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YVES AUBRY AND DIMITRIOS POULAKIS

Abstract. We obtain polynomial type bounds for the size of the integral solutions of Thue equations $F(X,Y) = m$ defined over a totally real number field K, assuming that $F(X,1)$ has at least a non real root and, for every couple of non real conjugate roots $(\alpha, \bar{\alpha})$ of $F(X,1)$, the field $K(\alpha, \bar{\alpha})$ is a CM-field. In case where $F(X,1)$ has also real roots, our approach gives polynomial type bounds that the Baker's method was not able to provide other than exponential bounds.

1. Introduction

Let $F(X,Y)$ be an irreducible binary form in $\mathbb{Z}[X,Y]$ with $\deg F \geq 3$ and $m \in \mathbb{Z} \setminus \{0\}$. In 1909, A. Thue [26] proved that the equation $F(X,Y) = m$ has only finitely many solutions $(x,y) \in \mathbb{Z}^2$. Thue's proof was ineffective and therefore does not provide a method to determine the integer solutions of this equation. Other non effective proofs of Thue's result can be found in [7, Chap. X] and [20, Chap. 23].

In 1968, A Baker [2], using his results on linear forms in logarithms of algebraic numbers, computed an explicit upper bound for the size of the integer solutions of Thue equations. Baker's result were improved by several authors (see for instance [6], [11], [21]) but the bounds remain of exponential type and thus, are not useful to compute integer solutions of such equations. Nevertheless, computation techniques for the resolution of Thue equations have been developed based on the above results [1], [12], [27] and the solutions of certain parameterized families of Thue equations have been obtained [13]. Furthermore, upper bounds for the number of integral solutions of Thue equations have been given [5], [8], [4].

In the case where all roots of the polynomial $F(X,1)$ are non real, we have a polynomial type bound provided by other methods [20, Theorem 2, page 186], [10] [22]. Győry's improvement in [10, Théorème 1] holds in the case where the splitting field of $F(X,1)$ is a CM-field i.e., is an imaginary quadratic extension of a totally real number field. In the same paper, Győry studied Thue equations defined over a CM-field L

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and also gave ([10, Théorème 2]) a polynomial upper bound for the size of their real algebraic integers solutions in L.

In this paper, we consider Thue equations $F(X,Y)=m$ defined over a totally real number field K. Following Győry's approach, we obtain (Theorem 1) polynomial type bounds for the size of their integral solutions over K, assuming that $F(X,1)$ has at least a non real root and, for every couple of non real conjugate roots $(\alpha,\bar{\alpha})$ of $F(X,1)$, the field $K(\alpha,\bar{\alpha})$ is a CM-field. In case where the splitting field is a CM-field we are in the situation of [10, Théorème 2]. Whenever all roots of the polynomial $F(X,1)$ are non real and $K\neq\mathbb{Q}$, we obtain much better bounds than those already known. Moreover, whenever $F(X,1)$ has a real and a non real root, we obtain polynomial type bounds that the Baker's method was not able to provide other than exponential bounds.

We illustrate our result by giving two examples of infinite families of Thue equations $F(X,Y)=m$ satisfying the hypothesis of Theorem 1. In the first, we consider Thue equations over some totally real subfields K of cyclotomic fields N such that the splitting field L of $F(X,1)$ over K is contained in N. In this case, L is an abelian extension of K. In the second , we consider some quartic Thue equations over $\mathbb{Q}$ whose splitting field N of $F(X,1)$ over $\mathbb{Q}$ has dihedral Galois group.

Finally, we prove two corollaries of Theorem 1. The first gives bounds for the size of integral solutions of a class of equations of the form $G(X,Y)=E(X,Y)$ where $G(X,Y)$ and $E(X,Y)$ are forms of different degrees. The second gives a partial effective version of a classic result on Diophantine approximation by means of the existing link between rational approximation and Thue equations ([17, Proposition 2.1]).

2. New bounds

We introduce a few notations. Let K be a number field. We consider the set of absolute values of K by extending the ordinary absolute value $|\cdot|$ of $\mathbb{Q}$ and, for every prime p, by extending the p-adic absolute value $|\cdot|_p$ with $|p|_p=p^{-1}$. Let $M(K)$ be an indexing set of symbols v such that $|\cdot|_v$, $v\in M(K)$, are all of the above absolute values of K. Given such an absolute value $|\cdot|_v$ on K, we denote by d_v its local degree. Let $\mathbf{x}=(x_0:\ldots:x_n)$ be a point of the projective space $\mathbb{P}^n(K)$ over K. We define the field height $H_K(\mathbf{x})$ of $\mathbf{x}$ by

$$H_K(\mathbf{x})=\prod_{v\in M(K)}\max\{|x_0|_v,\ldots,|x_n|_v\}^{d_v}.$$

Let d be the degree of K. We define the absolute height $H(\mathbf{x})$ by $H(\mathbf{x})=H_K(\mathbf{x})^{1/d}$. For $x\in K$, we put $H_K(x)=H_K((1:x))$ and $H(x)=H((1:x))$. If $G\in K[X_1,\ldots,X_m]$, then we define the field height $H_K(G)$ and the absolute height $H(G)$ of G as the field height and the absolute height respectively of the point whose coordinates are the coefficients of G (in any order). For an account of the properties of heights see [14], [16], [25]. Furthermore, we denote by O_K, D_K, R_K and N_K the ring of integers of K, the discriminant of K, the regulator of K and the norm of K, respectively. Finally, for every $z \in \mathbb{C}$ we denote, as usually, by $\bar{z}$ its complex conjugate.

We prove the following theorem:

Theorem 1. *Let K be a totally real number field and O_K its ring of integers. Let $b \in O_K \setminus \{0\}$ and $F(X, Y) \in O_K[X, Y]$ be a form of degree $n \geq 2$. Suppose that $F(X, 1)$ has at least a non real root and for every couple of non real conjugate roots $(\alpha, \bar{\alpha})$ of $F(X, 1)$ the field $K(\alpha, \bar{\alpha})$ is a CM-field. Then the solutions $(x, y) \in O_K^2$ of $F(X, Y) = b$ satisfy*

$$H(x) < \Omega_1 \quad \text{and} \quad H(y) < \Omega_2$$

for the following values of Ω_1 and Ω_2. If the coefficients of X^n and Y^n are ± 1, then

$$\Omega_1 = \Omega_2 = 2^8 H(F) H(b)^2.$$

If only the coefficient of X^n is ± 1, then

$$\Omega_1 = 2^{12} H(F)^2 H(b)^4 \quad \text{and} \quad \Omega_2 = 2^8 H(F) H(b)^2.$$

Otherwise, we have

$$\Omega_1 = 2^{12} H(b)^4 H(\Gamma)^{6n-4} \quad \text{and} \quad \Omega_2 = 2^8 H(b)^2 H(\Gamma)^{4n-2},$$

where Γ is a point of the projective space with 1 and the coefficients of $F(X, Y)$ as coordinates.

Notice that a non real algebraic number field L is a CM-field if and only if L is closed under the operation of complex conjugation and complex conjugation commutes with all the $\mathbb{Q}$ -monomorphisms of L into $\mathbb{C}$ ([3], [18, Lemma 2]).

When $K = \mathbb{Q}$ and the splitting field of $F(X, 1)$ over $\mathbb{Q}$ is an abelian totally imaginary extension, the hypothesis on complex conjugation is obviously satisfied. If the coefficient of X^n is ± 1, it is interesting to notice that our bounds are independent of the degree of the form $F(X, Y)$. Thus, in case where $H(F)$ and $H(b)$ are not too large, an exhaustive search can provide the integer solutions we are looking for.

Theorem 1 yields the following two corollaries.

Corollary 1. *Let K be a totally real number field and O_K its ring of integers. Let $G(X, Y) \in O_K[X, Y]$ be a form of degree $n \geq 2$ satisfying the hypothesis of Theorem 1. If $E(X, Y) \in O_K[X, Y]$ is a form relatively prime to $G(X, Y)$ with $\deg E = m < n$, then the solutions $(x, y) \in O_K$ to the equation $G(X, Y) = E(X, Y)$ satisfy*

$$\max\{H(x), H(y)\} < (m + n)^{17n(m+n)^2} H(\Omega)^{2n(8m+14n)(m+n)} |D_K|^{4n(m+n)^2/d} \exp\{8(m + n)n^2 C(d) R_K\}$$

where Ω is a point in a projective space with 1 and the coefficients of $G(X,Y)$ and $E(X,Y)$ as coordinates. Furthermore, $C(d)=0$ if $d=1$, $C(d)=1/2$ if $d=2$ and

$$C(d) = 29e(d-1)(d-1)!\sqrt{d-2}\log d$$

otherwise.

Corollary 2. *Let $f(X) \in \mathbb{Z}[X]$ be an irreducible polynomial of degree $d>1$ with a non real root and relatively prime coefficients. Furthermore, for every couple of non real conjugate roots $(\alpha, \bar{\alpha})$ of $f(X)$, suppose that the field $\mathbb{Q}(\alpha, \bar{\alpha})$ is a CM-field. Let k be a positive real number and α a root of $f(X)$. If p, $q>0$ are relatively prime integers satisfying*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{k}{q^d},$$

then

$$\max\{|p|,|q|\} < 2^{4d^2+12}\max\{k,1\}^{4d}H(\Gamma)^{4d^2+2d},$$

where Γ is a point of the projective space with 1 and the coefficients of $f(X)$ as coordinates.

3. Examples

In this section we give a few examples to illustrate our results. We denote by $F^*(X,Y)$ the homogenization of a polynomial $F(X) \in \mathbb{C}[X]$.

Example 1. Let p be a prime with $p \equiv 1 \pmod 4$ and ζ_p a p-th primitive root of unity in $\mathbb{C}$. Then the quadratic field $\mathbb{Q}(\sqrt{p})$ is a subfield of $\mathbb{Q}(\zeta_p)$. Then $\mathbb{Q}(\zeta_p)$ is a cyclic extension of $\mathbb{Q}$ with Galois group $G \simeq (\mathbb{Z}/p\mathbb{Z})^*$. Thus, every $\mathbb{Q}$-embedding of $\mathbb{Q}(\zeta_p)$ into $\mathbb{C}$ defines a $\mathbb{Q}$-automorphism of $\mathbb{Q}(\zeta_p)$ and so, it commutes with the complex conjugation.

Let $\alpha \in \mathbb{Z}[\zeta_p]$ be a primitive element of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{p})$ and $\alpha_1, \ldots, \alpha_m$, with $m=(p-1)/2$, all the distinct conjugates of α over $\mathbb{Q}(\sqrt{p})$. The largest real field contained in $\mathbb{Q}(\zeta_p)$ is $K_p = \mathbb{Q}(\zeta_p + \bar{\zeta}_p)$ which is a totally real number field. Let $\beta \in K_p$ be a primitive element of the extension $K_p/\mathbb{Q}(\sqrt{p})$ and $\beta_1, \ldots, \beta_n$, where $n=(p-1)/4$, all the distinct conjugates of β over $\mathbb{Q}(\sqrt{p})$. Then the polynomial

$$F(X) = (X-\alpha_1)\cdots(X-\alpha_m)(X-\beta_1)\cdots(X-\beta_n)$$

belongs to $\mathbb{Q}(\sqrt{p})[X]$ and have real and non real roots. Consequently, for every non zero $b \in \mathbb{Z}[(1+\sqrt{p})/2]$, the Thue equation $F^*(X,Y)=b$ satisfies the hypothesis of Theorem 1.

Furthermore, using [25, Theorem 5.9, page 211] and [25, Lemma 5.10, page 213], we obtain the following upper bound for the heights of solutions $x, y \in \mathbb{Z}[(1+\sqrt{p})/2]$:

$$H(x) < 2^{(3p+17)/4}H(\alpha)^{p-1}H(\beta)^{(p-1)/2}H(b)^4$$

and

$$H(y) < 2^{(3p+25)/4} H(\alpha)^{(p-1)/2} H(\beta)^{(p-1)/4} H(b)^2.$$

If $\Phi_p(X)$ is the p-th cyclotomic polynomial, then [10, Section 2] implies that the maximum of the absolute values of all algebraic integers $x, y \in K_p$ with $\Phi_p^*(x, y) = 1$ is $< 2^{(p-1)/2}$. Theorem 1 improves this result by yielding the bound 2^8.

If we want to apply Theorem 1, the Galois group $\mathrm{Gal}(N/K)$, where N is the splitting field over K of the polynomial F, needs not to be abelian as shown in the following example of a dihedral extension.

Example 2. Let m and n be two non negative rational integers such that $n \geq 5$ is not a square and $m < \sqrt{n}$. Consider the irreducible polynomial

$$F(X) = X^4 + 2mX^2 + m^2 - n$$

of $\mathbb{Q}[X]$. The roots of $F(X)$ are: $\pm i\sqrt{\sqrt{n}+m}$ and $\pm \sqrt{\sqrt{n}-m}$.

The field $L = \mathbb{Q}(i\sqrt{\sqrt{n}+m}) = (\mathbb{Q}(\sqrt{n}))(i\sqrt{\sqrt{n}+m})$ is a CM-field, and so we can apply Theorem 1 to the equation $F^*(X, Y) = b$, where b is a non zero integer .

On the one hand, it is worth noticing that in this case we cannot apply Runge's method [9], [28]. Additionally, since the polynomial $F(X)$ has real roots, we can neither apply [20, Theorem 2, page 186], nor [22, Theorem 1] which are the only known results which can provide polynomial bounds for the size of the integer solutions. On the other hand, Baker's method renders exponential bounds on the heights of the integer solutions of such an equation (see for example [11]). Moreover, since the splitting field N of $F(X)$ is not a CM field we cannot apply [10, Théorème 2].

Theorem 1 yields the following bounds for the size of the solutions $(x, y) \in \mathbb{Z}^2$ of the above equation:

$$|x| < 2^{12} n^2 b^4, \qquad |y| < 2^8 n b^2.$$

Finally, note that, by Theorem 3 of [15], the Galois group $\mathrm{Gal}(N/\mathbb{Q})$ is isomorphic to the dihedral group of order 8.

4. Proof of Theorem 1

Write

$$F(X, Y) = a_0(X - \alpha_1 Y) \cdots (X - \alpha_n Y).$$

First, we consider the case where $a_0 = \pm 1$. If $a_0 = -1$, we replace $F(X, Y)$ by $-F(X, Y)$ and b by $-b$ and then we may suppose that $a_0 = 1$. We denote by J the set of indexes j such that $\alpha_j \in \mathbb{C} \setminus \mathbb{R}$. Put $t = |J|$. By our hypothesis, we have $t > 0$. If $z \in \mathbb{C}$, we denote, as usual, by $\bar{z}$ the complex conjugate of z.

Let $x, y \in O_K$ such that $xy \neq 0$ and $F(x, y) = b$. We set

$$x - \alpha_j y = b_j \quad (j = 1, \ldots, n).$$

For every $j \in J$, we set $\rho_j = \bar{b}_j / b_j$. Since K is a totally real number field, we have

$$x - \bar{\alpha}_j y = \rho_j b_j.$$

Eliminating x and b_j from the above two equations, we get

$$y = \frac{b_j(1-\rho_j)}{\bar{\alpha}_j - \alpha_j}, \quad x = y\frac{\bar{\alpha}_j - \alpha_j \rho_j}{1-\rho_j}.$$

Let $j_0 \in J$. For $j \notin J$, we get

$$b_j = x - \alpha_j y = y\frac{(\bar{\alpha}_{j_0} - \alpha_j) + \rho_{j_0}(\alpha_j - \alpha_{j_0})}{1-\rho_{j_0}}.$$

Combining the above equalities, we obtain

$$y^n = b(1-\rho_{j_0})^{n-t} \prod_{j \in J} \frac{1-\rho_j}{\bar{\alpha}_j - \alpha_j} \prod_{j \notin J} \frac{1}{(\bar{\alpha}_{j_0} - \alpha_j) + \rho_{j_0}(\alpha_j - \alpha_{j_0})}.$$

Let $K_j = K(\alpha_j, \bar{\alpha}_j)$, $j \in J$. We denote by G_j the set of $\mathbb{Q}$-embedding $\sigma : K_j \to \mathbb{C}$. Therefore, for every $j \in J$, we have $\sigma(\bar{b}_j) = \overline{\sigma(b_j)}$, and so we get

$$|\sigma(\rho_j)| = \frac{|\sigma(\bar{b}_j)|}{|\sigma(b_j)|} = \frac{|\overline{\sigma(b_j)}|}{|\sigma(b_j)|} = 1.$$

The elements α_j, $\bar{\alpha}_j$ are algebraic integers and so, b_j, $\bar{b}_j$ are algebraic integers of L. Let $M_j(X)$ be the minimal polynomial of ρ_j over $\mathbb{Z}$ and m_j its leading coefficient. Since ρ_j is a root of the polynomial

$$\Pi_j(X) = \prod_{\sigma \in G_j} \sigma(b_j)(X - \sigma(\rho_j)),$$

which has integer coefficients, we have that $M_j(X)$ divides $\Pi_j(X)$ and thus we deduce that m_j divides

$$\prod_{\sigma \in G_j} \sigma(b_j) = N_{K_j}(b_j),$$

where N_{K_j} is the norm related to the extension $K_j/\mathbb{Q}$. It follows that m_j divides $N_{K_j}(b_j)$. As we saw above, all the conjugates of ρ_j are of absolute value 1. Therefore, by [16, page 54], for every $j \in J$ we have

$$H_{K_j}(\rho_j) = m_j \prod_{\sigma \in G} \max\{1, |\sigma(\rho_j)|\} \leq N_{K_j}(b_j) < N_{K_j}(b) < H_{K_j}(b).$$

Further, using elementary properties of heights and the above inequality, we deduce

$$H(y)^n < 2^{4n-2t} H(b)^{2n} H(\alpha_{j_0})^{2(n-t)} \prod_{i=1}^{n} H(\alpha_i)^2.$$

By [25, Theorem 5.9, page 211], we have

$$\prod_{i=1}^{n} H(\alpha_i) \leq 2^n H(F).$$

For every $j \in J$, the number α_j is not real, hence, using [19], we deduce that $H(\alpha_j) < 2H(F)^{1/2}$. Therefore, we obtain

$$H(y) < 2^8 H(b)^2 H(F).$$

And it follows that

$$H(x) \leq 4H(y)H(\alpha_j)^2 H(\rho_j)^2 \leq 16H(y)H(F)H(b)^2 < 2^{12}H(F)^2H(b)^4.$$

Suppose now that $a_0 \neq \pm 1$. Write $F(X,1) = a_0X^n + a_1X^{n-1} + \cdots + a_n$. Then $a_0\alpha_i$ is a root of $f(X) = X^n + a_1X^{n-1} + a_2a_0X^{n-2} + \cdots + a_na_0^{n-1}$ and thus $a_0\alpha_i$ is an algebraic integer. Denote by $F_1(X,Y)$ the homogenization of $f(X)$. If $(x,y) \in O_K^2$ is a solution to $F(X,Y) = b$, then (a_0x, y) is a solution to $F_1(X,Y) = ba_0^{n-1}$. Denote by Γ a point in the projective space with 1 and the coefficients of F as coordinates. Then we have $H(F_1) \leq H(\Gamma)^n$ and finally, we obtain

$$H(y) < 2^8 H(b)^2 H(\Gamma)^{3n-2} \quad \text{and} \quad H(x) < 2^{12} H(b)^4 H(\Gamma)^{6n-4}.$$

5. Proof of Corollary 1

By [24, Lemma 3.1], there are $\alpha, \beta \in O_K\setminus\{0\}$ and $A(T), B(T), C(T), D(T) \in O_K[T]$ with $\deg A, \deg C < m$, $\deg B$, $\deg D < n$ and

$$H(\alpha),\ H(\beta) < (m+n-1)!\ H(\Omega)^{m+n-1},$$

satisfying

$$A(X)G(X,1)+B(X)E(X,1) = \alpha \ \text{ and } \ C(Y)G(1,Y)+D(Y)E(1,Y) = \beta.$$

Now let $x, y \in O_K$ with $xy \neq 0$ and $G(x,y) = E(x,y)$. By [24, Lemma 3.2], there are $u, v \in O_K$ such that $x/y = u/v$ and the greatest common divisor $\mathcal{D}$ of ideals (u) and (v) satisfies $N_K(\mathcal{D}) \leq |D_K|^{1/2}$. We have

$$A(x/y)G(x/y,1) + B(x/y)E(x/y,1) = \alpha,$$

and

$$C(y/x)G(1,y/x) + D(y/x)E(1,y/x) = \beta.$$

Since $y^{n-m}G(x/y,1) = E(x/y,1)$ and $x^{n-m}G(1,y/x) = E(1,x/y)$ we get

$$A_1(u,v)G(u,v) + B_1(u,v)y^{n-m}G(u,v) = \alpha v^k$$

and

$$C_1(u,v)G(u,v) + D_1(u,v)x^{n-m}G(u,v) = \beta u^l,$$

where $k, l < m+n-1$ and $A_1(X,Y)$, $B_1(X,Y)$, $C_1(X,Y)$, $D_1(X,Y)$ are homogeneous polynomials with $A_1(X,1) = A(X)$, $B_1(X,1) = B(X)$, $C_1(1,Y) = C(Y)$ and $D_1(1,Y) = D(Y)$. It follows that the ideal $(G(u,v))$ divides (αv^k) and (βu^l) and so, $(G(u,v))$ divides $(\alpha\beta)\mathcal{D}^{m+n-1}$.

By [11, Lemma 3], there exists a unit $\epsilon \in O_K$ such that

$$H(G(\epsilon u, \epsilon v)) \leq N_K(G(u,v))^{1/d}\exp\{nC(d)R_K\},$$

where $C(d) = 0$ if $d = 1$, $C(d) = 1/2$ if $d = 2$ and

$$C(d) = 29e(d-1)(d-1)!\sqrt{d-2}\log d$$

otherwise. Then we obtain

$$H(G(\epsilon u, \epsilon v)) \leq (N_K(\alpha\beta))^{1/d} N_K(\mathcal{D}))^{(m+n-1)/d}\exp\{nC(d)R_K\},$$

whence we obtain

$$H(G(\epsilon u, \epsilon v)) \leq ((m+n-1)!)^2\, H(\Omega)^{2(m+n-1)}|D_K|^{(m+n-1)/2d}\exp\{nC(d)R_K\}.$$

Thus, Theorem 1 implies

$$\max\{H(\epsilon u), H(\epsilon v)\} < 2^{12}((m+n-1)!)^8\, H(\Omega)^{8m+14n-5}|D_K|^{2(m+n-1)/d}\exp\{4nC(d)R_K\}.$$

We have $y^{n-m}G(x/y,1) = E(x/y,1)$, whence we get

$$y^{n-m} = \frac{E(\epsilon u/\epsilon v, 1)}{G(\epsilon u/\epsilon v, 1)}.$$

Hence we obtain

$$\begin{aligned} H(y)^{n-m} &\leq H(E(\epsilon u/\epsilon v,1))H(G(\epsilon u/\epsilon v,1)) \\ &\leq (n+1)(m+1)H(\Omega)^2\max\{H(\epsilon u), H(\epsilon v)\}^{2(m+n)}. \end{aligned}$$

Combining the above estimates and using the inequality $k! < ((k+1)/2)^k$, we deduce

$$H(y)^{n-m} < (m+n)^{16(m+n)^2} H(\Omega)^{2(8m+14n-4)(m+n)} |D_K|^{4(m+n)(m+n-1)/d}\exp\{8(m+n)nC(d)R_K\}.$$

Finally, using [22, Lemma 7], we obtain the result.

6. Proof of Corollary 2

Let $f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d$. We denote by $F(X,Y)$ the homogenization of $f(X)$. Write

$$F(X,Y) = a_0\prod_{\sigma}(X - \sigma(\alpha)Y),$$

where σ in the product runs through the set of embeddings of the field $K = \mathbb{Q}(\alpha)$ into $\mathbb{C}$. Hence

$$|F(p,q)| = |a_0|q^d\left|\alpha - \frac{p}{q}\right|\prod_{\sigma\neq \mathrm{Id}}\left|\sigma(\alpha) - \frac{p}{q}\right|,$$

where Id is the inclusion of K into $\mathbb{C}$. For $\sigma \neq \mathrm{Id}$, we have

$$\left|\sigma(\alpha) - \frac{p}{q}\right| \leq |\sigma(\alpha) - \alpha| + \left|\alpha - \frac{p}{q}\right| \leq |\sigma(\alpha) - \alpha| + \frac{k}{q^d}.$$

Thus, we get

$$0 < |F(p,q)| \le |a_0| k \prod_{\sigma \neq Id} (|\sigma(\alpha)| + |\alpha| + k),$$

whence

$$0 < |F(p,q)| < \max\{k,1\}^d 2^{d-1} |a_0| \left(\prod_{\sigma} \max\{1, |\sigma(\alpha)|\} \right)^{d-1}.$$

By [16, page 54] and [19], we have

$$0 < |F(p,q)| \le \max\{k,1\}^d 2^{d^2} H(f)^{d(d-1)}.$$

Finally, using Theorem 1, we obtain the result.

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References

[1] Y. Bilu and G. Hanrot, Solving Thue equations of high degree, *J. Number Theory* 60 (1996), 373-392.

[2] A. Baker, Contribution to the theory of Diophantine equations, I. On representation of integers by binary forms, *Philos. Trans. Roy. Sot. London Ser.* A 263 (1968), 173-191.

[3] P. E. Blanksby and J. H. Loxton, A Note on the Characterization of CM-fields, *J. Austral. Math. Soc.* (Series A) 26 (1978), 26-30

[4] B. Brindza, Á. Pintér, A. van der Poorten and M. Waldschmidt, On the distribution of solutions of Thue's equations, Number theory in progress, Vol. 1 (Zakopane-Koscielisko, 1997), ed. K. Győry, H. Iwaniec and J. Urbanowicz, 35-46, de Gruyter, Berlin, 1999.

[5] B. Brindza, J.-H. Evertse and K. Győry, Bounds for the solutions of some Diophantine equations in terms of discriminants, *J. Austral. Math. Soc.* Ser. A 51 (1991), no. 1, 826.

[6] Y. Bugeaud and K. Győry, Bounds for the solutions of Thue-Mahler equations and norm form equations, Acta Arithmetica, LXXIV.3 (1996), 273-292.

[7] L. E. Dickson, *Introduction to the Theory of Numbers,* Dover, New York, 1957.

[8] J.-H. Evertse, The number of solutions of decomposable form equations, *Inventione Mathematicae,* 122, (1995) 559-601.

[9] A. Grytczuk and A. Schinzel, On Runge's Theorem about Diophantine Equations, *Colloq. Math. Soc. J. Bolyai* 60, (1992), 329-356.

[10] K. Győry, Représentation des nombres entiers par des formes binaires, *Publicationes Mathematicae Debrecen,* 24 (3-4) (1977), 363-375.

[11] K. Győry and K. Yu, Bounds for the solutions of S-unit equations and decomposable form equations, *Acta Arithmetica,* 123.1 (2006), 9-41.

[12] G. Hanrot, Solving Thue equations without the full unit group. Math. Comp., 69(229) (2000), 395-405.

[13] C. Heuberger, Parametrized Thue Equations A survey, Proceedings of the RIMS symposium Analytic Number Theory and Surrounding Areas, Kyoto, Oct 1822, 2004, RIMS Kkyroku 1511, August 2006, 8291.

[14] M. Hindry and J. H. Silverman, *Diophantine Geometry, An Introduction.* Springer-Verlag 2000.

[15] L-C. Kappe and B. Warren, An elementary test for the Galois group of a quartic polynomial, *The Amer. Math. Monthly*, Vol. 96, No. 2 (1989), 133-137.

[16] S. Lang, *Fundamentals of Diophantine Geometry,* Springer-Verlag, New York - Berlin, 1983.

[17] C. Levesque and M. Waldschmidt, Some remarks on Diophantine equations and Diophantine approximation, *Vietnam J. Math.* 39 (2011), no. 3, 343-368.

[18] S. Louboutin, R. Okazaki and M. Olivier, The class number one problem for some non-abelian normal CM-fields, *Trans. Amer. Math. Soc.* 349 (1997), no. 9, 3657-3678.

[19] M. Mignotte, An inequality of the greatest roots of a polynomial, *Elem. Math.* 46 (1991), 85-86.

[20] L. J. Mordell, *Diophantine equations,* Pure and Applied Mathematics, Vol. 30, Academic Press, London-New York 1969.

[21] D. Poulakis, Integer points on algebraic curves with exceptional units, *J. Austral. Math. Soc.* 63 (1997), 145-164.

[22] D. Poulakis, Polynomial Bounds for the Solutions of a Class of Diophantine Equations, *J. Number Theory* 66, No 2, (1997), 271-281.

[23] D. Poulakis, Bounds for the minimal solution of genus zero Diophantine equations, *Acta Arithm.,* 86, (1998), 51-90.

[24] D. Poulakis, Bounds for the size of integral points on curves of genus zero, *Acta Math. Hungar.* 93 (4) (2001), 327-346.

[25] J. H. Silverman, *Arithmetic of Elliptic Curves,* Springer Verlag 1986.

[26] A. Thue, Über Annäherungswerte algebraischer Zahlen, *J. Reine Angew. Mafh.* 135 (1909), 284-305.

[27] N. Tzanakis and B. M. M. de Weger, On the practical solution of the Thue equation. J. Number Theory, 31(2) (1989), 99-132.

[28] P.G. Walsh, A quantitative version of Runge's theorem on Diophantine equations, *Acta Arithm.* 62 (1992), 157-172.

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