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Luigi de Pascale, Jean Louet, Filippo Santambrogio. The Monge problem with vanishing gradient penalization: vortices and asymptotic profile. *Journal de Mathématiques Pures et Appliquées*, 2016, 106 (2), pp.237-279 10.1016/j.matpur.2016.02.009 . hal-01052294v1

**HAL Id: hal-01052294**

**<https://hal.science/hal-01052294v1>**

Submitted on 25 Jul 2014 (v1), last revised 30 Dec 2016 (v2)

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# A first analysis of the Monge problem with vanishing gradient penalization

Luigi De Pascale\*, Jean Louet†, Filippo Santambrogio†

July 25, 2014

## Abstract

We investigate the approximation of the Monge problem (minimizing  $\int_{\Omega} |T(x) - x| d\mu(x)$  among the vector-valued maps  $T$  with prescribed image measure  $T_{\#}\mu$ ) by adding a vanishing Dirichlet energy, namely  $\varepsilon \int_{\Omega} |DT|^2$ , where  $\varepsilon \rightarrow 0$ . We study the  $\Gamma$ -convergence as  $\varepsilon \rightarrow 0$ , proving a density result for Sobolev (or Lipschitz) transport maps in the class of transport plans. In a certain two-dimensional framework that we analyze in details, when no optimal plan is induced by an  $H^1$  map, we study the selected limit map, which is a new “special” Monge transport, different from the monotone one, and we find the precise asymptotics of the optimal cost depending on  $\varepsilon$ , where the leading term is of order  $\varepsilon |\log \varepsilon|$ .

**Keywords.** Monge problem, monotone transport,  $\Gamma$ -convergence

**AMS subject classification.** 49J30, 49J45

## Introduction

This paper investigates the following minimization problem: given  $\mu, \nu$  two - smooth enough - probability densities on  $\mathbb{R}^d$  with  $\mu$  supported in a domain  $\Omega$ , we study

$$\inf \{J_{\varepsilon}(T) : T_{\#}\mu = \nu\} \quad \text{where} \quad J_{\varepsilon}(T) = \int_{\Omega} |T(x) - x| d\mu(x) + \varepsilon \int_{\Omega} |DT|^2 dx.$$

Here  $\varepsilon$  is a vanishing positive parameter,  $|\cdot|$  is the usual Euclidean norm on  $\mathbb{R}^d$  or  $M_d(\mathbb{R})$ ,  $DT$  denotes the Jacobian matrix of the vector-valued map  $T$  and  $T_{\#}\mu$  is the image measure of  $\mu$  by the map  $T$  (defined by  $T_{\#}\mu(B) = \mu(T^{-1}(B))$  for any Borel set  $B \subset \mathbb{R}^d$ ). We aim to understand the behavior of the functional  $J_{\varepsilon}$  in the sense of  $\Gamma$ -convergence and of characterize the limits of the minimizers  $T_{\varepsilon}$ .

If we do not take in to account this gradient penalization, we recover the classical optimal transport problem originally proposed by Monge in the 18th century [27]. For this problem, the particular constraint  $T_{\#}\mu = \nu$  makes the existence of minimizers quite difficult to obtain by the direct method of the calculus of variations; when the Euclidean distance is replaced with nicer functions (usually strictly convex with respect to the difference  $x - T(x)$ ), strong progresses have been realized by Kantorovich in the '40s [20, 21] and Brenier in the late '80s [6]. In the Monge's case of the Euclidean distance, the existence results have been shown more recently by several techniques: we just mention the first approach by Sudakov [30] (later completed by Ambrosio [1]) and the differential equations methods by Evans and Gangbo [15], for the case of the Euclidean norm. More recently, this has been generalized to uniformly convex norms by approximation [11, 31]; finally, [12] generalizes this result for any generic norm in  $\mathbb{R}^d$ . We refer to [32] for a complete overview of the optimal transport theory, and, again, the lecture notes [1] for the particular case of the Monge problem.

Adding a Sobolev-like penalization is very natural in many applications, for instance in image processing, when these transport maps could model transformations in the space of colors, which are then required to avoid abrupt variations and discontinuities (see [16]). Also, these problems appear in mechanics (see [18]) and computer-science problems [22], where one looks for maps with minimal distance distortion (if possible, local isometries with prescribed image measure).

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However, in this paper we want to concentrate on the mathematical properties of this penalization. Notice first that this precisely allows to obtain very quickly the existence of optimal maps, since we get more compactness and we can this time use the direct method of the calculus of variations (see Prop. 1.4 below; actually, this is trickier in the case where the source measure is not regular, *cf.* [24] and [25, Chapter 1]; this requires to use the theory developed in [4]). Multiplying this penalization with a vanishing parameter  $\varepsilon$  is motivated by the particular structure of the Monge problem. It is known that the minimizers of  $J_0$  (that we will denote by  $J$  in the following) are not unique and are exactly the transport maps from  $\mu$  to  $\nu$  which also send almost any source point  $x$  to a point  $T(x)$  belonging to the same *transport ray* as  $x$  (see below the precise definition). Among these transport maps, selection results are particularly useful, and approximating with strictly convex transport costs  $c_\varepsilon(x, y) = |x - y| + \varepsilon|x - y|^2$  brings to the *monotone transport* (*i.e.* the unique transport map which is non-decreasing along each transport ray). Some regularity properties are known about it (continuity in the case of regular measures with disjoint and convex supports in the plane [17], uniform estimates on an approximating sequence under more general assumptions [23]). The question is to know which of these transport map is selected by the approximation through the gradient penalization that we propose here. This is a natural question, and, due to its higher regularity as other maps, one could wonder if it is again the monotone map in this case.

For this, we analyze the behavior of  $J_\varepsilon$  when  $\varepsilon$  vanishes in the sense of the  $\Gamma$ -convergence (see [5] for the definitions and well-known results about this notion). The convergence of  $J_\varepsilon$  to the transport energy  $J$  is straightforward but requires a density result of the set of Sobolev maps  $T \in H^1(\Omega)$  sending  $\mu$  to  $\nu$  among the set of transport maps. This result looks natural and can of course be used in other contexts, but was surprisingly not so much investigated so far:

**Theorem.** *Let  $\Omega, \Omega'$  be two bounded open star-shaped subsets of  $\mathbb{R}^d$  with sufficiently smooth boundaries. Let  $\mu \in \mathcal{P}(\Omega), \nu \in \mathcal{P}(\Omega')$  be two probability measures, both absolutely continuous with respect to the Lebesgue measure, with densities  $f, g$ ; assume that  $f, g$  belong to  $C^{0,\alpha}(\overline{\Omega}), C^{0,\alpha}(\overline{\Omega'})$  for some  $\alpha > 0$  and are bounded from above and below by positive constants. Then the set*

$$\{T \in \text{Lip}(\Omega) : T_{\#}\mu = \nu\}$$

*is non-empty, and is a dense subset of the set*

$$\{T : \Omega \rightarrow \Omega' : T_{\#}\mu = \nu\}$$

*endowed with the norm  $\|\cdot\|_{L^2(\Omega)}$ .*

*As a consequence, under these assumptions on  $\Omega, \Omega', \mu, \nu$ , we have  $J_\varepsilon \xrightarrow{\Gamma} J$  as  $\varepsilon \rightarrow 0$ .*

Notice that the above result would be completely satisfactory if the assumptions on the measures in order to get density of Lipschitz maps were the same as to get existence of at least one such a map. The assumptions that we used are likely not to be sharp, but are the most natural one if one wants to guarantee the existence of  $C^{1,\alpha}$  transport maps (and it is typical in regularity theory that Lipschitz regularity is not easy to provide, whereas Hölder results work better; notice on the contrary that a well-established  $L^p$  theory is not available in this framework). Anyway, from the proof that we give in section 2, it is clear that we are not using much more than the simple existence of Lipschitz maps.

Now a natural question concerns the behavior of the remainder with respect to the order  $\varepsilon$ . Namely, if we denote by  $W_1$  the optimal value of the Monge problem (*i.e.* the Wasserstein distance between  $\mu_1$  and  $\mu_2$ ), we need to consider  $\frac{J_\varepsilon - W_1}{\varepsilon}$ . The result is the most natural that we expect (this time with an almost a trivial proof):

$$\frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} \xrightarrow{\Gamma} \mathcal{H} \tag{1}$$

$$\text{where } \mathcal{H}(T) = \begin{cases} \int_{\Omega} |DT|^2 & \text{if } T \in \mathcal{O}_1(\mu, \nu) \cap H^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

where we have denoted by  $\mathcal{O}_1(\mu, \nu)$  the set of optimal maps for the Monge problem. Notice that this result allows immediately to build some examples of measures  $\mu, \nu$  for which the minimal value of the function  $\mathcal{H}$  is not attained by the monotone transport map from  $\mu$  to  $\nu$ , thanks to suitable analysis of the minimization of the  $H^1$ -norm among the set of transport maps on the real line (which has been very partially treated in [26]).

The convergence (1) gives immediately a first order approximation (meaning  $\inf J_\varepsilon = W_1 + \varepsilon \inf \mathcal{H} + o(\varepsilon)$ ) provided that  $\inf \mathcal{H} \neq +\infty$ , *i.e.* when there exists at least one map which minimizes the Monge cost and belongs to the Sobolev space  $H^1(\Omega)$ . If no such map exists, the question arises to investigate the order of convergence of  $\inf J_\varepsilon$  to  $W_1$ , the  $\Gamma$ -limit of the “rest” (provided that we are able to correctly define this rest), and which map is selected as  $\varepsilon$  goes to 0.

The rest of this paper is devoted to the complete study of this case in a particular example, which has very interesting properties in these issues. We take as source domain the quarter disk in the plane  $\Omega = \{x = (r, \theta) : 0 < r < 1, 0 < \theta < \pi/2\}$  and as target domain an annulus located between two regular curves in polar coordinates  $\Omega' = \{x = (r, \theta) : R_1(\theta) < r < R_2(\theta), 0 < \theta < \pi/2\}$ ; we endow these domains with two regular densities  $f, g$  satisfying the condition:

$$\text{for any } \theta, \quad \int_0^1 f(r, \theta) r \, dr = \int_{R_1(\theta)}^{R_2(\theta)} g(r, \theta) r \, dr$$

which means that the mass (with respect to  $f$ ) of the segment with angle  $\theta$  joining the origin to the quarter unit circle is equal to the mass (with respect to  $g$ ) of the segment with same angle joining the two boundaries of  $\Omega'$ . Under these assumptions, the transport rays are the lines starting from the origin and the optimal transport maps for the Monge problem send each  $x \in \Omega$  onto a point  $T(x)$  with same angle; in other words, any optimal  $T$  is written as

$$T(x) = \varphi(x) \frac{x}{|x|}$$

for some scalar function  $\varphi$  with  $\inf \varphi > 0$ . It then follows from an elementary computation that, for such a  $T$ ,  $\int_\Omega |DT|^2 = +\infty$ ; thus, this is precisely a framework where  $\mathcal{H} = +\infty$  and we cannot have  $\inf J_\varepsilon = W_1 + O(\varepsilon)$ . Indeed, we obtain a logarithmic term, namely

$$\inf J_\varepsilon = W_1 + \frac{K}{3} \varepsilon |\log \varepsilon| + O(\varepsilon) \quad \text{where} \quad K = \inf \left\{ \int_0^{\pi/2} (\varphi^2 + \varphi'^2) : R_1(\theta) \leq \varphi(\theta) \leq R_2(\theta) \right\} \quad (2)$$

which means that  $K$  is the smallest “cost” (in the sense of the  $H^1$ -norm on the interval  $[0, \pi/2]$ ) of a curve in polar coordinates belonging to the target domain  $\Omega'$ . In order to justify, a formal computation suggests that, if  $T_\varepsilon \rightarrow T$  for an optimal  $T$ ,

$$J_\varepsilon(T_\varepsilon) = W_1 + \frac{1}{3} \|\varphi(0, \cdot)\|_{H^1(0, \pi/2)}^2 \varepsilon |\log \varepsilon| + O(\varepsilon)$$

if  $T(x) = \varphi(r, \theta) \frac{x}{|x|}$ . In other words, some new phenomena appear:

- the approximation of the minimal value of the Monge problem has the order  $\varepsilon |\log \varepsilon|$ ;
- the main “rest” (namely,  $(J_\varepsilon(T_\varepsilon) - W_1)/(\varepsilon |\log \varepsilon|)$ ) involves only the behavior of  $T$  around the origin, which is, in this case, the only crossing point of all the transport rays (and also a common singular point for any element of  $\mathcal{O}_1(\mu, \nu)$ );
- it could be the case that this rest is not minimized (among the transport maps from  $\mu$  to  $\nu$ ) by the monotone transport map (if the function which realizes the infimum in (2), that we call  $\Phi$  in that follows, differs from the lower boundary  $R_1$ ).

These remarks are consequences of the following main result of this paper:

**Theorem.** *With the above notations, let us assume that  $R_1, R_2$  are Lipschitz functions of  $\theta$  and  $f, g$  are both Lipschitz and bounded from above and from below by positive constants on  $\overline{\Omega}, \overline{\Omega'}$ . We denote by*

$$F_\varepsilon = \frac{1}{\varepsilon} \left( J_\varepsilon - W_1 - \frac{K}{3} \varepsilon |\log \varepsilon| \right)$$

and by  $F$  the function of  $T$  which is equal to  $+\infty$  when  $T$  does not belong to  $\mathcal{O}_1(\mu, \nu)$  and such that, for  $T(x) = \varphi(r, \theta) \frac{x}{|x|}$  belonging to  $\mathcal{O}_1(\mu, \nu)$ , we have

$$F(T) = \int_0^1 \frac{\|\varphi(r, \cdot)\|_{H^1(0, \pi/2)}^2 - K}{r} \, dr + \int_0^1 \|\partial_r \varphi(r, \cdot)\|_{L^2(0, \pi/2)}^2 r \, dr$$

1. For any family of maps  $(T_\varepsilon)_\varepsilon$  such that  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is bounded, there exists a sequence  $\varepsilon_k \rightarrow 0$  and a map  $T$  such that  $T_{\varepsilon_k} \rightarrow T$  in  $L^2(\Omega)$ .
2. There exists a constant  $C$ , depending only on the domains  $\Omega$ ,  $\Omega'$  and on the measures  $f$ ,  $g$ , so that, for any family of maps  $(T_\varepsilon)_{\varepsilon>0}$  with  $T_\varepsilon \rightarrow T$  as  $\varepsilon \rightarrow 0$  in  $L^2(\Omega)$ , we have

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(T_\varepsilon) \geq F(T) - C$$

3. Moreover, there exists at least one family  $(T_\varepsilon)_{\varepsilon>0}$  such that  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is indeed bounded.

Notice that this result does not give precisely the  $\Gamma$ -limit of the functional  $F_\varepsilon$  but only a lower bound on the  $\Gamma$ -liminf. However, it is enough to obtain some important consequences on the behavior of the approximation. Indeed:

- the fact that there exists a family  $(T_\varepsilon)_\varepsilon$  such that  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is bounded implies that this is the case when we take a sequence  $T_\varepsilon$  of minimizers (and in this case we necessarily have  $T_\varepsilon \rightarrow T$  with  $F(T) < +\infty$ );
- it is possible to prove that  $\|\varphi(r, \cdot)\|_{H^1}^2 - K \geq \|\varphi(r, \cdot) - \Phi\|_{L^2}^2$  (see Lemma 3.1 below). This implies that with the above notations, if  $F(T) < +\infty$ , then  $\varphi(r, \cdot)$  is a continuous function of the variable  $r$  (and valued in the space  $L^2([0, \pi/2])$ ) and we have  $\varphi(0, \cdot) = \Phi$ , the optimizer for the Sobolev norm among the curves which are valued in  $\Omega'$ .

The fact that the  $\varepsilon |\log \varepsilon|$  term in the energy is due to the blow-up of the Sobolev norm at a single point suggests a formal but deep analogy with the Ginzburg-Landau theory (see for instance [3]), where we look at the minimization of

$$u \mapsto \frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u|^2) + \int_{\Omega} |\nabla u|^2$$

with Neumann boundary conditions. Here two terms are contradictory in the functional: the first one suggests to select unit vector fields parallel to the boundary, and the second requires  $H^1$ -regularity, which is impossible since the previous point creates a vortex. In our case, up to a  $90^\circ$  rotation, the situation is similar. The two contradictory phenomenas are the fact that  $T$  has to preserve the transport rays (to optimize the Monge problem) and that it has a finite Sobolev norm; this also leads to the creation of an explosion (a vortex, rotated by  $90^\circ$ ; here the origin, which is sent to a whole curve belonging to the target domain). The excess of order  $\varepsilon |\log \varepsilon|$  is a common feature of the two problems (but the analogy is likely to stop here).

**Plan of the paper.** Section 1 collects general notations and well-known facts about the Monge problem, basic notions of  $\Gamma$ -convergence and some useful and elementary results about the optimal transport with gradient term. In Section 2, we prove the density of the Sobolev transport maps among the transport maps and state the results of  $\Gamma$ -convergence with order 0 and with order 1 of  $(J_\varepsilon)_\varepsilon$  for generic (regular) domains and measures. In Section 3, we study precisely the example of  $\varepsilon |\log \varepsilon|$  approximation in the above framework; the main results and its interpretations are given in Paragraph 3.3, following a formal computation that we present in Paragraph 3.2. Section 4 is completely devoted to the rigorous proof of the main result of this paper (Theorem 3.1).

**Acknowledgements.** This work has mainly been developed during some visits of the second and third authors to the Università di Pisa, which have been possible thanks to the support of ANR project ANR-12-BS01-0014-01 GEOMETRYA and the PGMO project MACRO, of EDF and Fondation Mathématique Jacques Hadamard. This material also is based upon work supported by the National Science Foundation under Grant No. 0932078 000, while the second author was in residence at the Mathematical Science Research Institute in Berkeley, California, during the Fall term 2013. The research of the first author is part of the project 2010A2TFX2 “Calcolo delle Variazioni”, financed by the Italian Ministry of Research and is partially financed by the “Fondi di ricerca di ateneo” of the University of Pisa.

## 1 Preliminary notions

### 1.1 Known facts about the Monge problem

In this section, we recall some well-known facts and useful tools about the optimal transportation problem with the Monge cost  $c(x, y) = |x - y|$ , where  $|\cdot|$  is the Euclidean norm on  $\mathbb{R}^d$ .

Let  $\Omega, \Omega'$  be two bounded open set on  $\mathbb{R}^d$ ,  $\mu \in \mathcal{P}(\Omega)$ ,  $\nu \in \mathcal{P}(\Omega')$  and assume that  $\mu$  is absolutely continuous with respect to the Lebesgue measure with density  $f$ . Then we set

$$W_1(\mu, \nu) = \min \left\{ \int_{\Omega} |T(x) - x| d\mu(x) : T : \Omega \rightarrow \mathbb{R}^d, T_{\#}\mu = \nu \right\}$$

the minimal value of the Monge transport cost from  $\mu$  to  $\nu$ , and

$$\mathcal{O}_1(\mu, \nu) = \left\{ T : \Omega \rightarrow \mathbb{R}^d : T_{\#}\mu = \nu \text{ and } \int_{\Omega} |T(x) - x| d\mu(x) = W_1(\mu, \nu) \right\}$$

the set of optimal transport maps for the Monge cost (if there is no ambiguity, we will simply use the notations  $W_1$  and  $\mathcal{O}_1$ ). Notice that the above Monge problem is a particular issue of its Kantorovich formulation, i.e.

$$\min \left\{ \int_{\Omega \times \Omega'} |y - x| d\gamma(x, y) : \gamma \in \mathcal{P}(\Omega \times \Omega') : (\pi_x)_{\#}\gamma = \mu, (\pi_y)_{\#}\gamma = \nu \right\},$$

and the minimal value of this problem coincides with  $W_1(\mu, \nu)$  as well.

**Theorem 1.1** (Duality formula for the Monge problem). *We have the equality*

$$W_1(\mu, \nu) = \sup \left\{ \int_{\Omega'} u(y) d\nu(y) - \int_{\Omega} u(x) d\mu(x) : u \in \text{Lip}_1(\mathbb{R}^d) \right\}$$

where  $\text{Lip}_1(\mathbb{R}^d)$  denotes the set of 1-Lipschitz functions  $\mathbb{R}^d \rightarrow \mathbb{R}$ . Such optimal Lipschitz functions are called Kantorovich potentials. Moreover, the Kantorovich potential is unique provided that  $\Omega$  is a connected set and that the set  $\{f > 0\} \setminus \Omega'$  is a dense subset of  $\Omega$ .

As a direct consequence of the duality formula, if  $T : \Omega \rightarrow \mathbb{R}^d$  sends  $\mu$  to  $\nu$  and  $u \in \text{Lip}_1(\mathbb{R}^d)$ , we have the equivalence:

$$\begin{cases} T \in \mathcal{O}_1(\mu, \nu) \\ u \text{ is a Kantorovich potential} \end{cases} \iff \text{for } \mu\text{-a.e. } x \in \Omega, u(T(x)) - u(x) = |T(x) - x|$$

We now introduce the following crucial notion of *transport ray*.

**Definition 1.1** (Transport rays). *Let  $u$  be a Kantorovich potential and  $x \in \Omega, y \in \Omega'$ . Then:*

- *the open oriented segment  $(x, y)$  is called transport ray if  $u(y) - u(x) = |y - x|$ ;*
- *the closed oriented segment  $[x, y]$  is called maximal transport ray if any point of  $(x, y)$  is contained into at least one transport ray with same orientation as  $[x, y]$ , and if  $[x, y]$  is not strictly included in any segment with the same property.*

**Proposition 1.1** (Geometric properties of transport rays). *The set of maximal transport rays does not depend on the choice of the Kantorovich potential  $u$  and only depends on the source and target measures  $\mu$  and  $\nu$ . Moreover:*

- *any intersection point of two different maximal transport rays is an endpoint of these both maximal transport rays;*
- *the set of the endpoints of all the maximal transport rays is Lebesgue-negligible.*

These notions allow to prove the existence and to characterize the optimal transports:

**Proposition 1.2** (Existence and characterization of optimal transport maps). *The solutions of the Monge problem exist and are not unique; precisely, a map  $T$  sending  $\mu$  to  $\nu$  is optimal if and only if:*

- *for a.e.  $x \in \Omega$ ,  $T(x)$  belongs to the same maximal transport ray as  $x$ ;*
- *the oriented segment  $[x, T(x)]$  has the same orientation as this transport ray.*

We finish by recalling that, among all maps in  $\mathcal{O}_1(\mu, \nu)$ , there is a special one which has received much attention so far, and by quickly reviewing its properties.

**Proposition 1.3** (The monotone map and a secondary variational problem). *If  $\mu$  is absolutely continuous, there exists a unique transport map  $T$  from  $\mu$  to  $\nu$  such that, for each maximal transport ray  $S$ ,  $T$  is non-decreasing from the segment  $S \cap \Omega$  to the segment  $S \cap \Omega'$  (meaning that if  $x, x' \in \Omega$  belong to the same transport ray, then  $[x, x']$  and  $[T(x), T(x')]$  have the same orientation). Moreover,  $T$  solves the problem*

$$\inf \left\{ \int_{\Omega} |T(x) - x|^2 d\mu(x) : T \in \mathcal{O}_1(\mu, \nu) \right\}.$$

Notice that this solution is itself obtained as limit of minimizers of a perturbed variational problem, namely

$$\inf \left\{ \int_{\Omega} |T(x) - x| d\mu(x) + \varepsilon \int_{\Omega} |T(x) - x|^2 d\mu(x) : T_{\#}\mu = \nu \right\}$$

This very transport map is probably one of the most natural in  $\mathcal{O}_1(\mu, \nu)$  and one of the most regular. In particular, under some assumptions on the densities  $f, g$  and their supports  $\Omega, \Omega'$  (convex and disjoint supports in the plane and continuous and bounded by above and below densities), this transport has also been shown to be continuous [17], and [23] gives also some regularity results for the minimizer  $T_{\varepsilon}$  of an approximated problem where  $c$  is replaced with  $c_{\varepsilon}(x, y) = \sqrt{\varepsilon^2 + |x - y|^2}$  (local uniform bounds on the eigenvalues of the Jacobian matrix  $T_{\varepsilon}$ ).

## 1.2 Tools for optimal transport with gradient penalization

**Existence of solutions.** We begin by showing the existence of solutions for the penalized problem with an elementary proof:

**Proposition 1.4.** *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^d$  with Lipschitz boundary. Let  $\mu \in \mathcal{P}(\Omega)$  be absolutely continuous with density  $f$ , and  $\nu \in \mathcal{P}(\mathbb{R}^d)$  such that the set*

$$\{T \in H^1(\Omega) : T_{\#}\mu = \nu\}$$

*is non-empty. Then for any  $\varepsilon > 0$ , the problem*

$$\inf \left\{ \int_{\Omega} |T(x) - x| f(x) dx + \varepsilon \int_{\Omega} |DT(x)|^2 dx : T \in H^1(\Omega), T_{\#}\mu = \nu \right\} \quad (3)$$

*admits at least one solution.*

*Proof.* We use the direct method of the calculus of variations. Let  $(T_n)_n$  be a minimizing sequence; since this sequence is bounded in  $H^1(\Omega)$ , it admits, up to a subsequence, a limit  $T$  for the strong convergence in  $L^2(\Omega)$ ; moreover, we also can assume  $T_n(x) \rightarrow T(x)$  for a.e.  $x \in \Omega$ , which also implies that the convergence holds  $\mu$ -a.e.. This implies that  $T$  still satisfies the constraint on the image measure (since, for a continuous and bounded function  $\varphi$ , the a.e. convergence provides  $\int_{\Omega} \varphi \circ T_n d\mu \rightarrow \int_{\Omega} \varphi \circ T d\mu$ ) and is thus admissible for (3). On the other hand, it is clear that the functional that we are trying to minimize is lower semi-continuous with respect to the weak convergence in  $H^1(\Omega)$ . This achieves the proof.  $\square$

This existence result can be of course adapted to any functional with form  $\int_{\Omega} L(x, T(x), DT(x)) d\mu(x)$  under natural assumptions on the Lagrangian  $L$  (continuity with respect to the two first variables, convexity and coercivity with respect to the third one).

**Existence of smooth transport maps and Dacorogna-Moser's result.** In the above existence theorem, the existence of at least one admissible map  $T$  was an assumption. If the measures are regular enough, it can be seen as a consequence of the classical regularity results of the optimal transport maps for the quadratic cost [8, 9, 10, 14]. We recall here another result which provides the existence of a regular diffeomorphism which sends a given measure onto another one, which will be used several times in the paper. This result is due to Dacorogna and Moser [13] (this construction is nowadays regularly used in optimal transport, starting from the proof by Evans and Gangbo, [15]; it is also an important tool for the equivalence between different models of transportation, see [29]; notice that the transport that they consider is in some sense optimal for a sort of congested transport cost, as pointed out years ago by Brenier in [7]). The version of their result that we will use is the following:

**Theorem 1.2.** Let  $U \subset \mathbb{R}^d$  be a bounded open set with  $C^{3,\alpha}$  boundary  $\partial U$ . Let  $f_1, f_2$  be two positive Lipschitz functions on  $\overline{U}$  such that

$$\int_U f_1 = \int_U f_2$$

Then there exists a Lipschitz diffeomorphism  $T : \overline{U} \rightarrow \overline{U}$  satisfying

$$\begin{cases} \det \nabla T(x) = \frac{f_1(x)}{f_2(T(x))}, & x \in U \\ T(x) = x, & x \in \partial U \end{cases}$$

Moreover, the Lipschitz constant of  $T$  is bounded by a constant depending only on  $U$ , on the Lipschitz constants and on the lower bounds of  $f_1$  and  $f_2$ .

Notice that the equation satisfied by  $T$  exactly means (as  $T$  is Lipschitz and one-to-one) that it sends the measures with density  $f_1$  onto the measure with density  $f_2$ . The result of the original paper [13, Theorem 1'] deals with more general assumptions on the density  $f_1$  ( $f \in C^{k+3,\alpha}(\overline{U})$  and the result is a  $C^{k+1,\alpha}$  diffeomorphism) but only considered  $f_2 = 1$ . We gave here a formulation better suitable for our needs, easy to obtain from the original theorem. We do not claim this result to be ‘‘sharp’’, but it is sufficient in the case which we are interested in (see paragraph 4.2).

**The one-dimensional case.** We finish by giving some very partial results about the optimal transport problem with gradient term on the real line. First, we recall the classical result about the one-dimensional optimal transportation problem (we refer for instance to [28] for the proof):

**Proposition 1.5.** Let  $I$  be a bounded interval of  $\mathbb{R}$  and  $\mu \in \mathcal{P}(I)$  be atom-less and  $\nu \in \mathcal{P}(\mathbb{R})$ . Then there exists a unique map  $T : I \rightarrow \mathbb{R}$  which is non-decreasing and sends  $\mu$  onto  $\nu$ . Moreover, if  $h$  is a convex function  $\mathbb{R} \rightarrow \mathbb{R}$ , then  $T$  solves the minimization problem

$$\inf \left\{ \int_{\Omega} h(T(x) - x) d\mu(x) : T_{\#}\mu = \nu \right\}$$

with uniqueness provided that  $h$  is strictly convex.

Now we state the results concerning the optimality of this monotone map  $T$  for transport problems involving derivatives:

**Proposition 1.6.** Let  $I$  be a bounded interval of  $\mathbb{R}$ . Let  $\mu$  be a finite measure on  $I$  with positive density  $f$ , and  $\nu$  a positive measure on  $\mathbb{R}$  with same mass. Let  $h$  be a convex and non-decreasing function  $\mathbb{R} \rightarrow \mathbb{R}$ . Then, if  $f$  is constant on  $I$  (and  $\mu$  is therefore the uniform measure on  $I$ ), the monotone transport map from  $\mu$  to  $\nu$  is optimal for the problem

$$\inf \left\{ \int_{\Omega} h(|T'(x)|) d\mu(x) : T \in W^{1,1}(I), T_{\#}\mu = \nu \right\}.$$

The same stays true if  $f$  is not constant but satisfies  $\frac{\inf f}{\sup f} \geq \frac{1}{2}$ .

On the other hand, the same is not true for arbitrary  $f$ , if  $\frac{\inf f}{\sup f}$  is too small. Indeed, define  $U$  on  $[0, 1]$  as being the triangle function

$$U(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

and, for  $\alpha > 0$ , take  $f = \alpha$  on  $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$  and  $f = 1$  elsewhere, thus obtaining a measure  $\mu_{\alpha}$ . If  $T_{\alpha}$  is the unique non-decreasing map sending  $\mu_{\alpha}$  onto  $U_{\#}\mu_{\alpha}$ , then

$$\int_0^1 |U'|^2 < \int_0^1 |T'_{\alpha}|^2 \quad \text{for } \alpha \text{ large enough.}$$

We refer to [26] and [25, Chapter 2] for proofs and details (and notice that this construction has been used to find counter-examples to the optimality of the monotone transport for other variational problem in [22]).



### 1.3 Definitions and basic results of $\Gamma$ -convergence

We finish this preliminary section by the tools of  $\Gamma$ -convergence that we will use throughout this paper. All the details can be found, for instance, in the classical Braides's book [5]. In that follows,  $(X, d)$  is a metric space.

**Definition 1.2.** Let  $(F_n)_n$  be a sequence of functions  $X \mapsto \overline{\mathbb{R}}$ . We say that  $(F_n)_n$   $\Gamma$ -converges to  $F$ , and we write  $F_n \xrightarrow[n]{\Gamma} F$  if, for any  $x \in X$ , we have

- for any sequence  $(x_n)_n$  of  $X$  converging to  $x$ ,

$$\liminf_n F_n(x_n) \geq F(x) \quad (\Gamma\text{-liminf inequality});$$

- there exists a sequence  $(x_n)_n$  converging to  $x$  and such that

$$\limsup_n F_n(x_n) \leq F(x) \quad (\Gamma\text{-limsup inequality}).$$

This definition is actually equivalent to the following equalities for any  $x \in X$ :

$$F(x) = \inf \left\{ \liminf_n F_n(x_n) : x_n \rightarrow x \right\} = \inf \left\{ \limsup_n F_n(x_n) : x_n \rightarrow x \right\}$$

The function  $x \mapsto \inf \left\{ \liminf_n F_n(x_n) : x_n \rightarrow x \right\}$  is called  $\Gamma$ -liminf of the sequence  $(F_n)_n$ , and the other one its  $\Gamma$ -limsup. A useful result is the following (which, for instance, implies that a constant sequence of functions does not  $\Gamma$ -converge to itself in general):

**Proposition 1.7.** The  $\Gamma$ -liminf and the  $\Gamma$ -limsup of a sequence of functions  $(F_n)_n$  are both lower semi-continuous on  $X$ .

The main interest of  $\Gamma$ -convergence is its consequences in terms of convergence of minima:

**Theorem 1.3.** Let  $(F_n)_n$  be a sequence of functions  $X \rightarrow \overline{\mathbb{R}}$  and assume that  $F_n \xrightarrow[n]{\Gamma} F$ . Assume moreover that there exists a compact and non-empty subset  $K$  of  $X$  such that

$$\forall n \in N, \inf_X F_n = \inf_K F_n$$

(we say that  $(F_n)_n$  is equi-mildly coercive on  $X$ ). Then  $F$  admits a minimum on  $X$  and the sequence  $(\inf_X F_n)_n$  converges to  $\min F$ . Moreover, if  $(x_n)_n$  is a sequence of  $X$  such that

$$\lim_n F_n(x_n) = \lim_n (\inf_X F_n)$$

and if  $(x_{\varphi(n)})_n$  is a subsequence of  $(x_n)_n$  having a limit  $x$ , then  $F(x) = \inf_X F$ .

We finish with the following result, which allows to focus on the  $\Gamma$ -limsup inequality only on a dense subset of  $X$  under some assumptions:

**Proposition 1.8.** Let  $(F_n)_n$  be a sequence of functionals and  $F$  be a functional  $X \rightarrow \overline{\mathbb{R}}$ . Assume that there exists a dense subset  $Y \subset X$  such that:

- for any  $x \in X$ , there exists a sequence  $(x_n)_n$  of  $Y$  such that  $x_n \rightarrow x$  and  $F(x_n) \rightarrow F(x)$ ;
- the  $\Gamma$ -limsup inequality holds for any  $x \in Y$ .

Then it holds for any  $x$  belonging to the whole  $X$ .

## 2 Generalities and density of Sobolev transport maps

In that follows, we consider two regular enough domains  $\Omega, \Omega'$  and two measures  $\mu \in \mathcal{P}(\Omega), \nu \in \mathcal{P}(\Omega')$  with positive and bounded from below densities  $f, g$ ; we assume moreover that the class of maps  $T \in H^1(\Omega)$  sending  $\mu$  onto  $\nu$  is non-empty (for instance, the assumptions of the Dacorogna-Moser's result are enough). The functional that we will study is defined, for  $\varepsilon > 0$ , by

$$J_\varepsilon : T \mapsto \int_\Omega |T(x) - x| d\mu(x) + \varepsilon \int_\Omega |DT(x)|^2 dx$$

and we denote by  $J$  the corresponding functional when  $\varepsilon = 0$  (which is thus the classical Monge's transport energy); moreover, we extend  $J_\varepsilon, J$  to the whole  $L^2(\Omega)$  by setting  $J_\varepsilon(T) = J(T) = +\infty$  for a map  $T$  which is not a transport map from  $\mu$  to  $\nu$ .

As usual in transport theory, we consider as a setting for our variational problems the set of transport plans  $\gamma$  which are probabilities on the product space  $\Omega \times \Omega'$  with given marginals  $(\pi_x)_\# \gamma = \mu$  and  $(\pi_y)_\# \gamma = \nu$  and all the  $\Gamma$ -limits that we consider in that follows are considered with respect to the weak convergence of plans as probability measures. However, due to our choices of the functionals that we minimize, most of the transport plan that we consider will be actually induced by transport maps, i.e.  $\gamma_T = (\text{id} \times T)_\# \mu$  with  $T_\# \mu = \nu$ . These maps are valued in  $\Omega'$ , which is bounded, and are hence bounded. We could also consider different notions of convergence, in particular based on the pointwise convergence of these plans, and we will actually do it often. For simplicity, we will use the convergence in  $L^2(\Omega; \Omega')$  (but, since these functions are bounded, this is equivalent to any other  $L^p$  convergence with  $p < \infty$ ). As the following lemma (which will be also technically useful later) shows, this convergence is equivalent to the weak convergence in the sense of measures of the transport plans:

**Lemma 2.1.** *Assume that  $\Omega, \Omega'$  are compact domains and  $\mu$  is a finite non-negative measure on  $\Omega$ . Let  $(T_n)_n$  be a sequence of maps  $\Omega \rightarrow \Omega'$ . Assume that there exists a map  $T$  such that  $\gamma_{T_n} \rightharpoonup \gamma_T$  in the weak sense of measures. Then  $T_n \rightarrow T$  in  $L^2_\mu(\Omega)$ .*

*Conversely, if  $T_n \rightarrow T$  in  $L^2_\mu(\Omega)$ , then we have  $\gamma_{T_n} \rightharpoonup \gamma_T$  in the weak sense of measures.*

*Proof.* If  $\gamma_{T_n} \rightharpoonup \gamma_T$  and  $\varphi \in C_b(\Omega)$  is a vector-valued function, we have

$$\int_\Omega \varphi(x) \cdot T_n(x) d\mu(x) = \int_{\Omega \times \Omega'} \varphi(x) \cdot y d\gamma_{T_n}(x, y) \rightarrow \int_{\Omega \times \Omega'} \varphi(x) \cdot y d\gamma_T(x, y) = \int_\Omega \varphi(x) \cdot T(x) d\mu(x)$$

which proves that  $T_n \rightharpoonup T$  weakly in  $L^2_\mu(\Omega)$ . On the other hand,

$$\int_\Omega |T_n(x)|^2 d\mu(x) = \int_{\Omega \times \Omega'} |y|^2 d\gamma_{T_n}(x, y) \rightarrow \int_{\Omega \times \Omega'} |y|^2 d\gamma_T(x, y) = \int_\Omega |T(x)|^2 d\mu(x)$$

$$\text{thus} \quad \|T_n\|_{L^2_\mu} \rightarrow \|T\|_{L^2_\mu}$$

and the convergence  $T_n \rightarrow T$  is actually strong.

Conversely, assume that  $T_n \rightarrow T$  in  $L^2_\mu(\Omega)$  and let  $(n_k)_k$  be such that the convergence  $T_{n_k}(x) \rightarrow T(x)$  holds for a.e.  $x \in \Omega$ , then for any  $\varphi \in C_b(\Omega \times \Omega')$  we have

$$\int_{\Omega \times \Omega'} \varphi(x, y) d\gamma_{T_{n_k}}(x, y) = \int_\Omega \varphi(x, T_{n_k}(x)) d\mu(x) \rightarrow \int_\Omega \varphi(x, T(x)) d\mu(x) = \int_{\Omega \times \Omega'} \varphi(x, y) d\gamma_T(x, y).$$

This proves  $\gamma_{T_{n_k}} \rightharpoonup \gamma_T$ , but, the limit being independent of the subsequence we easily get full convergence of the whole sequence.  $\square$

Since the set of transport plans between  $\mu$  to  $\nu$  is compact for the weak topology in the set of measures on  $\Omega \times \Omega'$ , a consequence of lemma 2.1 is that the equi-coercivity needed in Theorem 1.3 will be satisfied in all the  $\Gamma$ -convergence results that follows. Therefore, we will not focus on it anymore and still will consider that these results imply the convergence of minima and of minimizers.

### 2.1 Statements of the zeroth and first order $\Gamma$ -convergences

**Zeroth order  $\Gamma$ -limit.** The first step is to check that  $J_\varepsilon \xrightarrow{\Gamma} J$ . Here we must consider that  $J_\varepsilon$  is extended to transport plan by setting  $+\infty$  on those transport plans which are not of the form  $\gamma = \gamma_T$  for  $T \in H^1$ , and that  $J$  is defined as usual as  $J(\gamma) = \int |x - y| d\gamma$  for transport plans. This  $\Gamma$ -convergence actually requires a non-trivial result which states that the set of Sobolev transport maps is a dense subset of the set of transport maps from  $\mu$  to  $\nu$ . We prove this result in the next paragraph (see theorem 2.1 below) for Hölder densities and a large class of domains given by the following definition:

**Definition 2.1.** We call Lipschitz polar domain any open bounded subset  $\Omega$  having form

$$\Omega = \left\{ x \in \mathbb{R}^d : |x - x_0| < \gamma \left( \frac{x - x_0}{|x - x_0|} \right) \right\}$$

for some  $x_0 \in \Omega$  and a Lipschitz function  $\gamma : S^{d-1} \rightarrow (0, +\infty)$ . In particular, such a domain  $\Omega$  is star-shaped with Lipschitz boundary.

**Proposition 2.1** (Zeroth order  $\Gamma$ -limit). Assume that  $\Omega, \Omega'$  are both Lipschitz polar domains and that  $f, g$  are both  $C^{0,\alpha}$  and bounded from below. Then  $J_\varepsilon \xrightarrow{\Gamma} J$  as  $\varepsilon \rightarrow 0$ .

*Proof.* The  $\Gamma$ -liminf inequality is trivial (we have  $J_\varepsilon \geq J$  by definition, and  $J$  is continuous for the weak convergence of plans), and the  $\Gamma$ -limsup inequality is a direct consequence of the Prop. 1.8 and of the density of the set of Sobolev transports for the  $L^2$ -convergence.  $\square$

**First order  $\Gamma$ -limit.** We state it as follows, with this time a short proof:

**Proposition 2.2** (First order  $\Gamma$ -limit). Assume simply that  $f, g$  are bounded from below on  $\Omega, \Omega'$  and that  $\Omega$  has Lipschitz boundary. Then the functional  $\frac{J_\varepsilon - W_1}{\varepsilon}$   $\Gamma$ -converges, when  $\varepsilon \rightarrow 0$ , to

$$\mathcal{H} : T \mapsto \begin{cases} \int_\Omega |DT(x)|^2 dx & \text{if } T \in \mathcal{O}_1(\mu, \nu) \cap H^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where, again,  $\mathcal{H}$  is extend to plans which are not induced by maps by  $+\infty$ .

*Proof.*  $\Gamma$ -limsup inequality. If  $T \in \mathcal{O}_1(\mu, \nu)$ , then by choosing  $T_\varepsilon = T$  for any  $\varepsilon$  we obtain automatically  $\frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} = \int_\Omega |DT|^2$  for each  $\varepsilon$ . It remains to show that if  $T \notin \mathcal{O}_1(\mu, \nu)$ , then we have  $\frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} \rightarrow +\infty$  for any sequence  $(T_\varepsilon)_\varepsilon$  converging to  $T$ ; but since the map

$$T \mapsto \int_\Omega |T(x) - x| d\mu(x)$$

is continuous for the  $L^2$ -convergence, we have for such a  $(T_\varepsilon)_\varepsilon$

$$\liminf_\varepsilon \frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} \geq \liminf_\varepsilon \frac{1}{\varepsilon} \left( \int_\Omega |T(x) - x| d\mu(x) - W_1 \right)$$

which is  $+\infty$  since  $T$  is not optimal for the Monge problem.

$\Gamma$ -liminf inequality. We can concentrate on sequence of maps  $T_\varepsilon$  with equibounded values for  $\frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon}$ , which provides a bound on  $\int_\Omega |DT_\varepsilon|^2$ . Assuming the liminf to be finite, from

$$C \geq \frac{J_\varepsilon(T_\varepsilon) - W_1}{\varepsilon} = \frac{1}{\varepsilon} \left( \int_\Omega |T_\varepsilon(x) - x| d\mu(x) - W_1 \right) + \int_\Omega |DT_\varepsilon|^2 \geq \int_\Omega |DT_\varepsilon|^2$$

we deduce as above that  $T$  must belong to  $\mathcal{O}_1(\mu, \nu)$  and, since the last term is lower semi-continuous with respect to the weak convergence in  $H^1(\Omega)$  (which is guaranteed up to subsequences since  $(J_{\varepsilon_k}(T_{\varepsilon_k}))_k$  is bounded) we get the inequality we look for.  $\square$

Since the  $\Gamma$ -convergence implies the convergence of minima, we then have

$$\inf J_\varepsilon = \inf J + \varepsilon \inf \mathcal{H} + o(\varepsilon) = W_1 + \varepsilon \inf \mathcal{H} + o(\varepsilon)$$

provided that the infimum is finite, which means that there exists at least one transport map  $T$  also belonging to the Sobolev space  $H^1(\Omega)$ . In the converse case, and under the assumptions of the zeroth order  $\Gamma$ -convergence, we have

$$\inf J_\varepsilon \rightarrow W_1 \quad \text{and} \quad \frac{\inf J_\varepsilon - W_1}{\varepsilon} \rightarrow +\infty$$

which means that the lowest order of convergence of  $\inf J_\varepsilon$  to  $J$  is smaller as  $\varepsilon$ . The study of a precise example where this order is  $\varepsilon |\log \varepsilon|$  is the object of the section 3.

**What about the selected map ?** The first-order  $\Gamma$ -convergence and the basic properties of  $\Gamma$ -limits imply that, if  $T_\varepsilon$  minimizes  $J_\varepsilon$ , then  $T_\varepsilon \rightarrow T$  which minimizes the Sobolev norm among the set  $\mathcal{O}_1(\mu, \nu)$  of optimal transport maps from  $\mu$  to  $\nu$ . This gives a selection principle, via secondary variational problem (minimizing something in the class of minimizers), in the same spirit of what we presented for the monotone transport map along each transport ray. A natural question is to find which is this new “special” selected map, and whether it can coincide with the monotone one. Thanks to the non-optimality results of this map for the Sobolev cost on the real line, the answer is that they are in general different. We can look at the following explicit counter-example (where we have however  $\mathcal{O}_1(\mu, \nu) \cap H^1(\Omega) \neq \emptyset$ ).

Let us set  $\Omega = (0, 1)^2$ ,  $\Omega' = (2, 3) \times (0, 1)$  in  $\mathbb{R}^2$ . Let  $F, G$  be two probability densities on the real line, supported in  $(0, 1)$  and  $(2, 3)$  respectively; we now consider the densities defined by

$$f(x_1, x_2) = F(x_1) \quad \text{and} \quad g(x_1, x_2) = G(x_1)$$

Then, if  $t$  is a transport map from  $F$  to  $G$  on the real line and  $T(x_1, x_2) = (t(x_1), x_2)$ , it is easy to check that  $T$  sends the density  $f$  onto  $g$  and if  $u(x_1, x_2) = x_1$  we have

$$|T(x) - x| = |t(x_1) - x_1| = t(x_1) - x_1 = u(T(x_1, x_2)) - u(x_1, x_2)$$

This proves that  $T$  is optimal and  $u$ , which is of course 1-Lipschitz, is a Kantorovich potential, so that the maximal transport rays are exactly the segments  $[0, 3] \times \{x_2\}$ ,  $0 < x_2 < 1$ . As a consequence,

$$T \in \mathcal{O}_1(\mu, \nu) \quad \Longleftrightarrow \quad T(x_1, x_2) = (t(x_1, x_2), x_2) \quad \text{with} \quad \begin{array}{l} t(\cdot, x_2) \# F = G \\ \text{for a.e. } x_2 \end{array}$$

In particular, the monotone transport map along the maximal transport rays is  $x \mapsto (t(x_1), x_2)$ , where  $t$  is the non-decreasing transport map from  $F$  to  $G$  on the real line. For this transport map  $T$ , we have

$$\int_{\Omega} |DT(x)|^2 dx = \int_0^1 t'(x_1)^2 dx_1$$

Now, one can choose  $F, G$  such that the solution of

$$\inf \left\{ \int_0^1 U'(x_1)^2 dx_1 : U \# F = G \right\} \quad (4)$$

is not attained by the increasing transport map from  $F$ , to  $G$ . Thus, if  $\tilde{t}$  minimizes (4) and  $\tilde{T}(x_1, x_2) = (\tilde{t}(x_1), x_2)$ , we have

$$\int_{\Omega} |D\tilde{T}(x)|^2 dx = \int_0^1 \tilde{t}'(x_1)^2 dx_1 < \int_0^1 t'(x_1)^2 dx_1 = \int_{\Omega} |DT(x)|^2 dx$$

and  $\tilde{T}$  is also an optimal transport map for the Monge problem.

## 2.2 Proof of the density of Lipschitz transports

In this section, we prove that, under natural assumptions (domains which are diffeomorphic to the unit ball, Hölder and bounded from above and below densities), the transport maps from  $\mu$  to  $\nu$  which are Lipschitz continuous form a dense subset of the set of transport maps from  $\mu$  to  $\nu$ . Notice that, for the sake of the applications to the zero-th order  $\Gamma$ -convergence of the previous section, we only need the density of those maps belonging to the Sobolev space  $H^1(\Omega)$ ; also, we needed density in the set of plans, but since it is well known that transport maps are dense in the set of transport plans, for simplicity we will prove density in the set of maps. This result, that we need in this paper to study the  $\Gamma$ -convergence of the functional  $J_\varepsilon$ , could of course be useful for other aims; however, it was, to the best of our knowledge, not investigated until now.

**Theorem 2.1.** *Assume that  $\Omega, \Omega'$  are both Lipschitz polar domains and that  $f, g$  are both  $C^{0,\alpha}$  and bounded from below. Then the set*

$$\{T \in \text{Lip}(\Omega) : T \# \mu = \nu\}$$

*is non-empty, and is a dense subset of the set*

$$\{T : \Omega \rightarrow \Omega' : T \# \mu = \nu\}$$

*endowed with the norm  $\|\cdot\|_{L^2(\Omega)}$ .*

Notice that, exactly as for the density of transport maps inside the set of transport plans, the best possible result would be to show the density under the same assumptions guaranteeing the existence of at least such a map (for the density of maps into plans the assumption is that  $\mu$  must be atomless, which is the same as for the existence of at least a map  $T$ ). Here we are not so far since the already known results about regularity of some transport maps (by Caffarelli *et al.* or Dacorogna-Moser) deal with at least Hölder densities. Also, one can see from the proof below that we do not use much more of the existence of Lipschitz maps. Concerning the assumption on the domains, this is used to send them onto the unit ball in a Lipschitz-diffeomorphic way (in order to get sufficiently regular measures), as shown in the following lemma which is used several times:

**Lemma 2.2.** *If  $U$  is a Lipschitz polar domain with*

$$U = \left\{ x \in \mathbb{R}^d : |x - x_0| < \gamma \left( \frac{x - x_0}{|x - x_0|} \right) \right\}$$

*then there exists a map  $\alpha : U \rightarrow B(0, 1)$  such that:*

- $\alpha$  is a bi-Lipschitz diffeomorphism from  $U$  to  $B(0, 1)$ ;
- $\det D\alpha$  is Lipschitz and bounded from below (thus,  $\det D\alpha^{-1}$  is also Lipschitz);
- for any  $x \neq x_0$ ,  $\frac{\alpha(x)}{|\alpha(x)|} = \frac{x - x_0}{|x - x_0|}$

(We will not use the third property in the proof of Theorem 2.1, but it will be useful later in section 4.2).

*Proof.* Up to a translation and a dilation, we can assume  $x_0 = 0$  and  $\gamma \leq 1$  on  $S^{d-1}$ . Now we set

$$\alpha(x) = \begin{cases} x & \text{if } |x| \leq \frac{1}{2}\gamma \left( \frac{x}{|x|} \right) \\ \lambda(x) \frac{x}{|x|} & \text{otherwise} \end{cases}.$$

for a suitable choice of the function  $\lambda : U \rightarrow [0, +\infty)$ . Assuming that  $\lambda$  is Lipschitz, we compute  $D\alpha$  on the region  $\left\{ |x| > \frac{1}{2}\gamma(x/|x|) \right\}$ . Here we have  $D\alpha = x \otimes \nabla \left( \frac{\lambda(x)}{x} \right) + \frac{\lambda}{|x|} I_d$  thus, in an orthonormal basis whose first vector is  $e = \frac{x}{|x|}$ ,

$$D\alpha = \begin{pmatrix} |x| \partial_e \left( \frac{\lambda(x)}{|x|} \right) & |x| \partial_{e_2} \left( \frac{\lambda(x)}{|x|} \right) & \dots & |x| \left( \partial_{e_n} \frac{\lambda(x)}{|x|} \right) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} + \frac{\lambda}{|x|} I_n$$

$$\text{and } \det D\alpha = \left( |x| \partial_e \left( \frac{\lambda(x)}{|x|} \right) + \frac{\lambda(x)}{|x|} \right) \left( \frac{\lambda(x)}{|x|} \right)^{n-1}$$

We write  $\lambda = \lambda(r, e)$  where  $r = |x|$ , which leads to

$$\det D\alpha = \left( r \partial_r \left( \frac{\lambda}{r} \right) + \frac{\lambda}{r} \right) \left( \frac{\lambda}{r} \right)^{n-1} = \partial_r \lambda \left( \frac{\lambda}{r} \right)^{n-1}$$

At this time, we see that the following conditions on  $\lambda$  allow to conclude:

- $\lambda$  is Lipschitz on the domain  $\left\{ \frac{\gamma(x/|x|)}{2} \leq |x| \leq \gamma(x/|x|) \right\}$ ;
- if we fix  $e \in S^{d-1}$ , then  $\lambda(\cdot, e)$  is increasing on the interval  $\left[ \frac{\gamma(e)}{2}, \gamma(e) \right]$ ;
- $\lambda \left( \frac{\gamma(e)}{2}, e \right) = \frac{\gamma(e)}{2}$  and  $\lambda(\gamma(e), e) = 1$ ;
- $\partial_r \lambda \left( \frac{\lambda}{r} \right)^{n-1}$  is Lipschitz and is equal to 1 for  $r = \frac{\gamma(e)}{2}$ , which means  $\partial_r \lambda \left( \frac{\gamma(e)}{2}, e \right) = 2^{n-1}$ .

To satisfy these conditions, it is enough to choose for  $\lambda(\cdot, e)$  a second-degree polynomial function with prescribed values at  $\frac{\gamma(e)}{2}$ ,  $\gamma(e)$  and prescribed first derivative at  $\frac{\gamma(e)}{2}$ .  $\square$

*Proof of Theorem 2.1.* First, we assume  $\Omega = \Omega' = [-\frac{1}{2}, \frac{1}{2}]^d$  but we will also identify (for convolution purposes, see below) the cube  $\Omega$  with a Torus. We will generalize later to any pair of Lipschitz star-shaped domains. Thus, let  $T : \Omega \rightarrow \Omega$  sending  $f$  to  $g$ , and let us build a sequence  $(T_n)_n$  of Lipschitz maps such that

$$T_n \xrightarrow[n \rightarrow +\infty]{L^2(\Omega)} T \quad \text{and} \quad \forall n \in \mathbb{N}, (T_n)_\# \mu = \nu$$

*Step 1: regularization of the transport plan.* We denote by  $\gamma$  the transport plan associated to  $T$ ; let us recall that it is defined by

$$\iint_{\Omega^2} \varphi(x, y) d\gamma(x, y) = \int_{\Omega} \varphi(x, T(x)) f(x) dx$$

for any continuous and bounded function  $\varphi$  on  $\Omega \times \Omega'$ . We use the disintegration of measures (see Theorem 5.3.1 in [2]) to write

$$\iint_{\Omega^2} \varphi d\gamma = \int_{\Omega} \left( \int_{\Omega} \varphi(x, y) d\gamma_x(y) \right) d\mu(x)$$

for a family of measures  $(\gamma_x)_{x \in \Omega}$  on  $\Omega$ . Now we take a family of periodic convolution kernels  $(\rho_k)_k$ , we define, for each  $x$ , the measure  $\gamma_x^k$  on  $\Omega$  by  $\gamma_x^k = \rho_k \star \gamma_x$ , where the convolution product is taken in the distributional sense on the torus  $\Omega'$ , and set  $d\gamma_k(x, y) = d\gamma_x^k(y) \otimes d\mu(x)$ ; the fact that  $(\rho_k)_k$  is an approximate identity guarantees that  $\gamma_k \rightarrow \gamma$  for the weak convergence of measures.

On the other hand, notice that

$$\begin{aligned} \iint_{\Omega^2} \varphi(x, y) d\gamma_k(x, y) &= \iiint_{\Omega^3} \varphi(x, y) \rho_k(y - z) d\gamma_x(z) f(x) dx dy \\ &= \int_{\Omega} \left( \int_{\Omega^2} \varphi(x, y) \rho_k(y - z) d\gamma(x, z) \right) dy \\ &= \iint_{\Omega^2} \varphi(x, y) \rho_k(y - T(x)) f(x) dy dx \end{aligned}$$

for any test function  $\varphi$ . Thus,  $\gamma_k$  is absolutely continuous with respect to the Lebesgue measure on  $\Omega^2$  and its density is  $\rho_k(y - T(x))f(x)$ . In particular, this density is smooth with respect to  $y$  and bounded from below by a positive constant  $c_k$ . Moreover, if we denote by  $\nu_k$  the second marginal of  $\gamma_k$ , then  $\nu_k$  has also a Lipschitz and bounded from below density.

*Step 2: construction of a transport map corresponding to the regularized transport plan.* The goal of this step is to build, for each  $k$ , a family of regular maps  $(T_h^k)_h$  each sending the first marginal of  $\gamma_k$  onto the second one and such that, for a suitable diagonal extraction,  $T_{h_k}^k \rightarrow T$  for the  $L^2$ -convergence.

We fix  $k \in \mathbb{N}$  and a dyadic number  $h$ , and decompose  $\Omega = \bigcup_i Q_i$  into a union of cubes  $Q_i$ , each with size-length  $2h$ , via a regular grid. For each  $i$ , we set

$$\mu_i = \mu|_{Q_i} \quad \text{and} \quad \nu_i^k = (\pi_2)_\#(\gamma_k|_{(Q_i \times \Omega)})$$

Notice that  $\gamma_k$  has  $\mu$  as first marginal, thus we have also  $\mu_i = (\pi_1)_\#(\gamma_k|_{(Q_i \times \Omega)})$ ; in particular,  $\mu_i$  (which is a measure on the cube  $Q_i$ ) and  $\nu_i^k$  (which is a measure on the whole cube  $\Omega$ ) have both same mass. Moreover,  $\mu_i$  still has  $f$  as density and the density of  $\nu_i^k$  is

$$y \mapsto \int_{Q_i} \rho_k(T(x) - y) f(x) dx$$

which is smooth and bounded from below by a positive constant. We finish this step by building a map  $U_{i,h}^k : Q_i \rightarrow \Omega$  such that

- $U_{i,h}^k$  is Lipschitz continuous;
- $U_{i,h}^k(x) = x$  for  $x \in \partial Q_i$  (this is allowed since the source  $Q_i$  is included in the target  $\Omega$ );
- $U_{i,h}^k$  sends  $\mu_i$  onto  $\nu_i^k$ .

Let us notice that the two first points guarantee that the global map defined by  $T_h^k(x) = U_{i,h}^k(x)$  (which depends on the radius  $h$  and the original integer  $k$ ) for  $x \in Q_i$  will be globally Lipschitz on  $\Omega$  and the third point implies that  $T_h^k$  sends globally the measure  $\mu$  onto the measure  $\nu_k = (\pi_2)_\# \gamma_k$  (which is equal to the sum of the  $\nu_i^k$ ).

Now we explain very briefly the construction of such a  $U_{i,h}^k$ . For the sake of simplicity, we give the details only in the case where  $Q_i$  is not an extremal cube of  $\Omega$ , *i.e.* where  $\partial Q_i \cap \partial \Omega$  is empty, and claim that the extreme case can be treated in a similar way.

We denote by  $x_i$  the center of  $Q_i$  which has radius  $h$  (for the infinite norm in  $\mathbb{R}^d$ ), and by  $\delta_1$  and  $\delta_2$  the two positive numbers such that

$$\mu(\{x : \delta_1 \leq \|x - x_i\|_\infty \leq h\}) = \mu(\{x : \delta_2 \leq \|x - x_i\|_\infty \leq \delta_1\}) = \frac{1}{2} \nu_i^k(\Omega \setminus Q_i)$$

(such positive numbers exist by the intermediate value theorem since  $\delta \mapsto \mu(B_\infty(x_i, \delta))$  is continuous). Now we consider:

- a first Lipschitz map  $V_1$  from  $\{x : \delta_1 \leq \|x - x_i\|_\infty \leq h\}$  to  $\Omega \setminus Q_i$ , with  $(V_1)_\# \mu_i = \frac{1}{2} \nu_i^k(\Omega \setminus Q_i)$  and such that  $V_1(x) = x$  for  $\|x - x_i\| = h$ ;
- a second Lipschitz map  $V_2$  from  $\{x : \delta_2 \leq \|x - x_i\|_\infty \leq \delta_1\}$  to  $\Omega \setminus Q_i$ , such that  $(V_2)_\# \mu_i = \frac{1}{2} \nu_i^k(\Omega \setminus Q_i)$  and  $V_2(x) = V_1(x)$  for  $\|x - x_i\| = \delta_1$ ;
- a third Lipschitz map  $V_3$  from  $\{0 : \|x - x_i\|_\infty \leq \delta_2\}$  to  $Q_i$ , such that  $V_3(x) = V_2(x)$  on the boundary and  $(V_3)_\# \mu = \nu_i^k(Q_i)$ .

and we set  $U_{i,h}^k(x) = V_1(x)$  if  $\delta_1 \leq \|x - x_i\|_\infty \leq h$ ,  $V_2(x)$  if  $\delta_2 \leq \|x - x_i\|_\infty \leq \delta_1$  and  $V_3(x)$  if  $\|x - x_i\|_\infty \leq \delta_2$ . The constraints on  $(V_1)_\# \mu$ ,  $(V_2)_\# \mu$ ,  $(V_3)_\# \mu$  imply that  $U_{i,h}^k$  defined on the whole  $Q_i$  sends globally  $\mu|_{Q_i}$  onto  $\nu_i^k$ , and the constraints on the boundary on each domain guarantee that  $U_{i,h}^k$  is Lipschitz on  $Q_i$ . We construct  $V_1, V_2, V_3$  using Dacorogna-Moser's result (a simple adaptation is needed, since the source and target domains are not the same here, but are diffeomorphic to each other: either they are both cubes, or they both have the form "large cube minus small cube").

In particular,  $\gamma_k$  and  $\gamma_{T_h^k}$  give both same mass to each  $Q_i \times \Omega$  which have radius  $h$ . Now if  $u$  is a 1-Lipschitz function  $\Omega \times \Omega' \rightarrow \mathbb{R}$ , we compute

$$\int_{\Omega^2} u(x, y) d\gamma_{T_h^k}(x, y) - \int_{\Omega^2} u(x, y) d\gamma_k(x, y) = \sum_i \int_{Q_i \times \Omega} u(x, y) (d\gamma_{T_h^k} - d\gamma_k)(x, y)$$

Since  $\gamma_k$  and  $\gamma_{T_h^k}$  have both the same second marginal on each  $Q_i \times \Omega$ ,

$$\int_{Q_i \times \Omega} u(x_i, y) (d\gamma_{T_h^k}(x, y) - d\gamma_k(x, y)) = 0$$

where, for each  $i$ ,  $x_i$  denotes the center of  $Q_i$ . Thus

$$\begin{aligned} \int_{\Omega^2} u(x, y) d\gamma_{T_h^k}(x, y) - \int_{\Omega^2} u(x, y) d\gamma_k(x, y) &= \sum_i \int_{Q_i \times \Omega} (u(x, y) - u(x_i, y)) (d\gamma_{T_h^k} - d\gamma_k)(x, y) \\ &\leq \sum_i \int_{Q_i \times \Omega} |x - x_i| (d\gamma_{T_h^k} - d\gamma_k)(x, y) \end{aligned}$$

where in the last inequality we used the fact that  $u$  is supposed to be 1-Lipschitz. We deduce

$$\int_{\Omega^2} u(x, y) d\gamma_{T_h^k}(x, y) - \int_{\Omega^2} u(x, y) d\gamma_k(x, y) \leq h \iint_{\Omega^2} (d\gamma_{T_h^k} - d\gamma_k) \leq 2h$$

This inequality holds for any 1-Lipschitz function  $u : \Omega^2 \rightarrow \mathbb{R}$ . Thus, if  $W_1$  denotes the classical Wasserstein distance (on  $\Omega^2$ : this is just a technical trick to metrize the weak convergence of plans), we deduce  $W_1(\gamma_k, \gamma_{T_h^k}) \leq h$  for any  $k$ . Since we know that  $\gamma_k \rightarrow \gamma_T$  as  $k \rightarrow +\infty$ , we also have  $W_1(\gamma_k, \gamma_T) \rightarrow 0$  and if we set  $h = 1/2^k$  we have  $W_1(\gamma_{T_h^k}, \gamma_T) \rightarrow 0$ . Thanks to the lemma 2.1, we deduce  $T_h^k \rightarrow T$  for the  $L^2$ -norm in  $\Omega$ .

*Step 3: rearranging  $T_h^k$ .* It remains to compose each map  $T_h^k$  with a transport map  $U_k$  from  $\nu_k$  to  $\nu$ , which will ensure the obtained map to send  $\mu$  onto  $\nu$  as well. We use the following classical result of Caffarelli ([32], Theorem 4.14): if

- $X, Y$  are two bounded open sets of  $\mathbb{R}^d$ , uniformly convex and with  $C^2$  boundary;
- $f \in C^{0,\alpha}(\overline{X}), g \in C^{0,\alpha}(\overline{Y})$  are two probability densities

then the optimal transport map from  $f$  to  $g$  belongs to  $C^{1,\alpha}(\overline{X})$ . The assumption on the source domain is of course not satisfied here, thus we use the lemma 2.2 to get a bi-Lipschitz map  $\alpha : \Omega \rightarrow B(0, 1)$  such that  $\det D\alpha$  is also Lipschitz. We then denote by

$$\overline{g} = \alpha_{\#}g \quad \text{and} \quad \overline{g}_k = \alpha_{\#}g_k$$

where  $g_k$  is the density of  $\nu_k$  (we know that  $g_k$  is Lipschitz and bounded from below). The fact that  $\det D\alpha$  is Lipschitz and bounded guarantees that  $\overline{g}$  is Hölder and  $\overline{g}_k$  is Lipschitz as well.

We know that  $g_k \rightarrow g$  in the weak sense of measures, thus  $\overline{g}_k \rightarrow \overline{g}$  also, and

$$W_2(\overline{g}_k, \overline{g}) = \int_{B(0,1)} |U_k(x) - x|^2 \overline{g}_k(x) dx \rightarrow 0$$

where  $U_k$  is the optimal transport map from  $\overline{g}_k$  to  $\overline{g}$  for the quadratic cost; the Caffarelli's regularity result guarantees  $U_k$  to belong to  $C^{1,\alpha}(\overline{B}(0, 1))$ , and we deduce from the convergence  $W_2(\overline{g}_k, \overline{g}) \rightarrow 0$  that  $U_k(x) \rightarrow x$  for almost any  $x \in B(0, 1)$ . Now if we consider

$$\tilde{T}_k = \alpha^{-1} \circ U_k \circ \alpha \circ T_h^k$$

then we can check that  $\tilde{T}_k \rightarrow T$  in  $L^2(\Omega)$ , and we have  $(\tilde{T}_k)_{\#}\mu = \nu$  for each  $k$  by construction. The proof is complete in the case  $\Omega = \Omega' = [-\frac{1}{2}, \frac{1}{2}]^d$ .

*Step 4: generalization to any pair of domains.* If  $\Omega, \Omega'$  are now two star-shaped domains with Lipschitz boundaries and  $T$  is a transport maps from  $\mu$  to  $\nu$ , we consider two Lipschitz diffeomorphisms  $\alpha_1 : \Omega \rightarrow [0, 1]^d, \alpha_2 : \Omega' \rightarrow [0, 1]^d$  as in the Lemma 2.2. The regularity of  $\det D\alpha_1, \det D\alpha_2$  guarantees that the image measures  $(\alpha_1)_{\#}\mu, (\alpha_2)_{\#}\nu$  have both  $C^{0,\alpha}$  regularity, thus we are able to find a sequence  $(U_k)_k$  of transport maps from  $(\alpha_1)_{\#}\mu$  to  $(\alpha_2)_{\#}\nu$  converging for the  $L^2([0, 1]^d)$ -norm to  $(\alpha_2) \circ T \circ (\alpha_1)^{-1}$ , and it is now easy to check that  $T_k = (\alpha_2)^{-1} \circ U_k \circ \alpha_1$  sends  $\mu$  to  $\nu$  and converges to  $T$  as well  $\square$

### 3 An example of approximation of order $\varepsilon |\log \varepsilon|$

As we said in the above section, the convergence  $\frac{J_\varepsilon - W_1}{\varepsilon} \rightarrow \mathcal{H}$  allows to know the behavior of  $\inf J_\varepsilon$  and of any family  $(T_\varepsilon)_\varepsilon$  of minimizers in the case where  $\inf \mathcal{H} < +\infty$ . This paragraph is devoted to the study of an example where this assumption fails (*i.e.* where any optimal transport map for the Monge problem is not Sobolev).

#### 3.1 Notations and first remarks

In the rest of this paper, we set  $d = 2$  and we will denote by  $(r, \theta)$  the usual system of polar coordinates in  $\mathbb{R}^2$ . Our source and target domains will be respectively

$$\Omega = \left\{ x = (r, \theta) : 0 < r < 1, 0 < \theta < \frac{\pi}{2} \right\}$$

$$\text{and} \quad \Omega' = \left\{ x = (r, \theta) : R_1(\theta) < r < R_2(\theta), 0 < \theta < \frac{\pi}{2} \right\}$$

where  $R_1, R_2$  are two Lipschitz functions  $[0, \frac{\pi}{2}] \rightarrow (0, +\infty)$  with  $\inf R_1 > 1$  and  $\inf R_2 > \sup R_1$ . We also suppose  $R_1$  to be such that  $R_1'$  is a function of bounded variation. Notice that we can choose  $R_1, R_2$  such that the target domain  $\Omega'$  is convex (for instance if the curves  $r = R_1(\theta), r = R_2(\theta)$  are actually two lines in the quarter plane). We assume that  $f$  and  $g$  are two Lipschitz densities on  $\overline{\Omega}, \overline{\Omega'}$ , bounded from above and below by positive constants, with the following hypothesis:

$$\forall \theta \in \left(0, \frac{\pi}{2}\right), \int_0^1 f(r, \theta) r dr = \int_{R_1(\theta)}^{R_2(\theta)} g(r, \theta) r dr \quad (5)$$

which means that, for any  $\theta$ , the mass (with respect to  $f$ ) of the segment joining the origin to the boundary of  $\Omega$  and with angle  $\theta$  is equal to the mass (with respect to  $g$ ) of the segment with same angle joining the “above” and “below” boundaries of  $\Omega'$  (*i.e.* the curves  $r = R_1(\theta)$  and  $r = R_2(\theta)$ ).

Then the structure of the optimal maps for the Monge cost is given by the following:



**Proposition 3.1.** *Under these assumptions on  $\Omega, \Omega', f, g$ , the Euclidean norm is a Kantorovich potential and the maximal transport rays are the segments joining 0 to  $(R_2(\theta), \theta)$ . Consequently,*

$$T \in \mathcal{O}_1(\mu, \nu) \iff \begin{array}{l} T_{\#}f = g \\ \text{and } T(x) = \varphi(x) \frac{x}{|x|} \end{array}$$

for some function  $\varphi \in \Omega \rightarrow (\inf R_1, +\infty)$ .

*Proof.* For  $\theta \in (0, \frac{\pi}{2})$ , we denote by  $t(\cdot, \theta)$  a one-dimensional transport map from the measure  $r \mapsto rf(r, \theta)$  to the measure  $r \mapsto rg(r, \theta)$  (such a transport map exists since these two measures have same mass thanks to the equality (5)). It is then easy to check that the map

$$T : x = (r, \theta) \in \Omega \rightarrow t(r, \theta) \frac{x}{|x|}$$

is a transport map from  $f$  to  $g$ , and if we set  $u = |\cdot|$ ,  $u$  is of course 1-Lipschitz and

$$u(T(x)) - u(x) = \left| t(r, \theta) \frac{x}{|x|} \right| - |x| = (|t(r, \theta)| - |x|) \frac{x}{|x|} = \left| (t(r, \theta) - |x|) \frac{x}{|x|} \right| = |T(x) - x| \quad \text{for any } x \in \Omega.$$

We deduce that that  $u = |\cdot|$  is a Kantorovich potential. Consequently, a segment  $[x, y]$  is a transport ray if and only if

$$u(y) - u(x) = |y - x| \quad \text{i.e.} \quad |y - x| = |y| - |x|$$

Thus, we have  $y = \lambda x$  for a positive  $\lambda$ . In particular,  $y$  and  $x$  belong to the same line passing through the origin. In other words, the transport rays are included in the lines passing through the origin. Moreover, a transport map  $T$  belongs to  $\mathcal{O}_1(\mu, \nu)$  if and only if, for a.e.  $x \in \Omega$ ,  $|T(x) - x| = |T(x)| - |x|$  which again means that  $T(x) = \varphi(x) \frac{x}{|x|}$  for some positive function  $\varphi$ .  $\square$

**Corollary 3.1.** *Under the above assumptions on  $\Omega, \Omega', \mu, \nu$ , we have  $\mathcal{O}_1(\mu, \nu) \cap H^1(\Omega) = \emptyset$*

*Proof.* Let  $T \in \mathcal{O}_1(\mu, \nu)$ . By Prop. 3.1, we have  $T(x) = \varphi(r, \theta) \frac{x}{|x|}$  for  $x = (r, \theta)$ , where  $\varphi$  is a real-valued function; the fact that  $T$  sends  $\mu$  onto  $\nu$  implies  $T(x) \in \Omega'$  for any  $x$ , thus  $\varphi$  is bounded from below on  $\Omega$  by the lower bound of  $R_1$ . We now compute the Jacobian matrix of  $T$ . Denoting by  $x^\perp$  the image of  $x$  by the rotation with angle  $\pi/2$ , we have in the basis  $\left( \frac{x}{|x|}, \frac{x^\perp}{|x|} \right)$ :

$$DT(x) = \frac{x}{|x|} \otimes \nabla \varphi(x) - \frac{\varphi(x)}{|x|} I_d = \begin{pmatrix} \partial_r \varphi & 0 \\ \frac{\partial_\theta \varphi}{r} & \frac{\varphi}{r} \end{pmatrix}$$

$$\text{thus} \quad \int_{\Omega} |DT(x)|^2 dx \geq \int_0^1 \int_0^{\frac{\pi}{2}} \frac{\varphi(r, \theta)^2}{r^2} r d\theta dr \geq \int_0^1 \frac{\pi}{2} (\inf R_1)^2 \frac{dr}{r} = +\infty \quad \square$$

## 3.2 Heuristics

In this paragraph, we give a preliminary example of analysis of the behavior of  $J_\varepsilon(T_\varepsilon)$  when  $\varepsilon \rightarrow 0$  and  $T_\varepsilon$  approaches an optimal map  $T$  for the Monge problem; this will not lead directly to a rigorous proof of the general result, but gives an idea of which quantities will appear.

Assume that  $T \in \mathcal{O}_1(\mu, \nu)$  with  $T(x) = \varphi(r, \theta) \frac{x}{|x|}$ , and let us build an approximation  $(T_\varepsilon)_\varepsilon$  defined by

$$T_\varepsilon(x) = \begin{cases} S(x) & \text{if } x \in \Omega_\delta \\ T(x) & \text{otherwise} \end{cases}$$

where  $\delta$  will be fixed (depending on  $\varepsilon$ ),  $\Omega_\delta = B(0, \delta) \cap \Omega$  and  $S$  will be build to send  $(f \cdot \mathcal{L}^d)|_{\Omega_\delta}$  onto the same image measure that the original  $T$  has on  $\Omega_\delta$ . Notice that  $S$  depends indeed both on  $\delta$  and on  $\varepsilon$ , but we omit this dependance. In this case, we have

$$J_\varepsilon(T_\varepsilon) - W_1 = \int_{\Omega} |T_\varepsilon(x) - x| f(x) dx - W_1 + \int_{\Omega_\delta} |DS|^2 + \int_{\Omega \setminus \Omega_\delta} |DT|^2$$

Since  $T$  is optimal for the Monge problem and coincides with  $T_\varepsilon$  outside of  $\Omega_\delta$ , we have

$$\int_{\Omega} |T_\varepsilon(x) - x| f(x) dx - W_1 = \int_{\Omega} (|T_\varepsilon(x) - x| - |T(x) - x|) f(x) dx = \int_{\Omega_\delta} (|S(x) - x| - |T(x) - x|) f(x) dx$$

We now claim that

$$\int_{\Omega_\delta} |T(x) - x| f(x) dx = \int_{\Omega_\delta} (|S(x)| - |x|) f(x) dx$$

Indeed, we still have the equality  $|T(x) - x| = |T(x)| - |x|$ , and the image measures of  $(f \cdot \mathcal{L}^d)|_{\Omega_\delta}$  by  $T$  and  $S$  are the same. As a consequence,

$$\int_{\Omega} |T_\varepsilon(x) - x| f(x) dx - W_1 = \int_{\Omega_\delta} (|S(x) - x| - |S(x)| + |x|) f(x) dx$$

and, by the triangle inequality,  $|S(x) - x| - |S(x)| + |x| \leq |(S(x) - x) - S(x)| + |x| \leq 2|x|$  so that

$$\int_{\Omega} |T_\varepsilon(x) - x| f(x) dx - W_1 \leq \int_{\Omega_\delta} 2|x| dx \leq 2 \int_0^{\frac{\pi}{2}} \int_0^\delta r^2 dr d\theta \leq \frac{\pi \delta^3}{3}$$

In order to estimate the norm of the Jacobian matrix  $DT$  outside of  $\Omega_\delta$ , we recall that, in the basis  $\left(\frac{x}{|x|}, \frac{x^\perp}{|x|}\right)$ ,

$$DT(x) = \begin{pmatrix} \partial_r \varphi & 0 \\ \frac{\partial_\theta \varphi}{r} & \frac{\varphi}{r} \end{pmatrix}$$

$$\text{so that } \int_{\Omega \setminus \Omega_\delta} |DT(x)|^2 dx = \int_\delta^1 \int_0^{\frac{\pi}{2}} \left( \frac{\varphi(r, \theta)^2 + \partial_\theta \varphi(r, \theta)^2}{r} + r \partial_r \varphi(r, \theta)^2 \right) d\theta dr$$

Now we note that, for  $\theta \in (0, \pi/2)$ , the one-dimensional map  $\varphi(\cdot, \theta)$  sends the density  $rf(\cdot, \theta)$  onto the density  $rg(r, \theta)$ . The first density is bounded from above (but vanishes around  $r = 0$ ), and the second one from below (also from above). Thus, one can assume that  $\partial_r \varphi(\cdot, \theta)$  is bounded (by the way, if  $T$  is the monotone map along the transport rays, we also have  $\partial_r \varphi(r, \theta) \rightarrow 0$  as  $r \rightarrow 0$ ), so that

$$\int_0^{\frac{\pi}{2}} \int_\delta^1 \partial_r \varphi(r, \theta)^2 r dr d\theta \leq C$$

where  $C$  is a constant independent of  $\delta$ . On the other hand,

$$\int_\delta^1 \int_0^{\frac{\pi}{2}} \frac{\varphi(r, \theta)^2 + \partial_\theta \varphi(r, \theta)^2}{r} d\theta dr = \int_\delta^1 \|\varphi(r, \cdot)\|_{H^1(0, \pi/2)}^2 \frac{dr}{r}$$

In this last integral, we make the change of variable  $r = \delta^t$ , which gives

$$\int_\delta^1 \int_0^{\frac{\pi}{2}} \frac{\varphi(r, \theta)^2 + \partial_\theta \varphi(r, \theta)^2}{r} d\theta dr = |\log \delta| \int_0^1 \|\varphi(\delta^t, \cdot)\|_{H^1(0, \pi/2)}^2 dt$$

We moreover assume that  $\|\varphi(\delta^t, \cdot)\|_{H^1(0, \pi/2)}^2 \rightarrow \|\varphi(0, \cdot)\|_{H^1(0, \pi/2)}^2$  as  $\delta \rightarrow 0$ . This leads to

$$\varepsilon \int_{\Omega \setminus \Omega_\delta} |DT|^2 \underset{\varepsilon, \delta \rightarrow 0}{\sim} \varepsilon |\log \delta| \|\varphi(0, \cdot)\|_{H^1(0, \pi/2)}^2 + O(\delta^3 + \varepsilon)$$

It remains to estimate the  $L^2$ -norm of the Jacobian matrix of  $S$  on  $\Omega_\delta$ . We recall that  $S$  has to be built so that  $T_\varepsilon$ , defined on the whole  $\Omega$ , is still a transport map from  $\mu$  to  $\nu$  with finite Sobolev norm; thus, the map  $S$ , defined on  $\Omega_\delta$ , must send  $\Omega_\delta$  onto its original image  $S(\Omega_\delta)$  in a regular way and keep the constraint on the image measures:

$$S\#(\mu|_{\Omega_\delta}) = T\#(\mu|_{\Omega_\delta})$$

Moreover, the regularity of the global map  $T_\varepsilon$  implies a compatibility condition at the boundary:

$$S(x) = T(x) \quad \text{for } |x| = \delta$$

Thanks to the Dacorogna-Moser's result, we are indeed able to build such a map  $S$ . Yet, the diameter of  $\Omega_\delta$  is  $\sqrt{2}\delta$ ; on the other hand,  $T(\Omega_\delta)$  contains the whole curve  $\theta \mapsto \varphi(0, \theta)$ , so that its diameter is bounded from below by a positive constant independent of  $\delta$ . Thus, the best estimate that one can expect is

$$\text{Lip } S \leq \frac{C}{\delta}$$

For a reasonable transport map  $T$ , one can show that such a map  $S$  can be found with moreover  $S(x) = T(x)$  for  $|x| = \delta$  (see the paragraph 4.2 below). In this case, the global map  $T_\varepsilon$  still sends  $\mu$  to  $\nu$  and we have

$$\int_{\Omega_\delta} |DT|^2 \leq \int_{\Omega_\delta} \left(\frac{C}{\delta}\right)^2 \leq C^2$$

Finally,

$$J_\varepsilon(T_\varepsilon) - W_1 = \varepsilon |\log \delta| \|\varphi(0, \cdot)\|_{H^1(0, \pi/2)} + O(\delta^3 + \varepsilon)$$

If we choose  $\delta = \varepsilon^{1/3}$ , we obtain

$$J_\varepsilon(T_\varepsilon) = W_1 + \varepsilon |\log \varepsilon| \frac{1}{3} \|\varphi(0, \cdot)\|_{H^1(0, \pi/2)}^2 + O(\varepsilon)$$

In particular:

- the first order of convergence of  $J_\varepsilon(T_\varepsilon)$  to  $W_1$  is not anymore  $\varepsilon$ , but  $\varepsilon |\log \varepsilon|$ ;
- the first significative term only involves the behavior of  $\varphi$  around 0, which is the crossing point of all the transport rays (and also the singularity point of the measure restricted to any transport ray, since it vanishes at 0). This suggests that  $T_\varepsilon \rightarrow T$ , where  $T$  at  $r = 0$  minimizes  $\|\varphi(r, \cdot)\|_{H^1}$ .

### 3.3 Main result and consequences

The analysis in the above paragraph suggests to introduce the minimal value of  $\|\varphi(0, \cdot)\|_{H^1(0, \pi/2)}$  among the functions  $\varphi$  such that

$$x \mapsto \varphi(r, \theta) \frac{x}{|x|}$$

is a transport map from  $\mu$  to  $\nu$ . In particular, for such a  $\varphi$  and for any  $x \in \Omega$ , the point  $\varphi(r, \theta) \frac{x}{|x|}$  still belongs to the target domain  $\Omega'$ ; thus, its value at  $r = 0$  verifies

$$\text{for a.e. } \theta \in (0, \pi/2), \quad R_1(\theta) \leq \varphi(0, \theta) \leq R_2(\theta)$$

We will thus set

$$K = \min \left\{ \int_0^{\frac{\pi}{2}} (\varphi(\theta)^2 + \varphi'(\theta)^2) d\theta : \varphi \in H^1\left(0, \frac{\pi}{2}\right), R_1(\theta) \leq \varphi(\theta) \leq R_2(\theta) \right\}$$

and, following the ideas of the last sub-section,

$$F_\varepsilon : T \mapsto \frac{1}{\varepsilon} \left( J_\varepsilon(T) - W_1 - \frac{K}{3} \varepsilon |\log \varepsilon| \right)$$

Notice here that the assumption  $\inf R_2 > \sup R_1$  allows to remove the constraint  $\varphi(\theta) \leq R_2(\theta)$  in the minimization problem defining  $K$  (indeed, if  $\varphi > R_2$  on a subset of  $(0, \pi/2)$ , one can replace it with  $\min(\varphi, \inf R_2)$  on this subset and this will decrease its Sobolev norm): in other words, we have

$$K = \min \left\{ \int_0^{\frac{\pi}{2}} (\varphi(\theta)^2 + \varphi'(\theta)^2) d\theta : \varphi \in H^1\left(0, \frac{\pi}{2}\right), R_1(\theta) \leq \varphi(\theta) \right\}$$

The main result of this paper is the following:

**Theorem 3.1.** *Let us denote by*

$$G : \varphi \in H^1(0, \pi/2) \mapsto \|\varphi\|_{H^1(0, \pi/2)}^2 - K$$

and

$$F(T) = \begin{cases} +\infty & \text{if } T \notin \mathcal{O}_1(\mu, \nu) \\ \int_0^1 G(\varphi(r, \cdot)) \frac{dr}{r} + \int_0^1 \|\partial_r \varphi(r, \cdot)\|_{L^2}^2 r \, dr & \text{if } T \in \mathcal{O}_1(\mu, \nu), T(x) = \varphi(r, \theta) \frac{x}{|x|} \end{cases}$$

1. *For any family of maps  $(T_\varepsilon)_\varepsilon$  such that  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is bounded, there exists a sequence  $\varepsilon_k \rightarrow 0$  and a map  $T$  such that  $T_{\varepsilon_k} \rightarrow T$  in  $L^2(\Omega)$ .*
2. *There exists a constant  $C$ , depending only on the domains  $\Omega, \Omega'$  and of the measures  $f, g$ , so that, for any family of maps  $(T_\varepsilon)_{\varepsilon>0}$  with  $T_\varepsilon \rightarrow T$  as  $\varepsilon \rightarrow 0$  in  $L^2(\Omega)$ , we have*

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(T_\varepsilon) \geq F(T) - C$$

3. *Moreover, there exists at least one family  $(T_\varepsilon)_{\varepsilon>0}$  such that  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is indeed bounded.*

Notice that we have not stated here a true  $\Gamma$ -convergence result, but we only provide an estimate on the  $\Gamma$ -liminf, and the existence of a sequence with equibounded energy. We conjecture indeed that the  $\Gamma$ -limit of the sequence  $F_\varepsilon$  is of the form  $F - C$  for a suitable constant  $C$  depending on the shape of  $\Omega'$  and  $f(0)$  (again, the main important region is that around  $x = 0$  in  $\Omega$ , which must be sent on the curve  $\Phi$ ); precisely, the constant  $C$  that we expect is

$$C = \lim_{\lambda \rightarrow +\infty} \left( \inf \left\{ \lambda^3 f(0) \int_{\Omega} \left( |y| - y \cdot \frac{S(y)}{|S(y)|} \right) dy + \int_{\Omega} |DS(y)|^2 dy - K \log \lambda : S : \Omega_1 \rightarrow \Omega' \right\} \right)$$

with may be some additional constraint in the minimization problem for each  $\lambda$  (see below a justification at the end of the paragraph 4.1).

However, we do not prove it here and we only use the estimate that we are really able to prove to get some consequences on the minima and the minimizers of  $F_\varepsilon$ .

**Consequences on the minimal value of  $J_\varepsilon$ .** If we apply the Theorem 3.1 to a sequence  $(T_\varepsilon)_\varepsilon$  where each  $T_\varepsilon$  minimizes  $J_\varepsilon$  (which is equivalent with minimizing  $F_\varepsilon$ ), we obtain that the sequence  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is bounded and

$$\inf J_\varepsilon = F_\varepsilon(T_\varepsilon) = W_1(\mu, \nu) + \frac{K}{3} \varepsilon |\log \varepsilon| + O(\varepsilon)$$

which is both the order in  $\varepsilon$  and the constant  $K$  which appeared at the end of the above paragraph. Notice that a full knowledge of the  $\Gamma$ -limit would allow to compute the constant in the term  $O(\varepsilon)$ .

**Consequences on the behavior of  $(T_\varepsilon)_\varepsilon$ .** We first note the following:

**Lemma 3.1.** *The problem which defines the constant  $K$ , namely*

$$\inf \left\{ \int_0^{\frac{\pi}{2}} (\varphi(\theta)^2 + \varphi'(\theta)^2) \, d\theta : \varphi \in H^1 \left( 0, \frac{\pi}{2} \right), R_1(\theta) \leq \varphi(\theta) \leq R_2(\theta) \right\}, \quad (6)$$

*admits a unique minimizer  $\Phi$ . Moreover, if  $\varphi \in H^1(0, \pi/2)$  verifies  $R_1 \leq \varphi$ , then*

$$G(\varphi) \geq \|\varphi - \Phi\|_{H^1}^2$$

*Proof.* Let us denote by  $\mathcal{C}$  the set of functions  $\varphi \in H^1(0, \pi/2)$  verifying the constraint  $R_1 \leq \varphi$  on  $(0, \pi/2)$ . Notice that  $\mathcal{C}$  is a convex closed subset of  $H^1(0, \pi/2)$ , so that (6) admits as well a unique minimizer  $\Phi$ , which is the orthogonal projection of 0 onto  $\mathcal{C}$  in the Hilbert space  $H^1(0, \pi/2)$ . This implies

$$\forall \varphi \in \mathcal{C}, \langle \Phi, \varphi - \Phi \rangle \geq 0 \quad (7)$$

If now  $\varphi \in \mathcal{C}$ , then

$$G(\varphi) - \|\varphi - \Phi\|_{H^1}^2 = \|\varphi\|_{H^1}^2 - \|\Phi\|_{H^1}^2 - \|\varphi - \Phi\|_{H^1}^2 = 2 \langle \Phi, \varphi - \Phi \rangle$$

which is non-negative thanks to the inequality (7).  $\square$

As a consequence, we obtain:

**Proposition 3.2.** *Let  $T \in \mathcal{O}_1(\mu, \nu)$ ,  $T(x) = \varphi(r, \theta) \frac{x}{|x|}$ , such that  $F(T) < +\infty$ . Then  $r \mapsto \varphi(r, \cdot)$  is continuous from  $[0, 1]$  to  $L^2(0, \pi/2)$ , and we have  $\varphi(0, \cdot) = \Phi$ .*

This, combined with Theorem 3.1, implies that that if  $(T_\varepsilon)_\varepsilon$  has an equi-bounded energy (meaning that  $F_\varepsilon(T_\varepsilon)$  is uniformly bounded in  $\varepsilon$ ), then it has, up to a subsequence, a limit  $T = \varphi(r, \theta) \frac{x}{|x|}$  where  $\varphi(r, \theta)$  is continuous with respect to  $r$  and has  $\Phi(\theta)$  as limit as  $r \rightarrow 0$ . In other words,  $T$  sends 0 onto the curve  $r = \Phi(\theta)$  which has the best  $H^1$ -norm among the curves with values in the target domain  $\Omega'$ . This is in particular true if each  $T_\varepsilon$  minimizes  $J_\varepsilon$  (thus  $F_\varepsilon$ ).

*Proof of Prop. 3.2.* The assumption on  $T$  implies that the integrals

$$\int_0^1 \|\partial_r \varphi(r, \cdot)\|_{L^2(0, \pi/2)}^2 r \, dr \quad \text{and} \quad \int_0^1 \|\partial_r \varphi(r, \cdot)\|_{L^2(0, \pi/2)}^2 r \, dr$$

are both controlled by some finite constant  $A$ . Now we have for  $\theta \in (0, \pi/2)$ :

$$\begin{aligned} |\varphi(r_1, \theta) - \varphi(r_2, \theta)| &= \int_{r_1}^{r_2} \partial_r \varphi(r, \theta) \, dr \leq \left( \int_{r_1}^{r_2} \partial_r \varphi(r, \theta)^2 r \, dr \right)^{1/2} \left( \int_{r_1}^{r_2} \frac{dr}{r} \right)^{1/2} \\ \text{thus} \quad \int_0^{\pi/2} |\varphi(r_1, \theta) - \varphi(r_2, \theta)|^2 \, d\theta &\leq \left( \int_{r_1}^{r_2} \|\partial_r \varphi(r, \cdot)\|_{L^2(0, \pi/2)}^2 r \, dr \right) \left( \int_{r_1}^{r_2} \frac{dr}{r} \right) \\ \text{so that} \quad \|\varphi(r_1, \cdot) - \varphi(r_2, \cdot)\|_{L^2(0, \pi/2)}^2 \, d\theta &\leq A \log \frac{r_2}{r_1} \end{aligned} \tag{8}$$

This proves the continuity of  $r \mapsto \varphi(r, \cdot)$ .

On the other hand, thanks to the Lemma 3.1, we have

$$A \geq \int_0^1 G(\varphi(r, \cdot)) \frac{dr}{r} \geq \int_0^1 \|\varphi(r, \cdot) - \Phi\|_{L^2(0, \pi/2)}^2 \frac{dr}{r}$$

By setting  $r = e^{-t}$ , we obtain

$$A \geq \int_0^{+\infty} \|\varphi(e^{-t}, \cdot) - \Phi\|_{L^2}^2 \, dt$$

But, for  $t_1 < t_2 \in (0, +\infty)$ , we have

$$\begin{aligned} \|\varphi(e^{-t_1}, \cdot) - \Phi\|_{L^2}^2 - \|\varphi(e^{-t_2}, \cdot) - \Phi\|_{L^2}^2 &= |\langle \varphi(e^{-t_1}, \cdot) - \varphi(e^{-t_2}, \cdot), \Phi \rangle_{L^2}| \\ &\leq \|\varphi(e^{-t_1}, \cdot) - \varphi(e^{-t_2}, \cdot)\| \|\Phi\| \\ &\leq A \log \frac{e^{-t_2}}{e^{-t_1}} = A(t_2 - t_1) \end{aligned}$$

where the last inequality comes from (8). Thus, the function  $t \mapsto \|\varphi(e^{-t}, \cdot) - \Phi\|_{L^2}^2$  is Lipschitz and belongs to  $L^1(0, +\infty)$ . This implies that it vanishes at  $+\infty$ , so that  $\varphi(r, \cdot) \rightarrow \Phi$  in  $L^2(0, \pi/2)$  as  $r \rightarrow 0$ .  $\square$

## 4 Proof of Theorem 3.1

### 4.1 $\Gamma$ -liminf estimate

Let  $(T_\varepsilon)_\varepsilon$  be a family of maps and let us begin by writing precisely the expression of  $F_\varepsilon(T_\varepsilon)$  for any  $\varepsilon > 0$  and any transport map  $T_\varepsilon$ . We have

$$F_\varepsilon(T_\varepsilon) = \frac{1}{\varepsilon} \left( \int_\Omega |T_\varepsilon(x) - x| f(x) \, dx - W_1 \right) + \int_\Omega |DT_\varepsilon(x)|^2 \, dx - \frac{K}{3} |\log \varepsilon|$$

We set  $\delta = \varepsilon^{1/3}$  and notice that

$$\frac{K}{3} |\log \varepsilon| = K |\log \delta| = K \int_\delta^1 \frac{dr}{r}$$

We decompose  $T_\varepsilon$  into radial and tangential components:

$$T_\varepsilon(x) = \varphi_\varepsilon(r, \theta) \frac{x}{|x|} + \psi_\varepsilon(r, \theta) \frac{x^\perp}{|x|}$$

and compute

$$DT_\varepsilon = \frac{x}{|x|} \otimes \nabla \varphi_\varepsilon(x) + \frac{\varphi_\varepsilon(x)}{|x|} \left( I_d - \frac{x}{|x|} \otimes \frac{x}{|x|} \right) + \frac{x^\perp}{|x|} \otimes \nabla \psi_\varepsilon(x) + \frac{\psi_\varepsilon(x)}{|x|} \left( R - \frac{x^\perp}{|x|} \otimes \frac{x}{|x|} \right)$$

where  $R$  denotes the rotation with angle  $\pi/2$  and we still set  $x^\perp = Rx$ . Thus, the matrix of  $DT_\varepsilon$  in the basis  $\left( \frac{x}{|x|}, \frac{x^\perp}{|x|} \right)$  is

$$DT_\varepsilon(x) = \begin{pmatrix} \partial_r \varphi_\varepsilon & \partial_r \psi_\varepsilon \\ \frac{\partial_\theta \varphi_\varepsilon - \psi_\varepsilon}{r} & \frac{\varphi_\varepsilon + \partial_\theta \psi_\varepsilon}{r} \end{pmatrix}$$

so that

$$|DT_\varepsilon|^2 = \partial_r \varphi_\varepsilon^2 + \partial_r \psi_\varepsilon^2 + \frac{(\partial_\theta \varphi_\varepsilon - \psi_\varepsilon)^2}{r^2} + \frac{(\varphi_\varepsilon + \partial_\theta \psi_\varepsilon)^2}{r^2}$$

We obtain

$$\int_\Omega |DT_\varepsilon|^2 = \int_{\Omega_\delta} |DT_\varepsilon|^2 + \int_\delta^1 (\|\partial_r \varphi_\varepsilon(r, \cdot)\|_{L^2}^2 + \|\partial_r \psi_\varepsilon\|_{L^2}^2) r \, dr + \int_\delta^1 (\|\partial_\theta \varphi_\varepsilon - \psi_\varepsilon\|_{L^2}^2 + \|\varphi_\varepsilon + \partial_\theta \psi_\varepsilon\|_{L^2}^2) \frac{dr}{r}$$

On the other hand, we already know that

$$\int_\Omega |T_\varepsilon(x) - x| f(x) \, dx - W_1 = \int_\Omega (|T_\varepsilon(x) - x| - |T_\varepsilon(x)| + |x|) f(x) \, dx$$

Finally, the complete expression of  $F_\varepsilon$  is the following:

$$\begin{aligned} F_\varepsilon(T_\varepsilon) &= \frac{1}{\varepsilon} \int_\Omega (|T_\varepsilon(x) - x| - |T_\varepsilon(x)| + |x|) f(x) \, dx + \int_{\Omega_\delta} |DT_\varepsilon|^2 \\ &+ \int_\delta^1 (\|(\partial_\theta \varphi_\varepsilon - \psi_\varepsilon)(r, \cdot)\|_{L^2}^2 + \|(\varphi_\varepsilon + \partial_\theta \psi_\varepsilon)(r, \cdot)\|_{L^2}^2 - K) \frac{dr}{r} + \int_\delta^1 (\|\partial_r \varphi_\varepsilon(r, \cdot)\|_{L^2}^2 + \|\partial_r \psi_\varepsilon\|_{L^2}^2) r \, dr \end{aligned}$$

thus, if we denote by  $H(\varphi, \psi) = \int_0^{\pi/2} (\varphi'(\theta) - \psi(\theta))^2 + (\varphi(\theta) + \psi'(\theta))^2 \, d\theta$  for  $\varphi, \psi \in H^1(0, \pi/2)$ , we have

$$\begin{aligned} F_\varepsilon(T_\varepsilon) &= \frac{1}{\varepsilon} \int_\Omega (|T_\varepsilon(x) - x| - |T_\varepsilon(x)| + |x|) f(x) \, dx + \int_{\Omega_\delta} |DT_\varepsilon|^2 \\ &+ \int_\delta^1 (H(\varphi_\varepsilon(r, \cdot), \psi_\varepsilon(r, \cdot)) - K) \frac{dr}{r} + \int_\delta^1 (\|\partial_r \varphi_\varepsilon(r, \cdot)\|_{L^2}^2 + \|\partial_r \psi_\varepsilon\|_{L^2}^2) r \, dr \quad (9) \end{aligned}$$

The following lemma collects some properties of the function  $H$ .

**Lemma 4.1.** *We recall that*

$$H : (\varphi, \psi) \in H^1(0, \pi/2) \times H^1(0, \pi/2) \mapsto \|\varphi' - \psi\|_{L^2}^2 + \|\varphi + \psi'\|_{L^2}^2$$

for  $\varphi, \psi \in H^1(0, \pi/2)$ . Then:

- The function  $H$  is lower semi-continuous with respect to the strong  $L^2$ -convergence.
- Assume that  $(\varphi, \psi)$  satisfies, for any  $\theta$ ,

$$\varphi(\theta) \hat{x}(\theta) + \psi(\theta) \hat{x}^\perp(\theta) \in \Omega'$$

where  $\hat{x}(\theta) = (\cos \theta, \sin \theta)$ . We denote by  $\tilde{\varphi}(\theta) = \max(\varphi(\theta), R_1(\theta))$ . Then we have the inequality

$$H(\varphi, \psi) \geq K + \frac{1}{2} \|\tilde{\varphi} - \Phi\|_{L^2}^2 - B \|\psi\|_{L^2(0, \pi/2)}^{2/3} \quad (10)$$

for some positive constant  $B$  which only depends on  $\Omega'$ .

*Proof. Step 1: the semi-continuity of  $H$ .* We take a sequence  $(\varphi_n, \psi_n)_n$  converging to some  $(\varphi, \psi)$  for the  $L^2$ -norm. Up to subsequences, we can assume that

$$\liminf_{n \rightarrow +\infty} H(\varphi_n, \psi_n) = \lim_{n \rightarrow +\infty} H(\varphi_n, \psi_n)$$

and we also assume that  $(H(\varphi_n, \psi_n))_n$  is bounded. Now we remark that

$$H(\varphi_n, \psi_n) = \|\varphi'_n - \psi_n\|_{L^2}^2 + \|\varphi_n + \psi'_n\|_{L^2}^2 \geq (\|\varphi'_n\|_{L^2} - \|\psi_n\|_{L^2})^2 + (\|\varphi'_n\|_{L^2} - \|\psi_n\|_{L^2}^2)^2$$

$$\text{thus } \|\varphi'_n\|_{L^2} \leq \sqrt{H(\varphi_n, \psi_n)} + \|\psi_n\|_{L^2} \quad \text{and} \quad \|\psi'_n\|_{L^2} \leq \sqrt{H(\varphi_n, \psi_n)} + \|\varphi_n\|_{L^2}$$

We deduce that  $(\varphi_n)_n, (\psi_n)_n$  are bounded in  $H^1(0, \pi/2)$  so that the convergence  $(\varphi_n, \psi_n) \rightarrow (\varphi, \psi)$  actually holds, up to a subsequence, weakly in  $H^1(0, \pi/2)$ . Now the convexity of  $(\varphi, \varphi', \psi, \psi') \mapsto (\varphi' - \psi)^2 + (\varphi + \psi')^2$  implies that  $H$  is lower semi-continuous with respect to the weak convergence in  $H^1(0, \pi/2)$ , which allows to conclude.

Now we pass to the proof of the inequality (10). We begin by a kind of “sub-lemma” which will be useful several times in the proof.

*Step 2: preliminary estimates.* We recall that  $\tilde{\varphi} = \max(R_1, \varphi)$  and denote by  $h = \tilde{\varphi} - \varphi \geq 0$ . Now we claim that:

- for any  $t \in (0, \pi/2)$ , we have the inequality

$$0 \leq h(t) \leq B_1 |\psi(t)| \tag{11}$$

for some constant  $B_1$  depending only on  $\Omega'$ .

- we have the inequality

$$|\langle \tilde{\varphi}, h \rangle_{H^1}| \leq B_2 \|h\|_\infty \tag{12}$$

for some constant  $B_2$  depending only on  $\Omega'$ ;

- the above both inequalities lead to the estimate

$$\|\varphi\|_{H^1}^2 \geq K + \|\tilde{\varphi} - \Phi\|_{H^1}^2 - B_3 \|\psi\|_\infty \tag{13}$$

for some constant  $B_3$  depending only on  $\Omega'$ .

First, we remark that the constraint  $\varphi(\theta)\hat{x}(\theta) + \psi(\theta)\hat{x}^\perp(\theta) \in \Omega'$  implies

$$R_1(\theta')^2 < \varphi(\theta)^2 + \psi(\theta)^2 < R_2(\theta')^2 \quad \text{where} \quad \theta' = \theta + \arcsin \frac{\psi(\theta)}{\sqrt{\varphi(\theta)^2 + \psi(\theta)^2}}$$

Thus, we have

$$\begin{aligned} h(\theta) &= R_1(\theta) - \varphi(\theta) \\ &= R_1(\theta) - R_1(\theta') + R_1(\theta') - \varphi(\theta) \\ &\leq (\text{Lip } R_1)|\theta - \theta'| + \sqrt{\varphi^2(\theta) + \psi^2(\theta)} - \varphi(\theta) \\ &\leq (\text{Lip } R_1) \arcsin \frac{|\psi(\theta)|}{R_1(\theta')} + |\psi(\theta)| \\ &\leq \left( \frac{\pi}{2} \frac{\text{Lip } R_1}{\inf R_1} + 1 \right) |\psi(\theta)| \end{aligned}$$

which is (11) with  $B_1 = \frac{\pi}{2} \frac{\text{Lip } R_1}{\inf R_1} + 1$ .

Second, we recall that  $h = (R_1 - \varphi)^+$ , thus  $\varphi + h = R_1$  on any point where  $h \neq 0$ . This leads to

$$\left| \int_0^{\pi/2} (\varphi + h)h \right| = \left| \int_0^{\pi/2} R_1 h \right| \leq \frac{\pi}{2} (\sup R_1) \|h\|_\infty$$

$$\text{and} \quad \left| \int_0^{\pi/2} (\varphi + h)'h' \right| = \left| \int_0^{\pi/2} R_1' h' \right| = \left| [R_1' h]_0^{\pi/2} - \int_0^{\pi/2} R_1'' h \right| \leq (2 \sup R_1' + \|R_1''\|_1) \|h\|_\infty$$

We get (12) with  $B_2 = \left( \frac{\pi}{2} \sup R_1 + 2 \text{Lip } R_1 + \|R_1''\|_1 \right)$ .

Third, we write

$$\|\varphi\|_{H^1}^2 = \|\tilde{\varphi}\|_{H^1}^2 + \|h\|_{H^1}^2 + 2\langle \tilde{\varphi}, h \rangle \quad (14)$$

Since  $\tilde{\varphi} \geq R_1$  on  $(0, \pi/2)$  and thanks to the Lemma 3.1, we have  $\|\tilde{\varphi}\|_{H^1}^2 \geq \|\tilde{\varphi} - \Phi\|_{H^1}^2 + K$ . On the other hand, by using (11) and (12), we have

$$\langle \tilde{\varphi}, h \rangle \geq -B_2 \|h\|_\infty \geq -B_1 B_2 \|\psi\|_\infty$$

We insert into (14) and skip  $\|h\|_{H^1}^2$  since it is nonnegative to get

$$\|\varphi\|_{H^1}^2 \geq K + \|\tilde{\varphi} - \Phi\|_{H^1}^2 - 2B_1 B_2 \|\psi\|_\infty$$

thus (13) holds with  $B_3 = 2B_1 B_2$ .

*Step 3: the inequality (10) holds if  $\|\varphi'\|_{L^2}$  is large enough.* We start from

$$H(\varphi, \psi) = \|\varphi\|_{H^1}^2 + \|\psi\|_{H^1}^2 + 2 \int_0^{\pi/2} \varphi \psi' - 2 \int_0^{\pi/2} \psi \varphi' = \|\varphi\|_{H^1}^2 + \|\psi\|_{H^1}^2 - 4 \int_0^{\pi/2} \psi \varphi' - 2[\varphi \psi]_0^{\pi/2}$$

First, the condition on  $(\varphi, \psi)$  implies that  $\|\varphi\|_\infty, \|\psi\|_\infty \leq \sup R_2$  so that

$$|[\varphi \psi]_0^{\pi/2}| \leq 2(\sup R_2)^2$$

On the other hand,

$$\left| \int_0^{\pi/2} \psi \varphi' \right| \leq \|\psi\|_\infty \sqrt{\frac{\pi}{2}} \|\varphi'\|_{L^2} \leq \sup R_2 \sqrt{\frac{\pi}{2}} \|\varphi'\|_{L^2}$$

This leads to

$$\begin{aligned} H(\varphi, \psi) &\geq \|\varphi\|_{H^1}^2 + \|\psi\|_{H^1}^2 - 4 \sup R_2 \sqrt{\frac{\pi}{2}} \|\varphi'\|_{L^2} - 4(\sup R_2)^2 \\ &\geq \frac{1}{2} \|\varphi\|_{H^1}^2 + \left( \frac{1}{2} \|\varphi'\|_{L^2}^2 - 4 \sup R_2 \sqrt{\frac{\pi}{2}} \|\varphi'\|_{L^2} - 4(\sup R_2)^2 \right) \end{aligned}$$

By using (13), we obtain

$$H(\varphi, \psi) \geq \frac{1}{2} (K + \|\tilde{\varphi} - \Phi\|_{H^1}^2 - B_3 \|\psi\|_\infty) + \left( \frac{1}{2} \|\varphi'\|_{L^2}^2 - 4 \sup R_2 \sqrt{\frac{\pi}{2}} \|\varphi'\|_{L^2} - 4(\sup R_2)^2 \right)$$

and, since  $|\psi| \leq \sqrt{\varphi^2 + \psi^2} \leq R_1$ , we have

$$H(\varphi, \psi) \geq \frac{1}{2} (K + \|\tilde{\varphi} - \Phi\|_{H^1}^2) + \left( \frac{1}{2} \|\varphi'\|_{L^2}^2 - 4 \sup R_2 \sqrt{\frac{\pi}{2}} \|\varphi'\|_{L^2} - \left( 4(\sup R_2)^2 + \frac{B_3}{2} \sup R_2 \right) \right)$$

The announced estimate (10) holds provided that the term in brackets is greater than  $\frac{K}{2}$ , which is true provided that  $\|\varphi'\|_{L^2} \geq B_4$  where  $B_4$  is the largest root of the polynomial

$$\frac{1}{2} X^2 - 4 \sup R_2 \sqrt{\frac{\pi}{2}} X - \left( 4(\sup R_2)^2 + \frac{B_3}{2} \sup R_2 + \frac{K}{2} \right)$$

and  $B_4$  only depends of  $\Omega'$ .

*Step 4: case  $\|\varphi'\|_{L^2} \leq B_4$ .* In this case, we still have

$$H(\varphi, \psi) = \|\varphi\|_{H^1}^2 + \|\psi\|_{H^1}^2 - 4 \int_0^{\pi/2} \psi \varphi' - 2[\varphi \psi]_0^{\pi/2}$$

$$\text{with } \left| \int_0^{\pi/2} \psi \varphi' \right| \leq \|\psi\|_\infty \sqrt{\frac{\pi}{2}} \|\varphi'\|_{L^2} \leq \sqrt{\frac{\pi}{2}} B_4 \|\psi\|_\infty$$

$$\text{and } |[\varphi \psi]_0^{\pi/2}| \leq 2 \|\varphi\|_\infty \|\psi\|_\infty \leq 2 \sup R_2 \|\psi\|_\infty$$

This leads to

$$H(\varphi, \psi) \geq \|\varphi\|_{H^1}^2 + \|\psi\|_{H^1}^2 - \left( \sqrt{\frac{\pi}{2}} B_4 + 2 \sup R_2 \right) \|\psi\|_\infty$$



$$\geq K + \|\tilde{\varphi} - \Phi\|_{H^1}^2 + \|\psi\|_{H^1}^2 - B_5 \|\psi\|_\infty$$

where we have again used (13) and set  $B_5 = (\sqrt{\frac{\pi}{2}}B_4 + 2 \sup R_2) + B_3$ , which only depends on  $\Omega'$ .

It now remains to estimate  $\|\psi\|_{H^1}^2 - B_5 \|\psi\|_\infty$  from below with  $-\|\psi\|_{L^2}^{2/3}$ . The condition on  $(\varphi, \psi)$  implies that  $\psi(0) \geq 0$  and  $\psi(\pi/2) \leq 0$ , so that there exists  $t_0$  such that  $\psi(t_0) = 0$ . We then have

$$\psi^2(t) = \int_{t_0}^t \frac{d}{dt}(\psi^2) = \int_{t_0}^t 2\psi\psi' \leq 2\|\psi\|_{L^2}\|\psi'\|_{L^2} \quad \text{thus} \quad \|\psi\|_\infty \leq \sqrt{2}\sqrt{\|\psi\|_{L^2}\|\psi'\|_{L^2}}$$

We use the Young inequality

$$ab \leq \frac{(\varepsilon a)^p}{p} + \frac{(b/\varepsilon)^q}{q} \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad \varepsilon > 0$$

with  $p = 4$ ,  $q = 4/3$ ,  $a = \sqrt{\|\psi'\|_{L^2}}$  and  $b = \sqrt{\|\psi\|_{L^2}}$ , to get

$$\|\psi\|_\infty \leq \frac{\sqrt{2}\varepsilon^4}{4}\|\psi'\|_{L^2}^2 + \frac{3\sqrt{2}}{4\varepsilon^{4/3}}\|\psi\|_{L^2}^{2/3}$$

We deduce

$$\|\psi'\|_{H^1}^2 - B_5 \|\psi\|_\infty \geq \left(1 - \frac{\sqrt{2}B_5\varepsilon^4}{4}\right)\|\psi'\|_{L^2} - \frac{3\sqrt{2}B_5}{4\varepsilon^{4/3}}\|\psi\|_{L^2}^{2/3}$$

By choosing  $\varepsilon$  such that  $\frac{\sqrt{2}B_5\varepsilon^4}{4} = 1$ , we obtain

$$\|\psi\|_{H^1}^2 - B_5 \|\psi\|_\infty \geq -B\|\psi\|_{L^2}^{2/3}$$

where  $B = \frac{3\sqrt{2}B_5}{4\varepsilon^{4/3}}$  only depends on  $\Omega'$  and  $K$ . This achieves the proof.  $\square$

We also will need the following estimate on the first term of the expression (9).

**Lemma 4.2.** *Let  $T$  be a transport map from  $\mu$  to  $\nu$ . We write*

$$T(x) = \varphi(x) \frac{x}{|x|} + \psi(x) \frac{x^\perp}{|x|}$$

Then, for a.e.  $x$ ,

$$|T(x) - x| - |T(x)| + |x| \geq A|x|\psi^2(x) \tag{15}$$

for some constant  $A$  which only depends on  $\Omega'$ .

*Proof.* We compute:

$$|T(x) - x| - |T(x)| + |x| = \frac{|T(x) - x|^2 - |T(x)|^2}{|T(x) - x| + |T(x)|} + |x|$$

We have  $|T(x) - x|^2 = (\varphi(x) - |x|)^2 + \psi(x)^2$  and  $|T|^2 = \varphi^2 + \psi^2$ , so that

$$|T(x) - x| - |T(x)| + |x| = \frac{|x|^2 - 2|x|\varphi(x)}{|T(x) - x| + |T(x)|} + |x| = |x| \frac{|x| + |T(x) - x| + |T(x)| - 2\varphi(x)}{|T(x) - x| + |T(x)|}$$

We remark that

$$|T(x) - x| - \varphi(x) + |x| = \sqrt{(\varphi(x) - |x|)^2 + \psi(x)^2} - (\varphi(x) - |x|) \geq 0$$

$$\text{thus} \quad |T(x) - x| - |T(x)| + |x| \geq |x| \frac{|T(x)| - \varphi(x)}{|T(x) - x| + |T(x)|} = |x| \frac{|T(x)|^2 - \varphi(x)^2}{(|T(x) - x| + |T(x)|)(|T(x)| + \varphi(x))}$$

Since  $x \in \Omega$  and  $T(x) \in \Omega'$ , we have

$$|T(x) - x| + |T(x)| \leq 2|T(x)| + |x| \leq 2 \sup R_2 + 1$$

$$\text{and} \quad |T(x)| + \varphi(x) \leq 2|T(x)| \leq 2 \sup R_2$$

On the other hand,  $|T(x)|^2 - \varphi(x)^2 = \psi(x)^2$ . This leads to the result with  $A = \frac{1}{(2 \sup R_2 + 1)(2 \sup R_2)}$   $\square$

The estimate (15) leads to

$$\int_{\Omega} (|T_{\varepsilon}(x) - x| - |T_{\varepsilon}(x)| + |x|) f(x) dx \geq A \inf f \int_0^1 \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r dr$$

and the estimate (10) to

$$\int_{\delta}^1 (H(\varphi_{\varepsilon}(r, \cdot), \psi_{\varepsilon}(r, \cdot)) - K) \frac{dr}{r} \geq \int_{\delta}^1 \left( -B \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^{2/3} + \frac{1}{2} \|\tilde{\varphi}_{\varepsilon}(r, \cdot) - \Phi\|_{H^1}^2 \right) \frac{dr}{r}$$

where we again have set  $\tilde{\varphi}_{\varepsilon} = \max(R_1, \varphi_{\varepsilon})$ . By inserting into (9), we have

$$\begin{aligned} F_{\varepsilon}(T_{\varepsilon}) &\geq \frac{A \inf f}{\varepsilon} \int_0^1 \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r^2 dr - B \int_{\delta}^1 \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^{2/3} \frac{dr}{r} \\ &\quad + \frac{1}{2} \int_{\delta}^1 (H(\varphi_{\varepsilon}(r, \cdot), \psi_{\varepsilon}(r, \cdot)) - K + B \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^{2/3}) \frac{dr}{r} + \int_{\delta}^1 \|\partial_r \varphi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r dr \end{aligned} \quad (16)$$

Let us denote by  $X_{\varepsilon} = \frac{1}{\varepsilon} \int_0^1 \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r^2 dr$ . By the Hölder inequality applied with respect to the measure with density  $1/r$  on  $(\delta, 1)$ , we have

$$\begin{aligned} \int_{\delta}^1 \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^{2/3} \frac{dr}{r} &= \int_{\delta}^1 (\|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r^3)^{1/3} \frac{1}{r} \frac{dr}{r} \leq \left( \int_{\delta}^1 \|\psi_{\varepsilon}\|_{L^2}^2 r^3 \frac{dr}{r} \right)^{1/3} \left( \int_{\delta}^1 \frac{1}{r^{4/3}} \frac{dr}{r} \right)^{3/4} \\ \text{with } \int_{\delta}^1 \|\psi_{\varepsilon}\|_{L^2}^2 r^3 \frac{dr}{r} &\leq \varepsilon X_{\varepsilon} \quad \text{and} \quad \int_{\delta}^1 \frac{1}{r^{4/3}} \frac{dr}{r} = \frac{3}{4} \left( \frac{1}{\delta^{4/3}} - 1 \right) \leq \frac{3}{2\delta^{4/3}} \end{aligned}$$

which leads to

$$\int_{\delta}^1 \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^{2/3} \frac{dr}{r} \leq (\varepsilon X_{\varepsilon})^{1/3} \left( \frac{3}{2\delta^{4/3}} \right)^{3/4} = \sqrt{\frac{3\sqrt{3}}{2\sqrt{2}}} X_{\varepsilon}^{1/3}$$

since  $\delta = \varepsilon^{1/3}$ . We insert into (16) to obtain

$$\begin{aligned} F_{\varepsilon}(T_{\varepsilon}) &\geq (A \inf f) X_{\varepsilon} - B' X_{\varepsilon}^{1/3} \\ &\quad + \int_{\delta}^1 (H(\varphi_{\varepsilon}(r, \cdot), \psi_{\varepsilon}(r, \cdot)) - K + B \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^{2/3}) \frac{dr}{r} + \int_{\delta}^1 \|\partial_r \varphi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r dr \end{aligned} \quad (17)$$

where  $B' = \sqrt{\frac{3\sqrt{3}}{2\sqrt{2}}} B$ .

Let us assume that  $(F_{\varepsilon}(T_{\varepsilon}))_{\varepsilon}$  is bounded by a positive constant  $M$ . This implies that  $(X_{\varepsilon})_{\varepsilon}$  is bounded by some constant  $M'$  (otherwise the term  $(A \inf f) X_{\varepsilon} - B' X_{\varepsilon}^{1/3}$  would tend to  $+\infty$ , and the other term is positive), thus

$$\int_0^1 \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r^2 dr \leq M' \varepsilon$$

and  $\psi_{\varepsilon} \rightarrow 0$  a.e. on  $\Omega$ . Since  $(X_{\varepsilon})_{\varepsilon}$  and  $(F_{\varepsilon}(T_{\varepsilon}))_{\varepsilon}$  are bounded, (17) provides

$$\int_{\delta}^1 (H(\varphi_{\varepsilon}(r, \cdot), \psi_{\varepsilon}(r, \cdot)) - K + B \|\psi_{\varepsilon}(r, \cdot)\|_{L^2}^{2/3}) \frac{dr}{r} + \int_{\delta}^1 \|\partial_r \varphi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r dr \leq M''$$

for some constant  $M''$  which does not depend on  $\varepsilon$ . We now use the estimate (10) to get

$$\frac{1}{2} \int_{\delta}^1 \|\tilde{\varphi}_{\varepsilon} - \Phi\|_{H^1}^2 \frac{dr}{r} + \int_{\delta}^1 \|\partial_r \varphi_{\varepsilon}(r, \cdot)\|_{L^2}^2 r dr \leq M''$$

We thus have a  $L_{loc}^2$ -loc bound  $\partial_r \varphi_{\varepsilon}$  and on  $\partial_{\theta} \tilde{\varphi}_{\varepsilon}$ , but since  $\tilde{\varphi}(r, \theta) = \max(R_1(\theta), \varphi(r, \theta))$ , the bound on  $\partial_r \varphi_{\varepsilon}$  implies a bound on  $\partial_r \tilde{\varphi}_{\varepsilon}$ . Thus, the family  $(\tilde{\varphi}_{\varepsilon})_{\varepsilon}$  is bounded in  $H_{loc}^1(\Omega)$  and there exists  $\varepsilon_k \rightarrow 0$  and  $\tilde{\varphi}$  such that  $\tilde{\varphi}_{\varepsilon_k} \rightarrow \tilde{\varphi}$  a.e. on  $\Omega$ . But we recall that the estimation (11) still holds and provides

$$|\varphi_{\varepsilon_k} - \tilde{\varphi}_{\varepsilon_k}| \leq B_1 |\psi_{\varepsilon_k}| \rightarrow 0$$

which leads  $\varphi_{\varepsilon_k} \rightarrow \tilde{\varphi}$  a.e. on  $\Omega$ . If we set now  $T(x) = \tilde{\varphi} \frac{x}{|x|}$ , we have proved that  $T_{\varepsilon_k} \rightarrow T$  a.e. on  $\Omega$ . This convergence also holds in  $L^2(\Omega)$  since  $|T_\varepsilon| \leq \sup R_2$  for any  $\varepsilon$ . This proves the first statement of the Theorem 3.1.

Assume now that  $(T_\varepsilon)_\varepsilon$  is a family of transport maps converging to some  $T$  for the  $L^2$ -norm on  $\Omega$ . We deduce from (17) that, if we set  $C = -\inf_{X>0} (A \inf f X^3 - B' X)$ , which only depends on  $\Omega'$ , we have

$$F_\varepsilon(T_\varepsilon) \geq -C + \int_\delta^1 (H(\varphi_\varepsilon(r, \cdot), \psi_\varepsilon(r, \cdot)) - K + B \|\psi_\varepsilon(r, \cdot)\|_{L^2}^{2/3}) \frac{dr}{r} + \int_\delta^1 \|\partial_r \varphi_\varepsilon(r, \cdot)\|_{L^2}^2 r dr$$

Assuming that  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is bounded, the above computations give a  $H^1$ -loc bound for  $(\varphi_\varepsilon)_\varepsilon$ , thus

$$\liminf_{\varepsilon \rightarrow 0} \int_\delta^1 \|\partial_r \varphi_\varepsilon(r, \cdot)\|_{L^2}^2 r dr \geq \int_0^1 \|\partial_r \varphi(r, \cdot)\|_{L^2}^2 r dr$$

since this functional is lower semi-continuous for the weak convergence in  $H^1(\Omega)$ . On the other hand, the estimate (10) shows also that

$$H(\varphi_\varepsilon(r, \cdot), \psi_\varepsilon(r, \cdot)) - K + B \|\psi_\varepsilon(r, \cdot)\|_{L^2}^{2/3} \geq 0$$

for any  $\varepsilon$  and  $r$ , thus we can apply the Fatou lemma since the semi-continuity of  $H$  provides

$$\liminf_{k \rightarrow +\infty} (H(\varphi_\varepsilon(r, \cdot), \psi_\varepsilon(r, \cdot)) - K + B \|\psi_\varepsilon(r, \cdot)\|_{L^2}^{2/3}) \geq G(\varphi(r, \cdot))$$

for a.e.  $r \in (0, 1)$ . Thus

$$\liminf_{\varepsilon \rightarrow 0} \geq -C + \int_0^1 G(\varphi(r, \cdot)) \frac{dr}{r} + \int_0^1 \|\partial_r \varphi(r, \cdot)\|_{L^2}^2 r dr$$

as announced.

**Remark.** If we choose to set  $\delta = \lambda \varepsilon^{1/3}$ , where  $\lambda$  is to be fixed (and possibly sent to 0 or  $\infty$ , or taken depending on  $\varepsilon$ ), the precise expression (9) becomes

$$\begin{aligned} F_\varepsilon(T_\varepsilon) &= \frac{1}{\varepsilon} \int_{\Omega_\delta} (|T_\varepsilon(x) - x| - |T(x)| + |x|) f(x) dx + \int_{\Omega_\delta} |DT_\varepsilon|^2 - K \log \lambda \\ &\quad + \frac{1}{\varepsilon} \int_{\Omega \setminus \Omega_\delta} (|T_\varepsilon(x) - x| - |T(x)| + |x|) f(x) dx - B \int_\delta^1 \|\psi_\varepsilon(r, \cdot)\|_{L^2}^{2/3} r dr \\ &\quad + \int_\delta^1 (H(\varphi_\varepsilon(r, \cdot), \psi_\varepsilon(r, \cdot)) - K + B \|\psi_\varepsilon(r, \cdot)\|_{L^2}^{2/3}) \frac{dr}{r} + \int_\delta^1 \|\partial_r \varphi_\varepsilon(r, \cdot)\|_{L^2}^2 r dr \end{aligned} \quad (18)$$

By using the above estimates (10) and (15), we get that the second line is this time bounded from below by

$$A(\inf f) X_\varepsilon - \frac{B'}{\lambda} (X_\varepsilon)^{1/3}$$

which is itself bounded from below by  $-C_\lambda = \inf\{A(\inf f)X - \frac{B'}{\lambda} X^{1/3}\}$ , which goes to 0 as  $\lambda \rightarrow +\infty$ . Thus, these terms disappear if we allow  $\lambda \rightarrow +\infty$ . Let us now compute the first line of (18):

$$\begin{aligned} &\frac{1}{\varepsilon} \int_{\Omega_\delta} (|T_\varepsilon(x) - x| - |T(x)| + |x|) f(x) dx + \int_{\Omega_\delta} |DT_\varepsilon|^2 - K \log \lambda \\ &= \frac{\delta^2}{\varepsilon} \int_{\Omega} (|U_\varepsilon(y) - \delta y| - |U_\varepsilon(y)| + |\delta y|) f(\delta y) dy + \int_{\Omega} |DU_\varepsilon|^2 - K \log \lambda \end{aligned}$$

where we have set  $x = \delta y$  and  $U_\varepsilon(y) = T_\varepsilon(\delta y)$ . Now we use the following expansion

$$|U_\varepsilon(y) - \delta y| - |U_\varepsilon(y)| = \delta \left( |y| - y \cdot \frac{U_\varepsilon(y)}{|U_\varepsilon(y)|} \right) + o(\delta^2)$$

and recall that  $\delta = \lambda \varepsilon^{1/3}$ , to get

$$\begin{aligned} \frac{\delta^2}{\varepsilon} \int_{\Omega} |U_{\varepsilon}(y) - \delta y| - |U_{\varepsilon}(y)| + |\delta y| f(\delta y) \, dy + \int_{\Omega} |DU_{\varepsilon}|^2 - K \log \lambda \\ = \int_{\Omega} \lambda^3 \left( |y| - y \cdot \frac{U_{\varepsilon}(y)}{|U_{\varepsilon}(y)|} \right) f(\delta y) \, dy + \int_{\Omega} |DU_{\varepsilon}|^2 - K \log \lambda + o\left(\lambda^4 \varepsilon^{4/3}\right) \end{aligned}$$

We can bound this term from below (up to the rest with order  $\lambda^4 \varepsilon^{1/3}$ , that we can make going to 0 with a good choice of  $\lambda$ , and another rest taking account of  $f(\delta y) \rightarrow f(0)$ ) with

$$-C'_{\lambda} = \inf \left\{ f(0) \int_{\Omega} \lambda^3 \left( |y| - y \cdot \frac{U(y)}{|U(y)|} \right) \, dy + \int_{\Omega} |DU|^2 - K \log \lambda \right\}$$

where the constraint on  $U$  has to be precised. For a good choice of  $\lambda$ , as  $\varepsilon \rightarrow 0$ , we skip the first constant  $-C_{\lambda}$  and get, since the third line of (18) is lower semi-continuous,

$$\liminf_{\varepsilon \rightarrow 0} F_{\varepsilon} \geq F(T) - \lim_{\lambda \rightarrow +\infty} C'_{\lambda}$$

which corresponds to the result that we conjectured in order to make more precise the Theorem 3.1.

## 4.2 Construction of family of transport maps with equi-bounded energy

The last point of the proof of Theorem 3.1 consists in building a family of maps  $(T_{\varepsilon})_{\varepsilon}$  such that  $(F_{\varepsilon}(T_{\varepsilon}))_{\varepsilon}$  is as well bounded. The sketch of the proof is the following: we start from a transport map  $T = \varphi \frac{x}{|x|}$  with  $\varphi(0, \cdot) = \Phi$  (that we call “the original  $T$ ” in the following), assume that  $\varphi$  is regular except around the origin, and modify  $T$  only on  $\Omega_{\delta}$ .

**Step 1: construction of the original transport map.** We set  $T(x) = \varphi(r, \theta) \frac{x}{|x|}$ , where  $\varphi$  is built as follows:

- $\varphi(0, \theta) = \Phi(\theta)$ , and  $\varphi(\cdot, \theta)$  is increasing and sends the one-dimensional measure  $\mu_{\theta}$  (the starting measure  $\mu$  concentrated on the transport ray with angle  $\theta$ ) onto  $\nu_{\theta}/2$  (where  $\nu_{\theta}$  is the target measure on the same transport ray), until the radius  $\rho_1$  such that  $\varphi(\rho_1, \theta) = R_2(\theta)$ ;
- starting from this radius  $\rho_1$ ,  $\varphi(\cdot, \theta)$  is decreasing with the same source and target measure, until the radius  $\rho_2$  such that, again,  $\varphi(\rho_2, \theta) = \Phi(\theta)$ . Therefore, on the interval  $(\rho_1, \rho_2)$ ,  $\varphi(\cdot, \theta)$  sends  $\mu_{\theta}$  onto  $\nu_{\theta}|_{(\Phi(\theta), R_2(\theta))}$ ;
- on the last interval (if it is non-empty, which corresponds to  $\Phi(\theta) > R_1(\theta)$ ),  $\varphi$  is still decreasing and sends  $\mu_{\theta}$  onto  $\nu_{\theta}|_{(R_1(\theta), \Phi(\theta))}$

Precisely, we fix  $\theta$  and the expressions of  $\mu_{\theta}$ ,  $\nu_{\theta}$  are

$$d\mu_{\theta}(r) = r f(r, \theta) \, dr \quad \text{and} \quad d\nu_{\theta}(r) = r g(r, \theta) \, dr$$

which have both same mass on  $(0, 1)$  and  $(R_1(\theta), R_2(\theta))$  respectively. Now we define successively  $\rho_1(\theta)$  and  $\rho_2(\theta)$  by

$$\int_0^{\rho_1(\theta)} d\mu_{\theta} = \int_{\Phi(\theta)}^{R_2(\theta)} \frac{1}{2} d\nu_{\theta} \quad \text{and} \quad \int_{\rho_1(\theta)}^{\rho_2(\theta)} d\mu_{\theta} = \int_{\Phi(\theta)}^{R_2(\theta)} \frac{1}{2} d\nu_{\theta}$$

which are proper definitions thanks to the intermediate value theorem, and imply

$$\int_{\rho_2(\theta)}^1 d\mu_{\theta} = \int_0^{\Phi(\theta)} d\nu_{\theta}$$

Thus, we have the equality between masses:

$$\mu_{\theta}(0, \rho_1(\theta)) = \mu_{\theta}(\rho_1(\theta), \rho_2(\theta)) = \frac{1}{2} \nu_{\theta}(\Phi(\theta), 1) \quad \text{and} \quad \mu_{\theta}(\rho_2(\theta), 1) = \nu_{\theta}(0, \Phi(\theta))$$

and the measures  $\mu_{\theta}$ ,  $\nu_{\theta}$  are absolutely continuous on these intervals. We now define the function  $\varphi(\cdot, \theta)$  as being:

- on the interval  $(0, \rho_1(\theta))$ , the unique increasing map  $(0, \rho_1(\theta)) \rightarrow (\Phi(\theta), 1)$  sending  $\mu_\theta$  onto  $\frac{1}{2}\nu_\theta$ ;
- on the interval  $(\rho_1(\theta), \rho_2(\theta))$ , the unique decreasing map  $(\rho_1(\theta), \rho_2(\theta)) \rightarrow (\Phi(\theta), 1)$  sending  $\mu_\theta$  onto  $\frac{1}{2}\nu_\theta$ ;
- on the interval  $(\rho_2(\theta), 1)$  (if this interval is not empty), the unique decreasing map  $(\rho_2(\theta), 1) \rightarrow (0, \Phi(\theta))$  sending  $\mu_\theta$  to  $\nu_\theta$

It is easy to check that  $\varphi(\cdot, \theta)$ , defined on the whole  $(0, 1)$ , sends globally  $\mu_\theta$  onto  $\nu_\theta$ . As a consequence, the two-dimensional valued function

$$T : x = (r, \theta) \in \Omega \mapsto \varphi(r, \theta) \frac{x}{|x|} \in \Omega'$$

is a transport map from  $\mu$  to  $\nu$ .

**Step 2: estimates on  $\varphi$  around the origin.** As above, we set  $\delta = \varepsilon^{1/3}$  and we aim to modify  $T$  only on  $\Omega_\delta = \Omega \cap B(0, \delta)$ . In order to obtain as well a transport map from  $\mu$  to  $\nu$ , we have to guarantee that the new map  $S$  sends the domain  $\Omega_\delta$  on its image  $T(\Omega_\delta)$ , with the constraint of image measure. For this, the following estimates on the original transport  $T$  will be useful:

**Proposition 4.1.** *The above function  $\varphi$  has Lipschitz regularity on  $(0, 1) \times (0, \pi/2)$ . Moreover, there exists some positive constants  $c, C$  depending only on  $\Omega', f, g$  such that*

$$cr^2 \leq \varphi(r, \theta) - \Phi(\theta) \leq Cr^2 \quad (19)$$

for any  $r$  small enough and  $\theta \in (0, \pi/2)$ , and

$$\text{Lip}(\varphi(r, \cdot) - \Phi) \leq Cr^2 \quad (20)$$

for any  $r \in (0, 1)$ .

*Proof.* The Lipschitz regularity of  $\varphi$  is actually a consequence of its definition and of the inverse function theorem. First, let us recall that  $\rho_1(\theta)$  is defined by

$$\tilde{F}(\rho_1(\theta), \theta) = \frac{1}{2}(G(R_2(\theta), \theta) - \tilde{G}(\Phi(\theta), \theta))$$

$$\text{where } \tilde{F}(R, \theta) = \int_0^R d\mu_\theta = \int_0^R r f(r, \theta) dr \quad \text{and} \quad \tilde{G}(R, \theta) = \int_{R_1(\theta)}^R d\nu_\theta = \int_{R_1(\theta)}^R r g(r, \theta) dr$$

Notice that we have necessary  $\rho_1(\theta) \geq \delta_0 > 0$ , where  $\delta_0$  verifies, for instance,

$$\int_{B(0, \delta_0) \cap \Omega} d\mu = \int_{\sup R_1 \leq r \leq \inf R_2} d\nu$$

(such a  $\delta_0$  exists since  $\delta \mapsto \mu(B(0, \delta) \cap \Omega)$  is continuous and increasing). Thus,  $\tilde{F}$  is  $C^1$  with respect to its first variable and its derivative is Lipschitz and bounded from below by  $\delta_0 \inf f$ , and the same holds for  $\tilde{G}$ ; moreover,  $\tilde{F}$  and  $\tilde{G}$  are both Lipschitz with respect to  $\theta$ . As a consequence of the inverse function theorem,  $\rho_1$  is well-defined and Lipschitz. The same reasoning shows that  $\rho_2$  is well-defined and Lipschitz.

Then we know that  $\varphi$  is defined if  $0 \leq r \leq \rho_1(\theta)$  by

$$\int_0^r d\mu_\theta = \int_{\Phi(\theta)}^{\varphi(r, \theta)} d\nu_\theta \quad \text{i.e.} \quad \tilde{F}(r, \theta) = \tilde{G}(\varphi(r, \theta), \theta) - G(\Phi(\theta), \theta)$$

Again, by the inverse function theorem,  $\varphi$  is  $C^1$  with respect to  $r$  and Lipschitz with respect to  $\theta$ . The same reasoning can be applied on the intervals  $(\rho_1(\theta), \rho_2(\theta))$  and, if it is non-empty,  $(\rho_2(\theta), 1)$ . We get that the function  $\varphi$  is Lipschitz with respect to its both variables  $(r, \theta)$ .

Notice also the following estimate on  $\varphi(r, \theta)$  for  $r$  small enough: we have  $\int_0^r d\mu_\theta = \int_{\Phi(\theta)}^{\varphi(r, \theta)} d\nu_\theta$ , with the inequalities

$$r^2 \inf f \leq \int_0^r d\mu_\theta \leq r^2 \sup f \quad \text{and} \quad (\inf g)(\varphi(r, \theta) - \Phi(\theta)) \leq \int_{\Phi(\theta)}^{\varphi(r, \theta)} d\nu_\theta \leq (\sup g)(\varphi(r, \theta) - \Phi(\theta))$$

This leads immediately to (19). On the other hand, we know that  $\varphi(r, \theta)$  has Lipschitz regularity with respect to  $\theta$  and

$$\int_{R_1(\theta)}^{\varphi(R, \theta)} r g(r, \theta) dr = \int_0^R r f(r, \theta) dr \quad (21)$$

If  $\theta$  is such that the equality (21) is differentiable with respect to  $\theta$  (which is true for a.e.  $\theta$  since all the considered functions are at least Lipschitz), we get

$$\partial_2 \varphi(R, \theta) \varphi(R, \theta) G(\varphi(R, \theta), \theta) - R'_1(\theta) R_1(\theta) G(\varphi(R, \theta), \theta) = \int_0^R r \partial_2 f(r, \theta) dr$$

thus

$$\partial_\theta (\varphi(R, \theta) - R_1(\theta)) \varphi(R, \theta) G(\varphi(R, \theta), \theta) + R'_1(\theta) (\varphi(R, \theta) - R_1(\theta)) G(\varphi(R, \theta), \theta) = \int_0^R r \partial_2 f(r, \theta) dr$$

and

$$|\partial_\theta (\varphi(R, \theta) - R_1(\theta))| \leq \frac{(\text{Lip } R_1)(\sup G)}{(\inf R_1)(\inf G)} |\varphi(R, \theta) - R_1(\theta)| + \frac{R^2}{2} \text{Lip } F$$

Thanks to the inequality (19), we get

$$|\partial_\theta (\varphi(R, \theta) - R_1(\theta))| \leq CR^2$$

for some constant  $C$  depending only on  $f, g, \Omega'$ , which proves (20).  $\square$

**Step 3: perturbation of the optimal  $T$ .** In that follows, we denote by

$$\Omega_\delta = \Omega_1 \cap B(0, \delta) = \{x = (r, \theta) : 0 < r < \delta \text{ and } 0 < \theta < \pi/2\}$$

$$\text{and } \Omega'_\delta = T(\Omega_\delta) = \{x = (r, \theta) : \Phi(\theta) < r < \varphi(\delta, \theta) \text{ and } 0 < \theta < \pi/2\}$$

We now denote by:

- $S_1 : x \in \Omega_\delta \mapsto \frac{x}{\delta} \in \Omega_1$  and  $f_\delta(x) = f(\delta x)$ . Notice that  $f_\delta = \frac{1}{\delta^2} (S_1)_\# (f|_{\Omega_\delta})$ ;
- $\Omega_2$  is the rectangle  $(0, 1) \times (0, \pi/2)$  and

$$S_2 : (\lambda, \theta) \in \Omega_2 \mapsto x = (\Phi(\theta) + \lambda(\varphi(\delta, \theta) - \Phi(\theta)), \theta) \in \Omega'_\delta$$

where  $x$  is here written in polar coordinates. We also denote by  $g_\delta = \frac{1}{\delta^2} \left( (S_2)^{-1} \# g|_{\Omega'_\delta} \right)$ .

As above,  $S_1$  and  $S_2$  actually depend on  $\delta$  but we omit the index  $\delta$  for the sake of simplicity of notations. We have of course  $\inf f \leq f_\delta(x) \leq \sup f$  for any  $x \in \Omega_1$ , and  $\text{Lip } f_\delta \leq \delta \text{Lip } f$ . On the other hand, since  $S_2$  is Lipschitz and one-to-one, the Monge-Ampère equation provides

$$\det DS_2(\lambda, \theta) = \frac{g_\delta(\lambda, \theta)}{g(S_2(\lambda, \theta))}$$

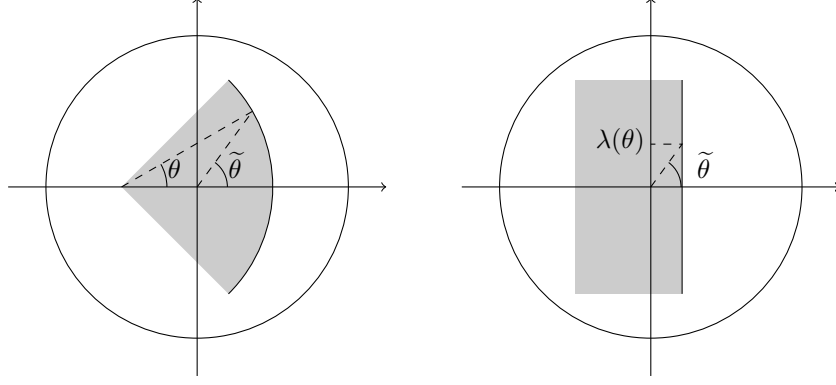
$$\text{thus } g_\delta(\lambda, \theta) = (\varphi(\delta, \theta) - \Phi(\theta)) \frac{1}{\delta^2} g(S_2(\lambda, \theta)) \quad (22)$$

We deduce from (19) and (22) that  $c \leq g_\delta \leq C$  for  $c, C$  positive and independent of  $\delta$ . Moreover, (22) provides

$$\text{Lip } g_\delta \leq \sup(\varphi_\delta - \varphi) \frac{1}{\delta^2} \text{Lip}(g \circ S_2) + \text{Lip}(\varphi_\delta - \varphi) \frac{1}{\delta^2} \sup g$$

Again, thanks to (19) and (22) and using that  $\text{Lip } S_2$  is uniformly bounded with respect to  $\delta$ , we get  $\text{Lip } g_\delta \leq C$  independent of  $\delta$ .

We would like to use the Theorem 1.2, but this result works *a priori* only to connect two measures defined on a same domain. Thus, an intermediate step consists in sending  $\Omega_\delta, \Omega'_\delta$  onto the unit ball, which can be done (with the regularity that we need) thanks to the Lemma 2.2. Moreover, let us notice that  $\Omega_1, \Omega_2$  are infinitely diffeomorphic to the following domains (which are obtained directly thanks to rotations/translations/dilations):



In the above picture, the upper right and lower right corners of the second domain are equal to the upper right and lower right corners of the first domain. Since the maps provided by the Lemma 2.2 from these two domains to the unit disk both preserve angles (with respect to the origin, see the angle  $\tilde{\theta}$  on the picture), if we denote by  $\alpha, \beta$  the corresponding maps starting from  $\Omega_1, \Omega_2$ , we have

$$\beta^{-1} \circ \alpha(x) = (1, \lambda(\theta)) \quad \text{if} \quad |x| = 1 \text{ and } x \text{ has } \theta \text{ for angle.}$$

where  $\lambda$  is a bi-Lipschitz map from the interval  $(0, \pi/2)$  onto  $(0, 1)$ ; by composing the second coordinate with  $\lambda^{-1}$ , one can actually assume that  $\beta^{-1} \circ \alpha(x) = (1, \theta)$ . On the other hand, we deduce from the regularity of  $\det D\alpha, \det D\beta$  that, if

$$\overline{f_\delta} = \alpha_\# f_\delta \quad \text{and} \quad \overline{g_\delta} = \beta_\# g_\delta$$

then  $\overline{f_\delta}, \overline{g_\delta}$  have also infimum, supremum and Lipschitz constant bounded independently of  $\delta$ . Now the Dacorogna-Moser result provides the existence of a bi-Lipschitz diffeomorphism  $u_\delta$ , whose Lipschitz constant is bounded independently of  $\delta$ , sending the density  $\overline{f_\delta}$  onto  $\overline{g_\delta}$ , and with  $u_\delta(x) = x$  for  $|x| = 1$ .

Finally, we consider

$$S = S_2 \circ \beta^{-1} \circ u_\delta \circ \alpha \circ S_1$$

If  $|x| = \delta$ , we have  $|S_1(x)| = 1$ , thus  $\alpha(S_1(x)) \in \partial B(0, 1)$  and

$$\beta^{-1}(u_\delta(\alpha(S_1(x)))) = \beta^{-1}(\alpha(S_1(x)))$$

so that  $\beta^{-1}(u_\delta(\alpha(S_1(x)))) = (1, \theta)$ , where  $\theta$  is the angle of  $x$ . Consequently,  $S_2(\beta^{-1}(u_\delta(\alpha(S_1(x))))))$  has  $(\varphi(\delta, \theta), \theta)$  as polar coordinates, so it is equal to  $T(x)$ .

To summarize,  $S$  and  $T$  coincide on the line  $\{|x| = \delta\}$ ,  $S$  is Lipschitz on  $\Omega_\delta$  and, thanks to the estimates on  $\varphi$ ,  $T$  is Lipschitz on  $\Omega \setminus \Omega_\delta$ . This implies that  $T_\varepsilon$  is globally Lipschitz on  $\Omega$ , thus it belongs to  $H^1(\Omega)$ . Moreover, we have

$$\text{Lip } u_\delta \leq C, \quad \text{Lip } S_2 \leq \text{Lip } \Phi \quad \text{and} \quad \text{Lip } S_1 = \frac{1}{\delta}$$

$$\text{thus} \quad \text{Lip } S \leq \frac{C}{\delta}$$

for some constant  $C$  which does not depend on  $\delta$ .

**Step 4: estimates on  $F_\varepsilon(T_\varepsilon)$ .** We restart from the expression (9), and use the facts that  $\psi_\varepsilon = 0$  and that  $T_\varepsilon = T$  outside of  $\Omega_\delta$ :

$$\begin{aligned} F_\varepsilon(T_\varepsilon) &= \frac{1}{\varepsilon} \int_{\Omega_\delta} (|S(x) - x| - |S(x)| + |x|) f(x) dx + \int_{\Omega_\delta} |DS(x)|^2 dx \\ &\quad + \int_\delta^1 (\|\varphi(r, \cdot)\|_{L^2}^2 + \|\partial_\theta \varphi(r, \cdot)\|_{L^2}^2 - K) \frac{dr}{r} + \int_\delta^1 \|\partial_r \varphi(r, \cdot)\|_{L^2}^2 r dr \end{aligned}$$

We still have  $|S(x) - x| - |S(x)| + |x| \leq 2|x|$  and  $|DS(x)| \leq C/\delta$ , so that

$$\frac{1}{\varepsilon} \int_{\Omega_\delta} (|S(x) - x| - |S(x)| + |x|) f(x) dx + \int_{\Omega_\delta} |DS(x)|^2 dx \leq \frac{\pi \sup f}{3} \frac{\delta^3}{\varepsilon} + \frac{C^2 \pi}{4}$$

which is bounded since  $\delta = \varepsilon^{1/3}$ . On the other hand,

$$\begin{aligned} \|\varphi(r, \cdot)\|_{L^2}^2 + \|\partial_\theta \varphi(r, \cdot)\|_{L^2}^2 - K &= (\|\varphi(r, \cdot)\|_{L^2}^2 - \|\Phi\|_{L^2}^2) + (\|\partial_\theta \varphi(r, \cdot)\|_{L^2}^2 - \|\Phi'\|_{L^2}^2) \\ &= \langle \varphi(r, \cdot) - \Phi, \varphi(r, \cdot) + \Phi \rangle_{L^2} + \langle \partial_\theta \varphi(r, \cdot) - \Phi', \partial_\theta \varphi(r, \cdot) + \Phi' \rangle_{L^2} \\ &\leq \|\varphi(r, \cdot) - \Phi'\|_{L^1} (\|\varphi(r, \cdot)\|_\infty + \|\Phi\|_\infty) + \|\partial_\theta \varphi(r, \cdot) - \Phi'\|_{L^1} (\text{Lip } \varphi + \text{Lip } \Phi) \end{aligned}$$

Since  $\Phi, \varphi(r, \cdot)$  are valued in  $\Omega'$ , their  $L^\infty$ -norm are controlled by  $\sup R_2$ . By combining this and the estimates (19) and (20), we obtain

$$0 \leq \|\varphi(r, \cdot)\|_{L^2}^2 + \|\partial_\theta \varphi(r, \cdot)\|_{L^2}^2 - K \leq Cr^2$$

for  $r$  small enough (and where  $C$  does not depend on  $r$ ). On the other hand, we know that  $\varphi, \Phi$  and their derivatives are globally bounded on  $(0, 1) \times (0, \pi/2)$ . This proves that

$$\int_0^1 (\|\varphi(r, \cdot)\|_{L^2}^2 + \|\partial_\theta \varphi(r, \cdot)\|_{L^2}^2 - K) \frac{dr}{r} < +\infty$$

$$\text{and} \quad \int_0^1 \|\partial_r \varphi(r, \cdot)\|_{L^2}^2 r \, dr < +\infty$$

and we conclude that  $(F_\varepsilon(T_\varepsilon))_\varepsilon$  is bounded as well.

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