same if we consider D_2 instead of D_1 because A_n does not contain critical values. By the same deductions on v_2 we prove that $\text{deg}_{f_{v_1}}(e) = \text{deg}_{f_{v_2}}(e)$.

In particular, if $n_0 \neq 0$ then $\phi_{n,v_1}(D_{1,n})$ contains an attaching point of an edge; thus every preimage of an edge attaching point is the attaching point of an edge. As the image of an edge attaching point is an edge attaching point, $f_v : Y_v \rightarrow Z_{F(v)}$ is a cover. Moreover the critical points of f_v are the limits of the critical points of $\phi_{n,F(v)}^Z \circ f_n \circ (\phi_{n,v}^Y)^{-1}$ so they are attaching points of edges. \square

This concludes the proof of proposition 3.14 because as required we have

$$\mathcal{F}_n \xrightarrow{(\phi_n^Y, \phi_n^Z)} \mathcal{F}.$$

\square

Corollary 3.15. *The topology given by \mathbf{I} is compatible with the convergence notion defined on $\overline{\mathbf{Rev}}_{\mathbf{F}}$:*

$$f_n \rightarrow \mathcal{F} \quad \text{if and only if} \quad \mathbf{I}([f_n]) \rightarrow \mathbf{I}([\mathcal{F}]).$$

Proof. The implication is given by corollary 3.11. Recyprocallly if $\mathbf{I}([\mathcal{F}_n : \mathcal{T}_n^Y \rightarrow \mathcal{T}_n^Z]) \rightarrow \mathbf{I}([\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z])$ then according to proposition 3.14, \mathcal{F}_n converges to a cover between trees of spheres \mathcal{F}' so $\mathbf{I}([\mathcal{F}_n]) \rightarrow \mathbf{I}([\mathcal{F}'])$. We deduce that $\mathbf{I}([\mathcal{F}']) = \mathbf{I}([\mathcal{F}])$, thus $\mathcal{F} = \mathcal{F}'$ according to proposition 3.9. \square

We can also directly deduce the theorem admitted in [A1].

Corollary 3.16. *Let y_n, z_n be two sequences of spheres marked respectively by the finite sets Y and Z containing each one at least three elements and converging to the trees of spheres \mathcal{T}^Y and \mathcal{T}^Z .*

Every sequence of marked spheres covers $(f_n, y_n, z_n)_n$ of a given portrait converges to a cover between the trees of spheres \mathcal{T}^Y and \mathcal{T}^Z .

Proposition 3.17. *The set $\mathbf{I}(\overline{\mathbf{rev}}_{\mathbf{F}})$ is closed, in particular*

$$\mathrm{Ad}(\mathbf{I}(\mathbf{rev}_{\mathbf{F}})) \subseteq \mathbf{I}(\overline{\mathbf{rev}}_{\mathbf{F}}).$$

Proof. This result follows directly from proposition 3.14 and corollary 3.15. \square

Proposition 3.18. *The set $\mathbf{Rev}_{\mathbf{F}}$ is dense in $\overline{\mathbf{Rev}}_{\mathbf{F}}$. In particular we have*

$$\mathbf{I}(\overline{\mathbf{rev}}_{\mathbf{F}}) \subseteq \mathrm{Ad}(\mathbf{I}(\mathbf{rev}_{\mathbf{F}})).$$

Proof. Take $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ in $\overline{\mathbf{Rev}}_{\mathbf{F}}$. In this proof, for the spheres at the vertices of \mathcal{T}^Y and \mathcal{T}^Z we fix projective charts and we don't distinguish them. Take $1 > \varepsilon > 0$. Take an edge e between two vertices v_1, v_2 . Define by $v'_i := F(v_i)$ and denote by e' := $F(e)$ the edge between v'_1 and v'_2 .

Let A'_1 (resp. A'_2) be an annulus between the circles of radii ε^2 and ε centered on the attaching point $e'_{v'_1}$ (resp. $e'_{v'_2}$). Let $\phi_\varepsilon^{e'} : A'_1 \rightarrow A'_2$ be a biholomorphism that exchanges the borders of the two annuli (maps the circle of radius ε^2 on A'_1 to the one of radius ε on A'_2 and reciprocally).

Let A_i be the preimage of A'_i on v_i . We consider an ε small enough such that the A_i are in neighborhoods of the e_{v_i} that map with degree $\deg_{f_{v_i}}(e_{v_i})$ and such that each of these neighborhoods contain a unique edge attaching point. As \mathcal{F} is a covering between trees of spheres, we have $\deg_{f_{v_1}}(e_{v_1}) = \deg_{f_{v_2}}(e_{v_2}) =: d_e$. We choose one of the d_e biholomorphisms ϕ_ε^e that makes the following diagram commuting ;

$$\begin{array}{ccc} A_1 & \xrightarrow{\phi_\varepsilon^e} & A_2 \\ f_{v_1} \downarrow & & \downarrow f_{v_2} \\ A'_1 & \xrightarrow{\phi_\varepsilon^{e'}} & A'_2. \end{array}$$

As $F : E^Y \rightarrow E^Z$ is surjective, after repeating this process we obtain some families Φ of biholomorphisms associated to the edges between the internal vertices of T^Y and Φ' associated to the same one of T^Z . We suppose ε small enough such that all the annuli already defined don't have common pairwise intersections. For all internal vertex v of T^* , denote by $\mathbb{S}_{\varepsilon, v}^*$ the sphere \mathbb{S}_v^* minus some topological closed disks around the attaching points of edges connecting to internal vertices which are bordered by the A_i (resp. A'_i) as previously defined (but does not contain the A_i (resp. A'_i)). We use the notations

$$\mathbb{S}_\varepsilon^Y := \bigsqcup_{\Phi} \mathbb{S}_{\varepsilon, v}^Y \quad \text{and} \quad \mathbb{S}_\varepsilon^Z := \bigsqcup_{\Phi'} \mathbb{S}_{\varepsilon, v}^Z.$$

Every element y of Y is a vertex of T^Y which has a unique edge so it is adjacent to a unique internal vertex v_y of T^Y . Denote by e_y the attaching point of this edge on v_y . We define a family of injections $a_\varepsilon^Y : Y \rightarrow \mathbb{S}_\varepsilon^Y$ that associate v_y to y .

Lemma. *For ε small enough, \mathbb{S}_ε^Y with a^Y is a marked sphere $\mathcal{T}_\varepsilon^Y$ and if $\varepsilon \rightarrow 0$, we have*

$$\mathcal{T}_\varepsilon^Y \rightarrow \mathcal{T}^Y.$$

Proof. The set of internal vertices of T^Y and edges connecting them is a subtree T' of T^Y . Thus it satisfies $\text{card}V' = \text{card}E' + 1$ (see for example [Di, corollary 1.5.3]). In addition the Euler characteristic of \mathbb{S}_ε^Y is equal to the sum of the one of the $\check{\mathbb{S}}_v^\varepsilon$ because the one of an annulus is 0. But the $\check{\mathbb{S}}_v^\varepsilon$ are spheres minus a disk for each of the edge of $v \in T'$. So the Euler characteristic of \mathbb{S}_ε^Y is

$$\sum (2 - \text{card}E'_v) = 2\text{card}V' - 2\text{card}E' = 2(\text{card}V' - \text{card}E') = 2.$$

But \mathcal{T}^Y is connected, so \mathbb{S}_ε^Y is connected too and it follows that \mathbb{S}_ε^Y is a topological sphere. As Φ is a family of isomorphisms, \mathbb{S}_ε^Y is equipped of a complex structure. Thus we proved that \mathbb{S}_ε^Y together with a_ε^Y is a sphere marked by Y that we will denote by $\mathcal{T}_\varepsilon^Y$.

Moreover, for all $v \in IV^Y$, if we define $\phi_{\varepsilon,v}$ an isomorphism defined by the identity on $\check{\mathbb{S}}_v^\varepsilon$, then we have $\mathcal{T}_\varepsilon^Y \xrightarrow{\phi_\varepsilon} \mathcal{T}^Y$ as required because the $\check{\mathbb{S}}_v^\varepsilon$ tend to the \mathbb{S}_v . \square

Similarly we construct a family of injections $a_\varepsilon^Z : Z \rightarrow \mathbb{S}_\varepsilon^Z$ then the associated trees of spheres $\mathcal{T}_\varepsilon^Z$ and we have $\mathcal{T}_\varepsilon^Z \rightarrow \mathcal{T}^Z$.

We are now ready to prove that the maps $\mathcal{F}_\varepsilon := (\mathcal{F}|_{\mathbb{S}_\varepsilon^Y} : \mathcal{T}_\varepsilon^Y \rightarrow \mathcal{T}_\varepsilon^Z)$ form a family of covers between marked spheres (for ε small enough) and $[\mathcal{F}_\varepsilon] \rightarrow [\mathcal{F}]$.

Indeed, for ε small enough, the $\mathbb{S}_{\varepsilon,v}^Y$ for $v \in IV^Y$ form a cover of \mathbb{S}_ε^Y and the map \mathcal{F}_ε restricted on these ones is holomorphic, then f_ε is holomorphic. By definition $(F_\varepsilon|_Y, \text{deg}|_Y) = \mathbf{F}$ so f_ε is a cover on the edges. Thus, for ε small enough, \mathcal{F}_ε is spheres cover. In addition we have $[\mathcal{T}_\varepsilon^Y] \rightarrow [\mathcal{T}^Y]$ so by definition $[\mathcal{F}_\varepsilon] \rightarrow [\mathcal{F}]$. \square

4 Dynamics

4.1 Background

In this subsection we recall notions and notations introduced in [A1].

We suppose that $X \subseteq Y \cap Z$ and we will say that $(\mathcal{F}, \mathcal{T}^X)$ is a dynamical system between trees of spheres if :

- $\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z$ is a cover between trees of spheres,
- \mathcal{T}^X is a tree of spheres compatible with \mathcal{T}^Y and \mathcal{T}^Z , ie :

- $X \subseteq Y \cap Z$
- each internal vertex v of T^X is an internal vertex common to T^Y and T^Z ,
- $\mathcal{S}_v^X = \mathcal{S}_v^Y = \mathcal{S}_v^Z$ and
- $a_v^X = a_v^Y|_X = a_v^Z|_X$.

Dynamical covers between marked spheres can be naturally identified to dynamically marked rational maps:

Definition 4.1 (Dynamically marked rational map). *A rational map dynamically marked by (\mathbf{F}, X) is a rational map (f, y, z) marked by \mathbf{F} such that $y|_X = z|_X$.*

We denote by $\text{Rat}_{\mathbf{F}, X}$ the set of rational maps dynamically marked by (\mathbf{F}, X) . On this space we define the convergence notion of a sequence of dynamical systems between marked spheres to a dynamical system of marked trees of spheres as follows.

Definition 4.2 (Dynamical convergence). *Let $(\mathcal{F} : \mathcal{T}^Y \rightarrow \mathcal{T}^Z, \mathcal{T}^X)$ be a dynamical system of trees of spheres with portrait \mathbf{F} . A sequence $(\mathcal{F}_n, a_n^Y, a_n^Z)_n$ of dynamical systems of spheres marked by (\mathbf{F}, X) converges to $(\mathcal{F}, \mathcal{T}^X)$ if*

$$\mathcal{F}_n \xrightarrow[\phi_n^Y, \phi_n^Z]{} \mathcal{F} \quad \text{with} \quad \phi_{n,v}^Y = \phi_{n,v}^Z$$

for all vertex $v \in IV^X$.

4.2 Conjugacy and compactification

Definition 4.3 (Conjugated dynamical systems). *Two dynamical systems between trees of spheres $(\mathcal{F}^1, \mathcal{T}_1^X)$ and $(\mathcal{F}^2, \mathcal{T}_2^X)$ are conjugated if :*

$$\mathcal{F}^1 \sim_{(\mathcal{M}^Y, \mathcal{M}^Z)} \mathcal{F}^2 \quad \text{and} \quad \forall v \in IV^X, m_v^Y = m_v^Z.$$

We denote by $\mathbf{Dyn}_{\mathbf{F}, X}$ the set of dynamical systems between trees of spheres of portrait \mathbf{F} . We denote by $\mathbf{dyn}_{\mathbf{F}, X}$ their conjugacy classes. With this definition the set $\text{rat}_{\mathbf{F}, X}$ is naturally identified to the set of classes of dynamical systems between marked spheres.

Lemma 4.4. *The map that associate to every $[(\mathcal{F}, \mathcal{T}^X)] \in \mathbf{dyn}_{\mathbf{F}, X}$ the element $[\mathcal{F}] \in \overline{\text{rev}}_{\mathbf{F}}$ is an injection.*

Proof. Take $(\mathcal{F}_1, \mathcal{T}_1^X)$ and $(\mathcal{F}_2, \mathcal{T}_2^X)$ in $\mathbf{Dyn}_{\mathbf{F}, X}$ such that $\mathcal{F}^1 \sim_{(\mathcal{M}^Y, \mathcal{M}^Z)} \mathcal{F}^2$. We want to prove that $[(\mathcal{F}_1, \mathcal{T}_1^X)] = [(\mathcal{F}_2, \mathcal{T}_2^X)]$.

It is clear that $M^Y|_{T_1^X} = M^Z|_{T_1^X}$. Take v an internal vertex of T_1^X . As $(\mathcal{F}_2, \mathcal{T}_2^X) \in \mathbf{Dyn}_{\mathbf{F}, X}$ we have $a_{M^Y(v)}^X = a_{M^Y(v)}^Y|_X = a_{M^Z(v)}^Z|_X$ and as $(\mathcal{F}_1, \mathcal{T}_1^X) \in \mathbf{Dyn}_{\mathbf{F}, X}$ we have $a_v^X = a_v^Y|_X = a_v^Z|_X$. Thus we deduce that

$m_v^Y \circ (m_v^Z)^{-1}$ fixes $a_{M^Z(v)}^Z|_X$ which contains at least three elements so we have $m_v^Y = m_v^Z$. \square

According to this lemma we make an identification of $\mathbf{dyn}_{\mathbf{F},X}$ in $\overline{\mathbf{rev}}_{\mathbf{F}}$ and we define the topology of $\mathbf{dyn}_{\mathbf{F},X}$ as the restriction of the one in $\mathbf{rev}_{\mathbf{F}}$. First we prove that this topology is compatible with the dynamical convergence.

Lemma 4.5. *A sequence of dynamical systems converges to a dynamical system if and only if it dynamically converges to this limit.*

Proof. Suppose that $(\mathcal{F}_n, \mathcal{T}_n^X)_n$ is a sequence of dynamical systems converging to a dynamical system $(\mathcal{F}, \mathcal{T}^X)$:

$$\mathcal{F}_n \xrightarrow{(\phi_n^X, \phi_n^Z)} \mathcal{F}.$$

For all $t \in \text{Trip}_X$, we define $\tilde{\phi}_{n,t}^Y = \phi_{n,t}^X$ and $\tilde{\phi}_{n,t}^Z = \phi_{n,t}^X$ (see notations following remark 2.23). Then, for all triple $t \in \text{Trip}_Y - \text{Trip}_X$, we define $\tilde{\phi}_{n,t}^Y = \phi_{n,t}^Y$ and for $t \in \text{Trip}_Z - \text{Trip}_X$, $\tilde{\phi}_{n,t}^Z = \phi_{n,t}^Z$.

For all $t \in \text{Trip}_X$, $(\tilde{\phi}_{n,t}^Y)^{-1} \circ \tilde{\phi}_{n,t}^Y$ tends to the identity of \mathbb{S}_t because it converges to the identity on the three elements of t . Thus we have $(\mathcal{F}_n, \mathcal{T}_n^X)$ converges dynamically to $(\mathcal{F}, \mathcal{T}^X)$ with respect to the families of sequences $(\tilde{\phi}_n^Y)_n$ and $(\tilde{\phi}_n^Z)_n$. \square

With this topology we are going to prove the following theorem.

Theorem 4.6. *The space $\mathbf{dyn}_{\mathbf{F},X}$ is compact.*

We have $\text{Quad}_X \subset \text{Quad}_Y$. Denote by $\pi_{Y,X}$ the natural projection

$$\pi_{Y,X} : \mathbb{S}^{\text{Quad}_Y} \rightarrow \mathbb{S}^{\text{Quad}_X}.$$

In the following we define a map $\Pi_{Y,X}$ from the set of trees of spheres marked by Y to the one marked by X . We are interested in this map because of the following observation.

Lemma 4.7. *The tree \mathcal{T}^X is compatible with \mathcal{T}^Y if and only if*

$$\mathcal{T}^X = \Pi_{X,Y}(\mathcal{T}^Y).$$

Proof. Suppose that \mathcal{T}^X is compatible with \mathcal{T}^Y . Then each $t \in \text{Trip}_X$ is separated by a unique vertex v_t of $\Pi_{Y,X}(\mathcal{T}^X)$ and a unique vertex v'_t of \mathcal{T}^X . We have \mathcal{T}^X is compatible for $(\mathcal{T}^Y, \mathcal{T}^Z)$ if and only if $\forall t \in \text{Trip}_X, a_{v'_t}^X = a_{v'_t}^Y|_X = a_{v_t}$ if and only if $\mathcal{T}^X = \Pi_{X,Y}(\mathcal{T}^Y)$.

Reciprocally, if $\mathcal{T}^X = \Pi_{X,Y}(\mathcal{T}^Y)$, the vertices of \mathcal{T}^X are vertices of \mathcal{T}^Y and by construction we have $a_v^X = a_v^Y|_X$ for all $v \in IV^X$. \square

Now we prove that this new map well behave in the quotient by the natural isomorphism relation as a map $\Pi_{Y,X}$.

Definition 4.8. We denote by $\Pi_{Y,X}$ the map such that the following diagram commutes :

$$\begin{array}{ccc} \overline{\text{Mod}}_Y & \xrightarrow{\mathfrak{B}_Y} & \text{Quad}_Y \\ \Pi_{Y,X} \downarrow & & \downarrow \pi_{Y,X} \\ \overline{\text{Mod}}_X & \xrightarrow{\mathfrak{B}_X} & \text{Quad}_X \end{array}$$

The fact that this map is well defined and continuous will follow from lemma 4.10.

Let \mathcal{T}^Y be a tree of spheres marked by Y . Denote by \mathcal{P} the set of partitions of X associated to the vertices of Y separating three elements of X .

Lemma 4.9. *The set \mathcal{P} is an admissible set of partitions.*

Proof. 1. By definition the vertices for which we are considering the partitions separate three elements of X .

2. Let P be a partition corresponding to a vertex $v \in T^Y$ and $B \in P$. Either $B = \{x\}$, or $\text{card}B > 1$ and in this case, the branch on V corresponding to B contains at least an internal vertex separating two elements of X . Let v' be one of these vertices in this branch which are the closest to v (for the length of $[v, v']$). Let e' be the edge on v' connecting v to v' . Then $B_{v'}(e') = (X - B)$. Indeed, suppose that this is not the case, we find an element $x \in B \cap B_{v'}(e')$. Take $x_1 \in B - \{x\}$ and $x_2 \in X - B$. The vertex separating this triple (x_1, x, x_2) is between v and v' (because $x, x_2 \in B_{v'}(e')$ and $x, x_1 \in B$) which contradicts the minimality of v' .

3. Suppose by contradiction that we have v_1 and v_2 two vertices of T^Y for which the associated partitions of X are P_1 and P_2 and such that $P_1 \cap P_2 \ni B (\neq \emptyset)$. Let B_1 (resp. B_2) be the branch of v_1 (resp. v_2) corresponding to B . As $B \in B_1 \cap B_2$ we have $v_1 \in B_2$ (or $v_1 \in B_2$ which is a symmetric case). Let e_1 be the edge on v_1 connecting it to v_2 . Given that v_1 separate three elements of X , we find $x \in X - (B \cup B_{v_1}(e_1))$ which is absurd because $x \notin B_{v_1}(e_1)$ so $x \in B_2 \in X = B$. \square

According to corollary 2.8, the set \mathcal{P} determines a unique isomorphism class of combinatorial trees $[T^X]$. For all $t \in \text{Trip}_X$, we denote by v_t the vertex separating t in T^Y . Denote by \mathcal{T}^X the tree of spheres which combinatorial tree is the representative of $[T^X]$ for which each internal vertex associated to a triple t is v_t and for which the map associated to each internal vertex v defined by a triple t is $a_v := a_{v_t}|_X$. We use the notation $\Pi_{Y,X}(\mathcal{T}^Y) := \mathcal{T}^X$.

Lemma 4.10. *The map $\Pi_{Y,X}$ is continuous as the quotient of the map $\Pi_{Y,X}$ by the isomorphism equivalence relation on the marked trees of spheres.*

Proof. Indeed, if $\mathcal{T}_1^Y \sim_{\mathcal{M}} \mathcal{T}_2^Y$ then $\Pi(\mathcal{T}_1^Y) \sim_{\mathcal{M}} \Pi(\mathcal{T}_2^Y)$. The formula follows directly from the definition of $\Pi_{Y,X}$ and as $\pi_{Y,X}$ is continuous we deduce that the map is continuous too. Moreover, $\Pi_{Y,X}$ acts on the marked spheres by restricting the marking map so it is the map previously defined. \square

Proof. (Theorem 4.6) According to proposition 3.5 and the definition of its topology, the set $\overline{\text{rev}}_{\mathbf{F}}$ can be identified to a subspace of $\mathbf{Mod}_Y \times \mathbf{Mod}_Z$.

According to 4.7 and 4.10, we have

$$\mathbf{dyn}_{\mathbf{F},X} = \{([\mathcal{T}^Y], [\mathcal{T}^Z]) \in \overline{\text{rev}}_{\mathbf{F}} \mid \mathbf{\Pi}_{Y,X}([\mathcal{T}^Y]) = \mathbf{\Pi}_{Z,X}([\mathcal{T}^Z])\}.$$

So $\mathbf{dyn}_{\mathbf{F},X}$ is a closed set in $\overline{\text{rev}}_{\mathbf{F}}$ which is compact □

4.3 Dynamics and representatives

As a corollary of all of this we have the statement below that was assumed in [A1].

Corollary 4.11. *If $(\mathcal{F}_n)_n$ is a sequence in $\mathbf{Dyn}_{\mathbf{F},X}$, then after passing to a subsequence, there exists $(\mathcal{F}, \mathcal{T}^X) \in \mathbf{Dyn}_{\mathbf{F},X}$ such that $(\mathcal{F}_n, \mathcal{T}_n^X)_n$ converges dynamically to $(\mathcal{F}, \mathcal{T}^X)$.*

Proof. This corollary follows directly from Theorem 4.6 and lemma 4.5. □

Proposition 4.12. *We have the following inclusions:*

$$\text{Ad}(\text{rat}_{\mathbf{F},X}) \subsetneq \mathbf{dyn}_{\mathbf{F},X} \subsetneq \overline{\text{rev}}_{\mathbf{F}}.$$

Proof. In [A2] we give an example of element in $\mathbf{Dyn}_{\mathbf{F},X}$ that was not a dynamical limit of dynamical covers between marked spheres so we have $\text{Ad}(\text{rat}_{\mathbf{F},X}) \subsetneq \mathbf{dyn}_{\mathbf{F},X}$. □

Remark 4.13. If a dynamical system between trees of marked spheres (in $\overline{\text{rev}}_{\mathbf{F},X}$) satisfies the two annuli lemmas and the branches lemma, we can hope that it is in the adherence of $\text{rat}_{\mathbf{F}}$ but this is an open question.

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