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Yves Aubry, Dimitrios Poulakis. Thue equations and CM-fields. 2014. hal-01044876v3

HAL Id: hal-01044876

https://hal.science/hal-01044876v3

Preprint submitted on 9 Sep 2014 (v3), last revised 7 Jan 2015 (v5)

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THUE AND UNIT EQUATIONS OVER TOTALLY REAL NUMBER FIELDS

YVES AUBRY AND DIMITRIOS POULAKIS

Abstract. We obtain a polynomial type upper bound for the size of the integral solutions of Thue equations $F(X,Y) = m$ defined over a totally real number field K, assuming that $F(X,1)$ has at least a non real root and, for every couple of non real conjugate roots $(\alpha, \bar{\alpha})$ of $F(X,1)$, the field $K(\alpha, \bar{\alpha})$ is a CM-field. In case where $F(X,1)$ has also real roots, our approach gives polynomial type bounds that the Baker's method was not able to provide other than exponential bounds. Using this result, we derive an improved upper bound for the size of the solutions of the unit equation defined over a totally real number field, which allows us to deduce an upper bound for the size of the integral solutions of Thue equations defined over a totally real number field.

1. Introduction

Let $F(X,Y)$ be an irreducible binary form in $\mathbb{Z}[X,Y]$ with $\deg F \geq 3$ and $m \in \mathbb{Z} \setminus \{0\}$. In 1909, A. Thue [31] proved that the equation $F(X,Y) = m$ has only finitely many solutions $(x,y) \in \mathbb{Z}^2$. Thue's proof was ineffective and therefore does not provide a method to determine the integer solutions of this equation. Other non effective proofs of Thue's result can be found in [9, Chap. X] and [24, Chap. 23].

In 1968, A. Baker [2], using his results on linear forms in logarithms of algebraic numbers, computed an explicit upper bound for the size of the integer solutions of Thue equations. Baker's result were improved by several authors (see for instance [8], [13], [25]) but the bounds remain of exponential type and thus, are not useful to compute integer solutions of such equations. Nevertheless, computation techniques for the resolution of Thue equations have been developed based on the above results [1], [14], [32] and the solutions of certain parameterized families of Thue equations have been obtained [15]. Furthermore, upper bounds for the number of integral solutions of Thue equations have been given [7], [10], [6].

In the case where all roots of the polynomial $F(X,1)$ are non real, we have a polynomial type bound provided by other methods [24, Theorem 2, page 186], [12] [26]. Győry's improvement in [12, Théorème 1] holds in the case where the splitting field of $F(X,1)$ is a CM-field i.e., is an

2000 *Mathematics Subject Classification.* 11D59, 11G30, 11G50, 14H25.
Key words and phrases. Thue equations, integer points, unit equations.

imaginary quadratic extension of a totally real number field. In the same paper, Győry studied Thue equations defined over a CM-field L and also gave ([12, Théorème 2]) a polynomial upper bound for the size of their real algebraic integers solutions in L.

In this paper, we consider Thue equations $F(X,Y) = m$ defined over a totally real number field K. Following Győry's approach, we obtain (Theorem 1) polynomial type bounds for the size of their integral solutions over K, assuming that $F(X,1)$ has at least a non real root and, for every couple of non real conjugate roots $(\alpha, \bar{\alpha})$ of $F(X,1)$, the field $K(\alpha, \bar{\alpha})$ is a CM-field. In case where the splitting field is a CM-field we are in the situation of [12, Théorème 2]. Whenever all roots of the polynomial $F(X,1)$ are non real and $K \neq \mathbb{Q}$, we obtain much better bounds than those already known. Moreover, whenever $F(X,1)$ has a real and a non real root, we obtain polynomial type bounds that the Baker's method was not able to provide other than exponential bounds.

We illustrate our result by giving two examples of infinite families of Thue equations $F(X,Y) = m$ satisfying the hypothesis of Theorem 1. In the first, we consider Thue equations over some totally real subfields K of cyclotomic fields N such that the splitting field L of $F(X,1)$ over K is contained in N. In this case, L is an abelian extension of K. In the second , we consider some quartic Thue equations over $\mathbb{Q}$ whose splitting field N of $F(X,1)$ over $\mathbb{Q}$ has dihedral Galois group.

Effective bounds have been established for the size of the solutions of elliptic, hyperelliptic, superelliptic equations, Thue equations, equations defining genus 0 and 1 algebraic curves and other classes of curves (see e.g. [18, Chapter VI]) by reduction to the problem of the computation of a bound for the solutions of a unit equation. The computation of such bounds for the unit equation, except in [4, 5], are based on the Baker's method and its p-adic analogue as well as certain quantitative results concerning independent units. Using Theorem 1 and [26, Theorem 2] we deduce an improved explicit upper bound for the size of solutions of the unit equation in case where it is defined over a totally real number field. This result allows the computation of explicit upper bounds for the size of the integral solutions of elliptic, hyperelliptic, superelliptic equations, Thue equations, Mordell equations ([21, Proposition 3.1]) and consequently of equations defining genus 0 and 1 algebraic curves and other classes of curves which are defined over totally real number fields, without the contribution of Baker's theory. Finally, using the classical method we derive an upper bound for the size of the integral solutions of Thue equations defined over totally real number fields.

2. New bounds

We introduce a few notations. Let K be a number field. We consider the set of absolute values of K by extending the ordinary absolute value $|\cdot|$ of $\mathbb{Q}$ and, for every prime p, by extending the p-adic absolute value $|\cdot|_p$ with $|p|_p = p^{-1}$. Let $M(K)$ be an indexing set of symbols v such that $|\cdot|_v$, $v \in M(K)$, are all of the above absolute values of K. Given such an absolute value $|\cdot|_v$ on K, we denote by d_v its local degree. Let $\mathbf{x} = (x_0 : \ldots : x_n)$ be a point of the projective space $\mathbb{P}^n(K)$ over K. We define the field height $H_K(\mathbf{x})$ of $\mathbf{x}$ by

$$H_K(\mathbf{x}) = \prod_{v \in M(K)} \max\{|x_0|_v, \ldots, |x_n|_v\}^{d_v}.$$

Let d be the degree of K. We define the absolute height $H(\mathbf{x})$ by $H(\mathbf{x}) = H_K(\mathbf{x})^{1/d}$. For $x \in K$, we put $H_K(x) = H_K((1 : x))$ and $H(x) = H((1 : x))$. If $G \in K[X_1, \ldots, X_m]$, then we define the field height $H_K(G)$ and the absolute height $H(G)$ of G as the field height and the absolute height respectively of the point whose coordinates are the coefficients of G (in any order). For an account of the properties of heights see [16, 19, 30]. Furthermore, we denote by O_K, U_K, D_K, R_K and N_K the ring of integers of K, the unit group of O_K, the discriminant of K, the regulator of K and the norm of K, respectively. Finally, for every $z \in \mathbb{C}$ we denote, as usually, by $\bar{z}$ its complex conjugate.

We prove the following theorem:

Theorem 1. *Suppose that K is a totally real number field. Let $b \in O_K \setminus \{0\}$ and $F(X,Y) \in O_K[X,Y]$ be a form of degree $n \geq 2$. Suppose that $F(X,1)$ has at least a non real root and for every couple of non real conjugate roots $(\alpha, \bar{\alpha})$ of $F(X,1)$ the field $K(\alpha, \bar{\alpha})$ is a CM-field. Then the solutions $(x,y) \in O_K^2$ of $F(X,Y) = b$ satisfy*

$$H(x) < \Omega_1 \quad \text{and} \quad H(y) < \Omega_2$$

for the following values of Ω_1 and Ω_2. If the coefficients of X^n and Y^n are ± 1, then

$$\Omega_1 = \Omega_2 = 2^8 H(F) H(b)^2.$$

If only the coefficient of X^n is ± 1, then

$$\Omega_1 = 2^{12} H(F)^2 H(b)^4 \quad \text{and} \quad \Omega_2 = 2^8 H(F) H(b)^2.$$

Otherwise, we have

$$\Omega_1 = 2^{12} H(b)^4 H(\Gamma)^{6n-4} \quad \text{and} \quad \Omega_2 = 2^8 H(b)^2 H(\Gamma)^{3n-2},$$

where Γ is a point of the projective space with 1 and the coefficients of $F(X,Y)$ as coordinates.

Notice that a non real algebraic number field L is a CM-field if and only if L is closed under the operation of complex conjugation and

complex conjugation commutes with all the $\mathbb{Q}$ -monomorphisms of L into $\mathbb{C}$ ([3], [22, Lemma 2]).

When $K = \mathbb{Q}$ and the splitting field of $F(X, 1)$ over $\mathbb{Q}$ is an abelian totally imaginary extension, the hypothesis on complex conjugation is obviously satisfied. If the coefficient of X^n is ± 1, it is interesting to notice that our bounds are independent of the degree of the form $F(X, Y)$. Thus, in case where $H(F)$ and $H(b)$ are not too large, an exhaustive search can provide the integer solutions we are looking for.

Combining Theorem 1 with [26, Theorem 2] we have the following result.

Theorem 2. *Let K be a totally real number field of degree $d \geq 2$. Let $a, b, c \in O_K \setminus \{0\}$. Then the solutions $(u, v) \in U_K^2$ to the equation*

$$aX + bY = c$$

satisfy

$$\max\{H(u), H(v)\} < (H(a)H(b)H(c))^{128} \exp\{\omega(d)R_K\}$$

where $\omega(d) = 1446$ if $d = 2$ and

$$\omega(d) = 102795(d-1)((d-1)!)^2\sqrt{d-2}(\log d)2^{2-d},$$

otherwise.

As far as we know the best upper bound for the size of the solutions of the unit equation is given in [13, Theorem 1]. Theorem 2 is an improvement of this result for the case of totally real number fields. Using this result we obtain with the classical method an upper bound for the size of the integral solutions of Thue equations over totally real number field.

Corollary 1. *Let K be a totally real number field of degree d. Suppose that $m \in O_K \setminus \{0\}$ and*

$$F(X, Y) = (X - \alpha_1 Y)(X - \alpha_2 Y)(X - \alpha_3 Y)G(X, Y),$$

where $\alpha_1, \alpha_2, \alpha_3$ are distinct elements of O_K and $G(X, Y)$ a form of $O_K[X, Y]$. Then the solutions $(x, y) \in O_K^2$ to the equation $F(x, y) = m$ satisfy

$$\max\{H(x), H(y)\} < 2^{2313} H(F)^{1541} H(m)^{2/n} N_K(m)^{784/d} \exp\{\Omega_2(d)R_K\},$$

where

$$\Omega_2(d) = 104320(d-1)((d-1)!)^2\sqrt{d-2}(\log d).$$

3. Examples

In this section we give a few examples to illustrate our results. We denote by $F^*(X, Y)$ the homogenization of a polynomial $F(X) \in \mathbb{C}[X]$.

Example 1. Let p be a prime with $p \equiv 1 \pmod 4$ and ζ_p a p-th primitive root of unity in $\mathbb{C}$. Then the quadratic field $\mathbb{Q}(\sqrt{p})$ is a subfield of $\mathbb{Q}(\zeta_p)$. Then $\mathbb{Q}(\zeta_p)$ is a cyclic extension of $\mathbb{Q}$ with Galois group $G \simeq (\mathbb{Z}/p\mathbb{Z})^*$. Thus, every $\mathbb{Q}$-embedding of $\mathbb{Q}(\zeta_p)$ into $\mathbb{C}$ defines a $\mathbb{Q}$-automorphism of $\mathbb{Q}(\zeta_p)$ and so, it commutes with the complex conjugation.

Let $\alpha \in \mathbb{Z}[\zeta_p]$ be a primitive element of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{p})$ and $\alpha_1, \ldots, \alpha_m$, with $m = (p-1)/2$, all the distinct conjugates of α over $\mathbb{Q}(\sqrt{p})$. The largest real field contained in $\mathbb{Q}(\zeta_p)$ is $K_p = \mathbb{Q}(\zeta_p + \bar{\zeta}_p)$ which is a totally real number field. Let $\beta \in K_p$ be a primitive element of the extension $K_p/\mathbb{Q}(\sqrt{p})$ and $\beta_1, \ldots, \beta_n$, where $n = (p-1)/4$, all the distinct conjugates of β over $\mathbb{Q}(\sqrt{p})$. Then the polynomial

$$F(X) = (X - \alpha_1) \cdots (X - \alpha_m)(X - \beta_1) \cdots (X - \beta_n)$$

belongs to $\mathbb{Q}(\sqrt{p})[X]$ and have real and non real roots. Consequently, for every non zero $b \in \mathbb{Z}[(1+\sqrt{p})/2]$, the Thue equation $F^*(X,Y) = b$ satisfies the hypothesis of Theorem 1.

Furthermore, using [30, Theorem 5.9, page 211] and [30, Lemma 5.10, page 213], we obtain the following upper bound for the heights of solutions $x, y \in \mathbb{Z}[(1+\sqrt{p})/2]$:

$$H(x) < 2^{(3p+17)/4} H(\alpha)^{p-1} H(\beta)^{(p-1)/2} H(b)^4$$

and

$$H(y) < 2^{(3p+25)/4} H(\alpha)^{(p-1)/2} H(\beta)^{(p-1)/4} H(b)^2.$$

If $\Phi_p(X)$ is the p-th cyclotomic polynomial, then [12, Section 2] implies that the maximum of the absolute values of all algebraic integers $x, y \in K_p$ with $\Phi_p^*(x,y) = 1$ is $< 2^{(p-1)/2}$. Theorem 1 improves this result by yielding the bound 2^8.

If we want to apply Theorem 1, the Galois group $\mathrm{Gal}(N/K)$, where N is the splitting field over K of the polynomial F, needs not to be abelian as shown in the following example of a dihedral extension.

Example 2. Let m and n be two non negative rational integers such that $n \geq 5$ is not a square and $m < \sqrt{n}$. Consider the irreducible polynomial

$$F(X) = X^4 + 2mX^2 + m^2 - n$$

of $\mathbb{Q}[X]$. The roots of $F(X)$ are: $\pm i\sqrt{\sqrt{n}+m}$ and $\pm\sqrt{\sqrt{n}-m}$.

The field $L = \mathbb{Q}(i\sqrt{\sqrt{n}+m}) = (\mathbb{Q}(\sqrt{n}))(i\sqrt{\sqrt{n}+m})$ is a CM-field, and so we can apply Theorem 1 to the equation $F^*(X,Y) = b$, where b is a non zero integer .

On the one hand, it is worth noticing that in this case we cannot apply Runge's method [11], [33]. Additionally, since the polynomial $F(X)$ has real roots, we can neither apply [24, Theorem 2, page 186], nor [26, Theorem 1] which are the only known results which can provide polynomial bounds for the size of the integer solutions. On the other

hand, Baker's method renders exponential bounds on the heights of the integer solutions of such an equation (see for example [13]). Moreover, since the splitting field N of $F(X)$ is not a CM field we cannot apply [12, Théorème 2].

Theorem 1 yields the following bounds for the size of the solutions $(x, y) \in \mathbb{Z}^2$ of the above equation:

$$|x| < 2^{12} n^2 b^4, \qquad |y| < 2^8 n b^2.$$

Finally, note that, by [17, Theorem 3], the Galois group $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to the dihedral group of order 8.

4. Proof of Theorem 1

Write

$$F(X, Y) = a_0(X - \alpha_1 Y) \cdots (X - \alpha_n Y).$$

First, we consider the case where $a_0 = \pm 1$. If $a_0 = -1$, we replace $F(X, Y)$ by $-F(X, Y)$ and b by $-b$ and then we may suppose that $a_0 = 1$. We denote by J the set of indexes j such that $\alpha_j \in \mathbb{C} \setminus \mathbb{R}$. Put $t = |J|$. By our hypothesis, we have $t > 0$. If $z \in \mathbb{C}$, we denote, as usual, by $\bar{z}$ the complex conjugate of z.

Let $x, y \in O_K$ such that $xy \neq 0$ and $F(x, y) = b$. We set

$$x - \alpha_j y = b_j \quad (j = 1, \ldots, n).$$

For every $j \in J$, we set $\rho_j = \bar{b}_j / b_j$. Since K is a totally real number field, we have

$$x - \bar{\alpha}_j y = \rho_j b_j.$$

Eliminating x and b_j from the above two equations, we get

$$y = \frac{b_j(1 - \rho_j)}{\bar{\alpha}_j - \alpha_j}, \quad x = y \frac{\bar{\alpha}_j - \alpha_j \rho_j}{1 - \rho_j}.$$

Let $j_0 \in J$. For $j \notin J$, we get

$$b_j = x - \alpha_j y = y \frac{(\bar{\alpha}_{j_0} - \alpha_j) + \rho_{j_0}(\alpha_j - \alpha_{j_0})}{1 - \rho_{j_0}}.$$

Combining the above equalities, we obtain

$$y^n = b(1 - \rho_{j_0})^{n-t} \prod_{j \in J} \frac{1 - \rho_j}{\bar{\alpha}_j - \alpha_j} \prod_{j \notin J} \frac{1}{(\bar{\alpha}_{j_0} - \alpha_j) + \rho_{j_0}(\alpha_j - \alpha_{j_0})}.$$

Let $K_j = K(\alpha_j, \bar{\alpha}_j)$, $j \in J$. We denote by G_j the set of $\mathbb{Q}$-embedding $\sigma : K_j \to \mathbb{C}$. Therefore, for every $j \in J$, we have $\sigma(\bar{b}_j) = \overline{\sigma(b_j)}$, and so we get

$$|\sigma(\rho_j)| = \frac{|\sigma(\bar{b}_j)|}{|\sigma(b_j)|} = \frac{|\overline{\sigma(b_j)}|}{|\sigma(b_j)|} = 1.$$

The elements α_j, $\bar{\alpha}_j$ are algebraic integers and so, b_j, $\bar{b}_j$ are algebraic integers of L. Let $M_j(X)$ be the minimal polynomial of ρ_j over $\mathbb{Z}$ and m_j its leading coefficient. Since ρ_j is a root of the polynomial

$$\Pi_j(X) = \prod_{\sigma \in G_j} \sigma(b_j)(X - \sigma(\rho_j)),$$

which has integer coefficients, we have that $M_j(X)$ divides $\Pi_j(X)$ and thus we deduce that m_j divides

$$\prod_{\sigma \in G_j} \sigma(b_j) = N_{K_j}(b_j),$$

where N_{K_j} is the norm related to the extension $K_j/\mathbb{Q}$. It follows that m_j divides $N_{K_j}(b_j)$. As we saw above, all the conjugates of ρ_j are of absolute value 1. Therefore, by [19, page 54], for every $j \in J$ we have

$$H_{K_j}(\rho_j) = m_j \prod_{\sigma \in G} \max\{1, |\sigma(\rho_j)|\} \leq N_{K_j}(b_j) < N_{K_j}(b) < H_{K_j}(b).$$

Further, using elementary properties of heights and the above inequality, we deduce

$$H(y)^n < 2^{4n-2t} H(b)^{2n} H(\alpha_{j_0})^{2(n-t)} \prod_{i=1}^{n} H(\alpha_i)^2.$$

By [30, Theorem 5.9, page 211], we have

$$\prod_{i=1}^{n} H(\alpha_i) \leq 2^n H(F).$$

For every $j \in J$, the number α_j is not real, hence, using [23], we deduce that $H(\alpha_j) < 2H(F)^{1/2}$. Therefore, we obtain

$$H(y) < 2^8 H(b)^2 H(F).$$

And it follows that

$$H(x) \leq 4H(y)H(\alpha_j)^2 H(\rho_j)^2 \leq 16H(y)H(F)H(b)^2 < 2^{12} H(F)^2 H(b)^4.$$

Suppose now that $a_0 \neq \pm 1$. Write $F(X,1) = a_0 X^n + a_1 X^{n-1} + \cdots + a_n$. Then $a_0\alpha_i$ is a root of $f(X) = X^n + a_1 X^{n-1} + a_2 a_0 X^{n-2} + \cdots + a_n a_0^{n-1}$ and thus $a_0\alpha_i$ is an algebraic integer. Denote by $F_1(X,Y)$ the homogenization of $f(X)$. If $(x,y) \in O_K^2$ is a solution to $F(X,Y) = b$, then $(a_0 x, y)$ is a solution to $F_1(X,Y) = b a_0^{n-1}$. Denote by Γ a point in the projective space with 1 and the coefficients of F as coordinates. Then we have $H(F_1) \leq H(\Gamma)^n$ and finally, we obtain

$$H(y) < 2^8 H(b)^2 H(\Gamma)^{3n-2} \quad \text{and} \quad H(x) < 2^{12} H(b)^4 H(\Gamma)^{6n-4}.$$

5. Proof of Theorem 2

By [13, Lemma 2], there is a fundamental system $\{\epsilon_1, \dots, \epsilon_{d-1}\}$ of units satisfying

$$\max\{H(\epsilon_1), \dots, H(\epsilon_{d-1})\} \leq \exp\{c(d)R_K\},$$

where

$$c(d) = ((d-1)!)^2 29e\sqrt{d-2}(\log d)2^{2-d}$$

if $d \geq 3$ and $c(d) = 1/2$ if $d = 2$.

Let $(u, v) \in U_K^2$ be a solution to the equation

$$aX + bY = c.$$

Then, we write $u = u_1 u_2^4$ and $v = v_1 v_2^4$, where $u_2, v_2 \in U_K$ and $u_1 = \pm\epsilon_1^{r_1} \cdots \epsilon_{d-1}^{r_{d-1}}$, $v_1 = \pm\epsilon_1^{s_1} \cdots \epsilon_{d-1}^{s_{d-1}}$ with $r_i, s_i \in \{0, 1, 2, 3\}$. It follows that (u_2, v_2) is a solution to the Thue equation

$$au_1X^4 + bv_1Y^4 = c.$$

We have two cases:

(i) $bv_1/au_1 = -A$, with $A > 0$. Then, the roots of $X^4 - A$ are $\pm\sqrt[4]{A}$ and $\pm i\sqrt[4]{A}$. Further, we have $K(i\sqrt[4]{A}) = K(\sqrt{A})(\sqrt{-\sqrt{A}})$. Thus, Theorem 1 implies

$$\max\{H(u_2), H(v_2)\} < 2^{12}H(c)^4H(1, au_1, bv_1)^{20},$$

whence we get

$$\max\{H(u_2), H(v_2)\} < 2^{12}H(c)^4(H(a)H(b))^{20}\exp\{120(d-1)c(d)R_K\}.$$

Therefore, we obtain

$$\max\{H(u), H(v)\} < 2^{48}H(c)^{16}(H(a)H(b))^{80}\exp\{483(d-1)c(d)R_K\}.$$

(ii) $bv_1/au_1 = A > 0$. Then, the roots of $X^4 + A$ are $\pm\sqrt{2}(1 \pm i)\sqrt[4]{A}/2$ which all are not real. Thus [26, Theorem 2] yields

$$\max\{H(u_2), H(v_2)\} < 64^{64}(H(a)H(b)H(c))^{32}\exp\{192(d-1)c(d)R_K\},$$

whence we have

$$\max\{H(u), H(v)\} < 64^{256}(H(a)H(b)H(c))^{128}\exp\{771(d-1)c(d)R_K\}.$$

The result follows.

6. Proof of Corollary 1

Suppose that $(x, y) \in O_K^2$ with $y \neq 0$ and $F(x, y) = m$. It follows that $x - \alpha_i y$ divides m in K and so, [13, Lemma 3] implies that there exist $\beta_i \in O_L$ and $\epsilon_i \in U_L$ such that

$$x - \alpha_i y = \epsilon_i \beta_i \quad (i = 1, 2, 3)$$

with

$$H(\beta_i) \leq N_K(m)^{1/d}\exp\{C(d)R_K\},$$

where $C(d) = 0$ if $d = 1$, $C(d) = 1/2$ if $d = 2$ and

$$C(d) = 29e(d-1)(d-1)!\sqrt{d-2}\log d$$

otherwise.

Multiplying $x - \alpha_1 y$ by $\alpha_2 - \alpha_3$, $x - \alpha_2 y$ by $\alpha_3 - \alpha_1$, $x - \alpha_3 y$ by $\alpha_1 - \alpha_2$ and adding, we obtain

$$au + bv = c,$$

where $u = \epsilon_1/\epsilon_3$, $v = \epsilon_2/\epsilon_3$ and

$$a = (\alpha_2 - \alpha_3)\beta_1, \quad b = (\alpha_3 - \alpha_1)\beta_2, \quad c = (\alpha_2 - \alpha_1)\beta_3.$$

By [25, Lemma 4], we have $H(\alpha_i) < 2H(F)$ $(i = 1, 2, 3)$. Thus, we get

$$H(a) \leq 2H(\alpha_2)H(\alpha_3)H(\beta_1) < 8H(F)^2 N_K(m)^{1/d} \exp\{C(d)R_K\}.$$

Similarly, the quantities $H(b)$ and $H(c)$ satisfy the same inequality. By Theorem 2, we get

$$\max\{H(u), H(v)\} < (H(a)H(b)H(c))^{128} \exp\{\omega(d)R_d\}$$

where

$$\omega(d) = 102795(d-1)((d-1)!)^2\sqrt{d-2}(\log d)2^{2-d}.$$

Combining the above estimates, we have

$$\max\{H(u), H(v)\} < (8H(F)^2 N_K(m)^{1/d})^{384} \exp\{(384C(d) + \omega(d))R_K\}.$$

We set $\epsilon_1 = u\epsilon_3$ and $\epsilon_2 = v\epsilon_3$ into $x - \alpha_i y = \epsilon_i \beta_i$ $(i = 1, 2, 3)$ and eliminating ϵ_3 we deduce

$$(\beta_3 - \beta_1 u)x - (\beta_3\alpha_1 - \beta_1 u\alpha_3)y = 0,$$

$$(\beta_3 - \beta_2 v)x - (\beta_3\alpha_2 - \beta_2 v\alpha_3)y = 0.$$

Suppose that $\beta_3 = \beta_1 u$. Since $y \neq 0$, we have $\beta_3\alpha_1 = \beta_1 u\alpha_3$. Replacing β_3 by $\beta_1 u$, we get $\alpha_1 = \alpha_3$ which is a contradiction. Thus $\beta_3 - \beta_1 u \neq 0$ and so, we have $x = By$, where

$$B = \frac{\beta_3\alpha_1 - \beta_1 u\alpha_3}{\beta_3 - \beta_1 u}.$$

We have

$$H(B) \leq H(\beta_3\alpha_1 - \beta_1 u\alpha_3)H(\beta_3 - \beta_1 u) \leq 4H(\beta_3)^2 H(\beta_1)^2 H(\alpha_1)H(\alpha_3)H(u),$$

whence we get

$$H(B) < 2^{1156} H(F)^{770} N_K(m)^{392/d} \exp\{(\omega(d) + 388C(d))R_K\}.$$

Since $x = By$, we have $y^n F(B, 1) = m$, and so we deduce

$$H(y)^n \leq H(F(B, 1))H(m) \leq (n+1)H(F)H(B)^n H(m).$$

Furthermore, we have $H(x) \leq H(B)H(x)$. Combining the above estimates we deduce the result.

Acknowledgements. This work was done during the visit of the second author at the Department of Mathematics of the University of Toulon. The second author wants to thanks this Department for its warm hospitality and fruitful collaboration.

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