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Cut locus and heat kernel at Grushin points of 2 dimensional almost Riemannian metrics

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Abstract

This article deals with 2d almost Riemannian structures, which are generalized Riemannian structures on manifolds of dimension 2. Such sub-Riemannian structures can be locally defined by a pair of vector fields (X, Y) , playing the role of orthonormal frame, that may become colinear on some subset. We denote $\mathcal{D} = \text{span}(X, Y)$. After a short introduction, I first give a description of the local cut and conjugate loci at a Grushin point q (where \mathcal{D}_q has dimension 1 and $\mathcal{D}_q = T_q M$) that makes appear that the cut locus may have an angle at q . In a second time I describe the local cut and conjugate loci at a Riemannian point x in a neighborhood of a Grushin point q . Finally, applying results of [6], I give the asymptotics in small time of the heat kernel $p_t(x, y)$ for y in the same neighborhood of q .

1 Introduction and definitions

An almost Riemannian structure of dimension 2 (2-ARS for short) is a sub-Riemannian structure on a 2 dimensional manifold with a rank varying distribution. It is supposed to be locally defined by a pair of vector fields, playing the role of an orthonormal frame, that satisfies the Hörmander condition. It differs from Riemannian geometry by the fact that the pair may become colinear.

2-ARSs were first studied in the context of hypoelliptic operators [5, 19, 20]. They have applications to quantum control [14, 15, 16] and orbital transfer in space mechanics [10, 9].

The singular set \mathcal{Z} where the distribution \mathcal{D} has dimension 1 is generically a 1 dimensional embedded submanifold (see [3]). There are generically three types of points : Riemannian points where the distribution has dimension 2; Grushin points where \mathcal{D}_q has dimension 1, \mathcal{D}_q^2 has dimension 2 and \mathcal{D}_q is transverse to \mathcal{Z} ; Tangency points where $\mathcal{D}_q = \mathcal{D}_q^2$ has dimension 1, \mathcal{D}_q^3 has dimension 2 and \mathcal{D}_q is tangent to \mathcal{Z} . We denote \mathcal{T} the set of tangency points.

In [3, 18], a Gauss Bonnet formula is obtained for 2-ARS without tangency points. In [2], it is generalized in presence of tangency points. In [13] a necessary and sufficient condition for two 2-ARS to be Lipschitz equivalent is given in terms of labelled graphs associated to the structures. In [17] the authors show that the singular set \mathcal{Z} acts as a barrier for the heat flow and for quantum particles despite the fact that geodesics can pass through \mathcal{Z} .

A general fact in sub-Riemannian geometry is that, at points q where $\mathcal{D}_q \neq T_q M$, the conjugate and the cut loci of q do accumulate at q . In [11], the local cut locus of a tangency point is described:

it is an asymmetric cusp, tangent to the distribution at q . In [12], the cut locus of the wave front starting from \mathcal{Z} is described in the neighborhood of a tangency point q : it is the union of a curve starting from q and transversal to \mathcal{D}_q and of another one tangent to \mathcal{D}_q .

As surprising as it may appear, nothing has been done for what concerns the local cut locus at a Grushin point q . Certainly one easily imagines that it is a C^1 -curve transverse to \mathcal{D}_q . In this article, I prove at the contrary that, for a Grushin point q outside a discrete set of $\mathcal{Z} - \mathcal{T}$, the cut locus of q has an angle at q . Studying the cut and conjugate loci for a point q close to \mathcal{Z} and far from \mathcal{T} , I also give estimates of the asymptotics in small time for the heat kernel associated with both the 2-ARS and a smooth volume.

1.1 Definitions and basic properties

Definition 1 *A 2-dimensional almost-Riemannian structure (2-ARS, for short) is a triple $\mathcal{S} = (E, \mathfrak{f}, g)$ where:*

- E is a vector bundle of rank 2 over a 2 dimensional smooth manifold M ;
- g is an Euclidean metric on E , that is g_q is a scalar product on E_q smoothly depending on q ;
- $\mathfrak{f} : E \rightarrow TM$ is a morphism of vector bundles, that is \mathfrak{f} is linear from E_q to T_qM for any q .

Denoting by $\Gamma(E)$ the $C^\infty(M)$ -module of smooth sections on E , and by $\mathfrak{f}_* : \Gamma(E) \rightarrow \text{Vec}(M)$ the map $\sigma \mapsto \mathfrak{f}_*(\sigma) := \mathfrak{f} \circ \sigma$, we require that the submodule of $\text{Vec}(M)$ given by $\mathcal{D} = \mathfrak{f}_*(\Gamma(E))$ is bracket generating, i.e., $\text{Lie}_q(\mathcal{D}) = T_qM$ for every $q \in M$. Moreover, we require that \mathfrak{f}_* is injective.

If (σ_1, σ_2) is an orthonormal frame for g on an open subset Ω of M , an *orthonormal frame* on Ω is given by $(\mathfrak{f}_*\sigma_1, \mathfrak{f}_*\sigma_2)$ which forms a local generator of the submodule \mathcal{D} .

Admissible curves, sub-Riemannian length and distance, geodesics, spheres and wave front, conjugate and cut loci, are defined as in the classical sub-Riemannian setting (see for example [1]).

Under the following generic¹ assumption **(H0)**, only Riemannian, Grushin and tangency points can occur (see [3]).

- (H0)** (i) \mathcal{Z} is an embedded one-dimensional submanifold of M ;
- (ii) the points $q \in M$ where \mathcal{D}_q^2 is one-dimensional are isolated and at these points $\mathcal{D}_q = T_q\mathcal{Z}$;
- (iii) $\mathcal{D}_q^3 = T_qM$ for every $q \in M$.

At Grushin points, it exists a canonical local coordinate system such that an orthonormal frame (X, Y) is given by the normal form

$$(NF) \quad (X = \partial_x, \quad Y = xf(x, y)\partial_y)$$

where $f(0, y) = 1$ (see [3, 12]). For this normal form, the natural orders of x and y are respectively 1 and 2 (see [8]). The nilpotent approximation is the so called Grushin metric defined by the orthonormal frame

$$X_{-1} = \partial_x, \quad Y_{-1} = x\partial_y.$$

¹Generic means true for a residual subset of the set of morphisms \mathfrak{f} endowed with the C^∞ -Whitney topology

One proves easily that there is no abnormal extremal which implies, thanks to the Pontryagin maximum principle, that any geodesic is the projection on M of a trajectory of the Hamiltonian defined on T^*M by

$$H(\lambda, q) = \frac{1}{2}((\lambda.X(q))^2 + (\lambda.Y(q))^2)$$

where (X, Y) is an orthonormal frame of \mathcal{D} . See [21].

In the following, we use the notation \mathcal{A}_ℓ for a family of singularity. See [4] for their definition. A map f from \mathbb{R}^n to \mathbb{R}^n has singularity \mathcal{A}_2 at q if, up to a good choice of variables at q and coordinates at $f(q)$, it can be written $(x_1, \dots, x_n) \mapsto (x_1^2, x_2, \dots, x_n)$ close to $x_1 = \dots = x_n = 0$ (a fold). A map has singularity \mathcal{A}_3 at q if it can be written $(x_1, \dots, x_n) \mapsto (x_1^3 - x_1x_2, x_2, \dots, x_n)$ close to $x_1 = \dots = x_n = 0$ (a cusp).

1.2 Results

In this article, concerning the cut locus at a Grushin point, I prove in section 2

Theorem 2 *Close to a Grushin point q of a 2-ARS, its cut locus is the union of the disjoint supports of two smooth curves $\gamma_i :]0, \varepsilon[\rightarrow M$ ($i = 1, 2$), such that $\gamma_i(0) = q$, $\dot{\gamma}_i(0) \neq 0$. Generically, the two vectors $\dot{\gamma}_1(0)$ and $\dot{\gamma}_2(0)$ are not colinear except at isolated points of $\mathcal{Z} \setminus \mathcal{T}$. More precisely, in the normal coordinate system such that (NF) holds, the curves are given by*

$$\gamma_1(t) = t(-\frac{4}{3}a, \frac{\pi}{2}) + O(t^2), \quad \gamma_2(t) = t(-\frac{4}{3}a, -\frac{\pi}{2}) + O(t^2).$$

where $a = \frac{\partial f}{\partial x}(0, 0)$.

For what concerns the cut locus of a point close enough to the singular set \mathcal{Z} and far enough to the set \mathcal{T} , I prove in section 3

Theorem 3 *Let q_0 be a Grushin point of a 2-ARS. If q is a Riemannian point, sufficiently close to q_0 , then the cut locus of q is locally the union of the disjoint supports of two smooth curves $\gamma_i :]0, \varepsilon[\rightarrow M$ ($i = 1, 2$). The points $\gamma_i(0)$ belong also to the first conjugate locus, they are reached by only one optimal geodesic and the corresponding singularity of the exponential map Exp_q is of type \mathcal{A}_3 . The other points of the local cut locus are reached by two optimal geodesics and do not correspond to singularities of Exp_q .*

Considering the Riemannian volume, one can study the canonical heat equation associated with the almost Riemannian structure. As explained in [17], in the case of a compact orientable surface without tangency points, a quantum particle in such a structure cannot cross the singular set and the heat cannot flow through the singularity. This is really surprising since geodesics do cross the singular set.

Considering a smooth volume on M (which is not the case of the Riemannian volume along \mathcal{Z}), one can define a divergence. Together with the sub-Riemannian gradient, it allows to define a Laplacian by

$$\Delta f = \text{div}(\nabla_g f).$$

Under the additional hypothesis that the manifold is complete, one gets that Δ is hypoelliptic and has a symmetric heat kernel $p_t(x, y)$.

In [6], the authors prove in particular that if a geodesic γ between x and y is such that y is both in the cut locus and the conjugate locus along γ and if the corresponding singularity of \exp_x is of type \mathcal{A}_ℓ then the contribution to the heat kernel has the following asymptotics in small time

$$p_t(x, y) = \frac{C + O(t^{\frac{2}{\ell+1}})}{t^{\frac{n+1}{2} - \frac{1}{\ell+1}}} e^{-d^2(x, y)/4t}.$$

In [7], the authors prove that if a geodesic γ between x and y is not conjugated in y then

$$p_t(x, y) = \frac{C + O(t)}{t^{\frac{n}{2}}} e^{-d^2(x, y)/4t}.$$

With these results, one proves easily that Theorem 3 implies

Theorem 4 *Let q_0 be a Grushin point of a 2-ARS. If x is a Riemannian point, sufficiently close to q_0 , then if $y \neq x$ is still close to q_0 one gets that*

- *if no optimal geodesic between x and y is conjugated at y then it exists C such that*

$$p_t(x, y) = \frac{C + O(t)}{t} e^{-d^2(x, y)/4t},$$

- *otherwise, there is only one optimal geodesic between x and y , which is conjugated at y and it exists C such that*

$$p_t(x, y) = \frac{C + O(t^{\frac{1}{2}})}{t^{\frac{5}{4}}} e^{-d^2(x, y)/4t}.$$

2 Cut locus at a Grushin point

Let us use the normal form (NF) at a Grushin point given by $F_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $F_2 = \begin{pmatrix} 0 \\ xf(x, y) \end{pmatrix}$, with f smooth such that $f(0, y) = 1$.

Using the Pontryagin Maximum Principle one gets the equations

$$\begin{aligned} \dot{x} &= p_x, & \dot{p}_x &= -p_y^2 x f(x, y) (f(x, y) + x \partial_x f(x, y)), \\ \dot{y} &= p_y (x f(x, y))^2, & \dot{p}_y &= -p_y^2 x^2 f(x, y) \partial_y f(x, y). \end{aligned}$$

Setting $\bar{p} = \frac{p_x}{p_y}$, defining the new time $s = p_y t$, and writing $f(x, y) = 1 + ax + o(x, y)$, one gets the new equations

$$\begin{aligned} \dot{x} &= \bar{p}, \\ \dot{y} &= x^2 + 2ax^3 + x^2 o(x, y), \\ \dot{\bar{p}} &= -x - 3ax^2 + x o(x, y). \end{aligned}$$

Initial condition is $(x = 0, y = 0, \bar{p} = \pm \rho)$ where $\rho = \frac{1}{p_y(0)}$. We look at the developments of x , y and \bar{p} in the parameter ρ that is

$$\begin{aligned} x(\rho, s) &= \rho x_1(s) + \rho^2 x_2(s) + O(\rho^3), \\ y(\rho, s) &= \rho^2 y_2(s) + \rho^3 y_3(s) + O(\rho^4), \\ \bar{p}(\rho, s) &= \rho \bar{p}_1(s) + \rho^2 \bar{p}_2(s) + O(\rho^3). \end{aligned}$$

We get the equations

$$\begin{aligned} \dot{x}_1 &= \bar{p}_1, & \dot{x}_2 &= \bar{p}_2, & \dot{y}_2 &= x_1^2, \\ \dot{p}_1 &= -x_1, & \dot{p}_2 &= -3ax_1^2 - x_2, & \dot{y}_3 &= 2ax_1^3 + 2x_1x_2, \end{aligned}$$

with the initial condition $x_1(0) = x_2(0) = y_2(0) = y_3(0) = p_2(0) = 0$ and $p_1(0) = 1$. The solution is given by

$$\begin{aligned} x_1(s) &= \sin(s), & \bar{p}_1(s) &= \cos(s), & y_2(s) &= \frac{1}{4}(2s - \sin(2s)), \\ x_2(s) &= -4a \sin^4\left(\frac{s}{2}\right), & \bar{p}_2(s) &= -4a \sin^2\left(\frac{s}{2}\right) \sin(s), & y_3(s) &= \frac{8a}{3}(1 + 2 \cos(s)) \sin^4\left(\frac{s}{2}\right). \end{aligned}$$

If we compute the cut locus for the nilpotent approximation (order -1), we find $x = 0, y = \frac{\pi}{2}\rho^2$ for the upper part and $x = 0, y = -\frac{\pi}{2}\rho^2$ for the lower part. These formulae cannot be a priori stable in the sense that the following developments could make appear terms in ρ^2 in the x variable and hence change the "tangent at 0" of the cut locus.

To look for the upper part of the cut locus for the normal form at order 0, we look for the cut point reached at time $t = \pi\rho_0$. The corresponding geodesic starting with $p_1 = +1$ has ρ and s close to ρ_0 and π that is

$$\begin{aligned} \rho_+^+ &= \rho_0 + \alpha_+^+ \rho_0^2 + o(\rho_0^2), \\ s_+^+ &= \pi + \beta_+^+ \rho_0 + o(\rho_0). \end{aligned}$$

Since $\rho_+^+ s_+^+ = \rho_0 \pi$ we get immediately that $\beta_+^+ = -\alpha_+^+ \pi$ and

$$\begin{aligned} x_+^+ &= (\alpha_+^+ \pi - 4a)\rho_0^2 + O(\rho_0^3), \\ y_+^+ &= \frac{\pi}{2}\rho_0^2 + (\alpha_+^+ \pi - \frac{8a}{3})\rho_0^3 + O(\rho_0^4). \end{aligned}$$

The corresponding geodesic starting with $p_1 = -1$ has ρ and s

$$\begin{aligned} \rho_+^- &= \rho_0 + \alpha_+^- \rho_0^2 + o(\rho_0^2), \\ s_+^- &= \pi + \beta_+^- \rho_0 + o(\rho_0), \end{aligned}$$

Since $\rho_+^- s_+^- = \rho_0 \pi$ we get immediately that $\beta_+^- = -\alpha_+^- \pi$ and

$$\begin{aligned} x_+^- &= (-\alpha_+^- \pi - 4a)\rho_0^2 + O(\rho_0^3), \\ y_+^- &= \frac{\pi}{2}\rho_0^2 + (\alpha_+^- \pi + \frac{8a}{3})\rho_0^3 + O(\rho_0^4). \end{aligned}$$

Since (x_+^+, y_+^+) and (x_+^-, y_+^-) should be the same point we get

$$\begin{aligned} \alpha_+^+ \pi - 4a &= -\alpha_+^- \pi - 4a, \\ \alpha_+^+ \pi - \frac{8a}{3} &= \alpha_+^- \pi + \frac{8a}{3}, \end{aligned}$$

which implies $\alpha_+^+ = -\alpha_+^- = \frac{8a}{3\pi}$ and that the cut point is

$$(x_{cut}^+, y_{cut}^+) = \rho_0^2 \left(-\frac{4}{3}a, \frac{\pi}{2}\right) + O(\rho_0^3).$$

The same computation for the lower part of the cut locus gives

$$(x_{cut}^-, y_{cut}^-) = \rho_0^2 \left(-\frac{4}{3}a, -\frac{\pi}{2} \right) + O(\rho_0^3).$$

The two formulae for the upper and lower parts are stable in the sense that the terms that could be added by further developments would be of order at least 3 in ρ_0 and hence would not change the tangent at 0. We can now conclude that the cut locus has a corner at 0 when $a \neq 0$ and none when $a = 0$. Theorem 2 is proved.

3 Singularities of the exponential map from a point q close enough to a given Grushin point q_0 . Applications to the heat kernel at q

3.1 Study in the Grushin plane $(\partial_x, x\partial_y)$.

In this section I describe the cut and conjugate loci at the point $(-1, 0)$ in the Grushin plane. In the case of the Grushin plane, whose orthonormal frame is given by $(\partial_x, x\partial_y)$, one can compute explicitly the geodesics. It was done in [17], and the geodesics from $(-1, 0)$ are given by

$$\begin{aligned} x(\theta, t) &= -\frac{\sin(\theta - t \sin(\theta))}{\sin(\theta)}, \\ y(\theta, t) &= \frac{2t \sin(\theta) - 2 \cos(\theta) \sin(\theta) + \sin(2\theta - 2t \sin(\theta))}{4 \sin^2(\theta)}. \end{aligned}$$

if $\theta \neq 0[\pi]$ or

$$\begin{aligned} x(\theta, t) &= -1 + (-1)^{\theta/\pi} t, \\ y(\theta, t) &= 0, \end{aligned}$$

if $\theta = 0[\pi]$. We denote $\gamma(\theta, t) = (x(\theta, t), y(\theta, t))$. The geodesics parameterized by arclength are the $t \mapsto \gamma(\theta, t)$.

The cut locus.

For $\theta \neq 0[\pi]$:

$$\gamma \left(\theta, \frac{\pi}{|\sin(\theta)|} \right) = \gamma \left(\pi - \theta, \frac{\pi}{|\sin(\theta)|} \right) = \left(1, \frac{\pi}{2 \sin(\theta) |\sin(\theta)|} \right).$$

These geodesics being normal geodesics, it implies that their cut time is less or equal to $\frac{\pi}{|\sin(\theta)|}$. Moreover, it is very easy to prove that any geodesic corresponding to $\theta = 0[\pi]$ is optimal until any time t . Let us prove that the cut time is in fact $\frac{\pi}{\sin(\theta)}$ if $\theta \neq 0[\pi]$.

The sphere at time $t > \pi$ is contained in $\{\gamma(\theta, t) \mid \theta \in \Theta_t\}$ where

$$\Theta_t = \left[-\arcsin\left(\frac{\pi}{t}\right), \arcsin\left(\frac{\pi}{t}\right) \right] \cup \left[\pi - \arcsin\left(\frac{\pi}{t}\right), \pi + \arcsin\left(\frac{\pi}{t}\right) \right]$$

since a geodesic with $t \geq \frac{\pi}{|\sin(\theta)|}$ is no more optimal. We are going to prove that $\theta \mapsto x(\theta, t)$ is strictly decreasing on $[0, \arcsin(\frac{\pi}{t})]$ and on $[\pi - \arcsin(\frac{\pi}{t}), \pi]$, which correspond to initial conditions constructing the upper part of the synthesis. If one add the fact the $\gamma(\arcsin(\frac{\pi}{t}), t) = \gamma(\pi - \arcsin(\frac{\pi}{t}), t)$

it will have proved that the upper part of the sphere, being connex, is the set $\{\gamma(\theta, t) \mid \theta \in \Theta_t\}$. Which implies that the cut locus of $(-1, 0)$ is attained at time $\frac{\pi}{\sin(\theta)}$ and hence that the cut locus is

$$\{(x, y) \mid x = 1 \text{ and } |y| \geq \pi/2\}.$$

In order to finish the proof, let us compute for $\frac{\partial x}{\partial \theta}(\theta, t)$ and prove that for $\theta \in]0, \pi[$ and $t \in [0, \frac{\pi}{\sin(\theta)}]$ it is not positive.

$$\begin{aligned} -\sin^2(\theta) \frac{\partial x}{\partial \theta}(\theta, t) &= \sin(t \sin(\theta)) - t \cos(\theta) \sin(\theta) \cos(\theta - t \sin(\theta)) \\ &= \sin(u) - u \cos(\theta) \cos(\theta - u) \\ &= \sin(u)(1 - u \cos(\theta) \sin(\theta)) - \cos(u)u \cos^2(\theta) \end{aligned}$$

where $u = t \sin(\theta)$. For $t = 0$, $\frac{\partial x}{\partial \theta}(\theta, t) = 0$. And for t small $\frac{\partial x}{\partial \theta}(\theta, t) \sim -t \sin(\theta) < 0$. Now we can conclude by proving that $\frac{\partial x}{\partial \theta}(\theta, t) \neq 0$ for $0 < t < \frac{\pi}{\sin(\theta)}$.

With the last equation we can see that $\frac{\partial x}{\partial \theta}(\theta, t) = 0$ if and only if $(\cos(u), \sin(u))$ is parallel to $(1 - u \cos(\theta) \sin(\theta), u \cos^2(\theta))$. We just have to prove now that with $\theta \in]0, \pi[$ and $u \in]0, \pi[$ it is not possible. This is a relatively simple exercise of geometry. Let us make the proof for $\theta \in]0, \pi/2[$, the proof being the same for $\theta \in]\pi/2, \pi[$ and very easy for $\theta = \pi/2$.

Let us fix the angles of the two vectors to be zero for $u = 0$. The first vector, $(\cos(u), \sin(u))$ has for angle u . The second vector $(1 - u \cos(\theta) \sin(\theta), u \cos^2(\theta)) = (1, 0) - u \cos(\theta)(\sin(\theta), -\cos(\theta))$ has norm larger than $\cos(\theta)$ if $u \neq \sin(\theta)$ and a derivative with respect to u of norm $\cos(\theta)$. Hence the derivative of its angle with respect to u is less than 1 (and positive) for $u \neq \sin(\theta)$ which allows to prove that its angle is positive and less than u for $u > 0$. As a conclusion, if $u < \pi$, the angles of the two vectors cannot be equal modulo π . Which finishes the proof.

Remark 5 One computes easily that $\frac{\partial \gamma}{\partial t}(\theta, \frac{\pi}{\sin(\theta)})$ and $\frac{\partial \gamma}{\partial t}(\pi - \theta, \frac{\pi}{\sin(\theta)})$ are not parallel hence the wave front is transversal to itself along the cut locus.

The first conjugate locus. The Jacobian of the map γ is

$$Jac(\theta, t) = \frac{t \cos(\theta) \cos(\theta - t \sin(\theta)) \sin(\theta) - \sin(t \sin(\theta))}{\sin^3(\theta)}.$$

when $\theta \neq 0[\pi]$. One proves easily that if $\theta \neq 0[\pi]$, then there is a conjugate time t_θ . Moreover at $t = t_\theta$ one has that the vectors $(\cos(t_\theta \sin(\theta)), \sin(t_\theta \sin(\theta)))$ and $(1 - t_\theta \cos(\theta) \sin^2(\theta), t_\theta \cos^2(\theta) \sin(\theta))$ are parallel since

$$0 = Jac(\theta, t_\theta) = \frac{t_\theta \cos^2(\theta) \sin(\theta) \cos(t_\theta \sin(\theta)) - (1 - t_\theta \cos(\theta) \sin^2(\theta)) \sin(t_\theta \sin(\theta))}{\sin^3(\theta)}.$$

One can compute

$$\frac{\partial \gamma}{\partial \theta}(\theta, t) = \frac{1}{\sin^3(\theta)} Jac(\theta, t) (\sin(\theta), -\cos(\theta - t \sin(\theta))),$$

which proves that at the conjugate time $\frac{\partial \gamma}{\partial \theta}(\theta, t_\theta) = 0$.

In order to understand which singularity has the map γ at the conjugate time, let compute

$$\begin{aligned}\frac{\partial^2 \gamma}{\partial \theta^2}(\theta, t_\theta) &= \frac{1}{\sin^3(\theta)} \frac{\partial Jac}{\partial \theta}(\theta, t_\theta) (\sin(\theta), -\cos(\theta - t_\theta \sin(\theta))), \\ \frac{\partial \gamma}{\partial t}(\theta, t_\theta) &= \frac{1}{\sin(\theta)} (\cos(\theta - t_\theta \sin(\theta)) \sin(\theta), \sin(\theta - t_\theta \sin(\theta))^2).\end{aligned}$$

Now one can compute the determinant of these two vectors and find at $t = t_\theta$

$$\begin{aligned}Det(\theta) &= \frac{1}{\sin^4(\theta)} \frac{\partial Jac}{\partial \theta}(\theta, t_\theta) \begin{vmatrix} \sin(\theta) & -\cos(\theta - t_\theta \sin(\theta)) \\ \cos(\theta - t_\theta \sin(\theta)) \sin(\theta) & \sin(\theta - t_\theta \sin(\theta))^2 \end{vmatrix}, \\ &= \frac{1}{\sin^3(\theta)} \frac{\partial Jac}{\partial \theta}(\theta, t_\theta).\end{aligned}$$

As a consequence, one can conclude that the singularity is of type \mathcal{A}_2 if and only if $\frac{\partial Jac}{\partial \theta}(\theta, t_\theta) \neq 0$. But one can compute that

$$\begin{aligned}\frac{\partial Jac}{\partial \theta}(\theta, t_\theta) &= \frac{t_\theta \sin(\theta)}{4} (\sin(t_\theta \sin(\theta))(2 - 3t_\theta \cos(\theta) + 6 \cos(2\theta) - t_\theta \cos(3\theta)) \\ &\quad + \cos(t_\theta \sin(\theta))(t_\theta \sin(\theta) - 6 \sin(2\theta) + t_\theta \sin(2\theta))).\end{aligned}$$

which implies that $\frac{\partial Jac}{\partial \theta}(\theta, t_\theta) = 0$ if and only if

$$\begin{aligned}(t_\theta \cos^2(\theta) \sin(\theta))(2 - 3t_\theta \cos(\theta) + 6 \cos(2\theta) - t_\theta \cos(3\theta)) \\ + (1 - t_\theta \cos(\theta) \sin^2(\theta))(t_\theta \sin(\theta) - 6 \sin(2\theta) + t_\theta \sin(2\theta)) = 0\end{aligned}$$

Simplifying this last expression one finds

$$-(6 + t_\theta^2 - 6t_\theta \cos(\theta) + t_\theta^2 \cos(2\theta)) \sin(2\theta) = 0$$

One proves easily that this expression is zero if and only if $\theta = 0[\frac{\pi}{2}]$. Hence, for any value of $\theta \neq 0[\frac{\pi}{2}]$, the singularity at the conjugate time is of type \mathcal{A}_2 .

To understand completely the optimal synthesis from $(-1, 0)$, it remains to understand which singularity is for $\theta = \frac{\pi}{2}[\pi]$ at its conjugate time, which is equal to its cut-time $t_{\frac{\pi}{2}} = \pi$. In order to do that let us make the change of variables $\theta = \frac{\pi}{2} + \theta_1$, $s = \pi + s_1$ and still denote the map γ . Then the Taylor series up to order 3 of γ at $(\theta_1 = 0, s_1 = 0)$ is

$$\gamma(\theta_1, s_1) = (1 + -\frac{s_1^2}{2} + \theta_1 s_1 + \frac{1}{2} (\pi s_1 \theta_1^2 - \pi \theta_1^3), \frac{\pi}{2} + s_1 - \frac{s_1^3}{3} + \theta_1 s_1^2 - \frac{\theta_1^2 s_1}{2}) + o_3(\theta_1, s_1).$$

Making the change of coordinates $x_1 = x - 1 + \frac{(y - \frac{\pi}{2})^2}{2}$, $y_1 = y - \frac{\pi}{2}$, still denoting the map γ , we found

$$\gamma(\theta_1, s_1) = (\theta_1 s_1 + \frac{1}{2} (\pi s_1 \theta_1^2 - \pi \theta_1^3), s_1 - \frac{s_1^3}{3} + \theta_1 s_1^2 - \frac{\theta_1^2 s_1}{2}) + o_3(\theta_1, s_1).$$

By changing for the variables $\theta_2 = \theta_1 + \frac{1}{2} \pi \theta_1^2$, $s_2 = s_1 - \frac{s_1^3}{3} + \theta_1 s_1^2 - \frac{\theta_1^2 s_1}{2}$, still denoting the map γ , we found

$$\gamma(\theta_2, s_2) = (\theta_2 s_2 - \frac{1}{2} \pi \theta_2^3, s_2) + o_3(\theta_2, s_2).$$

Finally, making the changes $y_2 = -\frac{\pi}{2} y_1$, $x_2 = -\frac{\pi}{2} x_1$, $s_3 = \frac{\pi}{2} s_2$, $\theta_3 = \theta_2$, we get

$$\gamma(\theta_3, s_3) = (\theta_3^3 - \theta_3 s_3, s_3) + o_3(\theta_3, s_3),$$

which proves that the singularity at the first conjugate locus for $\theta = \frac{\pi}{2}[\pi]$ is \mathcal{A}_3 .

3.2 Local cut and conjugate loci of a Riemannian point lying in the neighborhood of a Grushin point

Assume that we are close enough to a Grushin point in order the normal form (F_1, F_2) applies. In that case we can compute the jets of the exponential map from $(-a, 0)$ with respect to the small parameter a . From the computation done in the case of the nilpotent approximation, we can deduce that γ_a the exponential map from $(-a, 0)$ has the following expression

$$\begin{aligned} x_a(\theta, t) &= -a \left(\frac{\sin(\theta - \frac{t}{a} \sin(\theta))}{\sin(\theta)} \right) + O(a^2), \\ y_a(\theta, t) &= a^2 \left(\frac{2\frac{t}{a} \sin(\theta) - 2 \cos(\theta) \sin(\theta) + \sin(2\theta - 2\frac{t}{a} \sin(\theta))}{4 \sin^2(\theta)} \right) + O(a^3), \end{aligned}$$

which proves that $(\theta, s) \mapsto (\frac{x_a}{a}, \frac{y_a}{a^2})(\theta, as) = (x, y)(\theta, s) + O(a)$. As seen before, the map γ has only stable singularities at its first conjugate locus and the front is transversal to itself at the cut locus outside the conjugate points. It implies that $(\theta, s) \mapsto (\frac{x_a}{a}, \frac{y_a}{a^2})(\theta, as)$, as one parameter family of perturbation of the map (x, y) , has the same singularities at the conjugate locus and have a wave front transversal to itself at the cut locus outside the conjugate locus, at least on any compact for a small enough. We can assume that the compact corresponds to $0 \leq s \leq 2\pi$, that way we are sure that the cut-conjugate points are in the interior of the compact set. Hence the map $(\theta, t) \mapsto (\frac{x_a}{a}, \frac{y_a}{a^2})(\theta, t)$ has the same singularities at the conjugate locus and have a wave front transversal to itself at the cut locus outside the conjugate locus, for a small enough and $0 \leq t \leq 2\pi a$. Which implies that γ_a has for local cut locus and local conjugate locus the same picture as the map γ : only two cut-conjugate points, where the singularity is \mathcal{A}_3 , other first conjugate points with singularity \mathcal{A}_2 and the cut locus is the union of two disjoint curves issued from the cut-conjugate points along which the wave front is transversal to itself. This finishes the proof of theorem 3.

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