Path-Driven Orientation of Mixed Graphs

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Abstract

We consider in this paper two graph orientation problems. The input of both problems is (i) a mixed graph G whose vertex set is V and edge set (resp. arc set) is E (resp. A) and (ii) a set $\mathcal{P} \subseteq V \times V$ of source-target pairs. The first problem, called S-GO, is a decision problem introduced by Hassin and Megiddo (Linear Algebra and its Applications 114 (1989): 589-602) and defined as follows: is it possible to find an orientation of G that replaces each edge $(u, v) \in E$ by a single arc (either uv or vu) in such a way that, for each $(s, t) \in \mathcal{P}$, there exists a directed path from s to t ? Our second problem, called MIN-D-GO, is a minimization problem that can be seen as a variant of S-GO, in which we allow some edges $(u, v) \in E$ to be doubly oriented. The goal is then to find an orientation of G that replaces each edge $(u, v) \in E$ by uv and/or vu in such a way that (i) there exists a directed path from s to t for each $(s, t) \in \mathcal{P}$ and (ii) the number of doubly oriented edges is minimized. We investigate the complexity of S-GO and MIN-D-GO by considering some restrictions on the input instances (such as the maximum degree of G or the cardinality of $\mathcal{P}$). We provide several polynomial time algorithms, hardness and inapproximability results that together give an extensive picture of tractable and intractable instances for both problems.

Keywords: algorithmic complexity, graph orientation, mixed graphs, biological networks

1. Introduction

A mixed graph is a triple $G = (V, E, A)$ that consists of a set V of n vertices, a set E of edges and a set A of arcs. The orientation of mixed graphs (possibly with $A = \emptyset$) has applications in the design of urban networks [18, 8, 12] and communication networks [1]. Orienting these networks consists in the assignment of directions to edges in order to fulfill a given set of reachability or communication requests.

Recently, the orientation of mixed graphs found applications in biology, more specifically in the study of physical networks [21], which are mixed graphs representing functional relationships between proteins. External cellular events are transmitted

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into the nucleus via cascades of activation/deactivation of proteins, that correspond to paths in the physical network from a source protein (cause) to a target protein (effect). A key problem in biology is to infer the direction of these paths by combining causal information on cellular events [14, 19].

Let $G = (V, E, A)$ be a mixed graph and $\mathcal{P} \subseteq V \times V$ be a set of source-target vertices. We first consider the problem that we call here SIMPLE GRAPH ORIENTATION (abbreviated S-GO), introduced by Hassin and Megiddo [12]. The S-GO problem is the following: given G and $\mathcal{P}$, is it possible to replace each edge in G by a single arc in such a way that *all* the pairs in $\mathcal{P}$ are connected by a directed path ? When it is not possible to find such an orientation, one can try to satisfy a maximum number of pairs. This variant leads to the MAXIMUM GRAPH ORIENTATION problem (M-GO), see for instance [14, 19]: given $G = (V, E, A)$ and $\mathcal{P}$, replace each edge $(u, v) \in E$ by a single arc, so that in the new graph, there exists a directed path for a maximum number of source-target pairs.

The M-GO variant is not adequate when all the pairs in $\mathcal{P}$ are known to be relevant, and consequently each of them should be satisfied. In order to deal with this situation, we introduce and study the complexity of a minimization problem, called MIN-D-GO, that – unlike the M-GO problem – aims at satisfying *all* the pairs by allowing some edges of G to be *doubly oriented* (i.e., replaced by two arcs in opposite directions). However, the number of such edges has to be as small as possible, for the following reasons. In communication networks, a doubly oriented edge means the creation of a two-directional link that allows communication in both directions, simultaneously. This type of links is costly, because it needs the allocation of two different frequencies: transmitting on one frequency and receiving on another [9]. In the context of biology, a doubly oriented edge reflects the presence of a reversible reaction. Furthermore, in a dynamic biological system, most reactions tend to be irreversible [13]. Following these motivations, MIN-D-GO asks that the number of doubly oriented edges be minimized.

The S-GO problem has been shown to be polynomial-time solvable on undirected graphs [12] and when the number of pairs is constant [5]. In contrast, Arkin and Hassin [2] showed that the problem becomes NP-complete when G is a general mixed graph.

The M-GO problem is polynomially solvable when G is a path, but NP-complete even when G is a star (that is, a tree whose number of vertices with degree 2 or more is exactly one) [14]. See [10, 19] and [6] for recent approximation and parameterized complexity results for M-GO, respectively.

In this paper, we focus on the problems S-GO and MIN-D-GO, in which *all* the pairs in $\mathcal{P}$ must be satisfied. We study their complexity by considering some restrictions on the input instances (e.g., maximum degree of G or $|\mathcal{P}|$).

Our paper is organized as follows. We first formulate the S-GO and the MIN-D-GO problems in Section 2. Then, we show in Section 3 that for both problems we can always assume, without loss of generality, that G is a Mixed Acyclic Graph (MAG). In Section 4, we provide complexity results for S-GO. We study the complexity of MIN-D-GO in Section 5. Section 6 is the conclusion, together with several open questions.

2. Problems and Results

Throughout this paper, $G = (V, E, A)$ denotes a mixed graph without loops and with simple edges and arcs, where $V(G)$ (resp. $E(G), A(G)$) is the vertex set (resp. edge set, arc set) of G. An edge between two vertices u and v is noted by (u, v), and an arc from u to v is noted by uv. The *underlying* graph of G, denoted G^*, is a simple undirected graph defined as follows: $V(G^*) = V(G)$ and $E(G^*) = E(G) \cup \{(u, v) : uv \in A(G)\}$. Finally, $\Delta(G^*)$ is the maximum degree over all vertices in G^*.

A *path* P in $G = (V, E, A)$ from vertex v_1 to vertex v_m is a subgraph $P = v_1v_2 \ldots v_{m-1}v_m$ induced in G by the set of vertices $v_i \in V$, where all the v_is are pairwise distinct, and where for all $1 \leq i \leq m-1$, $(v_i, v_{i+1}) \in E$ or $v_iv_{i+1} \in A$. A *cycle* C in G is a path $v_1v_2 \ldots v_{m-1}v_m$ such that $v_1 = v_m$. A *circuit* in G is a special case of cycle $v_1v_2 \ldots v_{m-1}v_1$ where $v_iv_{i+1} \in A$ for all $1 \leq i \leq m-1$. A *Mixed Acyclic Graph* [19] (or *MAG*) is a mixed graph that contains no cycle (and therefore no circuit).

An *orientation* G' of G is a directed graph G' obtained from G by replacing each edge $(u, v) \in E$ by an arc uv, or an arc vu, or by uv and vu simultaneously. An edge (u, v) replaced by both arcs uv and vu is called a *doubly oriented* edge. Any orientation G' of G that contains no doubly oriented edge will be called a *simple orientation*. A pair of vertices $(u, v) \in V \times V$ is said to be *satisfied* by the orientation G' of G if there is a (directed) path from u to v in G'. Let $P = v_1v_2 \ldots v_{m-1}v_m$ be a path in G. In the following, we will often write *the orientation of P from v_1 towards v_m* to refer to the orientation that replaces every edge of the form (v_i, v_{i+1}), $1 \leq i \leq m-1$, by the arc v_iv_{i+1}.

Definition 1. *[2] Let $G = (V, E, A)$ be a mixed graph and let $\mathcal{P} \subseteq V \times V$ be a set of source-target pairs of vertices. The graph G is said to be $\mathcal{P}$-*connected *if for all $(u, v) \in \mathcal{P}$, there is a path in G from u to v.*

Definition 2. *[2] Let $G = (V, E, A)$ be a mixed graph and let $\mathcal{P} \subseteq V \times V$ s.t. G is $\mathcal{P}$-connected. A $\mathcal{P}$-*orientation *G' of G is a simple orientation of G that satisfies all pairs in $\mathcal{P}$.*

We call S-GO the problem of deciding whether a graph G admits a $\mathcal{P}$-orientation.

S-GO [12, 2] **Instance** : A mixed graph $G = (V, E, A)$ and $\mathcal{P} \subseteq V \times V$ s.t. G is $\mathcal{P}$-connected. **Question**: Does G admit a $\mathcal{P}$-orientation ?

Analogously to a $\mathcal{P}$-orientation, we define a $(\mathcal{P}, k)$-*D-orientation* as follows.

Definition 3. *Let $G = (V, E, A)$ be a mixed graph and let $\mathcal{P} \subseteq V \times V$ s.t. G is $\mathcal{P}$-connected. Let $k \geq 0$ be an integer. A $(\mathcal{P}, k)$-D-orientation G' of G satisfies the two following conditions: (i) G' is an orientation of G satisfying all the pairs in $\mathcal{P}$ and (ii) G' contains exactly k doubly oriented edges.*

We are now able to formulate the MIN-D-GO problem.

	$\Delta(G^*) = 2$	$\Delta(G^*) = 3$		
		$B = 2$	$B = 3$	B unbounded
S-GO	P [Cor. 2]	P [Th. 1]	NPC [Th. 4]	NPC [Th. 4]
MIN-D-GO	P [Th. 5]	APX-h [Th. 11]	NPC [Cor. 8], Non-approx. [Cor. 9]	NPC [Cor. 8], Non-approx. [Cor. 9], W[1]-h [Th. 10]

Table 1: Complexity of S-GO and MIN-D-GO when G is a MAG and G^* is a bounded degree graph. Recall that $B = \max\{n_i, 1 \leq i \leq |\mathcal{P}|\}$, where n_i is the number of distinct paths in G from s_i to t_i. Note also that the result provided in Theorem 1 remains valid even when $\Delta(G^*)$ is unbounded.

MIN-D-GO
Instance : A mixed graph $G = (V, E, A)$, $\mathcal{P} \subseteq V \times V$ s.t. G is $\mathcal{P}$-connected.
Question: Find a $(\mathcal{P}, k)$-D-orientation of G that minimizes k.

In this paper, we investigate MIN-D-GO (Section 4) and S-GO (Section 5), thus proposing several results that are summarized below. In order to simplify the approach, it is not only possible (as shown in the next section) but also suitable to assume that for any instance $(G, \mathcal{P})$ of MIN-D-GO (resp. S-GO), $G = (V, E, A)$ is a MAG and G is $\mathcal{P}$-connected. Note that we can also assume G^* to be connected. Otherwise, we can consider separately each graph $G_1, G_2, \ldots, G_r$ induced, in G, by the vertices of the connected components of G^*.

Therefore, let $G = (V, E, A)$ be a MAG and $\mathcal{P} = \{(s_i, t_i) \in V \times V : 1 \leq i \leq m\}$ be a set of pairs of vertices. For each i, $1 \leq i \leq m$, we denote by n_i the number of distinct paths in G from s_i to t_i. Throughout this paper, the integer B is defined as $B = \max\{n_i : 1 \leq i \leq m\}$. As it will be seen, B defines the border between easy and difficult instances of S-GO. As a consequence, we study the complexity of S-GO and MIN-D-GO by considering different constraints on the three following parameters: $\Delta(G^*)$, B and $|\mathcal{P}|$.

Remark that when $B = 1$ we can easily solve the S-GO and MIN-D-GO problems. Indeed, G is $\mathcal{P}$-connected and G^* is connected, thus if in addition we have $B = 1$, then for each pair $(s_i, t_i) \in \mathcal{P}$ there is a unique path P_i in G from s_i to t_i. Consequently, in order to satisfy the pair (s_i, t_i) we must orient P_i from s_i towards t_i. Then, we orient each remaining edge (u, v) in G following an arbitrarily direction. Obviously, in this orientation we create a minimum number of doubly oriented edges, and thus MIN-D-GO is optimally solved. If there is no doubly oriented edge at the end of the process, then we obtain a $\mathcal{P}$-orientation of G. Otherwise, G has no $\mathcal{P}$-orientation. The case $\Delta(G^*) = 1$ is obvious too.

The complexity results, when $B \geq 2$ and $\Delta(G^*) \geq 2$, are summarized in Tables 1 and 2. Interestingly, Table 1 shows that parameter B defines the border between easy ($B = 2$) and difficult ($B = 3$) instances of S-GO, even when G^* is of small maximum degree ($\Delta(G^*) = 3$). We also note that the parameter $\Delta(G^*)$ defines the border between easy ($\Delta(G^*) = 2$) and difficult ($\Delta(G^*) = 3$) instances of MIN-D-GO, even when $B = 2$. In Table 2, we show the complexity of both problems considering different values of $|\mathcal{P}|$. It should also be noted that, in the particular case where $\mathcal{P}$ is the set of all possible pairs of vertices of G (noted as $|\mathcal{P}| = n^2$ in Table 2 for simplicity), S-GO and MIN-D-GO are in P. These results come from Boesch and Tindell [4], who proved that a mixed graph G is strongly connected iff G^* has no bridge. Concerning MIN-D-GO, it can be easily seen that since G is, by hypothesis, $\mathcal{P}$-connected, then each bridge in G^* must be an (undirected) edge, and consequently the sought minimum number of doubly oriented edges is exactly the number of bridges in G^*.

	$\lvert\mathcal{P}\rvert \leq 2$	$\lvert\mathcal{P}\rvert \geq 3$ (and $\lvert\mathcal{P}\rvert = \mathcal{O}(1)$)		$\lvert\mathcal{P}\rvert$ unbounded	$\lvert\mathcal{P}\rvert = n^2$
		$B = \mathcal{O}(1)$	B unbounded	$B \geq 3$	
S-GO	P [2]	P$^{(*)}$ [Th. 3]	P [5]	NPC [2]	P [4]
MIN-D-GO	P [Th. 6]	P$^{(*)}$ [Th. 7]	Open	Non-approx. [Cor. 9]	P [4]

Table 2: Complexity of S-GO and MIN-D-GO for different values of $|\mathcal{P}|$ when G is a MAG. (*) The problem is FPT parameterized by B and $|\mathcal{P}|$.

3. Reduction to Mixed Acyclic Graphs

It has been shown in [19] that starting with any instance $(G_1, \mathcal{P}_1)$ of the M-GO problem (defined in Section 1), one can construct an equivalent instance $(G_2, \mathcal{P}_2)$ s.t. G_2 is a MAG.

Property 1. *[19] Let $G_1 = (V_1, E_1, A_1)$ be a mixed graph and let $\mathcal{P}_1 \subseteq V_1 \times V_1$. One can construct a MAG $G_2 = (V_2, E_2, A_2)$ and a set $\mathcal{P}_2 \subseteq V_2 \times V_2$ with $|\mathcal{P}_2| = |\mathcal{P}_1|$ s.t. for every integer $r \geq 0$, there exists a simple orientation of G_1 satisfying r pairs in $\mathcal{P}_1$ iff there exists a simple orientation of G_2 satisfying r pairs in $\mathcal{P}_2$.*

Applying Property 1 with $r = |\mathcal{P}_1|$, we obtain that G_1 admits a $\mathcal{P}_1$-orientation iff G_2 admits a $\mathcal{P}_2$-orientation. Hence, in the S-GO problem, wlog we can always consider the input mixed graph to be a MAG.

We say that two instances $(G_1, \mathcal{P}_1)$ and $(G_2, \mathcal{P}_2)$ of MIN-D-GO are *equivalent* if there exists an optimal solution of $(G_1, \mathcal{P}_1)$ that creates k doubly oriented edges iff there exists an optimal solution of $(G_2, \mathcal{P}_2)$ that creates k doubly oriented edges.

We will show that, in the MIN-D-GO problem, wlog we can always consider the input mixed graph to be a MAG. We first show in Property 2 that, for any $\mathcal{P}$-connected mixed graph G and for any cycle C in G, we can find an optimal solution for MIN-D-GO, with inputs G and $\mathcal{P}$, in which C becomes a circuit. We then show in Property 4 that the vertices of any circuit can be contracted into a single vertex without changing the nature of the problem. This proof closely follows the one in [19].

Property 2 (Orientation of cycles)**.** *Let $G = (V, E, A)$ be a mixed graph and let $\mathcal{P} \subseteq V \times V$. Assume that G is $\mathcal{P}$-connected. Let C be a cycle in G. Let G' be the mixed graph obtained from G by orienting all the edges in C in the same direction (such a direction is arbitrarily chosen when $A(C) = \emptyset$) to obtain a circuit C'. The instances $(G, \mathcal{P})$ and $(G', \mathcal{P})$, of the* MIN-D-GO *problem, are equivalent.*

Proof. We show the direct implication in the equivalence, the indirect one can be shown easily in the same way.

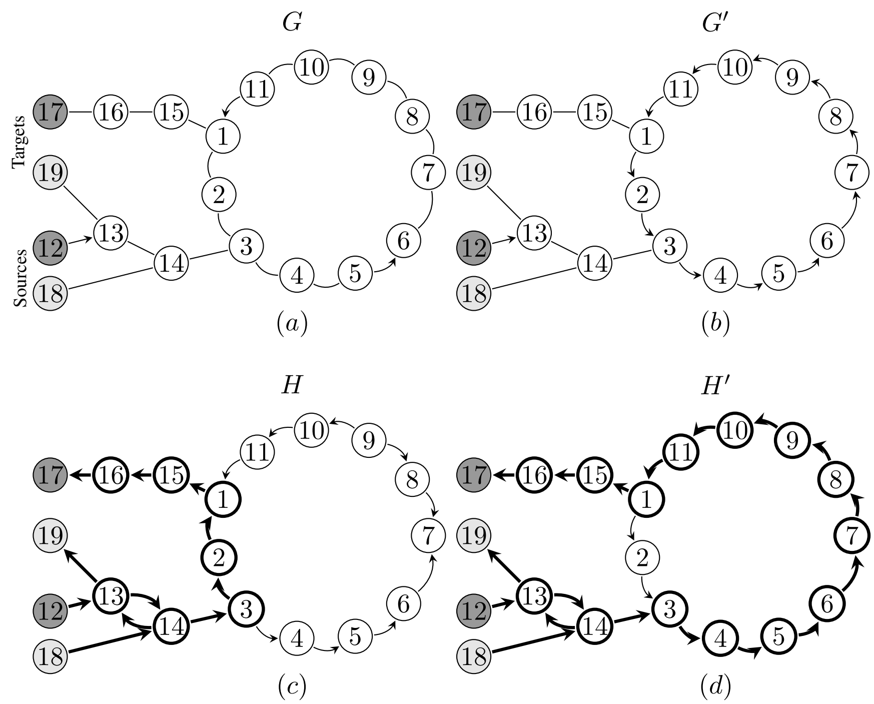

Figure 1: (a) $(G, \mathcal{P})$ is an instance of MIN-D-GO with $\mathcal{P} = \{(12, 17), (18, 19)\}$, C is a cycle in G with vertex set $\{1, 2, \ldots, 11\}$ (b) G' is a mixed graph obtained from G by orienting the edges of the cycle C in the same direction to obtain a circuit (c) H is an optimal solution of the instance $(G, \mathcal{P})$ that creates one doubly oriented edge. (d) H' is an optimal solution of the instance $(G', \mathcal{P})$ that creates also one doubly oriented edge. Gray vertices in H (resp. H') induce a path satisfying the pair $(12, 17)$.

First, since G is $\mathcal{P}$-connected, there must exist an optimal solution H for the instance $(G, \mathcal{P})$. Let H' be an orientation of G' s.t. the edges in $E(G') \setminus E(C)$ are replaced by the same arcs as in H (see Figure 1 for an illustration). We now show that H' is an optimal solution for the instance $(G', \mathcal{P})$ that has the same number of doubly oriented edges as H. Let $(u, v) \in \mathcal{P}$. If $u, v \in V(C)$ then obviously the pair (u, v) is satisfied in H' by a path in C'. If $u \notin V(C)$ or $v \notin V(C)$, then let us consider a path $P = a_1 a_2 \ldots a_m$ in H, from u to v ($u = a_1$ and $v = a_m$), that satisfies the pair (u, v). Let $x = \min\{i : a_i \in V(C)\}$ and let $y = \max\{i : a_i \in V(C)\}$. Then the pair (u, v) is satisfied in H' by the path $a_1 a_2 \ldots a_{x-1} Q a_{y+1} a_{y+2} \ldots a_m$ s.t. Q is the subpath in C' going from a_x to a_y (see for example Figure 1, in which $a_x = 3$ and $a_y = 1$). Let k (resp. k') denote the number of doubly oriented edges in H (resp. in H'). Clearly $k' \leq k$, because the circuit C' contains no doubly oriented edge and $A(H') \setminus A(C') \subset A(H)$. Now, because any orientation of G' is also an orientation of G, the optimality of k implies that $k' = k$ and that H' is also an optimal solution of the instance $(G', \mathcal{P})$. $\square$

Let $G = (V, E, A)$ be a mixed graph and let $\mathcal{P} \subseteq V \times V$. Let $G_1 = (V_1, E_1, A_1)$ be the mixed graph obtained from G by the following procedure.

1. $G_1 := G$.
2. While (there is a cycle C in G_1) do
   - Replace the edges in C by arcs s.t. C becomes a circuit.
   - Call G_1 the resulting graph.

According to Property 2, $(G, \mathcal{P})$ and $(G_1, \mathcal{P})$ are equivalent for all graphs G built during the procedure, including the final one. Let also $\mathcal{P}_1$ denote the set obtained from $\mathcal{P}$ by removing each pair $(u, v) \in \mathcal{P}$ s.t. there is a directed path in G_1 from u to v. In that case, the instance $(G_1, \mathcal{P}_1)$ of MIN-D-GO obtained from $(G, \mathcal{P})$ will be called a *reduced instance*. Clearly, $(G_1, \mathcal{P}_1)$ and $(G_1, \mathcal{P})$ are equivalent, and thus the following property holds.

Property 3 (Reduced instances)**.** *Let $(G_1, \mathcal{P}_1)$ be a reduced instance of* MIN-D-GO *obtained from instance $(G, \mathcal{P})$. Then $(G, \mathcal{P})$ and $(G_1, \mathcal{P}_1)$ are equivalent.*

Property 4 (Contraction of circuits)**.** *Let $(G_1, \mathcal{P}_1)$ be a reduced instance of* MIN-D-GO*, and let C' be a circuit in G_1. Let $(G_2, \mathcal{P}_2)$ be the instance of* MIN-D-GO *defined as follows: (i) $\mathcal{P}_2 = \mathcal{P}_1$ and (ii) G_2 is the graph obtained from G_1 by contracting the vertices of C' into a single vertex x_0. Then, $(G_1, \mathcal{P}_1)$ and $(G_2, \mathcal{P}_2)$ are equivalent.*

Proof. Suppose that $G_1 = (V_1, E_1, A_1)$ and $G_2 = (V_2, E_2, A_2)$. Let us show that the two instances $(G_1, \mathcal{P}_1)$ and $(G_2, \mathcal{P}_2)$ are equivalent. Let $G_1' = (V_1, A_1')$ be an optimal solution of the instance $(G_1, \mathcal{P}_1)$ that creates k_1 doubly oriented edges. We construct an orientation G_2' of G_2 as follows. Let $(u, v) \in E_2$. If $u \neq x_0$ and $v \neq x_0$, then (u, v) is oriented in G_2' as in G_1'. If $u = x_0$ (or similarly when $v = x_0$), then there is a vertex $w \in V(C')$ s.t. $(w, v) \in E_1$. If $wv \in A_1'$ (resp. $vw \in A_1'$) we replace in G_2 the edge (x_0, v) by the arc x_0v (resp. vx_0). Let k_2 denote the number of doubly oriented edges in G_2'. Clearly, $k_1 = k_2$, because the circuit C' contains no doubly oriented edge. Let $(u, v) \in \mathcal{P}_1$ and let $P_1' = a_1 a_2 \dots a_m$ be a directed path in G_1' from u to v ($u = a_1$ and $v = a_m$) satisfying the pair (u, v). Let $\alpha = \min\{i : a_i \in V(C')\}$ and let $\beta = \max\{i : a_i \in V(C')\}$. Then the pair (u, v) is satisfied in G_2' by the path $a_1 a_2 \dots a_{\alpha-1} x_0 a_{\beta+1} \dots a_m$. Conversely, starting with any optimal solution of the instance $(G_2, \mathcal{P}_2)$, in the same way, one can construct an optimal solution of the instance $(G_1, \mathcal{P}_1)$ that creates the same number of doubly oriented edges. Hence, the property follows. $\square$

Now, using the previous properties, we are able to show that for MIN-D-GO we may, wlog, assume that the input mixed graph is a MAG.

Property 5 (Reduction to a MAG)**.** *Let $(G, \mathcal{P})$ be an instance of the* MIN-D-GO *problem. Then there exists an equivalent instance $(G_M, \mathcal{P}_M)$ of* MIN-D-GO *s.t. G_M is a MAG.*

Proof. We construct the graph G_M and the set $\mathcal{P}_M$ by applying the following process:

1. Construct the reduced instance $(G_1, \mathcal{P}_1)$ obtained from $(G, \mathcal{P})$.

2. Construct the graph G_2, obtained by contracting in G_1, every circuit into a single vertex.
3. If G_2 is a MAG then set $G_M = G_2$ and $\mathcal{P}_M = \mathcal{P}_1$. Otherwise, set $G = G_2$ and $\mathcal{P} = \mathcal{P}_1$, and return to step 1.

Properties 3 and 4 ensure that $(G_M, \mathcal{P}_M)$ is equivalent to $(G, \mathcal{P})$, which proves the property. $\square$

Therefore, we will always assume, in the remainder of the paper, that for any instance $(G, \mathcal{P})$ of MIN-D-GO (resp. S-GO), $G = (V, E, A)$ is a MAG and G is $\mathcal{P}$-connected. Recall that we can also assume G^* to be connected.

4. Complexity of the S-GO problem

In this section, we investigate the complexity of the S-GO problem for MAGs with bounded $\Delta(G^*)$ and/or bounded B (see Table 1), and for bounded $|\mathcal{P}|$ (see Table 2).

4.1. Easy cases

Theorem 1. *The* S-GO *problem is polynomial-time solvable when G is a MAG and $B = 2$.*

Proof. For each pair $(s_i, t_i) \in \mathcal{P}$ there are at most two paths from s_i to t_i in G, and such paths can be computed in polynomial time [11].

If for a pair $(s_i, t_i) \in P$, there is only one path from s_i to t_i, then we orient it from s_i towards t_i and we remove the pair (s_i, t_i) from the set $\mathcal{P}$. We continue this process until (1) G is no longer $\mathcal{P}$-connected or (2) $\mathcal{P} = \emptyset$ or (3) for each pair $(s_i, t_i) \in \mathcal{P}$ there are exactly two paths from s_i to t_i. The first case implies that G has no $\mathcal{P}$-orientation. In the second case we arbitrarily orient each edge, in the resulting graph, in a unique direction to obtain a $\mathcal{P}$-orientation. Finally, in the last case we have an instance of the S-GO problem in which there are in G exactly two paths from s_i to t_i for all $(s_i, t_i) \in \mathcal{P}$.

We denote by X_{i1} and X_{i2} the two paths in G from s_i to t_i. Given i, j in the set $\{1, 2, \ldots |\mathcal{P}|\}$, $i \neq j$, and $a, b \in \{1, 2\}$, we say that the two paths X_{ia} and X_{jb} are in *conflict* if orienting X_{ia} from s_i towards t_i and X_{jb} from s_j towards t_j, creates a doubly oriented edge. Now, we construct an instance $(\mathcal{X}, \mathcal{C})$ of the problem 2-SAT as follows. Let $\mathcal{X} = \{x_{i1}, x_{i2} \; : \; 1 \leq i \leq |\mathcal{P}|\}$ be the variable set. For all $i \in \{1, 2, \ldots, |\mathcal{P}|\}$, we add the clause $c_i = (x_{i1} \vee x_{i2})$. For all $i, j \in \{1, 2, \ldots, |\mathcal{P}|\}$, $i \neq j$, and $a, b \in \{1, 2\}$, we add the clause $(\overline{x_{ia}} \vee \overline{x_{jb}})$ if paths X_{ia} and X_{jb} are in conflict.

Let us show that there is an assignment of the variables in $\mathcal{X}$ that satisfies all the clauses in $\mathcal{C}$ iff G has a $\mathcal{P}$-orientation.

Consider a truth assignment of clauses in $\mathcal{C}$ and let x_{ih_i}, $1 \leq h_i \leq 2$, be a true literal of clause c_i, $1 \leq i \leq |\mathcal{P}|$. We orient in G the path X_{ih_i} from s_i towards t_i, for all i, $1 \leq i \leq |\mathcal{P}|$. This orientation cannot create any doubly oriented edges. Otherwise, there are $i, j \in \{1, 2, \ldots |\mathcal{P}|\}$, $i \neq j$ such that the paths X_{ih_i} and X_{jh_j} are in conflict, implying that the clause $(\overline{x_{ih_i}} \vee \overline{x_{jh_j}})$ is unsatisfied. To complete the orientation of G, we orient arbitrarily the remaining edges in G without creating any doubly oriented edge.

Next we show the reverse implication. We consider the set $\{Y_{1h_1}, Y_{2h_2}, \ldots, Y_{|\mathcal{P}|h_{|\mathcal{P}|}}\}$ s.t. Y_{ih_i} is a directed path from s_i to t_i, in a $\mathcal{P}$-orientation of G. Each path Y_{ih_i} is the orientation (from the source towards the target vertex) of a mixed path $X_{ih_i} = G[V(Y_{ih_i})]$, $h_i \in \{1,2\}$, for all $1 \leq i \leq |\mathcal{P}|$. We set to $true$ the variable set $\{x_{ih_i} : 1 \leq i \leq |\mathcal{P}|\}$ and we set to $false$ the remaining variables. Obviously, this assignment satisfies the clause set $\{c_i : 1 \leq i \leq |\mathcal{P}|\}$. For the sake of contradiction, assume now that some clause $(\overline{x_{ia}} \vee \overline{x_{jb}})$ is not satisfied. Then $x_{ia} = true$ and $x_{jb} = true$. Consequently, in the resulting $\mathcal{P}$-orientation of G, the path X_{ia} (resp. X_{jb}) is oriented from s_i towards t_i (resp. from s_j towards t_j). This leads to a a contradiction, because a $\mathcal{P}$-orientation cannot use, simultaneously, two paths that are in conflict.

As the problem 2-SAT is polynomial-time solvable [3], we deduce that one can solve in polynomial-time the S-GO problem when graph G is a MAG s.t. there are in G at most two paths from s_i to t_i, for all $(s_i, t_i) \in \mathcal{P}$. □

Corollary 2. *The* S-GO *problem is polynomial-time solvable when G is a MAG and $\Delta(G^*) = 2$.*

Proof. The graph G^* is connected. Thus when $\Delta(G^*) = 2$, the graph G^* must be a path or a cycle, and consequently $B \leq 2$. If $B = 1$ the S-GO problem is trivial. If $B = 2$, we deduce from the previous result (Theorem 1) that the S-GO problem is polynomial-time solvable. □

Now, we first show that the S-GO problem is FPT parameterized by B and $|\mathcal{P}|$ when G is a MAG. Then, one can deduce that the problem is polynomial-time solvable when both parameters B and $|\mathcal{P}|$ are bounded.

Theorem 3. *The* S-GO *problem is* FPT *parameterized by B and $|\mathcal{P}|$ when G is a MAG.*

Proof. Let $G = (V, E, A)$ with $n = |V|$, and let $\mathcal{P} = \{(s_i, t_i), 1 \leq i \leq m\}$. Recall that for each pair $(s_i, t_i) \in \mathcal{P}$ there are at most B paths from s_i to t_i in G. Let χ_i denote the set of paths in G going from s_i to t_i. One can compute the set χ_i in polynomial time. Indeed, we first create a directed graph G' from G by replacing each edge $(u, v) \in E$ by the two arcs uv and vu, and then we compute the set χ'_i of the B shortest simple paths in G' from s_i to t_i (this can be done in $\mathcal{O}(B(n(|E| + |A|) + n^2 \log\log n))$ time [11]). Finally, the set χ_i is obtained from χ'_i by replacing, in each directed path $P' \in \chi'_i$, any arc $uv \in A(P') \backslash A$ by the edge (u, v).

Moreover, to satisfy the pair (s_i, t_i) we must choose a path $P_i \in \chi_i$ and orient its edges to create a directed path from s_i to t_i. The orientation of P_i can be done in $\mathcal{O}(n + |E| + |A|)$ time. Since $|\chi_i| \leq B$, we can consider all the possible combinations (we choose one path per pair) in $\mathcal{O}(B^m)$ time. Then we orient (from the source towards the target vertex) the m obtained paths in $\mathcal{O}(m(n + |E| + |A|))$ time. Thus the choice of paths and their orientations can be done in $\mathcal{O}(B^m \times m(n + |E| + |A|))$ time. If at least one of these orientations has no doubly oriented edge, then such an orientation can be completed, by orienting each remaining edge in an arbitrary single direction, to obtain a $\mathcal{P}$-orientation of G. Otherwise, G has no $\mathcal{P}$-orientation. Thus the running time of our algorithm is $f(B, |\mathcal{P}|) \cdot n^{\mathcal{O}(1)}$, where f is a function depending only on B

and $|\mathcal{P}|$. Hence, the S-GO problem is FPT parameterized by B and $|\mathcal{P}|$ when G is a MAG. $\square$

4.2. Difficult cases

We showed in Theorem 1 that the S-GO problem is easy when $B = 2$. However, in the following theorem, we show that when $B = 3$ the problem S-GO becomes difficult.

Theorem 4. *The* S-GO *problem is* NP*-complete even when the graph G is a MAG, $\Delta(G^*) = 3$ and $B = 3$.*

Proof. Arkin and Hassin [2] provided an NP-completeness proof for the S-GO problem on general MAGs. Their proof is based on a reduction from the Satisfiability problem (SAT). Here, we modify the MAG G constructed from their reduction to ensure that $\Delta(G^*) = 3$. Following these motivations, we perform a reduction from the NP-complete problem 3-SAT-4 [20]: Given a collection $\mathcal{C}_m = \{c_1, \ldots c_m\}$ of m clauses, where each clause consists of a set of three literals over a finite set of n boolean variables $\mathcal{V}_n = \{x_1, \ldots x_n\}$ s.t. each variable appears in at most four clauses, is there a truth assignment of the variable in $\mathcal{V}_n$ that satisfies all the clauses in $\mathcal{C}_m$?

Let $(\mathcal{C}_m, \mathcal{V}_n)$ be an instance of 3-SAT-4. For all j, $1 \leq j \leq n$, the variable x_j satisfies the following condition: (1) x_j and $\overline{x_j}$ appear at most in four clauses. In addition, one may assume wlog that (2) for each variable x_j, there is at least one clause that contains x_j and at least one clause that contains $\overline{x_j}$. Otherwise, the variable x_j can be arbitrarily fixed to *true* or *false*. Now, let us construct an instance $(G, \mathcal{P})$ of the S-GO problem (see Figure 2 for an illustration). For each clause c_i, we create two vertices s_i and t_i, $1 \leq i \leq m$. For each variable x_j, we create these 14 vertices: $\{u_j, v_j\} \cup \{a_{jk}, b_{jk}, a'_{jk}, b'_{jk}\}_{1 \leq k \leq 3}$. Then, we add an edge (u_j, v_j) and the four following directed paths: $a_{j1}a_{j2}a_{j3}u_j$, $v_jb_{j3}b_{j2}b_{j1}$, $a'_{j1}a'_{j2}a'_{j3}v_j$ and finally $u_jb'_{j3}b'_{j2}b'_{j1}$, for all $1 \leq j \leq n$. For each variable x_j, there are k_j clauses containing x_j and k'_j clauses containing $\overline{x_j}$ s.t. $1 \leq k_j \leq 3$, $1 \leq k'_j \leq 3$ and $k_j + k'_j \leq 4$. Let $\{c_{i_1}, c_{i_2}, \ldots, c_{i_{k_j}}\}$ (resp. $\{c_{i'_1}, c_{i'_2}, \ldots, c_{i'_{k'_j}}\}$) be the set of clauses that contain x_j (resp. $\overline{x_j}$). We add an arc $s_{i_\alpha}a_{j\alpha}$ and an arc $b_{j\alpha}t_{i_\alpha}$, for all $\alpha \in \{1, 2, \ldots k_j\}$. Also, we add an arc $s_{i'_\beta}a'_{j\beta}$ and an arc $b'_{j\beta}t_{i'_\beta}$, for all $\beta \in \{1, 2, \ldots k'_j\}$. To finish our construction, we set $\mathcal{P} = \{(s_i, t_i), 1 \leq i \leq m\}$.

According to conditions (1) and (2), one can easily show that $\Delta(G^*) = 3$. In addition, for each pair (s_i, t_i) there are exactly three paths in G from s_i to t_i, because each clause in $\mathcal{C}_m$ contains exactly three literals. Thus $B = 3$.

We claim that there is an assignment satisfying all the clauses in $\mathcal{C}_m$ if and only if there exists a $\mathcal{P}$-orientation of G. Indeed, consider an assignment satisfying all the clauses in $\mathcal{C}_m$. Similar to the proof presented in [2], if $x_j = true$ (resp. $x_j = false$) then we orient the edge (u_j, v_j) from u_j to v_j (resp. from v_j to u_j). Let l_i be a true literal of clause c_i. Then, there is a variable x_j s.t. $l_i = x_j$ or $l_i = \overline{x_j}$. If $l_i = x_j$ (resp. $l_i = \overline{x_j}$) then there is an integer k_i, $1 \leq k_i \leq 3$, such that $s_ia_{jk_i}, b_{jk_i}t_i \in A(G)$ (resp. $s_ia'_{jk_i}, b'_{jk_i}t_i \in A(G)$). Thus, the pair (s_i, t_i) is satisfied by the path $s_ia_{jk_i}a_{j(k_i+1)} \ldots u_jv_jb_{j3} \ldots b_{jk_i}t_i$ (resp. $s_ia'_{jk_i}a'_{j(k_i+1)} \ldots v_ju_jb'_{j3} \ldots b'_{jk_i}t_i$).

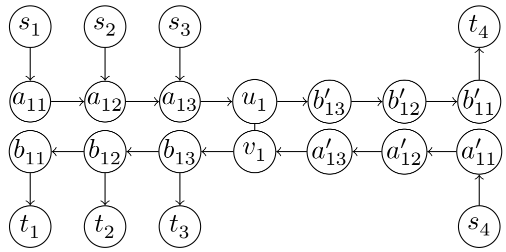

Figure 2: Construction of an instance $(G, \mathcal{P})$ of the S-GO problem, from an instance of 3-SAT (called 3-SAT-4) in which each variable appears at most in four clauses. Here, the variable set is $\mathcal{V} = \{x_j, 1 \le j \le 6\}$ and the clause set is $\mathcal{C} = \{c_i, 1 \le i \le 4\}$ s.t. $c_1 = (x_1 \vee \overline{x_2} \vee x_3), c_2 = (x_1 \vee x_4 \vee x_5), c_3 = (x_1 \vee \overline{x_4} \vee \overline{x_6})$ and $c_4 = (\overline{x_1} \vee \overline{x_5} \vee x_6)$. The set vertex pairs is $\mathcal{P} = \{(s_i, t_i), 1 \le i \le 4\}$. In this figure, we show only the subgraph corresponding to variable x_1.

Now, let us prove the reverse implication. Given a $\mathcal{P}$-orientation G' of G, we set the variable x_j to $true$ (resp. to $false$) if the arc $u_j v_j \in A(G')$ (resp. $v_j u_j \in A(G')$). Let c_i be a clause in $\mathcal{C}_m$. Then the pair (s_i, t_i) is satisfied by a directed path P in G', from s_i to t_i, going through an arc $u_j v_j$ or an arc $v_j u_j$. If P contains the arc $u_j v_j$ then the clause c_i must contain the literal x_j and thus c_i is satisfied. If P contains the arc $v_j u_j$ (consequently $x_j = false$) then the clause c_i must contain the literal $\overline{x_j}$ and thus c_i is also satisfied. □

5. Complexity of MIN-D-GO

The MIN-D-GO problem can be seen as a variant of the S-GO problem, investigated in the previous section (Section 4), in which we allow some edges to be doubly oriented. Hence, each $\mathcal{P}$-orientation of G is a solution of MIN-D-GO. However, if there is no $\mathcal{P}$-orientation of G, then we conclude just that at least one edge must be doubly oriented in a solution of MIN-D-GO, but in general that gives no information about the number of edges to be doubly oriented to solve the MIN-D-GO problem.

In this section, we study the complexity of MIN-D-GO when the input graph is a MAG (see Table 1 and Table 2). As in the previous section, we suppose that G is a $\mathcal{P}$-connected MAG. In the following, D-GO denotes the decision version of the minimization problem MIN-D-GO.

5.1. Easy cases

We first show that when G^* is a cycle (and thus $\Delta(G^*) = 2$ and $B \le 2$), the MIN-D-GO problem is polynomial-time solvable. This result is interesting since it allows to define a border (with respect to the parameter $\Delta(G^*)$) between the easy and the difficult instances of the MIN-D-GO problem. Indeed, in the next subsection (Corollaries 8 and 9), we will show that MIN-D-GO is difficult even when $\Delta(G^*) = 3$.

Let G be a MAG such that G^* is a cycle. Let $\mathcal{P}' \subseteq \mathcal{P}$ be the set of pairs for which there is a unique path from the source to the target vertex in G. Let also $\mathcal{P}''$ denote the set of pairs for which there are exactly two paths from the source to the

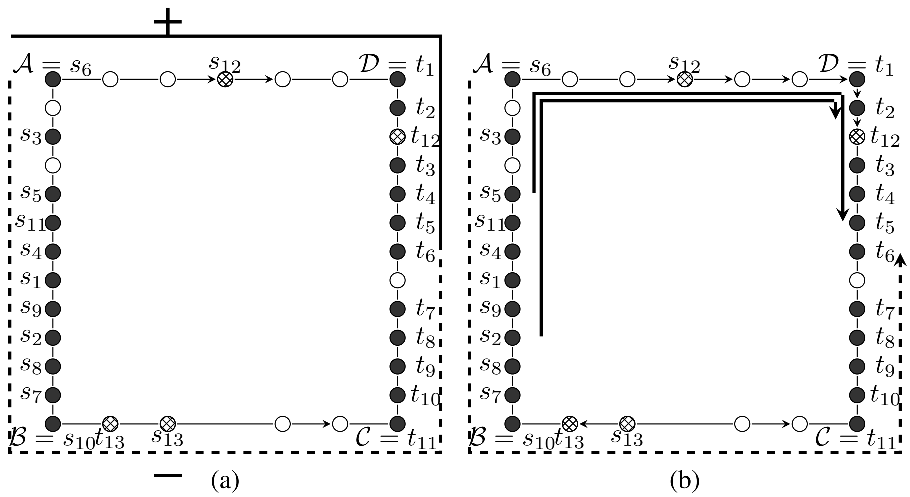

Figure 3: Problem MIN-D-GO when G^* is a cycle. (a) The set of hatched (resp. full) vertices is the set $\mathcal{P}'$ (resp. $\mathcal{P}''$) of pairs (s, t) s.t. G contains exactly one (resp. two) path(s) from s to t. The rectangle $\mathcal{ABCD}$ is the $\mathcal{P}''$-representation of G. The full path (resp. dotted path) is the path $G[s_6, t_6]^+$ (resp. $G[s_6, t_6]^-$) (b) The graph G' obtained from G after orienting the unique path from s to t for all $(s, t) \in \mathcal{P}'$. Orienting the three subgraphs $G'[s_6, t_6]^-$ (dotted line), $G'[s_5, t_5]^+$ and $G'[s_2, t_2]^+$, from the source towards the target vertex, results a feasible orientation.

target vertex in G. Let $sr(\mathcal{P}'')$ denote the set of source vertices in $\mathcal{P}''$, i.e., $sr(\mathcal{P}'') = \{s : \exists t \text{ s.t. } (s, t) \in \mathcal{P}''\}$. Similarly, we denote by $tr(\mathcal{P}'')$ the set of the target vertices in $\mathcal{P}''$. Let $m' = |\mathcal{P}'|$ and $m'' = |\mathcal{P}''|$. In the following we consider that $m'' \geq 2$ (when $m'' = 1$, the problem is trivial).

Property 6. *Let $(G, \mathcal{P})$ be an instance of* MIN-D-GO *such that G^* is a cycle. Then G is necessarily composed of the following four subgraphs:*

1. *An undirected path $P_1 = s_{i_1} X_1 s_{i_2} X_2 \ldots X_{(m''-1)} s_{i_{m''}}$ s.t. $\{s_{i_1}, s_{i_2}, \ldots, s_{i_{m''}}\} = sr(\mathcal{P}'')$ and for all k, $1 \leq k < m''$, X_k is an undirected path in G;*
2. *An undirected path $P_2 = t_{j_1} Y_1 t_{j_2} Y_2 \ldots Y_{(m''-1)} t_{j_{m''}}$ s.t. $\{t_{j_1}, t_{j_2}, \ldots, t_{j_{m''}}\} = tr(\mathcal{P}'')$ and for all k, $1 \leq k < m''$, Y_k is an undirected path in G;*
3. *A path P_3 going from s_{i_1} to t_{j_1}, with $A(P_3) \neq \emptyset$;*
4. *A path P_4 going from $s_{i_{m''}}$ to $t_{j_{m''}}$, with $A(P_4) \neq \emptyset$.*

We may also consider wlog that $j_1 = 1, j_2 = 2, \ldots, j_{m''} = m''$ (see Fig. 3 (a) in which $m' = 2$, $m'' = 11$, $i_1 = 6$ and $i_{11} = 10$).

Proof. Let $(s_1, t_1) \in \mathcal{P}''$. Let $p_1 = u_1 u_2 \ldots u_{k_1}$ and $p_2 = v_1 v_2 \ldots v_{k_2}$ denote the two distinct paths in G from s_1 to t_1. The graph G^* is a cycle, thus G is formed by the two paths p_1 and p_2, i.e., $V(G) = V(p_1) \cup V(p_2)$, $E(G) = E(p_1) \cup E(p_2)$ and $A(G) = A(p_1) \cup A(p_2)$. Since G is a MAG, there exists an integer $i \in \{1, \ldots, k_1 - 1\}$

and an integer $j \in \{1, \ldots, k_2 - 1\}$ s.t. $u_i u_{i+1} \in A(p_1)$ and $v_j v_{j+1} \in A(p_2)$. Let α_1 (resp. β_1) be the minimum (resp. maximum) integer for which $u_{\alpha_1} u_{\alpha_1+1} \in A(p_1)$ (resp. $u_{\beta_1} u_{\beta_1+1} \in A(p_1)$). Let also α_2 (resp. β_2) be the minimum (resp. maximum) integer for which $v_{\alpha_2} v_{\alpha_2+1} \in A(p_2)$ (resp. $v_{\beta_2} v_{\beta_2+1} \in A(p_2)$). Every vertex $s_i \in sr(\mathcal{P}'')$ must belong to the undirected path X in G going from u_{α_1} to v_{α_2}. Otherwise, due to the two arcs $u_{\alpha_1} u_{\alpha_1+1}$ and $v_{\alpha_2} v_{\alpha_2+1}$, the graph G could not contain two paths from s_i to t_i. Similarly, every vertex $t_i \in tr(\mathcal{P}'')$ must belong to the undirected path Y in G going from u_{β_1+1} to v_{β_2+1}.

Suppose that the path X (resp. Y) is written $X = a_1 a_2 \ldots a_{|X|}$ (resp. $Y = b_1 b_2 \ldots b_{|Y|}$) s.t. $a_1 = u_{\alpha_1}$ and $a_{|X|} = v_{\alpha_2}$ (resp. $b_1 = u_{\beta_1+1}$ and $b_{|X|} = v_{\beta_2+1}$). Let q_1 (resp. r_1) be the minimum (resp. maximum) integer s.t. $a_{q_1} \in V(X) \cap sr(\mathcal{P}'')$ (resp. $a_{r_1} \in V(X) \cap sr(\mathcal{P}'')$). Similarly, let q_2 (resp. r_2) be the minimum (resp. maximum) integer s.t. $b_{q_2} \in V(Y) \cap tr(\mathcal{P}'')$ (resp. $b_{r_2} \in V(X) \cap tr(\mathcal{P}'')$).

Let P_1 (resp. P_2) be the subpath of X (resp. Y) going from a_{q_1} (resp. b_{q_2}) to a_{r_1} (resp. b_{r_2}). The graph G is thus composed of (1) the path P_1 whose vertices in $sr(\mathcal{P}'')$ may be numbered from s_{i_1} to $s_{i_{m''}}$ s.t. $sr(\mathcal{P}'') = \{s_{i_1}, s_{i_2}, \ldots, s_{i_{m''}}\}$, (2) the path P_2 whose vertices in $tr(\mathcal{P}'')$ may be numbered from t_{j_1} to $t_{j_{m''}}$ s.t. $tr(\mathcal{P}'') = \{t_{j_1}, t_{j_2}, \ldots, t_{j_{m''}}\}$, (3) the path P_3 going from s_{i_1} to t_{j_1} and (4) the path P_4 going from $s_{i_{m''}}$ to $t_{j_{m''}}$. $\square$

Property 6 leads us to the following definition:

Definition 4. *Let G be a MAG such that G^* is a cycle, and let $\mathcal{P}$ be the set of (source, target)-pairs of vertices. The $\mathcal{P}''$-*representation *of G is a plane representation of G in the form of a rectangle $\mathcal{ABCD}$ with vertical segments $\mathcal{AB}$ and $\mathcal{DC}$ (from top to bottom), s.t. the pairs in $\mathcal{P}''$ are renumbered in such a way that:*

(i) *the source vertices in $\mathcal{P}''$ lie on the path $\mathcal{AB}$ of G^* in the order $\mathcal{A} = s_{i_1}, s_{i_2}, \ldots, s_{i_{m''}} = \mathcal{B}$.*

(ii) *the target vertices in $\mathcal{P}''$ lie on the path $\mathcal{DC}$ of G^* in the order $\mathcal{D} = t_1, t_2, \ldots, t_{m''} = \mathcal{C}$.*

For each $(u, v) \in sr(\mathcal{P}'') \times tr(\mathcal{P}'')$, the graph G (and similarly any orientation of G) can be decomposed into two subgraphs. The first (resp. second) one is composed of the vertices that lie on the vertical segment $u\mathcal{A}$, the horizontal segment $\mathcal{AD}$ and the vertical segment $\mathcal{D}v$ (resp. $u\mathcal{B}$, $\mathcal{BC}$ and $\mathcal{C}v$). We denote by $G[u, v]^+$ (resp. $G[u, v]^-$) the first subgraph (resp. the second subgraph). See an illustration in Fig. 3.

For each pair $(s_i, t_i) \in \mathcal{P}'$, there is a unique path P_i from s_i to t_i in G. In order to satisfy all the pairs in $\mathcal{P}'$, we must orient P_i from s_i towards t_i. In the following, we denote by G' the graph obtained from G after all such paths have been oriented as described above.

Consider the $\mathcal{P}''$-representation of G, and let $(s_i, t_i) \in \mathcal{P}''$. Remark that although $G[s_i, t_i]^+$ and $G[s_i, t_i]^-$ are paths in G, the subgraphs $G'[s_i, t_i]^+$ and $G'[s_i, t_i]^-$ are not necessarily paths. However, in order to satisfy the pair (s_i, t_i), we must orient at least one of them to obtain a directed path from s_i to t_i. Suppose that $G'[s_i, t_i]^+ = u_1 u_2 \ldots u_{k_1}$ s.t. $u_1 = s_i$, $u_{k_1} = t_i$ and $(u_j, u_{j+1}) \in E(G')$ or $u_j u_{j+1} \in A(G')$

or $u_{j+1}u_j \in A(G')$, for all $j \in \{1, 2, \ldots, k_1 - 1\}$. We will write *the orientation of* $G'[s_i, t_i]^+$ (or similarly of $G'[s_i, t_i]^-$) *from* s_i *towards* t_i when we refer to the following orientation: if $(u_j, u_{j+1}) \in E(G')$ then (u_j, u_{j+1}) is replaced by the arc $u_j u_{j+1}$; if $u_{j+1}u_j \in A(G')$ and $u_j u_{j+1} \notin A(G')$ then we add, in G', the arc $u_j u_{j+1}$ (i.e., we create a doubly oriented edge). We call a *partial* orientation of G' a mixed graph obtained from G' by orienting, for some pair $(s_i, t_i) \in \mathcal{P}''$, the subgraph $G'[s_i, t_i]^-$ and/or $G'[s_i, t_i]^+$ from s_i towards t_i.

For the ease of presentation we consider that all the source pairs in $\mathcal{P}''$ are distinct, and also that all the target pairs in $\mathcal{P}''$ are distinct. Such an hypothesis can be made without loss of generality.

Let H be a partial orientation of G'. The graph H is called a *feasible orientation* if, for each $(s_i, t_i) \in \mathcal{P}$, there is a directed path from s_i to t_i in H. The graph H is called an *optimal orientation* if it solves the MIN-D-GO problem. A feasible orientation is given in Fig. 3 (b).

Based on the $\mathcal{P}''$-representation of G, we propose a simple polynomial time algorithm (Algorithm MIN-CYCLE-ORIENTATION (MCO)) that optimally solves the MIN-D-GO problem when G is a MAG and G^* is a cycle. In MCO, we orient a path "-" and a path "+", form the source towards the target vertex, in such a way that all the source vertices and all the target vertices in $\mathcal{P}''$ are covered. We vary the beginning and the end of each one of these paths considering all possible cases, and finally we keep the feasible orientation that creates a minimum number of doubly oriented edges.

Theorem 5. *Algorithm* MCO *solves the* MIN-D-GO *problem when G is a MAG and* G^* *is a cycle.*

Proof. Let G_{res} be the output of Algorithm MCO. Let G_{opt} be an optimal orientation of G. We denote by n_{res} (resp. n_{opt}) the number of doubly oriented edges in G_{res} (resp. in G_{opt}). We will show that $n_{res} = n_{opt}$.

One can easily check that the graphs G_1 (at line 5), G_2 (at line 6) and G_3 (at line 11) are feasible orientations. Since the graph G_{res} is updated only at these lines, we deduce that G_{res} is also a feasible orientation. Now, let us show that $n_{res} \leq n_{opt}$ (and consequently $n_{res} = n_{opt}$). First of all, we identify the two following cases :

- $G_{opt}[s_{i_1}, t_1]^-$ is an oriented path going from s_{i_1} to t_1. In this case, $A(G_1) \subseteq A(G_{opt})$, where G_1 is the graph obtained at line 5.
- $G_{opt}[s_{i_{m''}}, t_{m''}]^+$ is a directed path going from $s_{i_{m''}}$ to $t_{m''}$. In this case, $A(G_2) \subseteq A(G_{opt})$, where G_2 is the graph obtained at line 6.

Furthermore, G_1 and G_2 are feasible orientations, and the arbitrary orientation of edges at line 15 does not increase the number of doubly oriented edges. Consequently, if one of the above two cases occurs, then we have $n_{res} \leq n_{opt}$.

Now, we suppose that none of the subgraphs $G_{opt}[s_{i_1}, t_1]^-$ or $G_{opt}[s_{i_{m''}}, t_{m''}]^+$ is a directed path. Let (w, y) be the pair of integers such that $G_{opt}[s_{i_w}, t_y]^+$ is the longest directed path, in the "+" direction, going from a source to a target vertex. Let also (x, z) be the pair of integers such that $G_{opt}[s_{i_x}, t_z]^-$ is the longest directed path, in the "-" direction, going from a source to a target vertex. Necessarily, the following inequalities hold: $w \geq x-1$ et $y \geq z-1$. Indeed, for example suppose by contradiction

Algorithm 1 $\texttt{MCO}$ $(G = (V, E, A), \mathcal{P} \subseteq V \times V)$

Require: A MAG $G = (V, A, E)$ s.t. G^* is a cycle, a set of pairs $\mathcal{P} \subseteq V \times V$.
Ensure: Find an optimal orientation of G.

1: /* $Update(G_{res}, G_i)$: if the number of doubly oriented edges of a feasible orientation G_i is strictly less than that of G_{res} then $G_{res} := G_i$
$Orient(s_\alpha, t_\beta, +, G_i)$: orientation of $G_i[s_\alpha, t_\beta]^+$ form s_α towards t_β
$Orient(s_\alpha, t_\beta, -, G_i)$: orientation of $G_i[s_\alpha, t_\beta]^-$ form s_α towards t_β */
2: Let $\mathcal{P}'$ (resp. $\mathcal{P}''$) be the set of pairs (s, t) s.t. there is in G exactly one (resp. two) path(s) from s to t.
3: Let G' be the graph obtained form G by orienting for each $(s, t) \in \mathcal{P}'$, the unique path between s and t, from s towards t.
4: $G_{res} := G'$; $Represent(\mathcal{P}'', G')$; /* construct the $\mathcal{P}''$-representation $\mathcal{ABCD}$ (Def. 4) */
5: $G_1 := G'$; $Orient(s_{i_1}, t_1, -, G_1)$; $Update(G_{res}, G_1)$;
6: $G_2 := G'$; $Orient(s_{i_{m''}}, t_{m''}, +, G_2)$; $Update(G_{res}, G_2)$;
7: **for all** (g, j) with $1 \leq g,j \leq m''$ and $g \geq j - 1$ **do**
8:     **for all** (k, l) with $1 \leq k,l \leq m''$ and $k \geq l - 1$ **do**
9:         $G_3 := G'$; $Orient(s_{i_g}, t_k, +, G_3)$; $Orient(s_{i_j}, t_l, -, G_3)$;
10:         **if** (G_3 is a feasible orientation) **then**
11:             $Update(G_{res}, G_3)$;
12:         **end if**
13:     **end for**
14: **end for**
15: Replace each edge $(u, v) \in E(G_{res})$ by the arc uv. /* arbitrary orientation of the remaining edges in G_{res} */
16: **return** G_{res}

that $w \leq x-2$. In this case, the pair $(s_{i_{(w+1)}}, t_{(w+1)})$ cannot be satisfied in G_{opt} neither by a "+" path (due to the maximality of $G_{opt}[s_{i_w}, t_y]^+$) nor by a "-" path (due to the maximality of the path $G_{opt}[s_{i_x}, t_z]^-$). Thus, we obtain a contradiction because G_{opt} is a feasible orientation.

The maximality of the paths $G_{opt}[s_{i_w}, t_y]^+$ and $G_{opt}[s_{i_x}, t_z]^-$ implies that the graph H obtained from G' (in its value at line 3) by orienting $G'[s_{i_w}, t_y]^+$ and $G'[s_{i_x}, t_z]^-$ from the source towards the target vertex, is a feasible orientation verifying : $A(H) \subseteq A(G_{opt})$. The graph H corresponds to the graph G_3 calculated by the algorithm at line 9 for $g = w$, $j = x$, $k = y$ and $l = z$. As the orientation at line 15 does not increase the number of doubly oriented edges, we deduce that $n_{res} \leq n_{opt}$.

□

Let $n = |V(G)|$, $m' = |\mathcal{P}'|$, $m'' = |\mathcal{P}''|$ and $m = m' + m''$. It is easy to check that the overall complexity of Algorithm $\texttt{MCO}$ has an upper bound of $\mathcal{O}((m' + m''^5) \cdot n)$. A more elaborated algorithm that achieves an upper bound of $\mathcal{O}((m' + m''^3) \cdot n)$ is presented in [15].

In the following theorem we show that, similar to the S-GO problem (proof by

Arkin and Hassin [2]), the MIN-D-GO problem is polynomial-time solvable for general MAGs when $|\mathcal{P}| \leq 2$.

Theorem 6. *The* MIN-D-GO *problem is polynomial-time solvable when G is a MAG and* $|\mathcal{P}| \leq 2$*.*

Proof. Remark that the case $|\mathcal{P}| = 1$ is obvious. Let $G = (V, E, A)$ be a MAG and $\mathcal{P} = \{(s_1, t_1), (s_2, t_2) \in V \times V\}$.

A $\mathcal{P}$-*essential* edge is an edge $e \in E$, with $e = (u, v)$ s.t. if we orient e in a single direction (either from u to v or from v to u), the graph G is no longer $\mathcal{P}$-connected. One can compute the $\mathcal{P}$-essential edges in polynomial-time [2].

Let E_{ess} (resp. E_{min}) be the set of $\mathcal{P}$-essential edges (resp. the set of doubly oriented edges in a solution of the MIN-D-GO problem).

We show that $E_{min} = E_{ess}$. Let $e \in E_{ess}$, with $e = (u, v)$. If we orient e in a unique direction (from u to v or from v to u) then, by definition of $\mathcal{P}$-essential edges, there is an integer i, $1 \leq i \leq 2$, s.t. there is no path in G from s_i to t_i. Thus, whatever the orientation of edges in $E - \{e\}$, the pair (s_i, t_i) would not be satisfied. Hence, we must replace in G each edge $e \in E_{ess}$ by a a doubly oriented one, which implies that $E_{ess} \subseteq E_{min}$. Conversely, let $G' = (V, E', A')$ denote the mixed graph obtained from G after replacing each $\mathcal{P}$-essential edge (u, v) by the arcs uv and vu, i.e., $V(G') = V(G)$, $E(G') = E(G) \backslash E_{ess}$ and $A' = A \cup \{uv, vu : (u, v) \in E_{ess}\}$. For this, Arkin and Hassin [2] showed that a mixed graph has $\mathcal{P}$-orientation iff it has no $\mathcal{P}$-essential edge. Thus the graph G' has a $\mathcal{P}$-orientation G''. Hence, G'' is an orientation of G that satisfies all the pairs in $\mathcal{P}$ and creates $|E_{ess}|$ doubly oriented edges, which implies that $|E_{ess}| \geq |E_{min}|$. Since we have already shown that $E_{ess} \subseteq E_{min}$, we conclude that $E_{min} = E_{ess}$.

Now, to solve the MIN-D-GO problem when $|\mathcal{P}| = 2$, we apply the following process.

1. Compute the $\mathcal{P}$-essential edges of G using the polynomial-time algorithm presented in [2].
2. Construct a mixed graph G' by replacing each $\mathcal{P}$-essential edge (u, v) in G by two arcs uv and vu.
3. Apply the polynomial-time algorithm presented in [2] in order to compute a $\mathcal{P}$-orientation of G'.

□

As in the S-GO problem, we show that the MIN-D-GO problem is FPT parameterized by B and $|\mathcal{P}|$ when G is a MAG. Then, one can deduce that the problem is polynomial-time solvable when both parameters $|\mathcal{P}|$ and $|B|$ are bounded.

Theorem 7. *The* MIN-D-GO *problem is* FPT *parameterized by* B *and* $|\mathcal{P}|$ *when* G *is a MAG.*

Proof. Let $G = (V, E, A)$ and let $\mathcal{P} = \{(s_i, t_i), 1 \leq i \leq m\}$. The proof is similar to that of Theorem 3. We consider all the possible combinations of paths to satisfy all the pairs in $\mathcal{P}$ (we choose one path per pair and we orient it from the source towards the target vertex), that leads to at most B^m combinations. Then, we keep the graph resulting from the orientation that creates a minimum number of doubly oriented edges. We complete the orientation of the obtained graph by arbitrarily orienting, in a single direction, each remaining edge. $\square$

5.2. *Difficult cases*

Recall that a $(\mathcal{P}, k)$-D-orientation is an orientation that satisfies all the pairs in $\mathcal{P}$ and creates exactly k doubly oriented edges. Thus a $(\mathcal{P}, 0)$-D-orientation is a $\mathcal{P}$-orientation. In addition, we showed in Theorem 4 that the S-GO problem is NP-complete even when the graph G is a MAG, $\Delta(G^*) = 3$ and $B = 3$. Hence, we may immediately deduce the following corollaries:

Corollary 8. *The problem* D-GO *is* NP*-complete when G is a MAG and $|\mathcal{P}|$ is unbounded even when $\Delta(G^*) = 3$ and $B = 3$.*

Corollary 9. *Unless* P = NP*, the* MIN-D-GO *problem is non-approximable when the graph G is a MAG and $|\mathcal{P}|$ is unbounded even when $\Delta(G^*) = 3$ and $B = 3$.*

Moreover, we show in the following theorem that MIN-D-GO is also W[1]-hard when $|\mathcal{P}|$ and B are unbounded even when $\Delta(G^*) = 3$.

Theorem 10. *The* MIN-D-GO *problem is* W[1]*-hard (parametrized by the number of doubly oriented edges) when G is a MAG and $|\mathcal{P}|$ and B are unbounded, even when $\Delta(G^*) = 3$.*

Proof. We propose a reduction from the problem MINIMUM SET COVER: given a set $\mathcal{X} = \{X_1, \ldots, X_n\}$ and a collection of sets $\mathcal{C} = \{S_1, \ldots, S_m\}$ s.t. $S_i \in 2^{\mathcal{X}}$, for all $1 \leq i \leq m$, the goal is to find a minimum set cover $\mathcal{C}'$, i.e., a set $\mathcal{C}' \subseteq \mathcal{C}$ s.t. $\mathcal{C} = \bigcup_{S_i \in \mathcal{C}'} S_i$ and $|\mathcal{C}'|$ is minimum.

We denote by α_j, $1 \leq j \leq n$, the number of sets in $\mathcal{C}$ containing X_j. Let us construct an instance $(G, \mathcal{P})$ of the MIN-D-GO problem. For each $X_j \in \mathcal{X}$, we add the vertex set $\{x_j\} \cup \{x_j^k, 1 \leq k \leq \alpha_j\}$, then we create the directed path $x_j^1 x_j^2 \ldots x_j^{\alpha_j} x_j$. For each S_i, we add the vertex set $\{s_i, s_i'\} \cup \{s_i^j, 1 \leq j \leq |S_i|\}$, and we add an edge (s_i', s_i) and a directed path $s_i s_i^1 s_i^2 \ldots s_i^{|S_i|}$.

Let $X_l \in \mathcal{X}$ be the $j-$th element in a set S_i. We add an arc from s_i^j towards *one* vertex in the set $\{x_l^k, 1 \leq k \leq \alpha_l\}$. Such a vertex is chosen in such a way that the indegree of each vertex x_l^k be exactly two, for all $2 \leq k \leq \alpha_l$, and 1 for $k = 1$. To finish the construction of G, we add a vertex r_1 connected by two arcs going from r_1 to the vertices s_1' and s_2'. Then, we add a new vertex r_2 connected by two arcs going from r_2 to the vertices r_1 and s_3'. We continue the creation of vertices r_i connected by arcs going from r_i to r_{i-1} and s_{i+1}', for all $3 \leq i \leq m-1$. The set of pairs to satisfy is $\mathcal{P} = \{(r_{m-1}, x_j), 1 \leq j \leq n\} \cup \{(s_i, s_i'), 1 \leq i \leq m\}$ with $m = |\mathcal{C}|$. An example of construction is illustrated in Figure 4. The degree of each vertex r_i in G is at most three and also each vertex s_i^j is connected to exactly one vertex $x_{i'}^{j'}$. Thus, one can easily check that $\Delta(G^*) = 3$.

We claim that, for every integer $k \geq 0$, there is a set cover of $\mathcal{C}$ of cardinality k if and only if there is a $(\mathcal{P}, k)$-D-orientation of G.

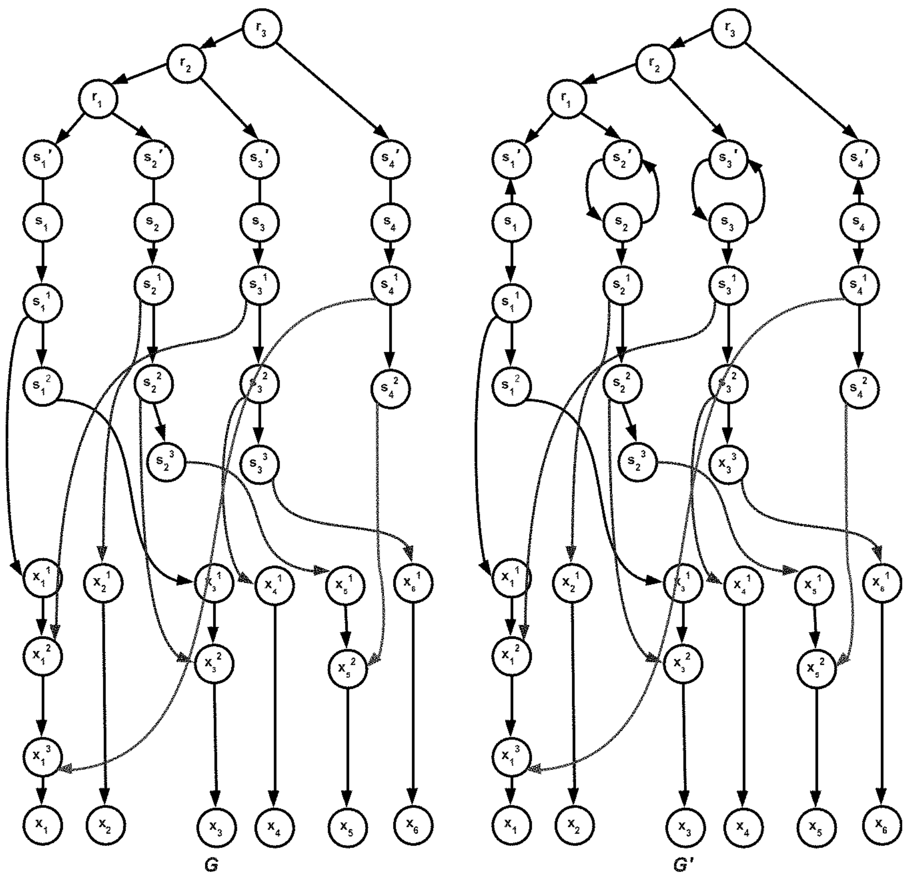

Figure 4: (a) Construction of an instance $(G, \mathcal{P})$ of the MIN-D-GO problem from an instance of MINIMUM SET COVER problem. Here, $\mathcal{X} = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ and $\mathcal{C} = \{S_1, S_2, S_3, S_4\}$ s.t. $S_1 = \{X_1, X_3\}$, $S_2 = \{X_2, X_3, X_5\}$, $S_3 = \{X_1, X_4, X_6\}$ and $S_4 = \{X_1, X_5\}$. The set of pairs is $\mathcal{P} = \{(s_1, s_1'), (s_2, s_2'), (s_3, s_3'), (s_4, s_4'), (r_3, x_1), (r_3, x_2), (r_3, x_3), (r_3, x_4), (r_3, x_5), (r_3, x_6)\}$. (b) The graph G' is a $(\mathcal{P}, 2)$-D-orientation of G corresponding to the set cover $\mathcal{C}' = \{S_2, S_3\}$.

$\Rightarrow$: Given a set cover $\{S_{i_1}, S_{i_2}, \ldots, S_{i_k}\}$, we doubly orient the edge (s_{i_j}, s'_{i_j}), for all $1 \leq j \leq k$. Then, we replace each edge (s_i, s'_i) by the arc $s_i s'_i$, for all $i \in \{1, 2, \ldots, m\} \backslash \{i_1, i_2, \ldots, i_k\}$.

$\Leftarrow$: Let E_D be the set of doubly oriented edges in a $(\mathcal{P}, k)$-D-orientation. Let $\mathcal{C}' = \{S_i : (s_i, s'_i) \in E_D\}$. Thus, $|\mathcal{C}'| \leq k$. We will show that the set $\mathcal{C}'$ is a set cover of $\mathcal{C}$. Suppose that there is $X_j \in \mathcal{X}$ s.t. $X_j \notin S_i$ for all $S_i \in \mathcal{C}'$. Let $\mathcal{C}_j$ denote the collection of the sets containing X_j, i.e., $\mathcal{C}_j = \{S \in \mathcal{C} : X_j \in S\}$. The graph G is constructed in such a way that, to satisfy any pair (r_{m-1}, x_j), we must add at least one arc $s'_i s_i$ s.t. $S_i \in \mathcal{C}_j$. On the other hand, we have to orient each edge (s'_i, s_i), from s_i towards s'_i, to satisfy the pair $(s_i, s'_i) \in \mathcal{P}$. Then the edge (s'_i, s_i) must be doubly oriented, which implies that $S_i \in \mathcal{C}'$. This is a contradiction, because $\mathcal{C}' \cap \mathcal{C}_j = \emptyset$.

Moreover, $|\mathcal{C}'| \leq k$. If $|\mathcal{C}'| \neq k$, then we add to $|\mathcal{C}'|$ a convenient number of arbitrarily sets from $\mathcal{C}$ to ensure that $|\mathcal{C}'| = k$ ($\mathcal{C}'$ remains a set cover of $\mathcal{C}$).

The above reduction preserves the parameter k (the cardinality of the set cover and the number of doubly oriented edges). Since the problem MINIMUM SET COVER is W[1]-hard when parametrized by k [17, 7], the MIN-D-GO problem is also W[1]-hard when parametrized by the number of doubly oriented edges. □

Now let us show that, unlike the S-GO problem, the MIN-D-GO problem remains difficult even when $B = 2$.

Theorem 11. *The problem* MIN-D-GO *is* APX*-hard when G is a MAG and $|\mathcal{P}|$ is unbounded, even when $\Delta(G^*) = 3$ and $B = 2$.*

Proof. Again, we use the previous reduction (proof of Theorem 10), but we consider the variant MINIMUM SET COVER-2 of the MINIMUM SET COVER problem in which each $X_j \in \mathcal{X}$ appears in exactly two sets in $\mathcal{C}$. For each pair of vertices $(s_i, s'_i) \in \mathcal{P}$ there is a unique path in G, from s_i to s'_i (that is the edge (s_i, s'_i)), for all $1 \leq i \leq m$. In addition, the fact that each $X_j \in \mathcal{X}$ appears in exactly two sets in $\mathcal{C}$, implies that for each pair (r_{m-1}, x_i) there are, in G, two paths from r_{m-1} to s_i, for all $1 \leq i \leq n$. Thus $B = 2$ and also the graph G is constructed so that we have $\Delta(G^*) = 3$.

As the proposed reduction is an L-reduction and the problem MINIMUM SET COVER-K is APX-hard [16], we conclude that MIN-D-GO is APX-hard when G is a MAG s.t. $\Delta(G^*) = 3$ and $|\mathcal{P}|$ is unbounded, even when $B = 2$. □

6. Conclusion

In this work, we considered two problems that concern the orientation of mixed graphs, both motivated, among others, by urban and communication network design and also by biological applications. We studied the complexity of both problems, and in particular we provided polynomial-time algorithms for some restricted instances, as well as hardness and inapproximability results. However, some interesting problems remain open so far, such as the following ones. First, since we showed that MIN-D-GO is easy when $|\mathcal{P}| \leq 2$, and difficult when $|\mathcal{P}|$ is unbounded, it is interesting to study the complexity of such a problem when $|\mathcal{P}| = 3$ (or more generally when $|\mathcal{P}|$ is a constant greater than or equal to 3), in order to identify the border (according to $|\mathcal{P}|$) between tractable and intractable instances. Investigating the complexity of S-GO and MIN-D-GO in terms of approximability (on specific graph classes) and fixed-parameterized tractable (FPT) algorithms is another interesting goal.

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