Generalized barycentric coordinates and Jensen type inequalities on convex polytopes

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GENERALIZED BARYCENTRIC COORDINATES AND JENSEN TYPE INEQUALITIES ON CONVEX POLYTOPES

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Abstract. In this paper we obtain some direct and converse new multidimensional Jensen's type inequalities on convex polytopes. Among the inequalities presented, we offer, as a particular case of our general results, a direct and converse multivariate extension of Mercer inequality. The main results are obtained with the aid of the generalized barycentric coordinates. For deriving such inequalities, we will also establish, analyze, and discuss links between barycentric codinates and certain class of partitions of unity. This method also allows us to derive continuous versions of various discrete inequalities established in our recent paper [7].

1. Introduction

One of the most popular results, establishing effective liaison between convexity and probability, is the so-called Jensen's inequality for the expectation of a convex real-valued function [23, p. 288]. It can be stated in the following way: Let $I \subseteq R$ be an interval, and let $f : I \to \mathbb{R}$ be a continuous convex function on I. Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $g : X \to I$ be a μ-integrable function over X. Then $E_\mu[g] \in I$, $E_\mu[f(g)]$ exists, and it holds that

$$f(E_\mu[g]) \leq E_\mu[f(g)], \tag{1.1}$$

where E_μ denotes (mathematical) expectation with respect to the probability measure μ on X.

This inequality has been refined, extended and applied in many areas such as probability and statistics (see, e g, [5, 15]). Its applications therein include the EM algorithm, Bayesian estimation and Bayesian inference to name a few, see, e g [3]. Among many extensions about Jensen's inequality the following are fundamental. Throughout this paper, X and L will be reserved exclusively to denote, respectively, a (nonempty) set and a subspace of the vector space of real-valued functions defined on X, such that

- L contains the constant functions.

We shall be concerned with a linear functional A which is defined on L, and is positive in the sense that it takes nonnegative values when applied to each nonnegative function in the set L. Furthermore, we will also assume that

- A is normalized in the sense that $A[1] = 1$.

Date: Received: January 17, 2012/ Accepted: March 23, 2013.

2000 *Mathematics Subject Classification.* Primary 41A36, 41A63, 41A80, 47A58; Secondary 26B25, 47B65.

Key words and phrases. Convex functions, Jensen's inequality, Barycentric coordinates, Farkas' lemma, Partitions of unity, Supporting hyperplane.

In the context of positive linear functionals the simplest generalization of Jensen's inequality (1.1) may be formulated as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous convex function on an interval I. Then for all $g \in L$ with $f(g) \in L$ we have that $A[g] \in I$ and

$$f\left(A[g]\right) \leq A[f(g)]. \tag{1.2}$$

The above inequality (1.1) was later generalized by McShane, see [22], as follows: Let $f : \Omega \subseteq \mathbb{R}^d \to \mathbb{R}$ be a continuous convex function on the polytope Ω. Let $g_i \in L, i = 1, \dots, d$, such that $\boldsymbol{g}(\boldsymbol{z}) := (g_1(\boldsymbol{z}), \dots, g_d(\boldsymbol{z})) \in \Omega$, for all $\boldsymbol{z} \in X$, and $f(\boldsymbol{g}) \in L$. Let us denote by $A[\boldsymbol{g}] := (A[g_1], \dots, A[g_d])$. Then $A[\boldsymbol{g}]$ is in $\Omega, f(A[\boldsymbol{g}])$ is defined and

$$f\left(A[\boldsymbol{g}]\right) \leq A[f(\boldsymbol{g})]. \tag{1.3}$$

Since Jensen's inequality is of great interest, it seems worthwhile to extend it to a very general setting to cover a wide variety of applications.
One of our basic aims here is to derive some new direct and converse multidimensional Jensen's type inequalities of the form (1.3). All the inequalities obtained here may be seen as continuous analogue versions of some discrete inequalities established in our recent paper [7], and hence the reader is referred to [7] for a more general discussion of these issues. The extensions are obtained in a natural way by using the generalized barycentric coordinates, which turn out to be appropriate to the more general setting of positive linear functionals. Another basic aim here is to obtain some fundamental properties of partitions of unity on (convex) polytopes. As we shall see, such results offer a rich insight into the structure of partitions of unity that are not barycentric systems. Among the inequalities presented, we derive, as an application of our results, a direct and converse multivariate extension of Mercer inequality [21, Theorem 1.2]. Recently, there has been considerable interest to look for refined inequalities of Mercer-type. In the one-dimensional case this inequality was generalized in many directions, for more details, we refer the interested reader to [1, 6, 18, 2, 19] and the references therein.
More concretely, the contributions and structure of the paper are as follows. In Section 2, we establish the main properties of partition of unity and barycentric coordinates, along with additional necessary background and notation. We also give a link between the two systems. In Section 3, after some pertinent results about the convexity preserving property and the convex hull property of a map, we apply the obtained results to establish direct and converse new multidimensional Jensen's type inequalities.

The following notations and conventions are used throughout the sequel. All linear spaces considered are understood to be real. The setting for this paper is d-dimensional Euclidean space $\mathbb{R}^d, (d \geq 1)$. By $\boldsymbol{x}$, $\boldsymbol{y}$, ..., we denote elements of $\mathbb{R}^d$. Throughout this paper, Ω will always denote a convex polytope with non-empty interior (that is, the convex hull of $(n+1)$, ($n \geq d \geq 1$), vertices $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ in $\mathbb{R}^d$). The vector $\boldsymbol{e} = \sum_{i=0}^n \boldsymbol{v}_i$ denotes the characteristic vector of the polytope Ω.

2. Partition of unity and barycentric coordinates

In this section, we establish the main structures of partitions of unity and barycentric coordinates. They will be used later in the section for the study of a tautological map that will play an important role in this paper. Particular attention is paid to the question of a link between these two systems.
Since perhaps not every reader of this paper is familiar with these coordinates, we wish to give a brief overview of the basic elements of barycentric coordinates in d dimensions, see, e. g., [17, pp. 132-135] for more details.
Suppose that n is a positive integer and $\Omega \subset \mathbb{R}^d$ be a polytope with vertices $V = \{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$. A partirion of unity on Ω is a family $\{p_i, i = 0, \ldots, n\}$ of *continuous* functions from Ω into $[0,1]$ such that at each $\boldsymbol{x} \in \Omega$, they sum to the constant function one; that is, for all $\boldsymbol{x} \in \Omega$

$$\sum_{i=0}^{n} p_i(\boldsymbol{x}) = 1. \tag{2.1}$$

The functions $\{p_i, i = 0, \ldots, n\}$ are called barycentric coordinates with respect to Ω or its set of vertices V, if they form a partition of unity and allow to write any point $\boldsymbol{x} \in \Omega$ as an affine combination of the vertices,

$$\boldsymbol{x} = \sum_{i=0}^{n} p_i(\boldsymbol{x})\boldsymbol{v}_i. \tag{2.2}$$

This last property is also sometimes referred to as linear precision since the coordinate functions can reproduce the linear function $\boldsymbol{x}$. The barycentric coordinates defined in this way satisfy nonnegativity, continuity, partition of unity and since $\boldsymbol{v}_i, i = 0, \ldots, n$, are extreme points of Ω, by (2.2) we have the Lagrange property

$$p_i(\boldsymbol{v}_j) = \delta_{ij}, \tag{2.3}$$

where δ_{ij} is Kronecker's delta. Due to the properties (2.1) and (2.2), barycentric coordinates may be used to decide if a point lies inside a polytoope, and to interpolate a function: indeed, it is clear from equations (2.2) that every point in the polytope is a convex combination of its vertices. In addition, the Lagrange property immediately implies vertex interpolation: the interpolation operator

$$S[f](\boldsymbol{x}) = \sum_{i=0}^{n} p_i(\boldsymbol{x}) f_i \tag{2.4}$$

interpolates the data f_i at the set of vertices V, and properties (2.1) and (2.2) further guarantee the reproduction of affine functions by the operator S. Obviously, the interpolant S inherits the continuity properties of the functions p_i.
Let $\boldsymbol{p} = \{p_i, i = 0, \ldots, n\}$ be a partition of unity on Ω defined as before. We say that $\boldsymbol{p}$ has a linear precision if for every affine function f defined on P,

$$f(\boldsymbol{x}) = \sum_{i=0}^{n} p_i(\boldsymbol{x}) f(\boldsymbol{v}_i). \tag{2.5}$$

We observe that any barycentric coordinates have the linear precision property.
Next we define a function that can be used to characterize such coordinates. The tautological map associated to the partition unity $\boldsymbol{p}$ is the transformation $T_{\boldsymbol{p}} : \Omega \to$

Ω defined by

$$T_{\boldsymbol{p}}(\boldsymbol{x}) = \sum_{i=0}^{n} p_i(\boldsymbol{x})\boldsymbol{v}_i. \tag{2.6}$$

This function will play a crucial role throughout the paper. The following proposition, whose easy proof is omitted, give us a precise connection between the two notions of partition of unity and barycentric coordinates.

Proposition 2.1. *A partition of unity forms a barycentric coordinate system if and only if its tautological map is the identity map of the domain polytope Ω.*

For a partition of unity $\{p_i, i = 0, \dots, n\}$, we simply write $\boldsymbol{p}$ and say that it is a *pu*-system. The collection of all *pu*-systems will be denoted by $\boldsymbol{P}_{n+1}$. Analogously, for a set of functions $\{b_i, i = 0, \dots, n\}$ that defines a barycentric coordinates system on Ω, we simply write $\boldsymbol{b}$ and say that it is a *bc*-system. The collection of all *bc*-systems will be denoted by $\boldsymbol{B}_{n+1}$.
If Ω is a nondegenerate simplex, then $n = d$, (e.g., a triangle in $2D$ or a tetrahedron in $3D$), with vertices $\boldsymbol{v}_0, \dots, \boldsymbol{v}_d \in \mathbb{R}^d$ that are affinely independent, then each point $\boldsymbol{x}$ of their convex hull Ω has a (*unique*) representation, that is there exist unique nonnegative real numbers $\{\lambda_i(\boldsymbol{x}), i = 0, \dots, d\}$ so that $\sum_{i=0}^{d} \lambda_i(\boldsymbol{x}) = 1$, and $\boldsymbol{x} = \sum_{i=0}^{d} \lambda_i(\boldsymbol{x})\boldsymbol{x}_i$. The barycentric coordinates $\lambda_0, \dots, \lambda_d$ are nonnegative affine functions on Ω, see [4, p. 288]. Note that a d-simplex is a special polytope given as the convex hull of $d+1$ vertices in d dimensions, each pair of which is joined by an edge.
Barycentric coordinates also exist for more general types of polytopes, see [16, Theorem 2]. The next lemma is due essentially to Kalman [16]. Our statements are stronger than the ones provided in [16], but the proof proceeds along the same lines as the proof of theorem 2 in [16], so we omit it.

Lemma 2.2. *Let Ω be a polytope in $\mathbb{R}^d$, $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ its vertices and $\boldsymbol{x}^*$ a given point in Ω with $\boldsymbol{x}^* = \sum_{i=0}^{n} \lambda_i^* \boldsymbol{v}_i$, $1 = \sum_{i=0}^{n} \lambda_i^*$, $\lambda_i^* \geq 0$. Then there are real continuous functions on Ω, $\{\lambda_0, \dots, \lambda_n\}$, such that*

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i, \quad \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1, \quad \textit{and} \quad \lambda_i(\boldsymbol{x}) \geq 0 \tag{2.7}$$

for each $\boldsymbol{x} \in \Omega$ and $\lambda_i(\boldsymbol{x}^*) = \lambda_i^*$.

The set of all affine functions $\Omega \to \mathbb{R}^d$ is a vector space denoted $A(\Omega)$. We use $\boldsymbol{P}_{n+1}^{\mathrm{Aff}}$ to denote the set of *pu*-systems whose tautological maps are in $A(\Omega)$. In the sequel we will need the following fact: for any $\boldsymbol{p} \in \boldsymbol{P}_{n+1}^{\mathrm{Aff}}$, there exist a matrix $M \in \mathbb{R}^{d,d}$ and a vector $\boldsymbol{a} \in \mathbb{R}^d$, such that the associated tautological map $T_{\boldsymbol{p}}$ can be expressed as follows:

$$\begin{aligned} T_{\boldsymbol{p}}(\boldsymbol{x}) &= M\boldsymbol{x} + \boldsymbol{a} \\ &= M\left(\sum_{i=0}^{n} b_i(\boldsymbol{x})\boldsymbol{v}_i\right) + \boldsymbol{a} \\ &= \sum_{i=0}^{n} b_i(\boldsymbol{x})M(\boldsymbol{v}_i) + \boldsymbol{a} \\ &= \sum_{i=0}^{n} b_i(\boldsymbol{x})(M(\boldsymbol{v}_i) + \boldsymbol{a}) \\ &= \sum_{i=0}^{n} b_i(\boldsymbol{x})T_{\boldsymbol{p}}(\boldsymbol{v}_i), \quad (\forall \boldsymbol{x} \in \Omega), \end{aligned}$$

where $\{b_0, b_1, \ldots, b_n\}$ is any *bc*-system. Therefore, for any *pu*-system $\boldsymbol{p}$ in $\boldsymbol{P}_{n+1}^{\mathrm{Aff}}$, we have the following representation formula for its tautological map:

$$T_{\boldsymbol{p}}(\boldsymbol{x}) \quad = \textstyle\sum_{i=0}^{n} b_i(\boldsymbol{x})T_{\boldsymbol{p}}(\boldsymbol{v}_i), \quad (\forall \boldsymbol{x} \in \Omega). \tag{2.8}$$

Note that the above representation formula is geometric in the sense that it does not depend on the choice of *bc*-coordinates, and in particular implies that an affine function is uniquely determined by its values at the vertices of the polytope.
In order to make a connection between *pu*- and *bc*-systems, we are concerned with the following problem:

- What are the necessary and sufficient conditions for $\boldsymbol{p} \in \boldsymbol{P}_{n+1}^{\mathrm{Aff}}$ to be a *bc*-system?

To give more complete results than those established in Proposition 2.1, the next theorem gives several characterizations of elements of $\boldsymbol{P}_{n+1}$ which have tautological maps in $A(\Omega)$. (Later on we will develop a stronger form of this Theorem.)

Theorem 2.3. *Let* $\boldsymbol{p}$ *be a pu-system belonging to* $\boldsymbol{P}_{n+1}^{\mathrm{Aff}}$. *Then, the following assertions are equivalent:*

(i) $\boldsymbol{p}$ *has the Lagrange property.*
(ii) $T_{\boldsymbol{p}}$ *has the vertices preserving property: that is* $T_{\boldsymbol{p}}(\boldsymbol{v}_i) = \boldsymbol{v}_i, i = 0, \ldots, n$.
(iii) *The tautological map of* $\boldsymbol{p}$ *is the identity map.*
(iv) $\boldsymbol{p}$ *is a bc-system.*

Proof. Let $\boldsymbol{p} = \{p_0, p_1, \ldots, p_n\}$ be any *pu*-system which belongs to $\boldsymbol{P}_{n+1}^{\mathrm{Aff}}$. Assume that $\boldsymbol{p}$ has the Lagrange property, then since $p_i(\boldsymbol{v}_j) = \delta_{ij}$ and $T_{\boldsymbol{p}}(\boldsymbol{x}) := \sum_{i=0}^{n} p_i(\boldsymbol{x})\boldsymbol{v}_i$, for all $\boldsymbol{x} \in \Omega$, we get by substituting $\boldsymbol{v}_j$ for $\boldsymbol{x}$

$$T_{\boldsymbol{p}}(\boldsymbol{v}_j) = \boldsymbol{v}_j, j = 0, \ldots, n,$$

this shows that $\boldsymbol{p}$ has the vertices preserving property.
Let $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ be the *bc*-system defined by Lemma 2.2. If $\boldsymbol{p}$ has the vertices preserving property, then by (2.8) we will have

$$T_{\boldsymbol{p}} = \sum_{i=0}^{n} \lambda_i \boldsymbol{v}_i.$$

Combining this with the fact that $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is a *bc*-system, we then deduce that the tautological map $T_{\boldsymbol{p}}$ is the identity map. The implication (iii) $\Rightarrow$ (iv) is trivial, due to the fact that $\boldsymbol{p}$ is already a $pu-$system. Finally, since any bc-system has the Lagrange property we have immediately the implication $(iv) \Rightarrow (i)$. □

Let us now discuss a more general subclass of the set of pu-systems. Of course, Lemma 2.2 shows that $\boldsymbol{B}_{n+1}$ is a nonempty set. Property (2.7) is often split into two equivalent properties: that the barycentric coordinates form a partition of unity, for each $\boldsymbol{x} \in \Omega$,

$$\sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1, \quad \text{and} \quad \lambda_i(\boldsymbol{x}) \geq 0, \tag{2.9}$$

and that any $\boldsymbol{x} \in \Omega$, $\boldsymbol{x}$ can be written as an affine combination of the polytopes vertices,

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) \boldsymbol{v}_i. \tag{2.10}$$

The function value $\lambda_i(\boldsymbol{x})$ can be viewed as the 'probability of influence of a vertex $\boldsymbol{v}_i$ at $\boldsymbol{x}$,' and the linear precision conditions (2.10) are the under-determined constraints. Such a representation is generally non-unique, but this is inconsequential for our purposes. It can also happen that $\boldsymbol{B}_{n+1}$ may be reduced to a singleton set: For example, this is the case when the polytope is a simplex. Moreover, since every bc-system is a pu-system and has the identity function as tautological map, then $\boldsymbol{B}_{n+1} \subseteq \boldsymbol{P}_{n+1}^{\mathrm{Aff}} \subseteq \boldsymbol{P}_{n+1}$. The next two Lemmas, which are central to our analysis, show that these inclusions are strict and the set $\boldsymbol{P}_{n+1}^{\mathrm{Aff}}$ contains a large class of pu-systems.

Lemma 2.4. *Let Ω be a polytope in $\mathbb{R}^d$. Then for any fixed positive real numbers α, β such that $\beta \leq \alpha$ and any bc-system $\{b_0, b_1, \ldots, b_n\}$ the functions defined by*

$$p_i^{\alpha,\beta} = \frac{\alpha - \beta b_i}{(n+1)\alpha - \beta}, i = 0, \ldots, n, \tag{2.11}$$

form a partition of unity on Ω.

Proof. Since, any bc-system froms a partition of unity, it is easy to check that under the condition $\beta \leq \alpha$, the system $\left\{p_0^{\alpha,\beta}, \ldots, p_n^{\alpha,\beta}\right\}$ is a set of continuous functions, from Ω to the unit interval $[0,1]$. □

Note that for any $0 < \alpha \leq \beta$, the partition of unity defined by Lemma 2.4 does not form a bc-system, since $\frac{1}{n+1} \sum_{i=0}^{n} \boldsymbol{v}_i$ is the only point of Ω such that $\sum_{i=0}^{n} p_i^{\alpha,\beta}(\boldsymbol{x}) \boldsymbol{v}_i = \boldsymbol{x}$.

Another simple but useful fact is the tautological map associated to pu-system $\boldsymbol{p}^{\alpha,\beta}$, defined by (2.11), can be easily expressed in terms of an affine function. For ease of notation, for any real numbers α, β such that $0 < \beta \leq \alpha$, we write $\boldsymbol{p} := \boldsymbol{p}^{\alpha,\beta}$ the pu-system defined in Lemma 2.4. More precisely:

Lemma 2.5. *For any real numbers α, β such that $0 < \beta \leq \alpha$, the tautological map associated to $\boldsymbol{p}$, $T_{\boldsymbol{p}}^{\alpha,\beta} : \Omega \to T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$ is a bijection and it can be written in the alternative form as:*

$$T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{x}) = \frac{\alpha \boldsymbol{e} - \beta \boldsymbol{x}}{(n+1)\alpha - \beta}, \tag{2.12}$$

for each $\boldsymbol{x} \in \Omega$.

Proof. To prove that $T_{\boldsymbol{p}}^{\alpha,\beta}$ is injective, assume that $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{x}) = T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{y})$, for two elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of Ω. Then, a computation based on (2.11) shows that this is equivalent to $T_{\boldsymbol{b}}(\boldsymbol{x}) = T_{\boldsymbol{b}}(\boldsymbol{y})$, where $\boldsymbol{b}$ is the *bc*-system used to define the partition of unity $\boldsymbol{p}$. But, by Proposition 2.1 the tautological map of the *bc*-system $\boldsymbol{b}$ is the identity map of Ω, therefore $\boldsymbol{x} = \boldsymbol{y}$.
To establish identity (2.12), we use the fact that if $\boldsymbol{b}$ is a *bc*-system, then equations (2.1) and (2.2) are satisfied for any $\boldsymbol{x} \in \Omega$. Indeed, by multiplying both sides of (2.11) by $\boldsymbol{v}_i$, and next summing over all i, it follows that, for any $\boldsymbol{x} \in \Omega$, the tautological map $T_{\boldsymbol{p}}^{\alpha,\beta}$ may also be written down exactly as in (2.12). □

For fixed positive real numbers $\beta \leq \alpha$, denote by $P_{n+1}^{\alpha,\beta}$ the class of pu-systems $\boldsymbol{p}^*$ that has as tautological map the affine function $T_{\boldsymbol{p}^*}^{\alpha,\beta}(\boldsymbol{x}) = \frac{\alpha\boldsymbol{e}-\beta\boldsymbol{x}}{(n+1)\alpha-\beta}$, for any $\boldsymbol{x} \in \Omega$. We obviously have $P_{n+1}^{\alpha,\beta} \subseteq \boldsymbol{P}_{n+1}^{\text{Aff}}$ and the set $P_{n+1}^{\alpha,\beta}$ contains all *pu*-systems defined by (2.11). Write P_{n+1}^{LP} the set of pu-systems that satisfy the Lagrange property (2.3). Since every *bc*-system satsisfy the Lagrange property we have $B_{n+1} \subseteq P_{n+1}^{LP}$. Thus, we always have the inclusions

$$B_{n+1} \subseteq P_{n+1}^{LP} \subseteq P_{n+1}. \tag{2.13}$$

We already know by Proposition 2.1 that every element in B_{n+1} is not contained in $P_{n+1}^{\alpha,\beta}$. The next result gives a connection between the two sets P_{n+1}^{LP} and $P_{n+1}^{\alpha,\beta}$.

Proposition 2.6. *Every element in* P_{n+1}^{LP} *is not contained in* $P_{n+1}^{\alpha,\beta}$.

Proof. Let $\boldsymbol{p}$ be a *pu*-system and $\boldsymbol{v}_i$ be any vertex of Ω. If $\boldsymbol{p}$ satisfies the Lagrange property then the tautological map $T_{\boldsymbol{p}}$ has $\boldsymbol{v}_i$ as fixed point. If $\boldsymbol{p}$ belongs to $P_{n+1}^{\alpha,\beta}$ it would contradict the fact that $\frac{\boldsymbol{e}}{n+1}$ is the unique fixed point of $T_{\boldsymbol{p}}$. Hence every element of P_{n+1}^{LP} cannot belong to $P_{n+1}^{\alpha,\beta}$. □

Equation (2.12) in Lemma 2.5 says that, instead of linear precision given by (2.2), the *pu*-system $\boldsymbol{p}^{\alpha,\beta}$ satisfy identity (2.12). Thus, by Proposition 2.1, the associated tautological map $T_{\boldsymbol{p}}^{\alpha,\beta}$ connot be the identity function. Let us note here that the alternative form given by equation (2.12) is independent of the *bc*-system used to define $\boldsymbol{p}^{\alpha,\beta}$. Since, for fixed α and β, using transformation (2.11), any *bc*-system generates the same tautological map $T_{\boldsymbol{p}}^{\alpha,\beta}$. Lemma 2.5 also says that $T_{\boldsymbol{p}}^{\alpha,\beta}$ maps Ω into a subset of itself, and it has the convex hull property, that is the image $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$ lies in the convex hull of the vertices of the polytope Ω.
Here and in the sequel, the symbol $\boldsymbol{cg}(C)$ denotes the center of gravity of a set $C \subset \mathbb{R}^d$, which is defined as

$$\boldsymbol{cg}(C) := \frac{\int_C \boldsymbol{x}\, d\boldsymbol{x}}{\int_C d\boldsymbol{x}},$$

assuming C is bounded and has nonempty interior. The vertex centroid of a polytope $C \subset \mathbb{R}^d$ with vertices $\{\boldsymbol{c}_0, \boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$ is defined as the average of the vertices in C:

$$\boldsymbol{vc}(C) = \frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{c}_i.$$

We recall that $\boldsymbol{cg}(C)$ and $\boldsymbol{vc}(C)$ are always located inside C, and, in general, they do not necessarily coincide. Equipped with the above results, we now show that the function $T_{\boldsymbol{p}}^{\alpha,\beta}$ also enjoys a number of additional, interesting properties that we elaborate on below.

Theorem 2.7. *Let Ω be a polytope in $\mathbb{R}^d$ and $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ its vertices. Let $\boldsymbol{p}$ be a pu-system defined as in the statement of Lemma 2.4. Then for any fixed positive real numbers $\beta \leq \alpha$, the associated tautological map $T_{\boldsymbol{p}}^{\alpha,\beta}$ satisfies the following properties:*

(i) *It maps vertices to vertices and preserves* any *barycentric coordinates.*

(ii) *It sends Ω to a polytope with vertices*

$$T_{\boldsymbol{p}}^{\alpha,\beta}(V) := \left\{T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_0), T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_1), \ldots, T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_n)\right\}.$$

(iii) *If the polytope Ω is a simplex, then its image $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$ is also a simplex.*

(iv) *It leaves the vertex centroid and the center of gravity of Ω invariant: $\boldsymbol{vc}(\Omega) = \boldsymbol{vc}(T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega))$ and $\boldsymbol{cg}(\Omega) = \boldsymbol{cg}(T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega))$.*

(v) *It preserves a point $\boldsymbol{x}$ of Ω if and only if $\boldsymbol{x}$ is the vertex centroid of Ω.*

(vi) *It preserves convexity: $f(T_{\boldsymbol{p}}^{\alpha,\beta})$ is convex if f is convex.*

(vii) *A vertex of Ω belongs to the image $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$ if and only if $d = n = 1$.*

Proof. Let us fix a *bc*-system $\{b_0, \ldots, b_n\}$. Start by observing that since $T_{\boldsymbol{p}}^{\alpha,\beta} \in A(\Omega)$, then by the representation formula derived in (2.8) we have

$$T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{x}) = \sum_{i=0}^{n} b_i(\boldsymbol{x}) T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_i). \tag{2.14}$$

From this it follows that $T_{\boldsymbol{p}}^{\alpha,\beta}$ preserves barycentric coordinates. Moreover, the same equation tells us that the image $T_{\boldsymbol{p}}(\Omega)$ lies in the convex hull of $T_{\boldsymbol{p}}^{\alpha,\beta}(V)$. Now, we show that $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$ has the set $T_{\boldsymbol{p}}^{\alpha,\beta}(V)$ as extreme points. To see this, assume that there exists a vertex $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_i)$ such that

$$T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_i) = \sum_{j=0}^{n} \alpha_j T_{\boldsymbol{p}}(\boldsymbol{v}_j), \tag{2.15}$$

where $\alpha_0, \ldots, \alpha_n$ are nonnegative real numbers that sum to 1. We may use the fact that $T_{\boldsymbol{p}}^{\alpha,\beta}$ is affine, to show that $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_i) = T_{\boldsymbol{p}}^{\alpha,\beta}(\sum_{j=0}^{n} \alpha_j \boldsymbol{v}_j)$. Since by Lemma 2.5, $T_{\boldsymbol{p}}^{\alpha,\beta}$ is injective, then clearly $\boldsymbol{v}_i = \sum_{j=0}^{n} \alpha_j \boldsymbol{v}_j$. The $\boldsymbol{v}_j$ are the extreme points of Ω then $\alpha_i = 1$ and $\alpha_j = 0$ if $i \neq j$. This means that $T_{\boldsymbol{p}}^{\alpha,\beta}(V)$ is exactly the set of the vertices of the polytope $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$.

Clearly, (iii) is an immediate consequence of the alternative representation of the function $T_{\boldsymbol{p}}^{\alpha,\beta}$ given by equation (2.14). To show that the vertex centroid is preserved, let $\lambda_i, i = 0, \ldots, n$ be the barycentric coordinates given by Lemma 2.2 such that $\lambda_i(\boldsymbol{vc}(\Omega)) = \frac{1}{n+1}$. Then (2.14) implies that

$$T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{vc}(\Omega)) = \frac{1}{n+1} \sum_{i=0}^{n} T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_i). \tag{2.16}$$

Since $T_{\boldsymbol{p}}^{\alpha,\beta}(V)$ is the set of the vertices of the polytope $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$, from the above equation we see that $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{vc}(\Omega)) = \boldsymbol{vc}(T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega))$. Moreover, equation (2.12) in Lemma 2.5 tells us that the vector $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{vc}(\Omega))$ can also be written as

$$T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{vc}(\Omega)) = \frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i := \boldsymbol{vc}(\Omega).$$

This shows that the vectors $\boldsymbol{vc}(\Omega)$ and $\boldsymbol{vc}(T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega))$ coincide. Using the change of variables $\boldsymbol{x'} = T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{x})$ and so the Jacobian of this transformation is $vol(\Omega)/vol(T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega))$, we then get $\boldsymbol{cg}(\Omega) = \boldsymbol{cg}(T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega))$. This establishes (v).
Let us assume now that $T_{\boldsymbol{p}}^{\alpha,\beta}$ has a fixed point $\boldsymbol{x} \in \Omega$. Then, $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{x}) = \boldsymbol{x}$, and consequently, it is equivalent to $\boldsymbol{x} = \boldsymbol{vc}(\Omega)$, completing the proof of (iv).
The convexity preserving of $f(T_{\boldsymbol{p}}^{\alpha,\beta})$ is an immediate consequence of the classical result on composition of a convex function with affine function.
To prove (vii), assume that there exists i such that $\boldsymbol{v}_i \in T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$. Then, there exists an $\boldsymbol{y}^*$ in Ω such that $\boldsymbol{v}_i = T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{y}^*)$. Consequently, by Lemma 2.5, equation (2.12), the vector $\boldsymbol{v}_i$ can be represented as follows:

$$\boldsymbol{v}_i = \sum_{j=0}^{n} \frac{\alpha - \beta\lambda_j(\boldsymbol{y}^*)}{(n+1)\alpha - \beta}\boldsymbol{v}_j.$$

Therefore, since $\boldsymbol{v}_i$ are extreme points of Ω, we have $\lambda_j(\boldsymbol{y}^*) = \frac{\alpha-((n+1)\alpha-\beta)\delta_{ij}}{\beta}$. It is now obvious that n must be equal to 1 and therefore $d = 1$, since n is assumed to satisfy $n \geq d \geq 1$. The inverse implication is immediate, thus the one-dimensional case is the only case in which the two polytopes Ω and its image $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$ are equal. The proof of Theorem 2.7 is now complete. □

Our final result in this section shows that for any $0 < \beta' \leq \alpha'$, the tautological map $T_{\boldsymbol{p}}^{\alpha',\beta'}$ also satisfies the inclusion property: All the polytopes $T_{\boldsymbol{p}}^{\alpha',\beta'}(\Omega)$ are contained in the polytope $T_{\boldsymbol{p}}^{1,1}(\Omega)$. This result follows easily from the following general fact:

Proposition 2.8. *Let $\alpha, \beta, \alpha', \beta'$ be positive real numbers satisfying $\beta \leq \alpha$, $\beta' \leq \alpha'$, and $\alpha/\beta \leq \alpha'/\beta'$. Then, the polytope $T_{\boldsymbol{p}}^{\alpha',\beta'}(\Omega)$ is contained in the polytope $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$.*

Proof. Let us fix $\boldsymbol{y}$ in $T_{\boldsymbol{p}}^{\alpha',\beta'}(\Omega)$. Then there exists $\boldsymbol{x}$ in Ω such that $T_{\boldsymbol{p}}^{\alpha',\beta'}(\boldsymbol{x}) = \boldsymbol{y}$ or equivalently

$$\boldsymbol{y} = \frac{\alpha'\boldsymbol{e} - \beta'\boldsymbol{x}}{(n+1)\alpha' - \beta'}.$$

Let us set $t := \alpha/\beta$, and $t' := \alpha'/\beta'$. Observe that t and t' are bigger than 1 and therefore, obviously, they are strictly bigger than $1/(n+1)$. Then, short calculations establish that $\boldsymbol{y}$ may also be written in the following form:

$$\boldsymbol{y} = \frac{\alpha\boldsymbol{e} - \beta\boldsymbol{z}}{(n+1)\alpha - \beta} \quad (:= T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{z})),$$

where

$$\boldsymbol{z} = \frac{(n+1)(t'-t)}{(n+1)t'-1}\boldsymbol{vc}(\Omega) + \frac{(n+1)t-1}{(n+1)t'-1}\boldsymbol{x}.$$

Then, it follows that $\boldsymbol{z} \in \Omega$, since it is immediate to verify, from the conditions satisfied by t and t', that $\boldsymbol{z}$ is written as a convex combination of two vetors belonging to Ω. Hence, $\boldsymbol{y}$ belongs also to the polytope $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$. Therefore we conclude that $T_{\boldsymbol{p}}^{\alpha',\beta'}(\Omega) \subseteq T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$. □

We would like to close this section by noting that, as already observed, any *pu*-system $\boldsymbol{p}$ belonging to $P_{n+1}^{\alpha,\beta}$ does not have the Lagrange property, see Proposition 2.6. We emphasize, however, that the later is always satisfied at the vertices $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{v}_i)$ of the transformed polytope $T_{\boldsymbol{p}}^{\alpha,\beta}(\Omega)$. Indeed, this result is an immediate consequence of the representation formula (2.14) available for any affine function, and the fact that any *bc*-system has the Lagrange property.

3. New inequalities related to the Jensen-type inequalities

The main objectives of this section are:

(i) to derive some pertinent results concerning convex hull and convexity preserving properties for a collection of functions of Ω into $\mathbb{R}^d$.

(ii) to establish some generalizations of inequality (1.3). The method developed yields direct and converse new multidimensional Jensen's type inequalities on convex polytopes. Among the inequalities presented, we offer, as a particular case of our general results, a direct and converse multivariate extension of Mercer inequality.

Here and in the rest of this section, $T_{\boldsymbol{p}}^{\alpha,\beta}$ is the tautological map associated to *pu*-system $\boldsymbol{p}$ defined in Lemma 2.4. We have seen in the preceding section the three fundamental properties of $T_{\boldsymbol{p}}^{\alpha,\beta}$, that we will use repeatedly in this section:

- $T_{\boldsymbol{p}}^{\alpha,\beta}$ is generated by a *pu*-system. That is there exists a *pu*-system $\boldsymbol{p} = \{p_i, = 0, \ldots, n\}$, such that for all $\boldsymbol{x} \in \Omega$, $T(\boldsymbol{x}) = \sum_{i=0}^n p_i(\boldsymbol{x})\boldsymbol{v}_i$. Hence, $T_{\boldsymbol{p}}^{\alpha,\beta}$ sends Ω into itself.
- *Affine representation*: For all $\boldsymbol{x} \in \Omega$, we have $T_{\boldsymbol{p}}^{\alpha,\beta}(\boldsymbol{x}) = \frac{\alpha\boldsymbol{e}-\beta\boldsymbol{x}}{(n+1)\alpha-\beta}$.
- *Convexity preserving:* $f(T_{\boldsymbol{p}}^{\alpha,\beta})$ is convex if f is convex.

First, we would like to study such properties for more general classes of functions. To be more precise, for T an arbitrary, but fixed, function defined on Ω with values in $\mathbb{R}^d$, let us say that T is generated by a *pu*-system if there exists a *pu*-system $\boldsymbol{p} = \{p_i, = 0, \ldots, n\}$, such that for all $\boldsymbol{x} \in \Omega$, $T(x) = \sum_{i=0}^n p_i(\boldsymbol{x})\boldsymbol{v}_i$. Note that if T is the identity map and T is generated by a *pu*-system $\boldsymbol{p}$, then this means exactly that $\boldsymbol{p}$ forms a *bc*-system.

Inspired by our earlier development in the case of discrete inequalities [7], it is natural to ask:

- Let $T : \Omega \to \mathbb{R}^d$ be an arbitrary function. What are the necessary and sufficient conditions for T to be generated by a *pu*-system?

The first contribution to this problem gives a sufficient condition for T to be generated by a *pu*-system.

Proposition 3.1. *Let $T : \Omega \to \mathbb{R}^d$ be an arbitrary function which maps Ω into Ω. Then, T is generated by a pu-system.*

Proof. Let $\lambda_i, i = 0, \dots n$, be the *bc*-system defined in Lemma 2.2. For every $i, i = 0, \dots n$ put $p_i = \lambda_i(T)$, so that p_i is well-defined because T maps Ω into Ω. Moreover, a simple inspection shows that the set $\boldsymbol{p} := \{p_i = \lambda_i(T), i = 0, \dots, n\}$ forms a *pu*-system, and since T also sends Ω, into itself, it follows from linear precision barycentric coordinates that for all $\boldsymbol{x} \in \Omega$

$$T(\boldsymbol{x}) = \sum_{i=0}^{n} p_i(\boldsymbol{x}) \boldsymbol{v}_i. \tag{3.1}$$

Hence, T is generated by the *pu*-system $\boldsymbol{p}$, as required. □

Our next result shows that the above sufficient condition is also a necessary condition. We shall formulate conditions similar to those given in the discrete cases, see [7]. More precisely, we have the following characterizations:

Theorem 3.2. *Let Ω be a convex polytope with vertices $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$, and let T be an arbitrary function defined on Ω with values in $\mathbb{R}^d$. For each real-valued convex function defined on $\mathbb{R}^d$ define $\widetilde{T}$ as follows:*

$$\widetilde{T}[f] = f(T). \tag{3.2}$$

Then the following statements are equivalent:

(i) *T is generated by a pu-system;*
(ii) *$\widetilde{T}$ is nonnegative for nonnegative affine functions;*
(iii) *T maps Ω into itself.*

Proof. Assume that (i) holds. Let us recall that the polytope Ω may also be defined by m inequalities:

$$\Omega = \left\{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{a}_k, \boldsymbol{x} \rangle + b_k \geq 0, k = 1, \dots, m\right\},$$

here $\boldsymbol{a}_k \in \mathbb{R}^d$ and $b_k \in \mathbb{R}$, see, e. g., [26]. Fix now a nonnegative affine function l on Ω. Then, by the so-called affine form of Farkas' lemma, l is a nonnegative affine combination of the affine forms used to define the polytope Ω, see [25]. Therefore, there exist some nonnegative values $\mu_k \geq 0$, for any $k = 0, \dots, m$ such that

$$l(\boldsymbol{x}) = \mu_0 + \sum_{k=1}^{m} \mu_k \left(\langle \boldsymbol{a}_k, \boldsymbol{x} \rangle + b_k \right), \quad (\boldsymbol{x} \in \Omega). \tag{3.3}$$

Then, since

$$\widetilde{T}[l](\boldsymbol{x}) = l(T(\boldsymbol{x})),$$

and T is generated by a *pu*-system $\{p_i, i = 0, \dots, n\}$ we get by an obvious calculation

$$\widetilde{T}[l](\boldsymbol{x}) = \sum_{i=0}^{n} p_i(\boldsymbol{x}) l(\boldsymbol{v}_i).$$

This permits us to rewrite $\widetilde{T}[l]$ as a convex combination of the values of l at the vertices. Consequently, it follows that $\widetilde{T}[l]$ is nonnegative on Ω since l takes nonnegative values. This shows that property (ii) holds.
Assume that the operator T sends Ω into itself, and on the contrary that there exists a $\boldsymbol{y} \in \Omega$ such that $T(\boldsymbol{y}) \notin \Omega$. Then, due to the Separation Theorem for closed

convex sets (see, e.g., [27, p. 65, Theorem 2.4.1]), there exists a point $\boldsymbol{x}^* \in \Omega$ such that the affine function

$$l(\boldsymbol{x}) := \langle T(\boldsymbol{y}) - \boldsymbol{x}^*, \boldsymbol{x} - \boldsymbol{x}^* \rangle \tag{3.4}$$

satisfies $l(\boldsymbol{x}) \leq 0$, for all $\boldsymbol{x} \in \Omega$. Here $\langle .,. \rangle$ denotes the usual scalar product in $\mathbb{R}^d$. Hence $\widetilde{T}[l] \leq 0$, since $l \leq 0$ and $\widetilde{T}$ is nonnegative for every nonnegative affine function. Consequently,

$$\widetilde{T}[l](\boldsymbol{y}) = l(T(\boldsymbol{y})) := \|T(\boldsymbol{y}) - \boldsymbol{x}^*\|^2 \leq 0,$$

where $\|.\|$ denotes the usual Euclidean norm on $\mathbb{R}^d$. This clearly implies $T(\boldsymbol{y}) = \boldsymbol{x}^*$, and contradicts the fact that $T(\boldsymbol{y}) \notin \Omega$. Hence this contradiction proves that assertion (iii) holds.
Finally, the implication (iii) $\Rightarrow$ (i) follows from Proposition 3.1. □

Remark 3.3. The nonnegative restriction condition (ii) in Theorem 3.2 for affine functions may seem too restrictive, it is often satisfied by many approximation operators, see [8, 9, 10, 11, 13, 14].

A function $T : \Omega \to \mathbb{R}^d$ will be said to preserve *bc*-systems if

$$T(\boldsymbol{x}) \quad = \textstyle\sum_{i=0}^{n} b_i(\boldsymbol{x}) T(\boldsymbol{v}_i), \quad (\forall \boldsymbol{x} \in \Omega), \tag{3.5}$$

holds for any *bc*-system $\{b_0, \ldots, b_n\}$. The collection of all functions from Ω to $\mathbb{R}^d$ that preserve *bc*-systems is denoted $H_{n+1}(\Omega)$. The representation formula (3.5) informs us that for any $T \in H_{n+1}(\Omega)$, we have $T(\Omega)$ is a subset of the set generated by the image of vertices of the polytope. Thus, if T maps $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ into Ω, then T carries Ω into a subset of Ω.
Identity (2.8) guarantees that any affine function preserves *bc*-systems. Consequently, we have the inclusion $A(\Omega) \subseteq H_{n+1}(\Omega)$.

Theorem 3.4. *Let Ω be a convex polytope with vertices $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$, and let T be an arbitrary function defined on Ω with values in $\mathbb{R}^d$. Assume that T is generated by a pu-system $\boldsymbol{p}$. Then, the following assertions are equivalent:*

(i) *$\boldsymbol{p}$ has the Lagrange property.*
(ii) *T has the vertices preserving property: that is $T(\boldsymbol{v}_i) = \boldsymbol{v}_i, i = 0, \ldots, n$.*

If, in addition, T preserves bc-systems then (i), (ii) and the following are all equivalent:

(iii) *T is the identity map.*
(iv) *$\boldsymbol{p}$ is a bc-system.*

Proof. Since T preserves *bc*-systems then T satisfies identity (3.5). Hence, the proof can be done by almost the same arguments as in Theorem 2.3, with the only modification that we use identity (3.5) instead of the representation formula proved in (2.8). □

We are now concerned with the following question:

- What are the necessary and sufficient conditions for $T \in H_{n+1}(\Omega)$ to be generated by a *pu*-system?

The next answer will give an easy and practical characterization of a function in $H_{n+1}(\Omega)$ that is generated by a *pu*-system. Indeed, we shall give simple conditions for a function in $H_{n+1}(\Omega)$, which can be generated by a *pu*-system by looking only the images of the vertices of the polytope.

Proposition 3.5. *Let Ω be a convex polytope with vertices $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$, and let T be an arbitrary function in $H_{n+1}(\Omega)$. Then the following statements are equivalent:*

(i) *T is generated by a pu-system.*
(ii) *T maps the set of vertices of Ω into Ω.*

Proof. The direct implication is a consequence of Theorem 3.2. For the converse implication, let us assume that (ii) holds. Then, since T belongs to $H_{n+1}(\Omega)$, it follows by the representation formula (3.5) that

$$T(\boldsymbol{x}) = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) T(\boldsymbol{v}_i),$$

where $\{\lambda_i, i = 0, \dots, n\}$ is the *bc*-system defined in Lemma 2.2. This means that T sends Ω into Ω, since for any point $\boldsymbol{x}$ in Ω, $T(\boldsymbol{x})$ is written as a convex combination of $(n+1)$ points belonging to Ω. In order to complete the proof of Proposition 3.5 we just use Proposition 3.1. □

We are now in a position to show an analogue of Lemma 2.4, which includes more general classes of functions. This result is a direct corollary of Proposition 3.5.

Corollary 3.6. *Let $T : \Omega \to \mathbb{R}^d$ be defined by*

$$T(\boldsymbol{x}) = \sum_{i=0}^{n} \alpha_i \boldsymbol{v}_i - \beta \boldsymbol{x}, \tag{3.6}$$

where $\alpha_i, i = 0, \dots, n$ and β are given positive real numbers such that

$$\sum_{i=0}^{n} \alpha_i - \beta = 1 \text{ and } \beta \leq \min_{0 \leq i \leq n} \alpha_i. \tag{3.7}$$

Then T is generates by a pu-system.

Proof. Clearly, since T is an affine function, consequently, $T \in H_{n+1}(\Omega)$, then in view of Proposition 3.5 we need to prove only that T maps all the vertices of Ω into Ω. It is immediate to verify that, for all $j = 0, \dots, n$, we have

$$\begin{aligned} T(\boldsymbol{v}_j) &= \sum_{i=0}^{n} \alpha_i \boldsymbol{v}_i - \beta \boldsymbol{v}_j \\ &= \sum_{i=0, i \neq j}^{n} \alpha_i \boldsymbol{v}_i + (\alpha_j - \beta) \boldsymbol{v}_j. \end{aligned}$$

Hence, $T(\boldsymbol{v}_j) \in \Omega$, since $T(\boldsymbol{v}_j)$ is written as convex combination of the vertices of the polytope Ω. □

Remark 3.7. Note that Corollary 3.6 applies to the tautological maps $T_{\boldsymbol{p}}^{\alpha,\beta}$ defined in Lemma 2.5.

Proposition 3.5 informs us that an affine function can be generated by a *pu*-system if and only if it sends every vertex in Ω to a point of Ω. For an arbitrary function T, which is not affine, it is easy to give an example to show that $T(\Omega)$ is not necessary a subset of Ω, even if the images of the vertices are transformed in

Ω. By Theorem 3.2, this means that T cannot be generated by a pu-system. To see this, let us consider the following simple one-dimensional example:

Example 3.8. As a trivial example, take $\Omega = [0,1]$ and consider the following function $T : [0,1] \to \mathbb{R}$, defined by

$$T(x) = 2x^2 - x. \tag{3.8}$$

For this particular function we have $T(0) = 0$ and $T(1) = 1$. Consequently, the endpoints are preserved, however, obvious verification shows that T does not map $[0,1]$ into itself. More precisely, it is easy to see that $T([0,1]) = [-\frac{1}{8}, 1]$, and so, by Theorem 3.2, this function cannot be generated by a pu-system.
There is another way of proving the above result. To this end, let l be the nonnegative affine function defined by $l(x) := 1 - x$ on $[0,1]$. It is not difficult to see that, for all $x \in [0,1]$, we have $T(l(x)) = (1-2x)l(x)$. Note, however, that the function $T(l)$ changes its sign in $[0,1]$. Hence by Theorem 3.2, (ii), T cannot be generated by a pu-system.

To state our next result, we need some more preparations. Let $T : \Omega \to \mathbb{R}^d$ be a given function, and assume that T is generated by a pu-system $\boldsymbol{p} = \{p_i, i = 0, \ldots, n\}$. Let us consider the associated function $\tilde{T} : \Omega \to \mathbb{R}^d$ defined by

$$\tilde{T}(\boldsymbol{x}) = \sum_{i=0}^{n} \alpha_i \boldsymbol{v}_i - \beta T(\boldsymbol{x}), \tag{3.9}$$

where $\alpha_i, i = 0, \ldots, n$ and β are given positive real numbers such that $\sum_{i=0}^{n} \alpha_i - \beta = 1$ and $\beta \leq \min_{0 \leq i \leq n} \alpha_i$. Now, completely analogous arguments to those in the proof of Lemma 2.4 show that $\tilde{T}$ can be generated by a pu-system $\tilde{\boldsymbol{p}}$ with

$$\tilde{p}_i = \alpha_i - \beta p_i, i = 0, \ldots n. \tag{3.10}$$

For a given continuous convex function on Ω, let us define the auxiliary function $\tilde{f}$ on Ω as follows:

$$\tilde{f} = \frac{1}{|\boldsymbol{\alpha}|} f\left(\tilde{T}\right) + \frac{\beta}{|\boldsymbol{\alpha}|} f(T), \tag{3.11}$$

where $|\boldsymbol{\alpha}| = \sum_{i=0}^{n} \alpha_i$. Note that $\tilde{f}$ is well-defined since T and $\tilde{T}$ are, respectively, generated by the pu-systems $\boldsymbol{p}$ and $\tilde{\boldsymbol{p}}$, therefore by Theorem 3.2 they map Ω in itself.
The next general lemma formulates a crucial observation, that allows us to obtain upper and lower bounds for the function $\tilde{f}$ over the polytope Ω.

Lemma 3.9. *Let $\alpha_i, i = 0, \ldots, n$ and β be defined as in Corollary 3.6. Let Ω be a polytope in $\mathbb{R}^d$, $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ its vertices, and $f : \Omega \to \mathbb{R}$ be a convex function. Then, the following inequalities always hold for all $\boldsymbol{x} \in \Omega$*

$$f\left(\sum_{i=0}^{n} \frac{\alpha_i}{|\boldsymbol{\alpha}|} \boldsymbol{v}_i\right) \leq \tilde{f}(\boldsymbol{x}) \leq \sum_{i=0}^{n} \frac{\alpha_i}{|\boldsymbol{\alpha}|} f(\boldsymbol{v}_i), \tag{3.12}$$

where $|\boldsymbol{\alpha}| = \sum_{i=0}^{n} \alpha_i$.

Proof. To prove Lemma 3.9 we shall make use the classical Jensen's discrete inequality. Indeed, to establish the right hand inequality, note that since the function $\tilde{T}$ is generated by the pu-system $\{\tilde{p}_i, i = 0, \ldots, n\}$, then, we have $\tilde{T}(\boldsymbol{x}) = \sum_{i=0}^{n} \tilde{p}_i(\boldsymbol{x})\boldsymbol{v}_i$, for all $\boldsymbol{x} \in \Omega$. Thus, using the classical Jensen's discrete inequality for f we have

$$f(\tilde{T}(\boldsymbol{x})) \leq \sum_{i=0}^{n} \tilde{p}_i(\boldsymbol{x}) f(\boldsymbol{v}_i). \tag{3.13}$$

Since $\tilde{p}_i = \alpha_i - \beta p_i$ where $\{p_0, \ldots, p_n\}$ is a pu-system, which generates T, then the right-hand side of the above equation can be rewritten as follows

$$\sum_{i=0}^{n} \tilde{p}_i(\boldsymbol{x}) f(\boldsymbol{v}_i) = \sum_{i=0}^{n} \alpha_i f(\boldsymbol{v}_i) - \beta \sum_{i=0}^{n} p_i(\boldsymbol{x}) f(\boldsymbol{v}_i).$$

From this, we get the following inequality

$$\tilde{f}(\boldsymbol{x}) \leq \sum_{i=0}^{n} \frac{\alpha_i}{|\boldsymbol{\alpha}|} f(\boldsymbol{v}_i) + \frac{\beta}{|\boldsymbol{\alpha}|} \left(f\left(T(\boldsymbol{x})\right) - \sum_{i=0}^{n} p_i(\boldsymbol{x}) f(\boldsymbol{v}_i) \right).$$

Due the convexity of f and the fact that $T(\boldsymbol{x}) = \sum_{i=0}^{n} p_i(\boldsymbol{x})\boldsymbol{v}_i$, Jensen's inequality shows that the term on the right is negative, then we get the right-hand side of (3.12).
To prove the left hand side inequality, note that, obviously, for all $\boldsymbol{x} \in \Omega$, by equation (3.9) we have the identity

$$\sum_{i=0}^{n} \frac{\alpha_i}{|\boldsymbol{\alpha}|} \boldsymbol{v}_i = \frac{1}{|\boldsymbol{\alpha}|} \tilde{T}(\boldsymbol{x}) + \frac{\beta}{|\boldsymbol{\alpha}|} T(\boldsymbol{x}). \tag{3.14}$$

The above shows that $\sum_{i=0}^{n} \frac{\alpha_i}{|\boldsymbol{\alpha}|} \boldsymbol{v}_i$ can be expressed, independently of $\boldsymbol{x}$, as a convex combination of elements belonging to Ω, $\tilde{T}(\boldsymbol{x})$ and $T(\boldsymbol{x})$. Then, by Jensen's inequality, we conclude

$$f\left(\sum_{i=0}^{n} \frac{\alpha_i}{|\boldsymbol{\alpha}|} \boldsymbol{v}_i \right) \leq \frac{1}{|\boldsymbol{\alpha}|} f(\tilde{T}(\boldsymbol{x})) + \frac{\beta}{|\boldsymbol{\alpha}|} f(T(\boldsymbol{x})), \tag{3.15}$$

which is clearly the desired result. □

Observe that if we assume that the function T is generated by a bc-system, then Proposition 2.1 tells us that T is the identity map of the domain polytope Ω. Hence by using the formula (3.9), we get that $\tilde{T}$ is the affine function defined in (3.6). One particular consequence is that the associated function $\tilde{f}$ defined by (3.11) is automatically convex, since composition with affine function: $f(\tilde{T})$ is convex if f is convex.

To further simplify the analysis, it will be assumed from now on that

$$\alpha_i = \frac{\alpha}{(n+1)\alpha - \beta}, i = 0, \ldots, n,$$

and β is replaced by $\frac{\beta}{(n+1)\alpha - \beta}$, where α and β are two positive real numbers. Hence as a corollary the main Lemma 3.9, we have the following result which is a crucial step in deriving our new inequalities.

Corollary 3.10. *Given two positive real numbers α, β such that $\beta \leq \alpha$. Let Ω be a polytope in $\mathbb{R}^d$, $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ its vertices, and $f : \Omega \to \mathbb{R}$ be a convex function. Then the function f^+ defined for all $\boldsymbol{x} \in \Omega$ by*

$$f^+(\boldsymbol{x}) = f\left(\frac{\alpha \boldsymbol{e} - \beta \boldsymbol{x}}{(n+1)\alpha - \beta}\right) + \frac{\beta}{(n+1)\alpha - \beta} f(\boldsymbol{x}), \tag{3.16}$$

is a convex function on Ω and the following inequalities always hold for all $\boldsymbol{x} \in \Omega$

$$\frac{(n+1)\alpha}{(n+1)\alpha - \beta} f(\boldsymbol{vc}(\Omega)) \leq f^+(\boldsymbol{x}) \leq \frac{\alpha}{(n+1)\alpha - \beta} \sum_{i=0}^{n} f(\boldsymbol{v}_i), \tag{3.17}$$

and both inequalities are attained for all affine functions.

Proof. Just apply Lemma 3.9 by replacing each α_i with $\frac{\alpha}{(n+1)\alpha-\beta}$, β with $\frac{\beta}{(n+1)\alpha-\beta}$, and then multiplying the expression of $\tilde{f}$ by $\frac{(n+1)\alpha}{(n+1)\alpha-\beta}$. Finally, the equality cases can be easily verified. □

Remark 3.11. We want to draw the reader's attention to the fact that there is, of course, an obvious extension of Corollary 3.10 to the situation when, as in Lemma 3.9, $\alpha_i, i = 0, \ldots, n$ and β are given positive real numbers such that equations (3.7) are fulfilled. The reader can state and prove the corresponding assertion himself if he notices that for any convex function f the function $\tilde{f}$ satisfies the equation (3.11). Furthermore, we should observe that when T is only generated by a *pu*-system, then the associated fucntion $\tilde{f}$ is not necessarily convex. Since the function $\tilde{T}$ is not generally affine.

The convexity-preserving transformation f^+ defined by (3.16) is the key to the subsequent development. As regards this transformation, we will be much more modest, by choosing the function $\tilde{T}$ $(= T_{\boldsymbol{p}}^{\alpha,\beta})$ as an affine function, but the next results shows that we cannot do more.
We shall denote the set of all convex functions defined on Ω by $K(\Omega)$ and let $B(\Omega)$ be the set of all real-valued functions defined on Ω. For an operator $S : K(\Omega) \to B(\Omega)$, if f is convex on Ω implies $S[f]$ is also convex on Ω, then we say that S preserves convexity.

Proposition 3.12. *Let $T : \Omega \to \Omega$ be an arbitrary function. Define the operator $\widetilde{T}$ on $K(\Omega)$ by*

$$\widetilde{T}[f] = f(T). \tag{3.18}$$

Then, the following assertions are equivalent:

(i) *$\widetilde{T}$ preserves convexity.*
(ii) *T is an affine function.*

Proof. Let us assume that $\widetilde{T}$ preserves convexity. First we note that to establish T is affine it suffices to show that for all $i = 1, \ldots, d$, $e_i(T)$ is affine, where $e_1, \ldots, e_d$, denote the projections $e_i : \boldsymbol{x} = (x_1, \ldots, x_d) \to x_i$. Since e_i and $-e_i$ are both convex, then it follows from the assumed convexity of $\widetilde{T}$ that $\pm e_i(T)$ are convex. Consequently, $e_i(T)$ is affine.
The inverse implication is obvious, since composition with affine function: $f(T)$ is convex if f is convex. □

From now on, if it is not explicitly mentioned, α and β represent positive real numbers such that $\beta \leq \alpha$.
Before going further, let us first observe the next result which gives some basic information about the '*best*' choices for the parameters α and β in (3.17). To this end, let us define the following function: $\mathcal{K}(t)$ from $[1, +\infty[$ to $\mathbb{R}$ by:

$$\mathcal{K}(t) := \frac{t}{(n+1)t - 1}. \tag{3.19}$$

It seems reasonable to choose and adjust the parameters α and β in such a way that they minimize the function $\mathcal{K}$. Indeed we have the following result:

Proposition 3.13. *Under the hypotheses of Corollary 3.10, let $\mathcal{K}$ be the function defined by (3.19). Then, for any positive real numbers α, β, with $\beta \leq \alpha$, and $\boldsymbol{x} \in \Omega$, it always holds that*

$$0 \leq f^+(\boldsymbol{x}) - \frac{(n+1)\alpha}{(n+1)\alpha - \beta} f(\boldsymbol{vc}(\Omega)) \leq \mathcal{K}(\alpha/\beta) \left(\sum_{i=0}^{n} f(\boldsymbol{v}_i) - (n+1) f(\boldsymbol{vc}(\Omega)) \right). \tag{3.20}$$

In addition, the function $\mathcal{K}$ is not increasing and

$$\mathcal{K}(1) = \frac{1}{n} = \sup_{t \geq 1} \mathcal{K}(t) \geq \frac{1}{n+1} = \inf_{t \geq 1} \mathcal{K}(t).$$

Proof. This is an immediate consequence of the estimates (3.17) given in Corollary 3.10. ☐

Remark 3.14. We would like to mention that for all $0 < \beta \leq \alpha, \mathcal{K}(\frac{\alpha}{\beta}) \leq \frac{1}{n}$, with equality if $\alpha = \beta = 1$. Hence, the values of the parameters $\alpha = \beta = 1$ yield the worst approximation, in the sense that they maximize the function $\mathcal{K}(\alpha/\beta)$ involved in the inequality (3.20). Note that, the minimum value $1/(n+1)$ is not attainable, however, we have $\mathcal{K}(\frac{\alpha}{\beta}) \to \frac{1}{n+1}$ when $\alpha \to +\infty$ and β is maintained fixed. Thus, the choice of appropriate parameters α, β, in inequality (3.17), can be archived by taking α as large as possible and β as small as possible.

The following applications are intended as examples of how the main results of the paper can be used. Before we state the next theorem, let us introduce more notation. Recall that, as mentioned in the Introduction, X is a (nonempty) set, L is assumed a subspace of the vector space of real-valued functions defined on X, that contains the constant functions. If $\boldsymbol{g} = (g_1, \ldots, g_d)$ is an d-tuple of functions in L, such that $\boldsymbol{g}$ carries X into Ω, we define

$$A[\boldsymbol{g}] := \big(A[g_1], \ldots, A[g_d]\big).$$

With the help of Corollary 3.10, we are now in a position to derive the following extension of Jensen-Mcshane's inequality.

Theorem 3.15. *Let Ω be a polytope in $\mathbb{R}^d$, $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ its vertices, and $f : \Omega \to \mathbb{R}$ be a convex function. Then, for any d-tuple of functions $\boldsymbol{g} = (g_1, \ldots, g_d)$ such that g_i, $f(\boldsymbol{g})$, $f\left(\frac{\alpha \sum_{i=0}^n \boldsymbol{v}_i - \beta A[\boldsymbol{g}]}{(n+1)\alpha - \beta}\right) \in L$, we have for any $0 < \beta \leq \alpha$, the following variant of Jensen's inequality*

$$f\left(\frac{\alpha \sum_{i=0}^n \boldsymbol{v}_i - \beta A[\boldsymbol{g}]}{(n+1)\alpha - \beta}\right) \leq \frac{\alpha}{(n+1)\alpha - \beta} \sum_{i=0}^{n} f(\boldsymbol{v}_i) - \frac{\beta}{(n+1)\alpha - \beta} A[f(\boldsymbol{g})]. \tag{3.21}$$

Proof. Since $\boldsymbol{g}(\boldsymbol{t}) \in \Omega$ for all $\boldsymbol{t} \in X$, it follows, by [22, theorem 1], that $A[\boldsymbol{g}] \in \Omega$ and then we have immediately by Mcshane's inequality (1.3)

$$\begin{aligned} f\left(\frac{\alpha\sum_{i=0}^{n}\boldsymbol{v}_i - \beta A[\boldsymbol{g}]}{(n+1)\alpha-\beta}\right) &= f\left(A\left[\frac{\alpha\sum_{i=0}^{n}\boldsymbol{v}_i - \beta \boldsymbol{g}}{(n+1)\alpha-\beta}\right]\right) \\ &\leq A\left[f\left(\frac{\alpha\sum_{i=0}^{n}\boldsymbol{v}_i - \beta \boldsymbol{g}}{(n+1)\alpha-\beta}\right)\right]. \end{aligned}$$

To obtain the desired inequality, it suffices to apply A on both sides of the right-hand inequality (3.17) and make use of the positivity and linearity of A. □

Our general result has the following corollary in the functional integral context:

Corollary 3.16. *Let $(X, \mathcal{A}, \mu)$ be a probability measure space, and let $\boldsymbol{g} : X \to \Omega$ be a measurable function. Then for any $0 < \beta \leq \alpha$, and any continuous convex function $f : \Omega \to \mathbb{R}$*

$$f\left(\frac{\alpha\sum_{i=0}^{n}\boldsymbol{v}_i - \beta\int_X \boldsymbol{g}d\mu}{(n+1)\alpha-\beta}\right) \leq \frac{\alpha}{(n+1)\alpha-\beta}\sum_{i=0}^{n} f(\boldsymbol{v}_i) - \frac{\beta}{(n+1)\alpha-\beta}\int_X f(\boldsymbol{g})d\mu. \tag{3.22}$$

Proof. This is a special case of the more general result established in Theorem 3.15. This comes immediately by taking $L := L^1(\mu)$ the set of real valued functions, defined on X, which are integrable on $(X, \mathcal{A}, \mu)$, and $A(\boldsymbol{g}) = \int_X \boldsymbol{g}d\mu$. □

Several known generalizations of Jensen's inequality follows from Corollary 3.16 by fixing the probability measure. We list some special cases. In particular, for the case of $d = 1$ (therefore $n = 2$) one-dimensional case, if in Corollary 3.16, we take $\Omega = [a, b], (a \neq b,)$ and μ a counting measure, then we obtain the corresponding discrete versions of (3.22):

Corollary 3.17. *If f is a convex function on a real interval $[a, b]$ containing the points $x_i, i = 1 \ldots, m$ and positive real numbers $\omega_i, i = 1 \ldots, m$ such that $\sum_{i=1}^{m} \omega_i = 1$, then for any $0 < \beta \leq \alpha$,*

$$f\left(\frac{\alpha(a+b) - \beta\sum_{i=1}^{m}\omega_i x_i}{2\alpha-\beta}\right) \leq \frac{\alpha\,(f(a)+f(b))}{2\alpha-\beta} - \frac{\beta}{2\alpha-\beta}\sum_{i=1}^{m}\omega_i f(x_i). \tag{3.23}$$

By using the previous inequality evaluated at $\alpha = \beta$, we obtain, in the present particular situation, an inequality due to Mercer [21, Theorem 1.2]:

$$f\left(a+b-\sum_{i=1}^{m} w_i x_i\right) \leq f(a)+f(b) - \sum_{i=1}^{m} w_i f(x_i). \tag{3.24}$$

Thus, inequality (3.21) can be seen as an extension in gereral settings of Mercer's inequality (3.24). However, it is important to observe that in order to obtain this result, the proof given there was different and the author in [21] has assumed that all points x_i are nonnegative, as a part of whole assumptions. But in his proof, we can see that this condition is needless to this theorem. We will show below that Mercer's result has a converse multivariate version.

The next theorem gives a natural converse inequality of (3.21). As in derivation of Theorem 3.15, the lower bound given in Corollary 3.10 might also be used as a starting point.

Theorem 3.18. *Let* Ω *be a polytope in* $\mathbb{R}^d$, $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ *its vertices, and* $f : \Omega \to \mathbb{R}$ *be a convex function. Then, for any d-tuple of functions* $\boldsymbol{g} = (g_1, \dots, g_d)$ *such that* g_i, $f(\boldsymbol{g})$, $f(\boldsymbol{g})$, $f\left(\frac{\alpha\sum_{i=0}^n \boldsymbol{v}_i - \beta A[\boldsymbol{g}]}{(n+1)\alpha-\beta}\right) \in L$, *we have for any* $0 < \beta \le \alpha$, *the following variant of the converse Jensen's inequality*

$$\frac{(n+1)\alpha}{(n+1)\alpha-\beta} f(\boldsymbol{vc}(\Omega)) - \frac{\beta}{(n+1)\alpha-\beta} A\left[f(\boldsymbol{g}\right] \le f\left(\frac{\alpha\sum_{i=0}^n \boldsymbol{v}_i - \beta A[\boldsymbol{g}]}{(n+1)\alpha-\beta}\right). \tag{3.25}$$

Proof. Since $\boldsymbol{g}(\boldsymbol{t}) \in \Omega$ for all $\boldsymbol{t} \in X$, again, from [22, theorem 1], it follows that $A[\boldsymbol{g}] \in \Omega$ and from the left inequality in Corollary 3.10 that

$$\frac{(n+1)\alpha}{(n+1)\alpha-\beta} f(\boldsymbol{vc}(\Omega)) - \frac{\beta}{(n+1)\alpha-\beta} f(A[\boldsymbol{g}]) \le f\left(\frac{\alpha\sum_{i=0}^n \boldsymbol{v}_i - \beta A[\boldsymbol{g}]}{(n+1)\alpha-\beta}\right). \tag{3.26}$$

Furthermore, by Mcshane's inequality (1.3), we have $-A[f(\boldsymbol{g})] \le -f(A[\boldsymbol{g}])$, then we immediately get the desired inequality. ☐

As a corollary of Theorem 3.18 we have the following result:

Corollary 3.19. *Let* $(X, \mathcal{A}, \mu)$ *be a probability measure space, and let* $\boldsymbol{g} : X \to \Omega$ *be a measurable function. Then for any* $0 < \beta \le \alpha$, *and any continuous convex function* $f : \Omega \to \mathbb{R}$

$$\frac{(n+1)\alpha}{(n+1)\alpha-\beta} f(\boldsymbol{vc}(\Omega)) - \frac{\beta}{(n+1)\alpha-\beta} \int_X f(\boldsymbol{g}) d\mu \le f\left(\frac{\alpha\sum_{i=0}^n \boldsymbol{v}_i - \beta \int_X \boldsymbol{g} d\mu}{(n+1)\alpha-\beta}\right). \tag{3.27}$$

In the univariate case, when Ω is the closed interval $[a, b]$ of $\mathbb{R}$, and p is a discrete probability distribution on the set $X := \{x_1 \dots, x_m\} \subset [a, b]$ with $p_i = p(x_i)$, by choosing $\alpha = \beta = 1$, our inequality (3.27) becomes:

$$2f(\frac{a+b}{2}) - \sum_{i=1}^m p_i f(x_i) \le f\left(a + b - \sum_{i=1}^m p_i x_i\right). \tag{3.28}$$

This particular inequality appears as a converse of Mercer's inequality (3.24).

Remark 3.20. We finally observe that the reader can easily reformulate all our results of Theorems 3.15, 3.18 and their Corollaries for the case where $\alpha_i, i = 0, \dots, n$ and β are given positive real numbers such that $\sum_{i=0}^n \alpha_i - \beta = 1$ and $\beta \le \min_{0 \le i \le n} \alpha_i$. The proofs are those of Theorems 3.15 and 3.18, except that Lemma 3.9 is used instead of its Corollary 3.10. Note that in Theorems 3.15, 3.18 and their Corollaries all the inequalities become equalities for every affine function. It should also be mentioned that discrete analogues of inequalities given in Corollaries 3.16 and 3.19 were obtained in [7]. These inequalities can be rediscovered by choosing T the identity map and taking the set X the polytope Ω.

Acknowledgements

This paper is dedicated to Professor Jean-Louis Gout on the occasion of his retirement. The author is sincerely grateful to Professor A. Jourani for his critical remarks, kind suggestions and a comment concerning the proof of Proposition 3.12.

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