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# THUE EQUATIONS AND CM-FIELDS

YVES AUBRY AND DIMITRIOS POULAKIS

ABSTRACT. We obtain a polynomial type upper bound for the size of the integral solutions of Thue equations  $F(X, Y) = b$  defined over a totally real number field  $K$ , assuming that  $F(X, 1)$  has a root  $\alpha$  such that  $K(\alpha)$  is a CM-field. Furthermore, we give an algorithm for the computation of the integral solutions of such an equation.

Keywords: Thue equations, integral solutions, CM-fields.  
MSC : 11D59; 11Y16 ; 11G30; 11G50.

## 1. INTRODUCTION

Let  $F(X, Y)$  be an irreducible binary form in  $\mathbb{Z}[X, Y]$  with  $\deg F \geq 3$  and  $b \in \mathbb{Z} \setminus \{0\}$ . In 1909, Thue [26] proved that the equation  $F(X, Y) = b$  has only finitely many solutions  $(x, y) \in \mathbb{Z}^2$ . Thue's proof was ineffective and therefore does not provide a method to determine the integer solutions of this equation. Other non effective proofs of Thue's result can be found in [7, Chap. X] and [21, Chap. 23].

In 1968, Baker [2], using his results on linear forms in logarithms of algebraic numbers, computed an explicit upper bound for the size of the integer solutions of Thue equations. Baker's result were improved by several authors (see for instance [6], [11], [22]) but the bounds remain of exponential type and thus, are not useful to compute integer solutions of such equations. Nevertheless, computation techniques for the resolution of Thue equations have been developed based on the above results [1], [13], [27] and the solutions of certain parameterized families of Thue equations have been obtained [14]. Furthermore, upper bounds for the number of integral solutions of Thue equations have been given [5], [9], [4].

In the case where all roots of the polynomial  $F(X, 1)$  are non real, we have a polynomial type bound provided by other methods [21, Theorem 2, page 186], [10] [23]. Györy's improvement in [10, Théorème 1] holds in the case where the splitting field of  $F(X, 1)$  is a CM-field i.e., is a totally imaginary quadratic extension of a totally real number field. In the same paper, Györy studied Thue equations defined over a CM-field  $L$  and also gave ([10, Théorème 2]) a polynomial upper bound for the size of their real algebraic integers solutions in  $L$ .

In this paper, we consider Thue equations  $F(X, Y) = b$  defined over a totally real number field  $K$ . Simplifying Györy's approach, we obtain (Theorem 1) polynomial type bounds for the size of their integral solutions

over  $K$ , assuming that  $F(X, 1)$  has a root  $\alpha$  such that the field  $K(\alpha)$  is a CM-field. In case where the splitting field is a CM-field we are in the situation of [10, Théorème 2]. Whenever all roots of the polynomial  $F(X, 1)$  are non real and  $K \neq \mathbb{Q}$ , we obtain much better bounds than those already known. Moreover, whenever  $F(X, 1)$  has a real and a non real root, we obtain polynomial type bounds that the Baker's method was not able to provide other than exponential bounds. Furthermore, the method of the proof of Theorem 1 provides us with an algorithm for the determination of the solutions of such equations.

We illustrate our result by giving two examples of infinite families of Thue equations  $F(X, Y) = b$  satisfying the hypothesis of Theorem 1. First we consider Thue equations over some totally real subfields  $K$  of cyclotomic fields  $N$  such that the splitting field  $L$  of  $F(X, 1)$  over  $K$  is contained in  $N$ . In this case,  $L$  is an abelian extension of  $K$ . Next, we give a family of equations  $F(X, Y) = b$  such that  $F(X, 1)$  has a root  $\alpha$  for which  $K(\alpha)$  is a biquadratic CM-field. These families contain equations such that  $F(X, 1)$  has also real roots and so the only method for having upper bound for the size of their solutions is Baker's method which provide only bounds of exponential type. Finally, we give two examples of determination of solutions of equations satisfying the hypothesis of Theorem 1, by using our algorithm.

## 2. NEW BOUNDS

We introduce a few notations. Let  $K$  be a number field. We consider the set of absolute values of  $K$  by extending the ordinary absolute value  $|\cdot|$  of  $\mathbb{Q}$  and, for every prime  $p$ , by extending the  $p$ -adic absolute value  $|\cdot|_p$  with  $|p|_p = p^{-1}$ . Let  $M(K)$  be an indexing set of symbols  $v$  such that  $|\cdot|_v$ ,  $v \in M(K)$ , are all of the above absolute values of  $K$ . Given such an absolute value  $|\cdot|_v$  on  $K$ , we denote by  $d_v$  its local degree. Let  $\mathbf{x} = (x_0 : \dots : x_n)$  be a point of the projective space  $\mathbb{P}^n(K)$  over  $K$ . We define the field height  $H_K(\mathbf{x})$  of  $\mathbf{x}$  by

$$H_K(\mathbf{x}) = \prod_{v \in M(K)} \max\{|x_0|_v, \dots, |x_n|_v\}^{d_v}.$$

Let  $d$  be the degree of  $K$ . We define the absolute height  $H(\mathbf{x})$  by  $H(\mathbf{x}) = H_K(\mathbf{x})^{1/d}$ . For  $x \in K$ , we put  $H_K(x) = H_K((1 : x))$  and  $H(x) = H((1 : x))$ . If  $G \in K[X_1, \dots, X_m]$ , then we define the field height  $H_K(G)$  and the absolute height  $H(G)$  of  $G$  as the field height and the absolute height respectively of the point whose coordinates are the coefficients of  $G$  (in any order). For an account of the properties of heights see [15, 17, 25]. Furthermore, we denote by  $O_K$  and  $N_K$  the ring of integers of  $K$  and the norm relative to the extension  $K/\mathbb{Q}$ , respectively. Finally, for every  $z \in \mathbb{C}$  we denote, as usually, by  $\bar{z}$  its complex conjugate.

We prove the following theorem:

**Theorem 1.** *Let  $K$  be a totally real number field of degree  $d$ . Let  $b \in O_K \setminus \{0\}$  and  $F(X, Y) \in O_K[X, Y]$  be a form of degree  $n \geq 2$ . Suppose that  $F(X, 1)$  has a root  $\alpha$  such that  $K(\alpha)$  is a CM-field. Then the solutions  $(x, y) \in O_K^2$  of  $F(X, Y) = b$  satisfy*

$$H(x) < \Omega_1 \quad \text{and} \quad H(y) < \Omega_2$$

for the following values of  $\Omega_1$  and  $\Omega_2$ . If the coefficients of  $X^n$  and  $Y^n$  are  $\pm 1$ , then

$$\Omega_1 = \Omega_2 = 32H(b)^{1/n} H(F)^{1+1/n} N_K(b)^{2/d}.$$

If only the coefficient of  $X^n$  is  $\pm 1$ , then

$$\Omega_1 = 2^9 H(b)^{1/n} H(F)^{2+1/n} N_K(b)^{4/d} \quad \text{and} \quad \Omega_2 = 32H(b)^{1/n} H(F)^{1+1/n} N_K(b)^{2/d}.$$

If both the coefficients of  $X^n$  and  $Y^n$  are  $\neq \pm 1$ , then

$$\Omega_1 = 2^9 H(b)^{1/n} H(\Gamma)^{2n+1} N_K(b)^{4/d} H(a_0)^{n-1} N_K(a_0)^{2(n-1)/d}$$

and

$$\Omega_2 = 32H(b)^{1/n} H(\Gamma)^{n+1} N_K(b)^{2/d} H(a_0)^{n-1} N_K(a_0)^{2(n-1)/d},$$

where  $a_0$  is the coefficient of  $X^n$  and  $\Gamma$  a point of the projective space with 1 and the coefficients of  $F(X, Y)$  as coordinates. Furthermore, the number of integral solutions to the equation  $F(X, Y) = b$  is at most

$$72 \cdot 4^{dn} N_K(b)^{2n}.$$

In case where  $b$  is a unit of  $O_K$ , this number is at most  $2wn$ , where  $w$  is the number of the roots of unity in  $K(\alpha)$ .

The proof of this result is relied on the following property of CM-fields. A non real algebraic number field  $L$  is a CM-field if and only if  $L$  is closed under the operation of complex conjugation and complex conjugation commutes with all the  $\mathbb{Q}$ -monomorphisms of  $L$  into  $\mathbb{C}$  ([3], [12, Théorème 1], [18, Lemma 2]).

When  $K = \mathbb{Q}$  and the splitting field of  $F(X, 1)$  over  $\mathbb{Q}$  is an abelian totally imaginary extension, the hypothesis on complex conjugation is obviously satisfied. If the coefficient of  $X^n$  is  $\pm 1$ , it is interesting to notice that our bounds are essentially independent of the degree of the form  $F(X, Y)$ . Thus, in case where  $H(F)$  and  $H(b)$  are not too large, an exhaustive search can provide the integer solutions we are looking for.

### 3. EXAMPLES

In this section we give two examples in order to illustrate our result. We denote by  $F^*(X, Y)$  the homogenization of a polynomial  $F(X) \in \mathbb{C}[X]$ .

**Example 1.** Let  $p$  be a prime with  $p \equiv 1 \pmod{4}$  and  $\zeta_p$  a  $p$ -th primitive root of unity in  $\mathbb{C}$ . Then the quadratic field  $\mathbb{Q}(\sqrt{p})$  is a subfield of  $\mathbb{Q}(\zeta_p)$ . The field  $\mathbb{Q}(\zeta_p)$  is a cyclic extension of  $\mathbb{Q}$  with Galois group  $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \simeq (\mathbb{Z}/p\mathbb{Z})^*$ .

Let  $\alpha \in \mathbb{Z}[\zeta_p]$  be a primitive element of the extension  $\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{p})$  and  $\alpha_1, \dots, \alpha_m$ , with  $m = (p-1)/2$ , all the distinct conjugates of  $\alpha$  over  $\mathbb{Q}(\sqrt{p})$ . The largest real field contained in  $\mathbb{Q}(\zeta_p)$  is  $K_p = \mathbb{Q}(\zeta_p + \bar{\zeta}_p)$  which is a totally real number field. Let  $\beta \in K_p$  be a primitive element of the extension  $K_p/\mathbb{Q}(\sqrt{p})$  and  $\beta_1, \dots, \beta_n$ , where  $n = (p-1)/4$ , all the distinct conjugates of  $\beta$  over  $\mathbb{Q}(\sqrt{p})$ . Then the polynomial

$$F(X) = (X - \alpha_1) \cdots (X - \alpha_m)(X - \beta_1) \cdots (X - \beta_n)$$

belongs to  $\mathbb{Q}(\sqrt{p})[X]$  and has real and non real roots. Furthermore, we have  $\mathbb{Q}(\sqrt{p})(a_i) = \mathbb{Q}(\zeta_p)$  which is a CM-field. Consequently, for every non zero  $b \in \mathbb{Z}[(1 + \sqrt{p})/2]$ , the Thue equation  $F^*(X, Y) = b$  satisfies the hypothesis of Theorem 1. Note that this equation satisfies also the hypothesis of [10, Théorème 2].

Then, using [25, Theorem 5.9, page 211] and [25, Lemma 5.10, page 213], Theorem 1 gives the following upper bound for the heights of solutions  $x, y \in \mathbb{Z}[(1 + \sqrt{p})/2]$ :

$$H(x) < 2^{(3p+17)/4} (H(\alpha)^2 H(\beta))^{(3p+1)/6} H(b)^{4/3(p-1)} N_{\mathbb{Q}(\sqrt{p})}(b)$$

and

$$H(y) < 2^{(3p+13)/2} (H(\alpha)^2 H(\beta))^{5(p-1)/6} H(b)^{4/3(p-1)} N_{\mathbb{Q}(\sqrt{p})}(b)^2.$$

If we consider the particular case where  $\Phi_p(X)$  is the  $p$ -th cyclotomic polynomial, then [10, Section 2] implies that the maximum of the absolute heights of all algebraic integers  $x, y \in K_p$  with  $\Phi_p^*(x, y) = 1$  is  $< 2^{(p-1)/2}$ . Theorem 1 improves this result by yielding the bound 32.

**Example 2.** Let  $d$  be a positive integer  $\geq 2$  and  $r = m + n\sqrt{d}$ , where  $m, n$  are integers such that  $m > 0$  and  $m^2 - n^2d > 0$ . The minimal polynomial of  $r$  over  $\mathbb{Q}$  is

$$M(X) = X^2 - 2mX + m^2 - dn^2.$$

Then, the polynomial

$$P(X) = M(-X^2) = X^4 + 2mX^2 + m^2 - dn^2$$

is the minimal polynomial of  $\sqrt{-r}$  over  $\mathbb{Q}$ . Since  $m > 0$  and  $m^2 - n^2d > 0$ , their roots are not real and so,  $\mathbb{Q}(\sqrt{d}, \sqrt{-r})$  is a CM-field. If  $Q(X) \in \mathbb{Z}[\sqrt{d}][X] \setminus \mathbb{Z}$ , then we put

$$F(X) = (X^2 + (m + n\sqrt{d})Y^2)Q(X).$$

Then for every nonzero  $b \in \mathbb{Z}[\sqrt{d}]$ , the Thue equation  $F^*(X, Y) = b$  over  $K = \mathbb{Q}(\sqrt{d})$  satisfies the hypothesis of Theorem 1. Suppose that  $Q(X)$  is monic and  $\deg Q = q > 0$ . By [15, Remark B.7.4], we have

$$H(F) \leq 4H(m + n\sqrt{d})H(Q).$$

Thus Theorem 1 yields the following upper bounds for the height of integral solutions of the above equations over  $K$ :

$$H(x) < 2^{13+1/q} H(b)^{1/2q} (H(m + n\sqrt{d})H(Q))^{2+1/q} N_K(b)^2,$$

$$H(y) < 2^{7+1/q} H(b)^{1/2q} (H(m + n\sqrt{d})H(Q))^{1+1/q} N_K(b).$$

Note that in case where the splitting field of  $F(X)$  is not a CM-field, [10, Théorème 2] cannot be applied. Furthermore, Baker's method can provide only bounds of exponential type.

#### 4. PROOF OF THEOREM 1

Write

$$F(X, Y) = a_0(X - \alpha_1 Y) \cdots (X - \alpha_n Y).$$

First, we consider the case where  $a_0 = \pm 1$ . If  $a_0 = -1$ , we replace  $F(X, Y)$  by  $-F(X, Y)$  and  $b$  by  $-b$  and then we may suppose that  $a_0 = 1$ . By our hypothesis, there is  $j$  such that  $K_j = K(\alpha_j)$  is a CM-field.

Let  $x, y \in O_K$  such that  $xy \neq 0$  and  $F(x, y) = b$ . We set  $b_j := x - \alpha_j y$ . Since  $K$  is a totally real number field, we have  $x - \bar{\alpha}_j y = \bar{b}_j$ . Setting  $\bar{b}_j = \rho_j b_j$ , we obtain the system

$$x - \alpha_j y = b_j, \quad x - \bar{\alpha}_j y = \rho_j b_j.$$

Eliminating  $b_j$  from the above two equations, we get

$$x = y \frac{\bar{\alpha}_j - \alpha_j \rho_j}{1 - \rho_j}.$$

Set

$$A = \frac{\bar{\alpha}_j - \alpha_j \rho_j}{1 - \rho_j}.$$

We have

$$H(A) \leq H(\bar{\alpha}_j - \alpha_j \rho_j) H(1 - \rho_j) \leq 4H(\alpha_j)^2 H(\rho_j)^2.$$

Since  $\alpha_j$  is not real, using [19], we deduce  $H(\alpha_j) < 2H(F)^{1/2}$ . It follows that

$$H(A) \leq 16H(F)H(\rho_j)^2.$$

Substituting in the equation  $F(x, y) = b$  we deduce that

$$y^n F(A, 1) = b,$$

and thus

$$H(y)^n \leq H(F(A, 1))H(b) \leq (n+1)H(F)H(A)^n H(b).$$

Using the bound for  $H(A)$  we obtain

$$(1) \quad H(y)^n \leq (n+1)16^n H(b)H(F)^{n+1}H(\rho_j)^{2n}.$$

Next, we shall compute a bound for the height of  $\rho_j$ . We denote by  $G_j$  the set of  $\mathbb{Q}$ -embeddings  $\sigma : K_j \rightarrow \mathbb{C}$ . Since  $K_j$  is a CM-field, [12, Théorème 1] yields that the complex conjugation commutes with all the elements of  $G_j$ . Further,  $K_j$  is closed under the operation of complex conjugation whence we get  $\bar{\alpha}_j \in K_j$  and so  $\bar{b}_j \in K_j$ . Thus, for every  $\sigma \in G_j$ , we have  $\sigma(\bar{b}_j) = \overline{\sigma(b_j)}$ . It follows that

$$|\sigma(\rho_j)| = \frac{|\sigma(\bar{b}_j)|}{|\sigma(b_j)|} = \frac{|\overline{\sigma(b_j)}|}{|\sigma(b_j)|} = 1.$$

Let  $M_j(X)$  be the minimal polynomial of  $\rho_j$  over  $\mathbb{Z}$  and  $m_j$  its leading coefficient. The elements  $\alpha_j, \bar{\alpha}_j$  are algebraic integers of  $K_j$  and so,  $b_j, \bar{b}_j$  are algebraic integers of  $K_j$ . It follows that the polynomial

$$\Pi_j(X) = \prod_{\sigma \in G_j} \sigma(b_j)(X - \sigma(\rho_j))$$

has integer coefficients. Since  $\rho_j$  is a root of  $\Pi_j(X)$ , we have that  $M_j(X)$  divides  $\Pi_j(X)$  and thus we deduce that  $m_j$  divides

$$\prod_{\sigma \in G_j} \sigma(b_j) = N_{K_j}(b_j),$$

where  $N_{K_j}$  is the norm relative to the extension  $K_j/\mathbb{Q}$ . It follows that  $m_j$  divides  $N_{K_j}(b_j)$ . As we saw above, all the conjugates  $\rho_{j1}, \dots, \rho_{j\mu}$  ( $\mu \leq dn$ ), of  $\rho_j$  are of absolute value 1. By [17, page 54], we have

$$H_{K_j}(\rho_j) = m_j \prod_{i=1}^{\mu} \max\{1, |\rho_{ji}|\} = m_j.$$

Since  $m_j$  divides  $N_{K_j}(b_j)$ , we have that  $m_j$  divides  $N_{K_j}(b)$ . Thus

$$(2) \quad H_{K_j}(\rho_j) \leq N_{K_j}(b).$$

Combining the inequalities (1) and (2), we get

$$H(y) \leq 32H(b)^{1/n} H(F)^{1+1/n} N_K(b)^{2/d}.$$

We have

$$H(x) \leq H(A)H(y) \leq 16H(F)H(\rho_j)^2 H(y)$$

whence we obtain

$$H(x) \leq 2^9 H(b)^{1/n} H(F)^{2+1/n} N_K(b)^{4/d}.$$

Suppose now that  $a_0 \neq \pm 1$ . Write  $F(X, 1) = a_0 X^n + a_1 X^{n-1} + \dots + a_n$ . Then  $a_0 \alpha_i$  is a root of  $f(X) = X^n + a_1 X^{n-1} + a_2 a_0 X^{n-2} + \dots + a_n a_0^{n-1}$  and thus  $a_0 \alpha_i$  is an algebraic integer. Denote by  $F_1(X, Y)$  the homogenization of  $f(X)$ . If  $(x, y) \in O_K^2$  is a solution to  $F(X, Y) = b$ , then  $(a_0 x, y)$  is a solution to  $F_1(X, Y) = b a_0^{n-1}$ . Denote by  $\Gamma$  a point in the projective space with 1 and the coefficients of  $F$  as coordinates. Then we have  $H(F_1) \leq H(\Gamma)^n$  and finally, we obtain

$$H(y) \leq 32H(b)^{1/n} H(\Gamma)^{n+1} N_K(b)^{2/d} H(a_0)^{n-1} N_K(a_0)^{2(n-1)/d}$$

and

$$H(x) \leq 2^9 H(b)^{1/n} H(\Gamma)^{2n+1} N_K(b)^{4/d} H(a_0)^{n-1} N_K(a_0)^{2(n-1)/d}.$$

Now suppose that  $b$  is a unit in  $O_K$ . Then inequality (2) implies that  $H(\rho_j) = 1$  and so Kronecker's theorem yields that  $\rho_j$  is a root of unity. Let  $w$  be the number of the roots of unity in  $K_j$ . Then we have  $w$  choices for  $A$  (for the roots of unity  $\neq \pm 1$ ) and, since  $y$  is real, the equation  $y^n F(A, 1) = b$  gives us at most  $2w$  choices for  $y$ . Considering also the solutions of the equation with  $xy = 0$ , we deduce that the number of integral solutions to

the equation  $F(X, Y) = b$  is at most  $2wn$ . Finally, suppose that  $b$  is not a unit in  $O_K$ . Using [24, Lemma 8B], we obtain that the number of elements  $\rho_j \in K_j$  with  $H(\rho_j) \leq N_K(b)^{1/d}$  is bounded by

$$36 \cdot 4^{dn} N_K(b)^{2n}$$

and so the result follows.

## 5. AN ALGORITHM

In this section we give an algorithm for the computation of the integral solutions to  $F(X, Y) = b$  based on the proof of Theorem 1.

### SOLVE-THUE-1

*Input:* A totally real number field  $K$ , a form  $F(X, Y) \in O_K[X, Y]$  with  $F(X, 1)$  monic,  $b \in O_K \setminus \{0\}$  and  $\alpha$  a root of  $F(X, 1)$  such that  $K(\alpha)$  is a CM-field.

*Output:* The integral solutions of  $F(X, Y) = b$  over  $K$ .

- (1) Compute the set  $\Lambda$  of all the elements  $\rho \in K(\alpha) \setminus K$  having the absolute values of all their conjugates equal to 1 and  $H_{K(\alpha)}(\rho) | N_{K(\alpha)}(b)$ . If  $b$  is a unit of  $O_K$ , then the set  $\Lambda$  consist of all the roots of unity of  $K(\alpha)$  which does not belong to  $K$ .
- (2) Compute the set  $\Xi$  of elements  $\xi$  of  $K$  of the form:

$$\xi = \frac{\bar{\alpha} - \alpha\rho}{1 - \rho},$$

where  $\rho \in \Lambda$ .

- (3) Compute the set  $S$  of elements  $y \in O_K$  such that there is  $\xi \in \Xi \cup \{(\bar{\alpha} + \alpha)/2\}$  with

$$y^n F(\xi, 1) = b.$$

- (4) Output the solutions  $(x, y) \in O_K^2$  to  $F(X, Y) = b$  with  $y \in S$  and the solutions  $(x, y) \in O_K^2$  with  $xy = 0$ .

*Proof of Correctness.* Let  $(x, y) \in O_K^2$  be a solution to  $F(X, Y) = b$  with  $xy \neq 0$ . We set  $x - \alpha y = \beta$  and  $\rho = \bar{\beta}/\beta$ . From the proof of Theorem 1 we have  $x = yA$ , where

$$A = \frac{\bar{\alpha} - \alpha\rho}{1 - \rho},$$

and so  $y^n F(A, 1) = b$ . Since  $x, y \in K$ , we get  $A \in K$ . Further,  $H_{K(\alpha)}(\rho)$  divides  $N_{K(\alpha)}(b)$ . Let  $\beta = \beta_1 + \beta_2 i$ , with  $\beta_1, \beta_2 \in \mathbb{R}$ . Suppose that  $\rho \in K$ . Then  $\rho \in \mathbb{R}$  and  $\bar{\beta} = \rho\beta = \rho\beta_1 + \rho\beta_2 i$ , whence we have  $\rho\beta_1 = \beta_1$  and  $\rho\beta_2 = -\beta_2$ . If  $\beta_1 \neq 0$ , then  $\rho = 1$  and so  $\bar{\beta} = \beta$  which is a contradiction (since  $\alpha$  is not real because  $K(\alpha)$  is CM). If  $\beta_1 = 0$ , then  $\rho = -1$ . In this case we have  $A = (\bar{\alpha} + \alpha)/2$ . Finally, if  $b$  is a unit, then we have that  $H(\rho) = 1$  and so  $\rho$  is a root of unity in  $K(\alpha)$ .



Note that there are algorithms for the computation of the elements of a number field of bounded height [8] and for the computation of roots of unity in a number field [20, Annexe C]. As far we know there are not implementations for such algorithms. The other computations can be carry out by a computational system such as MAGMA or MAPLE.

**Remark 1.** By [17, page 54], the leading coefficient  $m$  of the minimal polynomial of  $\rho$  is equal to  $H_{K(\alpha)}(\rho)$ . Thus,  $m\rho \in O_K$ .

Finally, we give two examples of Thue equations that satisfy the hypothesis of Theorem 1 for which we use the previous algorithm to determine all the integral solutions, the first one having a right-hand side a unit but not the second one.

**Example 3.** *The only solution of the equation*

$$(X^2 + Y^2)(X^2 - \sqrt{2}XY + Y^2) = 1$$

over  $\mathbb{Z}[\sqrt{2}]$  are  $(X, Y) = (\pm 1, 0), (0, \pm 1)$ .

*Proof.* The complex number  $i$  is a root of  $X^2 + 1$  and  $K = \mathbb{Q}(\sqrt{2}, i)$  is a CM-field. The roots of unity lying in  $K \setminus \mathbb{Q}(\sqrt{2})$  are  $\pm i$ . Next, we compute

$$\xi_{\pm} = \frac{-i - i(\pm i)}{1 - (\pm i)} = \pm 1.$$

Thus we have the equations  $y^4 2(2 \pm \sqrt{2}) = 1$ . If there is  $y \in \mathbb{Z}[\sqrt{2}]$  satisfying one of these equations, then 2 is a unit in  $\mathbb{Z}[\sqrt{2}]$ , which is a contradiction since its norm is not equal to  $\pm 1$ . Furthermore, the solutions  $(x, y) \in O_K^2$  with  $xy = 0$  are  $(\pm 1, 0)$  and  $(0, \pm 1)$ .

**Example 4.** *Consider the form*

$$F(X, Y) = (X^2 + (3 - 2\sqrt{2})Y^2)(X^2 - 4XY + \sqrt{2}Y^2) \in \mathbb{Z}[\sqrt{2}][X, Y].$$

*Then the only solutions of the equation  $F(X, Y) = 3\sqrt{2} - 4$  over  $\mathbb{Z}[\sqrt{2}]$  are  $(X, Y) = (0, \pm 1)$ .*

*Proof.* The given equation belongs to the family of equations of Example 2. Thus, we shall use the above algorithm for the determination of their solutions. First, we remark that the equation  $F(X, 0) = 3\sqrt{2} - 4$  has no solution over  $\mathbb{Z}[\sqrt{2}]$  and the only solutions of  $F(0, Y) = 3\sqrt{2} - 4$  over  $\mathbb{Z}[\sqrt{2}]$  are  $Y = \pm 1$ .

Set  $y = i\sqrt{3 - 2\sqrt{2}}$  and  $K = \mathbb{Q}(y)$ . We have  $N_K(6 - 4\sqrt{2}) = 16$ . We shall compute all the elements  $\rho \in K \setminus \mathbb{Q}(\sqrt{2})$  with  $H_K(\rho) | 16$  and having all the absolute values of their conjugates equal to 1.

If  $H_K(\rho) = 1$ , then  $\rho$  is a root of unity in  $K \setminus \mathbb{Q}(\sqrt{2})$ . Since there are not other roots of unity in  $K$  than  $\pm 1$ , we consider the case where  $H_K(\rho) > 1$ . Let  $H_K(\rho) = 2^\epsilon$ , where  $\epsilon = 1, 2$ . By Remark 1, we have  $\rho = \alpha/2^\epsilon$ , where  $\alpha \in O_K$ . Using MAGMA, we get the following integral base for  $K$ :

$$\omega_0 = 1, \quad \omega_1 = y, \quad \omega_2 = \frac{1}{2}(y^2 - 1), \quad \omega_3 = \frac{1}{4}(y^3 + y^2 - y - 1).$$

Since all the conjugates of  $\rho$  have absolute value 1, we obtain the two equalities

$$((a_0 - 2a_1) + a_1\sqrt{2})^2 + (2 - \sqrt{2})((a_2 - 2a_3) + a_3\sqrt{2})^2 = 2^{2\epsilon},$$

and

$$((a_0 - 2a_1) - a_1\sqrt{2})^2 + (2 + \sqrt{2})((a_2 - 2a_3) - a_3\sqrt{2})^2 = 2^{2\epsilon}.$$

It follows that

$$(3) \quad (a_0 - 2a_2)^2 + 2a_2^2 + 2(a_1 - 3a_3)^2 + 2a_3^2 = 2^{2\epsilon}$$

and

$$(4) \quad 2a_0a_2 - 4a_2^2 + 8a_1a_3 - 14a_3^2 - a_1^2 = 0.$$

From (3) and (4) we deduce that  $a_0, a_1, a_2$  and  $a_3$  are even. Furthermore, we have:  $4|a_1$ .

Suppose that  $\epsilon = 1$ . If  $a_2$  or  $a_3$  is not zero, then the left-hand side of (3) is  $> 4$  which is a contradiction. Hence  $a_2 = a_3 = 0$ . Similarly, we deduce that  $a_1 = 0$ . Then  $a_0 = \pm 2$  and so  $\rho \in \mathbb{Q}$  which is not the case.

Suppose next that  $\epsilon = 2$ . Putting  $a_i = a'_i$  ( $i = 0, 1, 2, 3$ ) we have

$$(5) \quad (a'_0 - 2a'_2)^2 + 2a'_2{}^2 + 2(a'_1 - 3a'_3)^2 + 2a'_3{}^2 = 4.$$

If  $a'_0 - 2a'_2 \neq 0$ , then (5) implies that  $a'_1 = a'_2 = a'_3 = 0$  and so  $\rho \in \mathbb{Q}$  which is a contradiction. Then  $a'_0 = 2a'_2$ . If  $a'_3 = 0$ , then (5) implies that  $a'_1 = \pm 1$  and so  $a_1 = \pm 2$ . Since  $4|a_1$  we obtain a contradiction. Thus  $a'_3 = \pm 1$ . If  $a'_1 - 3a'_3 = 0$ , then  $a_1 = \pm 6$  and so 4 does not divide  $a_1$  which is a contradiction. Finally suppose that  $a'_2 = 0$ . It follows that  $a'_1 - 3a'_3 = \pm 1$ . Thus we have

$$(a_0, a_1, a_2, a_3) = (0, 8, 0, 2), (0, -8, 0, -2), (0, 4, 0, -2), (0, -4, 0, 2).$$

We see that these values do not satisfy (4). Finally, we have  $(\bar{y} + y)/2 = 0$  and we see that the equation  $Y^4F(0, 1) = 3\sqrt{2} - 4$  has no solution in  $\mathbb{Q}(\sqrt{2})$ . The result follows.

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INSTITUT DE MATHÉMATIQUES DE TOULON, UNIVERSITÉ DE TOULON, FRANCE AND  
INSITUT DE MATHÉMATIQUES DE MARSEILLE, AIX-MARSEILLE UNIVERSITÉ, FRANCE  
*E-mail address:* `yves.aubry@univ-tln.fr`

ARISTOTLE UNIVERSITY OF THESSALONIKI, DEPARTMENT OF MATHEMATICS, THES-  
SALONIKI 54124, GREECE  
*E-mail address:* `poulakis@math.auth.gr`