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Jean Michel. ”Case-free” derivation for Weyl groups of the number of reflection factorisations of a Coxeter element. 2014. hal-01052484

HAL Id: hal-01052484
https://hal.science/hal-01052484v1

Preprint submitted on 26 Jul 2014

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“Case-free” derivation for Weyl groups of the number of reflection factorizations of a Coxeter element

Jean Michel

8th July 2014

Let $W \subset \mathrm{GL}(\mathbb{C}^n)$ be an irreducible well-generated complex reflection group, let $\mathcal{R}$ be the set of its reflections, $\mathcal{R}^*$ the set of its reflecting hyperplanes, and let c be a Coxeter element of W (see [CT, Remark 1.3]).

In [CT, Theorem 1.1], Chapuy and Stump obtain a very nice generating series for the number $N_l := |\{r_1, \ldots, r_l \in \mathcal{R}^l \mid r_1 \ldots r_l = c\}|$ of factorizations of c into the product of l elements of $\mathcal{R}$. Their formula is

$$\sum_{l \geq 0} \frac{t^l}{l!} N_l = \frac{1}{|W|}(e^{t|\mathcal{R}|/n} - e^{-t|\mathcal{R}^*|/n})^n.$$

Their method is to obtain a character-theoretic expression for N_l, that they proceed to evaluate case by case. My observation is that, in the case of Weyl group, a uniform evaluation of their character-theoretic expression can be done using properties of Deligne-Lusztig representations. I thank Christian Stump for making me aware of the problem, and for a careful reading of this text.

We start with [CT, formula above (10)] which states that if S is the element of the group algebra $\mathbb{C}W$ given by $\sum_{r \in \mathcal{R}} r$, then $|W|N_l = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1)\chi(S^l c^{-1})$. We observe that S is in the center of $\mathbb{C}W$, thus the formula can also be written

$$|W|N_l = \sum_{\chi \in \mathrm{Irr}(W)} \chi(S^l)\chi(c^{-1}) \qquad (1).$$

For $\chi \in \mathrm{Irr}(W)$ write the fake degree of χ as $x^{e_1} + \ldots + x^{e_{\chi(1)}}$ and define $N(\chi^*) = \sum_{i=1}^{\chi(1)} e_i$, where χ^* denotes the complex conjugate (see for example [BM, §4.B]). We will need the following property:

Lemma 1. *Assume that $|C_W(H)|$ has a common value e for all $H \in \mathcal{R}^*$. Then $\chi(S) = |\mathcal{R}|\chi(1) - N(\chi) - N(\chi^*)$.*

Proof. For $H \in \mathcal{R}^*$, whose fixator $C_W(H)$ is generated by a reflection s_H of hyperplane H and non-trivial eigenvalue $\zeta = \exp(2i\pi/e)$, let $m_{H,i}(\chi)$ be the

multiplicity of the eigenvalue ζ^i of s_H in the representation affording χ. Then we get

$$\chi(\sum_{j=1}^{e-1} s_H^j) = \sum_{i=0}^{e-1} m_{H,i}(\chi) \sum_{j=1}^{e-1} \zeta^{ij} = (e-1)m_{H,0}(\chi) - \sum_{i=1}^{e-1} m_{H,i}(\chi).$$

Now the formulae in [BM, Corollaire 4.2] read, in our case where $|C_W(H)|$ has a common value e:

$$\sum_{H\in\mathcal{R}^*} \sum_{i=1}^{e-1} m_{H,i}(\chi) = \frac{N(\chi)+N(\chi^*)}{e}$$

$$\sum_{H\in\mathcal{R}^*} m_{H,0}(\chi) = \frac{|\mathcal{R}|\chi(1)}{e-1} - \frac{N(\chi)+N(\chi^*)}{e}$$

whence the Lemma. □

Remark 2. Another expression valid without assuming $|C_W(H)|$ constant is $\chi(S) = |\mathcal{R}|\chi(1) - N(\chi) - N(\iota(\chi))$ where ι is *Opdam's involution*; to see this, differentiate with respect to x and then evaluate at $x = 1$ formula [Ma, 6.5]; see also [Ma, 6.8].

We now restrict to the case where W is the Weyl group of a connected reductive algebraic group $\mathbf{G}$ over an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$ with q elements. We assume that $\mathbf{G}$ is defined over $\mathbb{F}_q$ and denote by F the Frobenius endomorphism defining the corresponding $\mathbb{F}_q$-structure. Let $\mathbf{T}$ be an F-stable maximal torus lying in an F-stable Borel subgroup $\mathbf{B}$. We may identify W with $N_{\mathbf{G}}(\mathbf{T})$ and we assume $\mathbf{G}$ split, which means that F acts trivially on W. For $w \in W$, let us denote by R_w the (virtual) character of $\mathbf{G}^F$ defined by the Deligne-Lusztig induction $R_{\mathbf{T}_w}^{\mathbf{G}}(\mathrm{Id})$, where $\mathbf{T}_w$ is an F-stable maximal torus of type w (with respect to $\mathbf{T}$). Here R_w is a $\overline{\mathbb{Q}}_\ell$-character, for some prime number ℓ not dividing q, but we will consider it as a complex character by choosing a suitable embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$. The set $\mathcal{E}(\mathbf{G}^F, 1)$ of constituents of the various R_w is called the set of *unipotent characters* of $\mathbf{G}^F$.

The character R_1 identifies to that of $\mathrm{Ind}_{\mathbf{B}^F}^{\mathbf{G}^F}\mathrm{Id}$, and, since the commuting algebra of this representation is the Hecke algebra $\mathcal{H}$, isomorphic to the group algebra of W, we have a decomposition of the form $R_1 = \sum_{\chi\in\mathrm{Irr}(W)} \chi(1)U_\chi$ where the U_χ are certain characters of $\mathbf{G}^F$ called the *principal series unipotent characters*.

The set $\mathcal{E}(\mathbf{G}^F, 1)$ and the values $\langle R_w, \rho\rangle_{\mathbf{G}^F}$ for $\rho \in \mathcal{E}(\mathbf{G}^F, 1)$ are independent of q; they provide an additional combinatorial structure on W which can actually be entirely determined by the Hecke algebra. In the case where W is the symmetric group, the characters U_χ exhaust the set $\mathcal{E}(\mathbf{G}^F, 1)$ and $\langle R_w, U_\chi\rangle_{\mathbf{G}^F} = \chi(w)$, so Deligne-Lusztig combinatorics bring nothing new. We could characterize their role in the current proof as enabling the use for other Weyl groups of the same features which occur in the proof in the symmetric group case.

By linearity we attach to any $a \in \mathbb{C}W$ a class function R_a on $\mathbf{G}^F$, given if $a = \sum_{w \in W} a_w w$ by $R_a = \sum_w a_w R_w$. In the particular case where a is the idempotent $\frac{1}{|W|}\sum_{w \in W} \chi(w^{-1})w$ attached to $\chi \in \mathrm{Irr}(W)$ we denote by R_χ the corresponding class function. Inverting we get $R_w = \sum_\chi \chi(w) R_\chi$. By [Lu2, 3.19.2] we have $\langle R_\chi, R_\psi \rangle_{\mathbf{G}^F} = \delta_{\chi,\psi}$. It follows that for any two elements $a, b \in \mathbb{C}W$ we have $\langle R_a, R_b \rangle_{\mathbf{G}^F} = \sum_{\chi \in \mathrm{Irr}(W)} \chi(a)\overline{\chi(b)}$. Thus formula (1) becomes

$$|W| N_l = \langle R_{S^l}, R_c \rangle. \tag{1'}$$

Lemma 3. *We have* $\frac{\chi(S)}{\chi(1)} = |\mathcal{R}| - a_\chi - A_\chi$, *where* a_χ *(resp.* A_χ*) is the valuation (resp. the degree) of the generic degree of* $\mathcal{H}$ *attached to* χ *(see for example [BM, §2.B]).*

Proof. This is just a translation of Lemma 1 in terms of the invariants coming from the Hecke algebra. In the case of Coxeter groups in Lemma 1 we have $e = 2$ and $\chi = \chi^*$. Thus Lemma 1 for Coxeter groups becomes $\chi(S) = \chi(1)|\mathcal{R}| - 2N(\chi)$. We then conclude by [BM, formula 4.21] which states that in Coxeter groups $N(\chi) = \chi(1)\frac{a_\chi + A_\chi}{2}$. □

Lemma 4. *For* $\rho \in \mathcal{E}(\mathbf{G}^F, 1)$ *we have*

$$\langle R_{S^l}, \rho \rangle_{\mathbf{G}^F} = \begin{cases} \chi(S^l) & \text{if } \rho = U_\chi \text{ for some } \chi \in Irr(W) \\ 0 & \text{otherwise} \end{cases}.$$

Proof. The blocks of the matrix $\{\langle R_\chi, \rho \rangle_{\mathbf{G}^F}\}_{\chi \in \mathrm{Irr}(W), \rho \in \mathcal{E}(\mathbf{G}^F, 1)}$ are called the *Lusztig families*. They constitute thus a partition Ξ of $\mathcal{E}(\mathbf{G}^F, 1)$ such that for a "family" $\mathcal{F} \in \Xi$ we have:

- If $U_\chi \in \mathcal{F}$, $\rho \notin \mathcal{F}$ then $\langle R_\chi, \rho \rangle_{\mathbf{G}^F} = 0$.

Given a family $\mathcal{F}$, the invariants a_χ and A_χ take a constant value on the $\chi \in \mathrm{Irr}(W)$ such that $U_\chi \in \mathcal{F}$ (see, for example [Lu3, 4.23, 5.25 and 5.27]), thus $\chi(S)/\chi(1)$ takes a constant value that we will denote $c_\mathcal{F}$ on a family $\mathcal{F}$. Thus, for $\rho \in \mathcal{F}$ we have:

$$\begin{aligned} \langle R_{S^l}, \rho \rangle_{\mathbf{G}^F} &= \langle \sum_{\chi \in \mathrm{Irr}(W)} \chi(S^l) R_\chi, \rho \rangle_{\mathbf{G}^F} = \sum_{\{\chi | U_\chi \in \mathcal{F}\}} \chi(S^l) \langle R_\chi, \rho \rangle_{\mathbf{G}^F} \\ &= c_\mathcal{F}^l \sum_{\{\chi | U_\chi \in \mathcal{F}\}} \chi(1) \langle R_\chi, \rho \rangle_{\mathbf{G}^F} = c_\mathcal{F}^l \langle \sum_{\chi \in \mathrm{Irr}(W)} \chi(1) R_\chi, \rho \rangle_{\mathbf{G}^F} \\ &= \begin{cases} 0 & \text{unless } \rho \text{ is a } U_\chi \\ c_\mathcal{F}^l \chi(1) = \chi(S^l) & \text{if } \rho = U_\chi \end{cases} \end{aligned}$$

□

Lemma 5. *If* c *is a Coxeter element and* $\chi \in Irr(W)$ *then* $\langle R_c, U_\chi \rangle_{\mathbf{G}^F} = 0$ *unless* χ *is an exterior power of the reflection character ref of* W*. Moreover* $\langle R_c, U_{\wedge^i ref} \rangle_{\mathbf{G}^F} = (-1)^i$.

Proof. Let $\mathbf{X}_c$ be the Deligne-Lusztig variety defining R_c, so that for $g \in \mathbf{G}^F$ we have $R_c(g) = \sum_i (-1)^i \operatorname{Trace}(g \mid H^i_c(\mathbf{X}_c, \overline{\mathbb{Q}}_\ell))$. By [Lu1, 6.7 (ii)] applied with $I = \emptyset$ and $R_0 = \mathrm{Id}$, we know that in each cohomology group $H^i_c(\mathbf{X}_c, \overline{\mathbb{Q}}_\ell)$ with $i = n, n+1, \ldots, 2n$, there is exactly one irreducible representation $U_{\chi_i^W}$ in the principal series and that it has multiplicity 1. The fact that $\chi_i^W = \wedge^{2n-i} \operatorname{ref}$ is [Lu1, remark 7.8] applied with $I = \emptyset$ and $R_0 = \mathrm{Id}$. In this case the Hecke algebra $\mathcal{H}$ of loc. cit. is the same as our algebra $\mathcal{H}$ and the i-th power of the reflection representation of $\mathcal{H}$ defines precisely the character $U_{\wedge^i \operatorname{ref}}$. □

Using Lemmas 4 and 5 to evaluate (1′) we get:

$$|W|N_l = \sum_{i=0}^{n} (-1)^i \wedge^i \operatorname{ref}(S^l). \tag{1''}$$

Lemma 6. *For any well-generated irreducible complex reflection group, formula* (1″) *is equivalent to the Chapuy-Stump formula.*

Proof. Since the representations $\wedge^i \operatorname{ref}$ are irreducible by a result of Steinberg (see [Bou, §2 ex. 3]), formula (1″) can be written

$$|W|N_l = \sum_{i=0}^{n} (-1)^i (\wedge^i \operatorname{ref}(S)/ \wedge^i \operatorname{ref}(1))^l \wedge^i \operatorname{ref}(1).$$

Let us pick a reflecting hyperplane H, let $e = |C_W(H)|$ and $\zeta = \exp(2i\pi/e)$ and let s be a reflection with eigenvalue ζ which generates $C_W(H)$. Let us compute $\sum_{j=1}^{j=e-1} \wedge^i \operatorname{ref}(s^j)$. We may choose a basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$ such that $e_i \in H$ for $i = 1, \ldots, n-1$ and $se_n = \zeta e_n$. A basis of $\wedge^i \mathbb{C}^n$ consists of $e_I = e_{a_1} \wedge \ldots \wedge e_{a_i}$ where $I = \{a_1, \ldots, a_i\}$ with $a_1 < \ldots < a_i$ runs over all subsets of $\{1, \ldots, n\}$ of cardinality i. We have $\sum_{j=1}^{e-1} s^j(e_I) = \begin{cases} (e-1)e_I & \text{if } n \notin I \\ -e_I & \text{otherwise} \end{cases}$ whence $\sum_{j=1}^{j=e-1} \wedge^i \operatorname{ref}(s^j) = (e-1)\binom{n}{i} - e\binom{n-1}{i-1}$ and $\wedge^i \operatorname{ref}(S) = |\mathcal{R}|(\binom{n}{i} - \binom{n-1}{i-1}) - |\mathcal{R}^*|\binom{n-1}{i-1}$. We finally get $\wedge^i \operatorname{ref}(S)/ \wedge^i \operatorname{ref}(1) = |\mathcal{R}|(1 - \frac{i}{n}) - |\mathcal{R}^*|\frac{i}{n}$ and

$$|W|N_l = \sum_{i=0}^{n} (-1)^i (|\mathcal{R}|(1 - \frac{i}{n}) - |\mathcal{R}^*|\frac{i}{n})^l \binom{n}{i},$$

which is exactly what one gets when expanding the Chapuy-Stump formula. □

Note. I believe that Lemmas 4 and 5 hold for any Spetsial complex reflection group, with an appropriate definition of a formal set of unipotent characters (see [BMM]). I checked it by computer for the primitive irreducible Spetsial complex reflection groups using [Mi].

For Spetsial groups whose reflections have order 2 the proof of Lemma 4 remains formally valid since Lemma 3 remains true for such groups, thus also the fact that $\chi(S)/\chi(1)$ is constant on families. When reflections do not all have order 2, there are examples where $\chi(S)/\chi(1)$ is not constant on families so the proof of Lemma 4 has to change.

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