



Spontaneous Breaking of a Translational Symmetry and Analogue Gravity

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Spontaneous Breaking of a Translational Symmetry and Analogue Gravity

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Abstract

We study the ground state of Bose-Einstein condensates with a roton-maxon dispersion relation when the fluid velocity is larger than the Landau velocity. This leads to a phase breaking translational invariance, which we call “undulation”. Stability of the undulation strongly depends on the choice of fixed parameters. We find two stability conditions when fixing the condensate velocity, the total number of atoms and the wave-vector of the undulation. The first one constraints the condensate velocity, while the second one constraints the two-body interaction potential. The structure of the ground state is investigated numerically. We also comment on analogies with a Chern-Simons theory in 5 dimensions. This work is a first step towards what could be a more general study of such phases breaking translational invariance in different domains like hydrodynamics, condensed matter, cold atoms or quantum field theory in curved space-time.

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1 Introduction

At first sight, black holes physics and hydrodynamics seem to be two radically different domains of physics. While classical fluids are known from immemorial times and omnipresent in everyday life, black holes remain quite abstract objects because of the lack of direct observation and nontrivial mathematical concepts needed to describe them. However, General Relativity and Hydrodynamics share common concepts (as one can expect for two classical field theories) which provide us with bridges between them. These can be used to study some aspects of gravitational systems using techniques from hydrodynamics or conversely.

General Relativity is a classical theory in the sense that it does not include quantum effects, which are however expected to play a role in real gravitational systems. Indeed, since gravity is sourced by quantum matter fields, the gravitational field itself should be a quantum operator

instead of a classical quantity. Unfortunately, the standard procedure for quantizing a classical field does not apply to gravity, or leads to a non-renormalizable theory (see for instance [15] for a pedagogical argument), *ie* a quantum field theory containing infinite quantities which must be removed by adding an infinite number of so-called "counterterms". Such a theory is not predictive since the value of a quantity measured in some experiment can be made arbitrary by fine-tuning one of these counterterms, independently of the results of other experiments. Our knowledge of systems where gravity plays an important role, for instance black holes, is thus so far fundamentally limited by our being unable to take the quantum nature of the gravitational field into account, until a satisfactory quantum theory of gravity emerges. Hydrodynamics is free of this problem since the "fundamental" theory behind it is known: at a microscopic level, a field description breaks down and one must see the fluid as made of atoms or molecules. While General Relativity is not *UV complete*, in the sense that short-distance (or high-energy) phenomena should require a quantum theory to be understood, hydrodynamics (or rather an extension of hydrodynamics taking the molecular nature of the fluid into account) is. Modifications with respect to the hydrodynamical behaviour can thus be computed, and eventually shed some light on what differences a quantum theory of gravity could bring for analogous systems. Analogue black holes have been widely studied for this reason.

In particular, a lot of interest has been devoted to Hawking Radiation ([10, 2] and references therein): when coupled to quantum matter fields, black holes are predicted to emit a thermal radiation with a temperature inversely proportional to their mass. This prediction assumes gravity remains classical and that the standard equations for the quantum matter fields, for instance the Klein-Gordon equation for a scalar field, are valid at all scales. The question of what happens if one of these assumptions is relaxed is still open. Moreover, the very low temperature of this radiation ($6.4 \cdot 10^{-8} K$ for a black hole with the mass of the Sun, much smaller than the temperature of the Cosmic Microwave Background of $2.7 K$) prevents any direct observation, except if a very small black hole were close to the Earth. W. G. Unruh [16] showed that Hawking Radiation should also occur in transonic hydrodynamical flows. From the equations of motion for an irrotational fluid with a radial velocity, sound waves are found to behave exactly as scalar waves in a curved space-time with metric

$$ds^2 = \frac{\rho_0}{c} \left((c^2 - v_0^2) d\tau^2 - \frac{cd r^2}{c^2 - v_0^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right), \quad (1)$$

where ρ_0 is the background fluid density, v_0 its radial velocity and c the speed of sound. That is, in the long wave-length approximation, the hydrodynamical equations reduce to a massless Klein-Gordon equation in the metric (1). If the radial velocity v_0 crosses the speed of sound at $r = r_0$ with $-v_0 > c$ for $r < r_0$, then (1) is the Schwarzschild metric describing the near-horizon region of a neutral, non-rotating black hole for $r \approx r_0$. Thus the mechanism of Hawking radiation should occur just as in the gravitational case. This is an excellent system for investigating the effects of short-distance physics and its predictions may be tested experimentally.

One can also find hydrodynamical analogues of white holes, although gravitational ones may not exist in Nature: contrary to black holes, no mechanism producing a white hole is known. The question of white hole stability is rather subtle and involves spontaneous breaking of translational symmetry: a zero-frequency wave, which we shall call undulation since it is very close to the undulation solutions in hydrodynamics [3, 12, 17], grows and stabilizes the flow near the horizon [7, 4]. The mechanism making the undulation saturate is not yet fully understood. On the other hand, such spatially modulated phases are well-known in superfluids [11] and Bose-Einstein condensates

[1] and their saturation is well understood in these contexts. This is also an important question for experimental investigation of Hawking radiation. For instance, this undulation has been observed in experiments on Hawking radiation from white holes in [13, 17].

The present work aims at understanding better the general necessary or/and sufficient conditions for spontaneous breaking of a continuous translational symmetry to one of its subgroups, what makes the new phase stable and why the corresponding undulation saturates. We focus on quasi-one dimensional Bose-Einstein condensates for which the dispersion relation of perturbations has a particular, so-called “roton-maxon” structure, producing an undulation when the condensate flows above some critical velocity. We compare it to a previous model of symmetry breaking in superfluid Helium IV and study the stability conditions for the undulation. In particular, we find two stability conditions which, to our knowledge, were never discussed in the literature. The field equations are solved numerically and show an unexpected structure of the undulation. A comparison with the symmetry-breaking phase in a five-dimensional Maxwell-Chern-Simons theory in Minkowski space or an asymptotically anti de-Sitter space with black hole is made in order to see to what extent the underlying mechanisms are similar. Such symmetry-breaking phases have been discussed in [8, 9, 5].

Bose-Einstein condensation occurs when a gas of bosons (usually atoms, although Bose-Einstein condensation of molecules is theoretically possible [6]) is cooled below some critical temperature. A macroscopic number of atoms are then in their ground state, forming a Bose-Einstein condensate. This can be made quasi-one dimensional using an external potential which confines the gas in two directions, so that transverse excitations have a much larger energy than longitudinal ones and can be neglected.¹ Most of our work also applies directly to three-dimensional condensates which are uniform in two directions with a flow directed along the third one.

In all this work, it is very important to keep in mind which quantities are fixed and which can fluctuate. This is because, as we shall see, the thermodynamical ensembles one can use are not equivalent.

This report is organized as follows. In Section 2, we present the main part of our work, namely the study of symmetry-breaking phases in Bose-Einstein condensates with a roton-maxon structure. We begin with a theoretical model exhibiting a suitable form of the dispersion relation and discuss the conditions for appearance of an undulation in an initially homogeneous flow. We also see what happens when some hypotheses like homogeneity of the flow are removed. We conclude in Section 3. Appendix A presents the analogue instability in a Maxwell-Chern-Simons theory in 5 dimensions. Although the analogy between the two models may well be of primary importance for understanding the general phenomenon, we do not discuss it in the main text because our analysis is only qualitative and does not provide fundamentally new results. In Appendix B we exhibit a realistic Bose-Einstein condensate model showing the roton-maxon structure. Appendix C contains some remarks on Hawking radiation in the presence of an undulation.

¹Strictly speaking, Bose-Einstein condensation does not occur in one dimension. This is a particular case of the Landau argument against phase transitions in statistical one-dimensional systems with short-range interactions: the entropy associated with a defect grows logarithmically with the system size while the energy remains finite, so that defect formation is always thermodynamically favoured at any finite temperature in the limit of an infinite system. However, a macroscopic occupation number of the ground state can be achieved in a trapped gas [14].

2 Undulation in a Homogeneous Bose-Einstein Condensate

In this section we present the main part of our work, concerning the conditions for the existence and stability of an undulation in Bose-Einstein condensates with a unidirectional flow. It is more often called "supersolid" in this context. However, since the basic ingredients we shall use can be found in different systems, we shall stick to "undulation", which seems more general. We begin by introducing the model under study and give its main properties, following [1, 11]. We then detail the procedure to compute the spectrum of perturbations in a background with undulation and use it to obtain a first stability condition. With that in mind, we expand on Baym and Pethick's analysis based on the minimization of the relevant energy functional to find another stability condition, and comment on the role of fixed quantities. These results rely on the assumptions that the flow is homogeneous and the undulation has only one wave-vector. These hypotheses are relaxed in subsection 2.5, which can be seen as the first step of a more general study. Although we shall often talk about Bose-Einstein condensates for definiteness, everything applies equally to other systems exhibiting important quantum correlations and a suitable dispersion relation. For instance [11], on which part of our work is based, focuses on superfluid Helium IV instead, without any fundamental difference.

2.1 Presentation of the Models

Here we present the main specificities of the models we studied. The reader can refer to [1, 11] for more details. We also make the link between the analyses of these two articles explicit.

In [1], Baym and Pethick consider a weakly interacting Bose gas, *ie* a gas of bosons with small enough interactions, so that it is legitimate to consider only two-body interactions. They work at zero temperature and the condensate flows along one direction x . They first consider a homogeneous condensed gas and assume the dispersion relation of phonons has a so-called *roton-maxon* structure in the rest frame of the condensate, meaning it is linear for small momenta and has an inflexion point at a finite k , see Fig. 1. Note that [11] assumes there is a true minimum, but we shall not need this assumption. Consider doing a special Galilean transformation to go to the "laboratory" frame where the condensate has a velocity v . The wave-vector k of a perturbation is unchanged while its frequency changes as:

$$\omega' = \omega + vk \quad (2)$$

(for simplicity, we assume the wave-vector is in the x direction.) If v and k have opposite signs, the dispersion relation is tilted towards the k axis and eventually negative-energy modes appear if the velocity is larger than some critical velocity, known as the *Landau velocity* v_L (see Fig. 1). The first wave-vector to become unstable is called the *critical wave-vector* k_c . Intuitively, these negative-energy modes should make the condensate unstable and destroy its peculiar properties like superfluidity. This is the Landau argument: a stable condensate (or superfluid in [11]) could exist only below the Landau velocity. According to Pitaevskii, Baym and Pethick, this picture is not fully correct. The point is that a fraction of the fluid remains condensed/superfluid as long as the velocity is not too much above v_L , while the other part produces another condensate, with a different wave-vector. The total order parameter ψ (often called the "wave-function of the condensate") is then a superposition of plane waves, so that its square modulus, which gives the local density, has an oscillating term. This is what we call the "undulation". Under some conditions

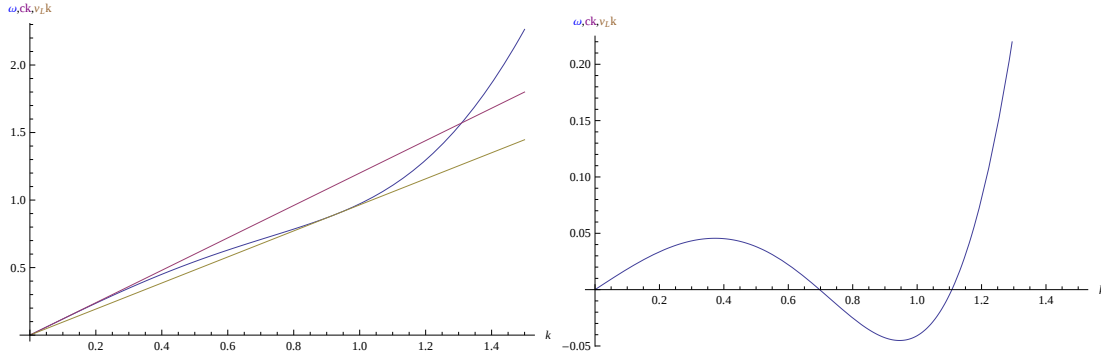


Figure 1: Left: Typical shape of the dispersion relation. The frequency ω of a perturbation as a function of its wave-vector k along the x direction in the rest frame of the fluid is shown in blue. $\omega = ck$ and $\omega = v_L k$, where c and v_L are the sound and Landau velocities, respectively, are represented respectively in purple and brown. Right: Dispersion relation of density fluctuations in a frame where the condensate has a velocity $-1.05v_L$. Wave-vectors corresponding to a negative energy are clearly seen between $k \approx 0.7$ and $k \approx 1.1$.

which we shall explain in this section, the undulation does not deplete the condensate entirely and increases the energy of modes close to k_c , so that the new spectrum is positive and the condensate with the undulation is stable.

We will have a lot to say about these results and the underlying hypotheses in the following subsections. For the moment, let us focus on the descriptions of the condensate given in [1, 11] and the questions they raise. The first non-trivial point, although standard in condensed matter, is to treat ψ as a classical field to minimize the energy functional and then quantize only the fluctuations around the solution. We know that a Bose-Einstein condensate is made of atoms and a purely quantum effect with no classical counterpart. So why do we use a field (which is clearly not a fundamental field like those used in particle physics)? and why a classical one?

The answer to the first question lies in the procedure of *second quantization*. To make a long discussion short, second quantization proceeds as follows. We consider the system made of all atoms and a complete set ψ_k of normalized eigenfunctions of the Hamiltonian. For each mode we define creation and annihilation operators $a_k^\dagger$ and a_k satisfying the commutation relations:

$$[a_k, a_{k'}] = 0 \quad (3)$$

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'} \quad (4)$$

and the Hilbert space is spanned by a vacuum state $|0\rangle$ annihilated by all the a_k 's, plus all the states obtained from $|0\rangle$ by acting with any finite number of creation operators. Note that although we will work with a complex field, there is no antiparticle. That is why the creation operators are hermitian conjugates of the annihilation operators. We define a field operator (in Schrödinger picture)

$$\psi(\vec{r}) = \sum_k \psi_k(\vec{r}) a_k. \quad (5)$$

If the basis of eigenfunctions we started from is correctly normalized, ψ satisfies:

$$\left[\psi(\vec{r}), \psi(\vec{r}')\right] = 0, \quad (6)$$

$$\left[\psi(\vec{r}), \psi(\vec{r}')^\dagger\right] = \hbar\delta^{(3)}(\vec{r} - \vec{r}'). \quad (7)$$

We can then go to the Heisenberg picture as usual, and the above relations become equal-time commutators.

The second question has to do with the assumption that a macroscopic number of atoms are in their ground state, which is the very definition of a Bose-Einstein condensate. In that case, commutators give a negligible contribution to the mean value of a product of fields and ψ can be treated as a complex number in a first approximation.

What Pitaevskii on one hand and Baym and Pethick on the other hand do seem at first sight quite different for two reasons: the Hamiltonians don't look exactly the same and the ansätze are different. In these conditions, how relevant are Pitaevskii's results in the case studied by Baym? And why does his simpler ansatz work? We now answer these two questions in three steps, which will also be the occasion to introduce important quantities, in particular the Hamiltonian. We shall first show that the quadratic parts of the two Hamiltonians for the perturbation over the homogeneous condensate (the undulation being for the moment included in the perturbations) are simply related by a linear transformation. We then show how the same transformation maps one ansatz onto the other, so that these two trial solutions are the same physical perturbation written in two different ways. Finally we will see that, for momenta close to k_c , the two full Hamiltonians are equivalent (but we stress this is not true far from k_c). Intuitively, by considering only momenta close to k_c we neglect mixing between modes and turns the (rather difficult) calculation of the renormalized spectrum in [1] to the (much simpler) case of [11]. In order to keep the algebra simple, we shall skip straightforward calculations and not write explicitly some quadratic terms which can be absorbed by a chemical potential. We work in one space dimension but generalization to any dimension is straightforward.

Let us start from the Hamiltonian in [1], maybe more fundamental:

$$H = \int dx \frac{\hbar^2}{2m} \partial_x \psi^\dagger \partial_x \psi + \frac{1}{2} \int dx dy g(x-y) \psi(x)^\dagger \psi(y)^\dagger \psi(x) \psi(y) \quad (8)$$

The corresponding equation of motion is a slight generalization of the Gross-Pitaevskii equation:

$$i\hbar \partial_t \psi(x, t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x, t) + V(x, t) \psi(x, t) + \int dy g(x, y) \psi(y, t) \psi(y, t)^\dagger \psi(x, t) \quad (9)$$

In (8), the first term is the kinetic energy while the last one describes two-body interactions. The energy of a perturbation on momentum k over a homogeneous condensate is given by:

$$\epsilon_0(k) = \sqrt{\frac{\hbar^2 k^4}{4m^2} + \frac{n_0}{m} \tilde{g}(k) k^2}, \quad (10)$$

where m is the mass of the atoms and n_0 their number density. We want it to show the desired roton-maxon structure. In most of this work we will not need any explicit form of g . Unless explicitly stated, numerical results are obtained with a second-order polynomial in k^2 in Fourier space:

$$\tilde{g}(k) \equiv \int dx e^{ikx} g(x) = \alpha - \beta k^2 + \gamma k^4 \quad (11)$$

with $\alpha, \beta, \gamma > 0$.

We want to describe a Bose-Einstein condensate with velocity $v = q/m$ plus some perturbations. To this end, we write $\psi = \psi_0 + \hat{\psi}$, where ψ_0 is the condensed part, containing the modes of momentum q . For simplicity, we first work in the fluid frame $q = 0$. Replacing ψ_0 by the complex number $\sqrt{n_0}$ and defining the Fourier transform of the perturbation as

$$\tilde{\psi}(k, t) \equiv \int dx \hat{\psi}(x, t) e^{ikx} \quad (12)$$

the quadratic part of the Hamiltonian for $\hat{\psi}$ is

$$H_Q = \int \frac{dk}{2\pi} \left[\left(\frac{\hbar^2 k^2}{2m} + n_0 \tilde{g}(k) \right) \tilde{\psi}(k, t)^\dagger \tilde{\psi}(k, t) + \frac{1}{2} n_0 \tilde{g}(k) \left(\tilde{\psi}(k, t) \tilde{\psi}(-k, t) + \tilde{\psi}(k, t)^\dagger \tilde{\psi}(-k, t)^\dagger \right) \right]. \quad (13)$$

We must now diagonalize it. Let

$$\tilde{\phi}(k) = A(k) \tilde{\psi}(k) + B(-k) \tilde{\psi}(-k)^\dagger. \quad (14)$$

Assuming $(A(k), B(k)) \in \mathbb{R}^2$, $A(k) = A(-k)$ and $B(k) = B(-k)$, we find this choice diagonalizes the above quadratic form provided

$$A(k) = \frac{\alpha}{2} \left(\sqrt{\frac{\hbar^2 k^2}{2m}} + \sqrt{\frac{\hbar^2 k^2}{2m} + \tilde{g}(k) 2n_0} \right) \quad (15)$$

$$B(k) = \frac{\alpha}{2} \left(-\sqrt{\frac{\hbar^2 k^2}{2m}} + \sqrt{\frac{\hbar^2 k^2}{2m} + \tilde{g}(k) 2n_0} \right) \quad (16)$$

for some $\alpha \in \mathbb{R}$. Then $A^2 - B^2 = \alpha^2 \epsilon_0(k)$, so the field is correctly normalized (in the sense that it satisfies the same equal-time commutation relations as $\tilde{\psi}$ does) for $\alpha = \frac{1}{\sqrt{\epsilon_0(k)}}$. And then

$$H_Q = \int \frac{dk}{2\pi} \epsilon_0(k) \tilde{\phi}(k)^\dagger \tilde{\phi}(k) = \int dx \phi(x)^\dagger \epsilon_0(\hat{p}) \phi(x), \quad (17)$$

where ϕ is the inverse Fourier transform of $\tilde{\phi}$. It is now straightforward to find the quadratic Hamiltonian in the laboratory frame. If the fluid moves with velocity v , $\partial_{t'} = \partial_t - v \partial_x$ and $\partial_{x'} = \partial_x$, showing the momentum operators in the two frames are the same while

$$H_{Q,lab} = H_{Q,fluid} + \int dx \phi^\dagger v \hat{p} \phi = \int dx \phi^\dagger (\epsilon_0(\hat{p}) + v \hat{p}) \phi. \quad (18)$$

Note that ϕ is still an operator defined in the fluid frame.

After a few straightforward lines of algebra, one shows that the linear transformation (14) sends the ansatz of Baym and Pethick (see equations (6) and (12) in [1]):

$$\psi_u(t, x) = e^{iqx} \xi \left[\cosh \frac{\eta}{2} e^{-ik_c x} - \sinh \frac{\eta}{2} e^{ik_c x} \right] \quad (19)$$

to that of Pitaevskii (equation (5) in [11]):

$$\phi_u(t', x') = \xi e^{-ik_c x'}. \quad (20)$$

In the above formulae, a subscript u stands for the part of the field responsible for the undulation, *ie* what we must add to the homogeneous condensate to minimize the energy functional.

In general, the transformation (14) makes the higher-order terms in the Hamiltonian more complicated. In order to simplify the analysis, we consider only modes with wave-vectors close to k_c . This is justified for v slightly above v_L because couplings between the modes k and $k \pm k_c$ or $k \pm 2k_c$ are smaller than the ones we considered above by a factor of order $\frac{\xi}{\sqrt{n_0}}$, as can be seen by plugging a sum of plane waves in (9), and because, having a different wave-vector, their contributions will be suppressed by a factor $\frac{\delta p}{k_c}$, if we are interested in modes within $[k_c - \delta p, k_c + \delta p]$. Under this assumption (which is also present in [11]), cubic terms vanish after integration over x . Using that A and B are even functions of k and considering only modes close to k_c , the quartic part of the Hamiltonian becomes after an easy calculation:

$$H_4 \approx \frac{1}{2\epsilon_0^2} \left[\frac{n_0^2}{2} \tilde{g}(k_c)^2 \tilde{g}(2k_c) + \left(\frac{\hbar^2 k_c^2}{2m} + n_0 \tilde{g}(k_c) \right)^2 \tilde{g}(0) \right] \int dz \phi(t, z)^\dagger \phi(t, z)^\dagger \phi(t, z) \phi(t, z). \quad (21)$$

We recover the form given by Pitaevskii. Notice that the coupling constant in (21) is different from the effective coupling constant γ in [1]. This is because we did not impose conservation of the total number of atoms in order to make the algebra simple. Also, this equivalence holds only if ξ and η have the values given in [1], which in turn is true provided the amplitude of the undulation is small before $\sqrt{n_0}$.

Before closing this subsection, let us point out two facts which will prove important in the following. First, $\psi_0 + \psi_u$ is an exact solution of the equation of motion coming from Pitaevskii's Hamiltonian, but only an approximate solution to (9). Second, the perturbations we consider by writing $\psi = \psi_0 + \hat{\psi}$ where $\hat{\psi}$ is small with respect to ψ_0 do not allow for a change δq in the condensate wave-vector q , since this would introduce a large perturbation where $\delta q x \approx \pi$.

2.2 Spectrum of Perturbation in the Symmetry-Breaking Phase

In this subsection compute the spectrum of perturbations on top of the undulation and use it to find a stability condition on the amplitude of the undulation and its wave-vector. We work under the assumptions of [11] to get analytical expressions. Some ideas to go further are discussed.

We start from a general Hamiltonian of the form:

$$H = \int \left(\psi^\dagger (\epsilon(\hat{p}) + \vec{v} \cdot \hat{p}) \psi + \frac{g}{2} \psi^\dagger \psi^\dagger \psi \psi \right) d^d x. \quad (22)$$

We write:

$$\hat{\psi} = \psi_0 + \psi', \quad (23)$$

where

$$\psi_0(\vec{x}) = \eta \exp \left(i \frac{\vec{q} \cdot \vec{x}}{\hbar} - i \omega t \right). \quad (24)$$

We shall come back to the assumption that the undulation contains only one wave-vector in subsection 2.5. For the time being, let us just state that it is natural if v is just slightly above v_L , so that all the unstable modes have a wave-vector close to k_c . (24) is a solution of the field equation provided

$$\hbar \omega = \epsilon(\vec{q}) + \vec{v} \cdot \vec{q} + g|\eta|^2 \quad (25)$$

Choosing the value of $|\eta|$ which minimizes the energy functional: $|\eta|^2 = -\frac{\epsilon(\vec{q}) + \vec{v} \cdot \vec{q}}{g}$ gives $\omega = 0$ and $\psi = \psi_0$. Neglecting terms which are linear in ψ' (they oscillate out if ψ' has no component of momentum $\vec{q}$) or of order ≥ 3 and subtracting the Hamiltonian with $\psi' = 0$ we get:

$$\delta H = \int \psi'^{\dagger} (\epsilon(\hat{p}) + \vec{v} \cdot \hat{p} + 2\eta^2 g) \psi' + \frac{g}{2} \left(\eta^2 \exp \left(2i \frac{\vec{q} \cdot \vec{x}}{\hbar} \right) (\psi'^{\dagger})^2 + \eta^2 \exp \left(-2i \frac{\vec{q} \cdot \vec{x}}{\hbar} \right) \psi'^2 \right) d^d x \quad (26)$$

We now write the operator ψ' as:

$$\psi'(t, \vec{x}) = \sum_{\nu} \left(u_{\nu}(\vec{x}) \hat{b}_{\nu} \exp \left(-i \frac{\epsilon_{\nu} t}{\hbar} \right) - v_{\nu}(\vec{x}) \hat{b}_{\nu}^{\dagger} \exp \left(i \frac{\epsilon_{\nu} t}{\hbar} \right) \right) \quad (27)$$

(The "sum" can involve integrals over continuous degrees of freedom.) The operators b_{ν} satisfy:

$$[\hat{b}_{\nu}, \hat{b}_{\mu}] = 0 \quad (28)$$

$$[\hat{b}_{\nu}, \hat{b}_{\mu}^{\dagger}] = \delta_{\mu\nu} \quad (29)$$

The mode equations on u_{ν} and v_{ν} are used to write down a closed equation on u_{ν} . After a straightforward calculation, we find:

$$\epsilon_{\nu} = \frac{1}{2} \left[\epsilon_0(\vec{k}) - \epsilon_0(2\vec{q} - \vec{k}) + 2\vec{v} \cdot (\vec{k} - \vec{q}) \pm \left(\left(\epsilon_0(\vec{k}) - \epsilon_0(2\vec{q} - \vec{k}) + 2\vec{v} \cdot (\vec{k} - \vec{q}) \right)^2 - 4 \left((\epsilon_0(\vec{q}) + \vec{v} \cdot \vec{q})^2 - \left(\epsilon_0(2\vec{q} - \vec{k}) + \vec{v} \cdot (2\vec{q} - \vec{k}) - 2(\epsilon_0(\vec{q}) + \vec{v} \cdot \vec{q}) \right) \left(\epsilon_0(\vec{k}) + \vec{v} \cdot \vec{k} - 2(\epsilon_0(\vec{q}) + \vec{v} \cdot \vec{q}) \right) \right) \right)^{1/2} \right] \quad (30)$$

For consistency in the limit $v \rightarrow v_L$, we must choose the "+" sign. The minus sign gives the opposite of the energy of the mode $2\vec{q} - \vec{k}$, ie the energy of the v mode of wave-vector $2\vec{q} - \vec{k}$ (as usual, this is not a physical energy since v modes have a negative norm). Equation (9) in [11] is recovered if we set $\vec{q} = k_c \vec{e}_x$ and $\vec{k} \approx k_c \vec{e}_x$.

Fig.2 shows the new dispersion relation for some values of the parameters, assuming the Fourier transform $\tilde{g}$ of g is a second-order polynomial in k^2 . We can use (30) to study the stability of the undulation: it is stable with respect to small perturbations if and only if the new spectrum is positive. If the undulation has a wave-vector k_c opposite to the direction of the flow, we find it is stable provided:

$$\boxed{v_L < v < v_L + k_c \epsilon_0''(k_c \vec{e}_x)} \quad (31)$$

where $\vec{e}_x$ is the unit vector in the direction of the flow. To our knowledge, this stability condition was never discussed in the literature. For larger velocities (but still small enough for the approximations we did to remain valid), stable undulations exist but have a larger momentum, see Fig. 3.

It is also insightful to consider the stability region in the plane $(\vec{q} \cdot \vec{e}_x, \eta)$. Note that plane-wave, time-independent exact solutions to the equation of motion from (22) exist throughout this plane provided the velocity is chosen accordingly, but not all of them are stable. We are interested in what happens when the roton depth is decreased, ultimately reaching zero. To do so, we send v_L to

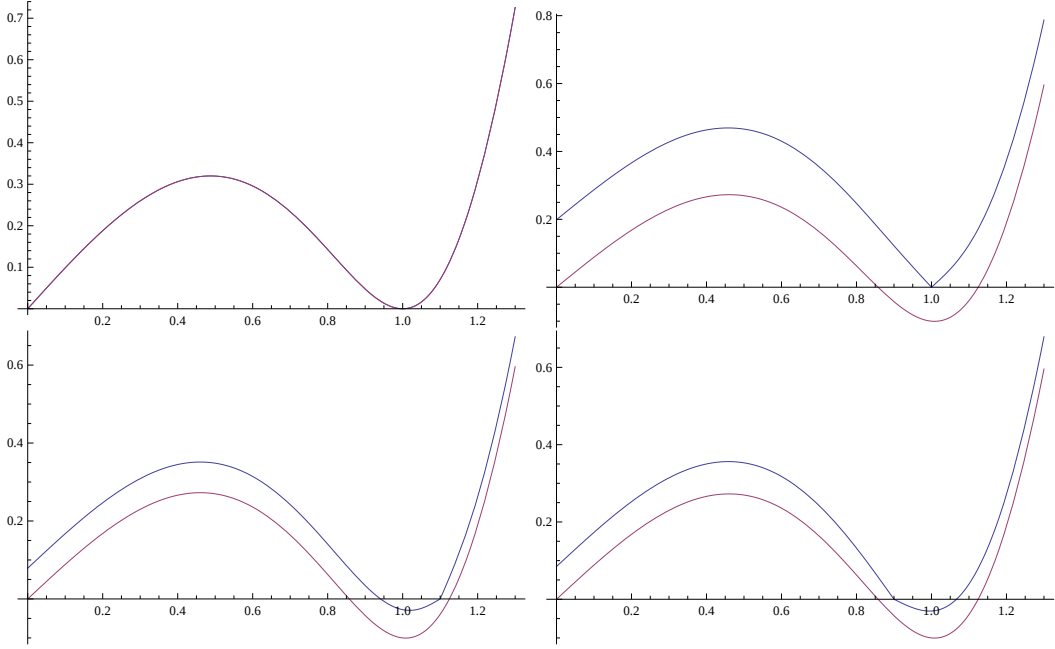


Figure 2: Dispersion relations on top of the homogeneous state (purple) and the undulation (blue) for $v_L = 1$, $c = 2$, $k_c = 1$, $v = -1$ (top left), $v = 1.1$ (top right and bottom) and $q = 1$ (top), $q = 1.1$ (bottom left) or $q = 0.9$ (bottom right). The last two spectra still contain negative-energy modes, thereby showing that the undulation is energetically unstable.

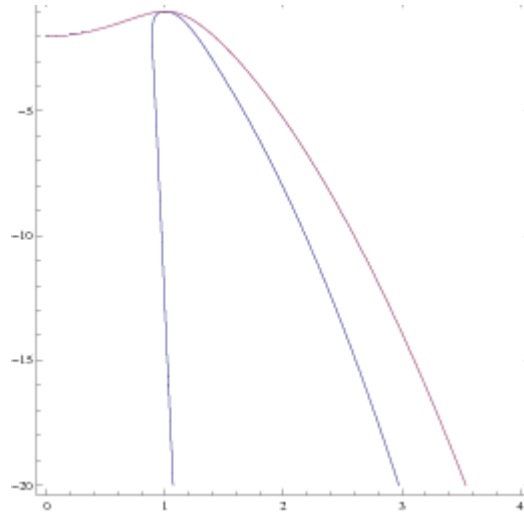


Figure 3: Regions in the plane $(\vec{q} \cdot \vec{e}_x, v - v_L)$ where the undulation exists (below the purple curve) and is stable in the sense of equation (31) (inside the blue curve.) for $k_c = 1$, $c = 2.0$ and $v_L = 2.0$. For $\tilde{g}$ we take a second-order polynomial in k^2 : $\tilde{g}(k) = \alpha - \beta k^2 + \gamma k^4$, with coefficients chosen to give the correct values of c , v_L and k_c .

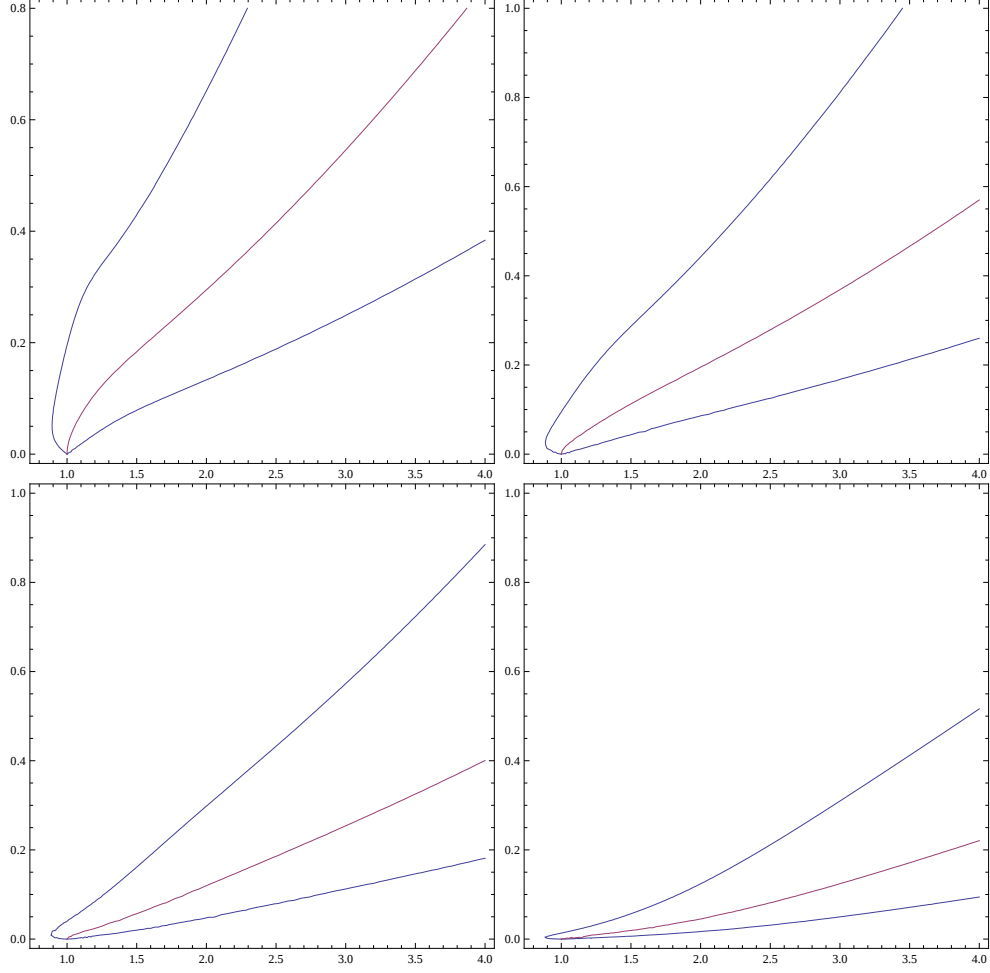


Figure 4: Domain of stability in the plane $(\vec{q} \cdot \vec{e}_x, \eta)$ for $c = 2.0$ and $v_L = 1.0$ (top left), $v_L = 1.5$ (top right), $v_L = 1.9$ (bottom left) and $v_L = 1.99$ (bottom right). Stable undulations exist between the two blue curves. Note the "kink" is always present although hardly visible because of the larger scale. The purple curves is the set of points where $\vec{q} \cdot \vec{e}_x = k_c$.

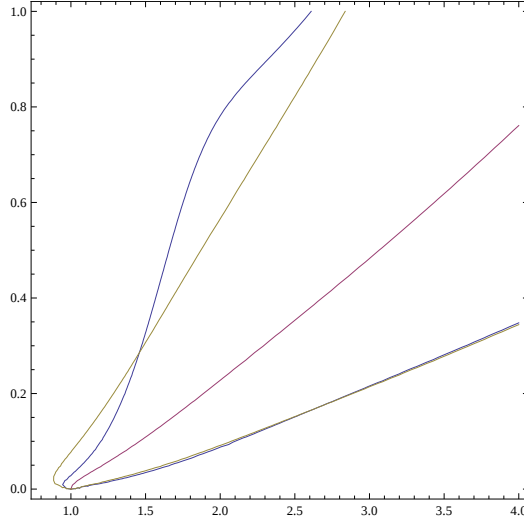


Figure 5: Domain of stability in the plane $(\vec{q} \cdot \vec{e}_x, \eta)$ for $k_c = 1$, $c = 2.0$ and $v_L = 1.9$. The blue curve is the limit of the stability domain computed using the model of [11], while the brown one uses the model in [1], neglecting couplings between modes k and $k \pm k_c/2$.

c at fixed k_c . We find the domain in which the undulation is stable shrinks when v_L goes to c , the two blue curves in Fig. 4 approaching one another. They merge with the $\eta = 0$ axis when $v_L = c$.

It would be interesting to relax the hypotheses of [11] and see what these stability conditions become in the model of [1]. We did this calculation neglecting couplings between the modes k and $k \pm k_c/2$ or $k \pm k_c$, but found only minor deviations from the above results (see Fig. 5). Although we could not perform the full calculation because of technical difficulties, it should be relatively easy to do it numerically. We plan to investigate this question in the future.

2.3 The minimization procedure

In this subsection we complete the analysis in [1], where the ground state is found by minimization of the energy functional. We first briefly explain some peculiar features of their method and the apparent breaking of Galilean invariance, before showing how the stability condition change when the assumption of small amplitude is relaxed. This new result will be tested numerically in subsection 2.4. In this subsection we set $\hbar = 1$.

The starting point of [1] is the energy functional:

$$\mathcal{E} = \int dx \left(\frac{1}{2m} \partial_x \psi^\dagger \partial_x \psi + (V(x) - \mu) \psi(x, t)^\dagger \psi(x, t) + \frac{1}{2} \int dy g(x, y) \psi(x, t)^\dagger \psi(y, t)^\dagger \psi(x, t) \psi(y, t) \right) \quad (32)$$

Minimizing $\mathcal{E}$ is equivalent to looking for solutions of the Gross-Pitaevskii-like equation (9) with frequency $\omega = \mu$. Since they are interested in stationary solutions, μ is set to zero. They choose an ansatz of the form:

$$\psi(x) = e^{iqx} (\sqrt{\rho_0} + \mathcal{U}e^{-ik_c x} - \mathcal{V}e^{ik_c x}) \quad (33)$$

The term $e^{iqx} \sqrt{\rho_0}$ is the initial condensate. $\mathcal{U}e^{-ik_c x}$ is the condensate of negative-energy modes and $\mathcal{V}e^{ik_c x}$ is added to account for two-body interactions which can produce excitations of wave-vector

$q + k_c$ via the scattering $q + q \rightarrow (q - k_c) + (q + k_c)$. They assume the total number of atoms, the condensate velocity and the perturbation wave-vector are fixed. These hypotheses are very important for this analysis. The last one is valid if the condensate velocity is just slightly above the Landau velocity, so that all the modes with negative energy have a wave-vector close to k_c .

At this point the astute reader may wonder what becomes of Galilean invariance. The existence of negative-energy modes, and thus the presence of the undulation (which is a measurable quantity) depends on the frame since, as noted above, the dispersion relation does not look the same in all frames. Does this mean the theory is not invariant under Galilean transformation? Or that some assumption we made breaks Galilean-invariance? One can easily check the theory is indeed invariant under the Galilean group, (the argument is the same as for the Schrödinger equation: contrary to the case of a scalar field in relativistic field theory, ψ gets a phase from a special Galilean transformation, but this has no observable consequence). Breaking of the Galilean invariance comes from the ansatz (33). More precisely, the density computed from (33) is not homogeneous, so it is time-independent only in one special frame. *A priori*, this choice of frame is arbitrary. Whatever the Galilean frame in which the calculation is done, minimizing the energy functional will give a solution of the field equation. However, we expect there is a physically “preferred” frame in which the undulation should be stationary or, which is equivalent, the energy functional (without adding a chemical potential) should be minimized. This is either the “laboratory” frame of a thermal bath with which the condensate is in equilibrium or the frame of anything which could introduce friction, like a pipe constraining the flow. These effects are taken into account indirectly by working in the right reference frame.

We now follow the analysis of [1] using slightly weaker assumptions, and show how their results are changed. Baym and Pethick want to minimize the energy density (15) in [1]:

$$\mathcal{E} = \zeta^2 \left(n \tilde{g}(k_c) e^{-\eta} + \frac{k_c^2}{2m} \cosh(\eta) - \frac{1}{m} q k_c \right) + \zeta^4 \left(\frac{\tilde{g}(2k_c)}{4} \sinh^2(\eta) - \tilde{g}(k_c) e^{-\eta} \cosh(\eta) \right), \quad (34)$$

under the assumption that ζ is “small”. Here ζ and η are related to $\mathcal{U}$ and $\mathcal{V}$ by:

$$\begin{aligned} \mathcal{U} &= \zeta \cosh \frac{\eta}{2} \\ \mathcal{V} &= \zeta \sinh \frac{\eta}{2} \end{aligned} \quad (35)$$

and n is the mean density. For the two terms in (34) to be of the same order, the coefficient of ζ^2 in the first one must be very small. η can thus be found by just minimizing the first term, giving the enlightening formula (17) in [1] which tells us that e^η is just the ratio of the total energy of an excitation over its kinetic energy. This gives a nice interpretation to η for very small undulations.

We will now see how relaxing the assumption that the amplitude of the undulation is small affects the stability condition given by equation (21) in [1].

It seems simpler to minimize first with respect to ζ . If the coefficients before ζ^2 and ζ^4 have the same sign, the minimum is reached either for $\zeta = 0$ (and then there is no undulation) or $\zeta = \infty$ (in which case some additional terms must be added to the Hamiltonian to understand the physics and make ζ finite). In the following we shall assume their signs are different. Also, we need the energy to be bounded from below, otherwise again another term must be added to E' . We thus assume that:

$$\frac{1}{4} \tilde{g}(2k_c) \sinh^2 \eta - \tilde{g}(k_c) e^{-\eta} \cosh \eta > 0, \quad (36)$$

and

$$\frac{k_c^2}{2m} \cosh \eta - \frac{k_c q}{m} + \tilde{g}(k_c) n e^{-\eta} < 0. \quad (37)$$

For any given η , the values of ζ which minimizes $\mathcal{E}$ satisfy:

$$\zeta^2 = -\frac{1}{2} \frac{\frac{k_c^2}{2m} \cosh \eta - \frac{k_c q}{m} + \tilde{g}(k_c) n e^{-\eta}}{\frac{1}{4} \tilde{g}(2k_c) \sinh^2 \eta - \tilde{g}(k_c) e^{-\eta} \cosh \eta}. \quad (38)$$

Plugging this expression into (34), the averaged energy density becomes:

$$\mathcal{E} = -\frac{1}{4} \frac{\left(\frac{k_c^2}{2m} \cosh \eta - \frac{k_c q}{m} + \tilde{g}(k_c) n e^{-\eta} \right)^2}{\frac{1}{4} \tilde{g}(2k_c) \sinh^2 \eta - \tilde{g}(k_c) e^{-\eta} \cosh \eta}. \quad (39)$$

Let us define $\mathcal{E}' = 4\mathcal{E}/n^2$ and rescale the momenta so that $mn = 1$. We get:

$$\mathcal{E}' = -\frac{\left(\frac{k_c^2}{2} \cosh \eta - k_c q + \tilde{g}(k_c) e^{-\eta} \right)^2}{\frac{1}{4} \tilde{g}(2k_c) \sinh^2 \eta - \tilde{g}(k_c) e^{-\eta} \cosh \eta}. \quad (40)$$

This can be rewritten in a nicer way by using $X \equiv e^{-\eta}$:

$$\mathcal{E}' = -\frac{\left(\frac{k_c^2}{4} \left(X + \frac{1}{X} \right) - k_c q + \tilde{g}(k_c) X \right)^2}{\frac{1}{16} \tilde{g}(2k_c) \left(X - \frac{1}{X} \right)^2 - \frac{1}{2} \tilde{g}(k_c) (X^2 + 1)} = -\frac{\left(\frac{k_c^2}{4} (X^2 + 1) - k_c q + \tilde{g}(k_c) X^2 \right)^2}{\frac{1}{16} \tilde{g}(2k_c) (X^2 - 1)^2 - \frac{1}{2} \tilde{g}(k_c) X^2 (X^2 + 1)}. \quad (41)$$

The denominator in (41) must not vanish and change sign, otherwise there is a point where $\mathcal{E}$ goes to $-\infty$ and ζ to $+\infty$, meaning the “undulation” (which will probably not be an undulation anymore) completely depletes the condensate. Let us study the sign of

$$\frac{\tilde{g}(2k_c)}{8} (X^2 - 1)^2 - \tilde{g}(k_c) X^2 (X^2 + 1) = \left(\frac{\tilde{g}(2k_c)}{8} - \tilde{g}(k_c) \right) X^4 - \left(\frac{\tilde{g}(2k_c)}{4} + \tilde{g}(k_c) \right) X^2 + \frac{\tilde{g}(2k_c)}{8} \quad (42)$$

The discriminant of this polynomial is

$$\Delta = \left(\frac{\tilde{g}(2k_c)}{4} + \tilde{g}(k_c) \right)^2 - \frac{\tilde{g}(2k_c)}{2} \left(\frac{\tilde{g}(2k_c)}{8} - \tilde{g}(k_c) \right) = \tilde{g}(k_c) (\tilde{g}(k_c) + \tilde{g}(2k_c)) \quad (43)$$

If both $\tilde{g}(k_c)$ and $\tilde{g}(k_c) + \tilde{g}(2k_c)$ are positive, we have two real roots (in X^2), and at least one of them is positive for $\frac{\tilde{g}(2k_c)}{4} + \tilde{g}(k_c) \geq 0$. So, we don't find any stable undulation. Moreover, the undulation can not have a negative energy if both $\tilde{g}(k_c)$ is positive while $\tilde{g}(2k_c)$ is negative. The only way out is to choose

$$\boxed{\tilde{g}(k_c) < 0} \quad (44)$$

Note that a straightforward calculation shows the amplitude of the undulation is non zero if and only if v is larger than the Landau velocity, as in the analysis in [1]. So, we get a new necessary condition

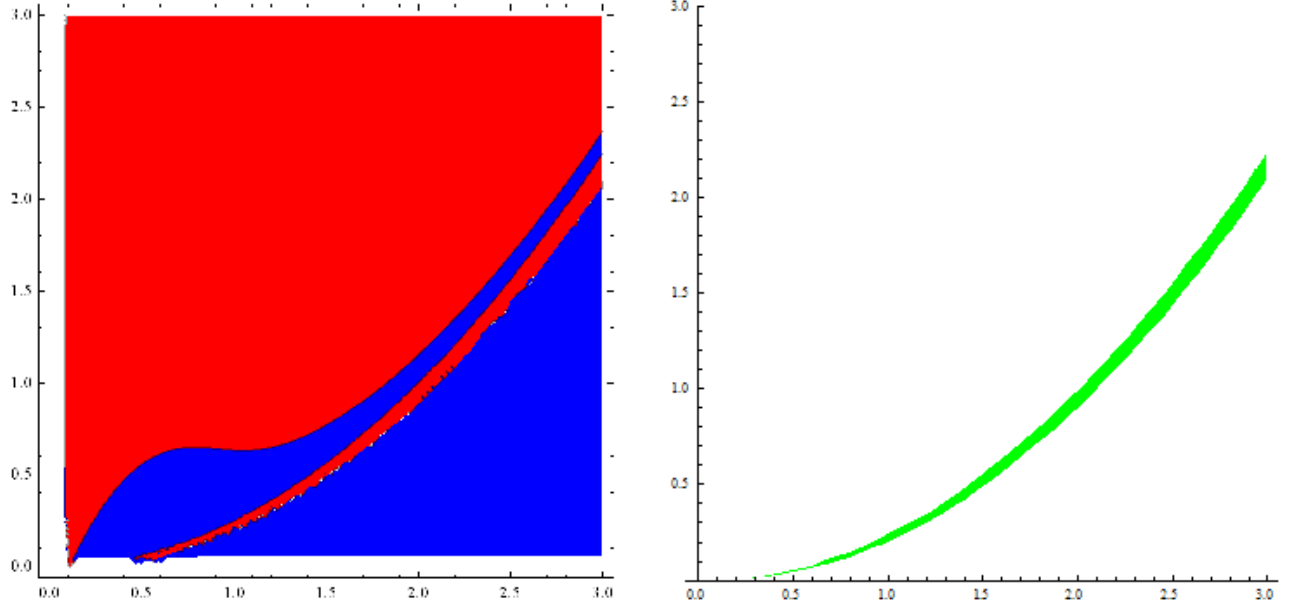


Figure 6: Comparison of the stability condition given in [1] and $\tilde{g}(k_c) < 0$ in the β, δ plane. Left: domain where the stability condition of [1] is satisfied (in red). Right: Domain where $\tilde{g}(k_c) < 0$ and the spectrum of perturbations remain real (green strip). Values of the parameters are $g_0 = 2.04$, $m = 1.45$ and $\rho = 1.37$. One can see (44) is much more restrictive.

for the existence of a stable undulation (44). As can be seen on Fig 6, this is much more restrictive than the condition $\gamma > 0$ given in [1]. Fig 6 compares the two conditions in the β, δ plane with the choice: $\tilde{g} = g_0(1 - \beta k^2 + \delta k^4)$. Note that we also impose $\epsilon_0(k_c) \in \mathbb{R}$ (which is also assumed in [1]), since it is not clear this analysis is valid when a *dynamical instability* (complex frequencies in the dispersion relation) is present. We focus only on the structures appearing from an energetical instability, when the spectrum of perturbations is not positive but remains real. The condition (44) is one of the main results of our work since it puts severe constraints on the shape of the interaction potential giving a stable undulation under the hypotheses of [1].

The condition (44) is compatible with the interpretation of $\tilde{g}(k_c)$ (up to some positive factor) as a mixing energy between the “fluids” consisting in the condensates of wave-vectors q and $q - k_c$. More precisely, if we neglect the modes $k_c + q$ for simplicity, the total energy density is found to be the sum of the energies of the two fluids alone, plus a term

$$\delta\mathcal{E} = n_0|\zeta|^2\tilde{g}(k_c) \quad (45)$$

Intuitively, a phase where the two fluids are mixed is stable if $\tilde{g}(k_c) < 0$.

For comparison with the numerical results below, it is important to know what changes when the hypotheses mentioned at the beginning of this subsection are relaxed. First, we assumed the total density n is fixed, so that n_0 changes with $\mathcal{U}$ and $\mathcal{V}$. If instead the condensate density n_0 is held fixed, the condition $\tilde{g}(k_c) < 0$ is no longer necessary. Also, assuming the mean flux of atoms is fixed instead of the condensate velocity, we find no undulation as long as the spectrum of perturbations is real. This is because a state with mean density n and an undulation of amplitude ζ has a smaller

energy than a homogeneous condensate with the same density, but a larger energy than another homogeneous configuration with a different number of atoms per unit length. From a physical point of view, the quantities which are held fixed will depend on the particular experimental setup, but choosing n and q seems the most natural ones. If one is more interested in the mathematical properties of equation (9), the energy functional must in principle be minimized without *a priori* assumption or constraint. This is of course more difficult and we resorted to numerical integration (see subsection 2.5).

2.4 Numerical Results: Homogeneous Case for $v \approx v_L$

In this subsection we explain how equation (9) is solved numerically under different assumptions and constraints in a homogeneous potential. The goal is to check our analytical results, in particular the new condition $\tilde{g}(k_c) < 0$. The present analysis relies on the specific form given above for the undulation, which must be valid for $v \approx v_L$. On the other hand, it may break down when v is finitely larger than v_L .

We wrote a C program for solving ordinary differential equations which can be reduced to a functional to extremize. We shall call this functional the “action” S . The first and second derivatives of S with respect to the degrees of freedom at each point are computed from an initial approximate solution, and used to get a better one. More precisely, if V is the vector whose components are the first derivatives of S , and M the matrix of its second derivatives, the quantity ϵ to be added to the solution satisfies:

$$M \cdot \epsilon = -V \quad (46)$$

In practice, one may multiply ϵ by some $\alpha < 1$ to make the code more stable. Iterating this procedure should in the end either diverge or give a good solution, the error being controlled by the relative variation of S between two iterations.

This scheme has two main advantages compared to a standard Euler or Runge-Kutta method:

*It can easily deal with so-called “boundary condition problems”, for which boundary conditions are not all imposed at the same point. There is no need for a shooting method and imposing boundary conditions at different points is very natural. Note that we can *a priori* add more boundary conditions than required by the equation, for instance to simulate an obstacle in the middle of the integration domain, or less than that (in which case the action is extremized over a subset of the space of solutions, which can be physically relevant). In the same spirit, the ability to choose the ansatz one starts with makes it easier to find a solution with some special property.

*It deals directly with the action (or thermodynamic potential depending on the problem), which is sometimes more fundamental or insightful than the equation itself. All the properties associated with the fact that this quantity is extremized are also automatically satisfied.

The main technical point consists in inverting the matrix M . This however remains relatively easy for a one-dimensional problem, where only a finite number of diagonals of M are non-vanishing, independantly of the number of points, using Gauss’s algorithm.

The functional to be extremized is, after some rescalings:

$$H = \int dx \left[(\partial_x \psi)^* (\partial_x \psi) - \mu \psi^* \psi + \frac{1}{2} |\psi|^2 (1 - \beta \partial_x^2 + \gamma \partial_x^4) |\psi|^2 \right] \quad (47)$$

ψ is written as

$$\psi(x) = r(x) e^{i \int_0^x q(y) dy} \quad (48)$$

where r and q are two real-valued functions.

The equations of motion impose that the momentum flow $\Pi = qr^2$ be a constant. Using this, the Hamiltonian is written in terms of r_0 alone:

$$H = \int dx \left[(\partial_x r)^2 - \mu r^2 + \frac{\Pi^2}{r^2} + \frac{1}{2} (r^4 - \beta (\partial_x r^2)^2 + \gamma (\partial_x^2 r^2)^2) \right] \quad (49)$$

Setting $\Pi = 0$ for simplicity, and choosing an ansatz of the form:

$$r^2(x) = A + B \cos(kx) \quad (50)$$

with $A \gg B \geq 0$, $k > 0$, we find the energy is minimum for $A = \mu$. Also, B is different from zero only if

$$\beta - \frac{1}{2\mu} < 2\sqrt{\gamma} \quad (51)$$

meaning the undulation is not expected to appear if the spectrum of perturbations over the homogeneous condensate is real, in accordance with the above results concerning the formation of an undulation at fixed flux.

Although this program is mostly interesting as a first step towards the numerical solution of equation (9) (the corresponding code presented in the next subsection works exactly the same way), it was also used to check that no undulation occurs at fixed density flux provided the spectrum of perturbations is real. Numerical results are consistent with (51).

Of greater interest for us is to check the condition $\tilde{g}(k_c) < 0$ is necessary to have a stable undulation. We want to be in the same conditions as those studied by Baym and Pethick, so that the results can not be seen as effects of different assumptions. We thus expand ψ in a Fourier series, the first terms of which are the three terms in (33), and minimize the energy functional with respect to the Fourier coefficients using the method described above. We first considered only three terms (so that we were exactly in the conditions of [1], except that the velocity could be made arbitrarily large) and found the term of wave-vector q is dominant for $v < v_L$, the other ones being equal to zero up to numerical errors as expected. For $v > v_L$ and $\tilde{g}(k_c) < 0$, two or three of these terms have a comparable amplitude, depending on the parameters. But for $v > v_L$ and $\tilde{g}(k_c) > 0$, only one of them (not the one of wave-vector q) is non-zero (see Fig. 7). Including more terms does not change this picture since the additional ones are negligible at the onset of instability, when v is only slightly above v_L . These results are consistent with the stability condition discussed in the previous subsection.

2.5 Numerical Results: Homogeneous case at finite $v - v_L$ and Inhomogeneous Flow.

In this subsection we show a few numerical results, which could be the starting point of a more complete study. We make more realistic assumptions than above in regions where the background flow is homogeneous and discuss the effects of inhomogeneities. We are especially interested in *Landau horizons*, where the background velocity crosses the Landau velocity. Study of a flow with a Landau horizon is in a sense the point where our work presented above converges and the very

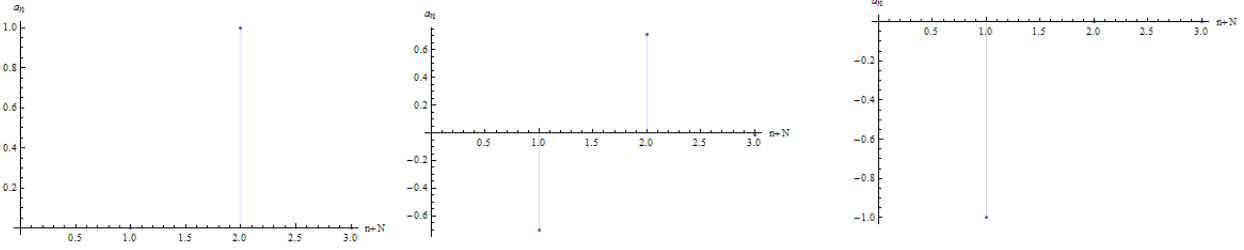


Figure 7: Amplitude of the three terms in (33). On the abscissa axis, 1 correspond to the mode $q - k_c$, 2 to q and 3 to $k + k_c$. Left: below the Landau velocity, the mode q dominates ($m = 0.5$, $\alpha = 1$, $\beta = 2.2$, $\gamma = 1.0$, $q = 0.2$). Middle: Above the Landau velocity, for $\tilde{g}(k_c) < 0$, at least two modes are macroscopically occupied ($m = 0.5$, $\alpha = 1$, $\beta = 2.2$, $\gamma = 1.0$, $q = 0.5$). Right: If $\tilde{g}(k_c) > 0$, only one mode is occupied even above the Landau velocity ($m = 0.5$, $\alpha = 1$, $\beta = 1.9$, $\gamma = 1.0$, $q = 0.5$).

first step of a more complete understanding of the instability mechanism. The aim is to catch both near-horizon and far-horizon effects to understand how the symmetry-breaking phase develops in space from a homogeneous flow and what makes it saturate.

The Hamiltonian we want to minimize is expressed in terms of the density n and phase $\theta = \text{Arg } \psi$ as (note that we use the phase of the field instead of the velocity, which would give a different equation of motion):

$$H = \int dx \left[\frac{\hbar^2}{2m} ((\partial_x \sqrt{n})^2 + n(\partial_x \theta)^2) + (V(x) - \mu)n + \frac{1}{2} (\alpha n^2 - \beta (\partial_x n)^2 + \gamma (\partial_x n)^4) \right]. \quad (52)$$

The potential V is chosen so that the hydrodynamical equations (which are the equations obtained by neglecting variations of the density) give a velocity profile:

$$v(x) = v_0 + A \tanh(\kappa x), \quad (53)$$

and we rescale the units of mass and time to set $\hbar = m = 1$. We could eliminate one more quantity (since three dimensions enter our problem, namely time, length and mass), but we choose to keep it in order to make comparison with our previous results more straightforward. Working with more parameters than necessary is not a problem since we do not perform a systematic study of the solutions of the Gross-Pitaevskii equation. We used an algorithm very similar to the one described in subsection 2.4, but turned to Mathematica, which has many built-in functions making technical points easier to handle.

We first studied the homogeneous case with a different emphasis as in subsection 2.4. Namely, instead of assuming a particular form of the condensate and undulation, we solved the equation of motion from (52) without additional constraints. As expected, we found an undulation only for $v > v_L$ or if the spectrum contains complex modes. From the results of [1, 11] and the shape of the dispersion relation, one could expect to find an order parameter similar to (33), giving an oscillating density with a wave-vector k_c . Instead, we found a superposition of the *two* wave-vectors corresponding to zero frequency. Going above the sound velocity, we found only one wave-vector, again corresponding to $\omega = 0$. This suggests the assumption that the undulation is given by the most unstable mode, *ie* the one with largest negative energy, is not valid except at the onset of

instability. For a finite value of $v - v_L$, or a finite range of dynamically unstable modes, we must take two wave-vectors into account. This is one of the main results of this work, along with the stability conditions above.

Adding a Landau or sonic horizon does not change these results in the asymptotic regions where the background flow is homogeneous. Whatever happens close to the horizon, we find far from it a superposition of modes with zero frequency. This may seem counter intuitive given the results of [1, 11]. Indeed, if only one mode is present (as we observe in the supersonic region), its amplitude goes to zero as $\omega \rightarrow 0$, and is maximized by the “most unstable” wave-vector, with the largest $-\omega$. Intuitively, one could have expected either an undulation with momentum equal to the most unstable one, no undulation at all (because of destructive interferences between modes), or something with no well-defined momentum. We argue that these three conjectures captures part of the reality, and that it explains the observed wave-vector(s).

Since, from the results of [1, 11], *each* unstable mode gives, if alone, an undulation with a finite amplitude, it seems reasonable to assume this remains true when considering *all* of the modes. More precisely, the modes between k and $k + dk$ should contribute by $f(k) dk$, where f is a smooth function which vanishes outside the domain where $\omega < 0$. Let k_1 and k_2 be the two wave-vectors corresponding to $\omega = 0$, with $0 < k_1 < k_2$. We assume f is a polynomial between k_1 and k_2 , and $f(k) = 0$ outside this interval. It also seems reasonable to assume f has a maximum at the most unstable wave-vector, but we shall not need this hypothesis. The question is: what is the result of this (continuous) superposition of modes? It is easy to show using integrations by parts that

$$\int_{k_1}^{k_2} dk k^n \exp(ikx) \quad (54)$$

is a linear combination of modes of wave-vectors k_1 and k_2 , multiplied by x^{-1-n} . So, the total amplitude will be of the form

$$\delta\psi(x) = F_1(x) \exp(ik_1x) + F_2(x) \exp(ik_2x) + F_3(x) \exp(-ik_1x) + F_4(x) \exp(-ik_2x) \quad (55)$$

where F_1, F_2, F_3 and F_4 are four polynomials in $1/x$. We include the opposite of unstable modes to account for interactions between excitations: two phonons with zero momentum can scatter and get momenta $+k, -k$.² In Fourier space, (55) gives a continuum plus Dirac deltas at $k \in \{-k_1, -k_2, +k_1, +k_2\}$ for n (and at sums of these momenta, but with a much smaller coefficient if $\delta\psi$ is small before ψ), which explains the observed wave-vectors on the undulation. The same argument works above the sound velocity, with $k_1 = 0$.

To illustrate this, we choose a function f of the form

$$\begin{aligned} f(k) &= f_0 (k - k_1) (k_2 - k) \text{ if } k \in [k_1, k_2] \\ f(k) &= 0 \text{ otherwise} \end{aligned} \quad (56)$$

and assume it gives directly the perturbation on n , not on ψ . This is not very realistic, but illustrates the above argument more directly. The important point is the emergence of k_1 and k_2

²This can also be seen from the fact that *negative*-norm modes of momenta between $-k_2$ and $-k_1$ have a positive energy, also leading to an energetical instability.

from the integration. We have:

$$\begin{aligned}
\delta n(x) &= f_0 \int_{k_1}^{k_2} dk (k - k_1) (k_2 - k) \cos(ikx) \\
&= -f_0 \int_{k_1}^{k_2} dk (k_2 + k_1 - 2k) \frac{1}{x} \sin(kx) \\
&= f_0 \left[\frac{k_2 + k_1 - 2k}{x^2} \cos(kx) \right]_{k=k_1}^{k=k_2} + 2f_0 \frac{1}{x^2} \int_{k_1}^{k_2} dk \cos(kx) \\
&= f_0 \left(\frac{k_1 - k_2}{x^2} (\cos(k_2x) + \cos(k_1x)) + \frac{2}{x^3} (\sin(k_2x) - \sin(k_1x)) \right) \tag{57}
\end{aligned}$$

The particular role of $x = 0$ comes from the implicit assumption that all the phases in $\delta\psi$ or δn are equal at $x = 0$. While the emergence of oscillations with wave-vectors k_1 and k_2 does not depend on the particular choice of f provided it is sufficiently regular, the x -dependence of the amplitude does and the factors $1/x^2$, $1/x^3$ above are probably not relevant.

We can now go back to our conjectures above and see why they fail. It is true that the most unstable mode contributes, possibly more than the others, but its effects are suppressed by the unstable modes around it. This continuum does give an undulation because of boundary effects in Fourier space: the fact that f vanishes outside $[k_1, k_2]$ prevents destructive interferences from completely removing the undulation. Modes close to k_1 and k_2 survive and are enhanced by the continuum. Note that, since $k_1 = 0$ above the speed of sound, the undulation has only one non-vanishing wave-vector k_2 .

Now that we know what modes make up the undulation and have a possible analytical motivation, we must understand which parameters fix the amplitude. As noted above, the method of [1, 11] can not be applied since it would give a vanishing amplitude, contrary to what is observed. Moreover, our numerical results do not show clearly the dependence of the amplitude in the model parameters. This is still an open question which we plan to investigate in a near future both numerically and analytically. We will first try to make the code more efficient, maybe returning to a compiled language in order to do a more systematic study, and then try to understand what are the relevant parameters and why they are important.

3 Conclusion

We studied instabilities in Bose-Einstein condensates leading to breaking of translational invariance. We first investigated stability of the resulting undulation and found two stability conditions which, to our knowledge, have not been stated before (see equations (31) and (44)). We then solved the equations of motion numerically and found the ground state is made of the zero-frequency modes, not the lowest-energy ones. This unexpected result should be understandable analytically, maybe by a refinement of the motivation we gave. Finally, we studied links with a Maxwell-Chern-Simons theory in 5 dimensions (see Appendix A) and conjecture a deep link between these models.

We see this work as a first step towards the understanding of both the undulation itself and its relations with other seemingly different phenomena, like undulations in hydrodynamics. This subject covers very different domains of physics, from cold atoms to classical hydrodynamics to quantum gravity theories, providing us with models which are studied and understood using very

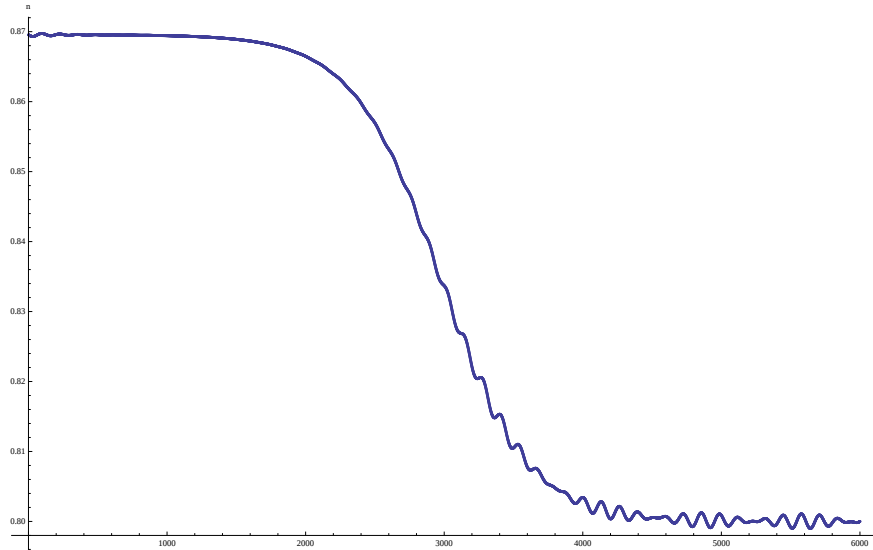


Figure 8: Density profile in the presence of a Landau horizon. Here we plot the density as a function of x . In abscissa is the number of integration points, so that “6000” corresponds to $x = 100$. The velocity goes from 0.46 at $x = 0$ to 0.5 at $x = 100$. The Landau velocity is equal to 0.48 and the sound velocity to 0.8. Small oscillations close to $x = 0$ are due to border effects.

different techniques and with different aims, but are possibly linked by the same underlying phenomenon. It seems to have the right balance between generality and specificity for the beginning of a thesis project. More specifically, we plan to expand on the results presented in subsection 2.5, which leave many important questions unanswered, concerning for instance the amplitude of the undulation. These problems may be solved by a combination of numerical analytic studies. The link with Maxwell-Chern-Simons black holes and undulations in classical hydrodynamics also deserve some attention, and can help exhibit general features under seemingly different models.

Acknowledgements

Foremost, I would like to express my sincere gratitude to my advisor Renaud Parentani for guidance as well as enlightening discussions about Bose-Einstein condensates, black holes, interpretations of Quantum Mechanics, cosmology and other domains. His patience, enthusiasm, great knowledge and ability to explain difficult ideas in a simple way have made working with him particularly pleasant and rewarding.

I am grateful to Gora Schlyapnikov for pointing out dipolar gas as an interesting illustration of our work and for giving me the tools to compute the two-body interaction potential under realistic assumptions.

My thanks also go to Antonin Coutant, Robin Zegers and Xavier Busch for interesting discussions.

4 Appendices

4.1 Appendix A: Maxwell-Chern-Simons Theory in 5 Dimensional Anti-de Sitter Space

As we said in the introduction, there exists an instability in a Maxwell-Chern-Simons theory which is qualitatively similar to the Landau instability in Bose-Einstein condensates we studied. A precise comparison would be most useful to understand what are the general conditions and features of such a symmetry-breaking phase under the specificities of each model. Here we very briefly present the Maxwell-Chern-Simons theory and the instability we are interested in (more details can be found in [8, 9]), and state the main similarities and differences with the Bose-Einstein condensate case. We did not get new results, but rather a qualitative understanding of these relations which paves the way for a more precise comparison. This lack of any precise result is the reason we did not devote part of the main text to this analysis.

Maxwell-Chern-Simons theory in 5 dimensions describes a vector field A_μ with a Lagrangian density:

$$\mathcal{L} = -\frac{1}{4}F_{IJ}F^{IJ} + \frac{\alpha}{3!\sqrt{-g}}\epsilon^{IJKLM}A_IF_{JK}F_{LM}, \quad (58)$$

where $F_{IJ} = \partial_IA_J - \partial_JA_I$ is the Maxwell tensor, g the determinant of the metric and ϵ^{IJKLM} is equal to -1 if $IJKLM$ is an even permutation of 01234, $+1$ if it is an odd permutation and 0 otherwise. The second term in (58) is called the Chern-Simons term, and appears for instance in supergravity theories [8].

Let us first consider this theory in Minkowski space. A constant electric field E is a solution of the equations of motion from (58), but it is unstable. More precisely, transverse modes of wave-vector $\vec{k}$ are tachyonic in the range

$$0 < |\vec{k}| < 4\alpha E. \quad (59)$$

This is similar to what happens in a Bose-Einstein condensate above the speed of sound, with or without the roton-maxon structure. The mode with largest negative squared mass, which would be the equivalent of the most unstable mode, is

$$k_m = 2\alpha E. \quad (60)$$

In [9], the non-linear solution at fixed k is found, exhibiting a symmetry-breaking phase analogous to our undulation. It is found to be unstable under a change in k except for $k = 0$, so that the lowest-energy solution is the trivial vacuum state with $F_{IJ} = 0$: the electric field of the “perturbation” just cancels the applied one. One can argue the total charge (which is equal to zero here) is analogous to the density flow in the Bose-Einstein case, in the sense that fixing them does not lead to a symmetry-breaking phase. On the other hand, the electric field E would be analogous to the condensate velocity, since this is the quantity which makes the homogeneous phase unstable.

This breaks down when considering a charged black hole in anti-de Sitter space. It is found that, depending on the values of α , the temperature T and the chemical potential μ of the black hole, a perturbation with finite wave-vector can be stable. An important question is whether this

is due to the presence of a charged black hole or to the asymptotically anti-de Sitter geometry. The effect of the black hole is easy to understand if we assume E is still analogous to the condensate velocity. Since a charged black hole has a non-vanishing electric field on its horizon, the velocity of the corresponding condensate should be above the critical one, either v_L or c . The black hole is the source of the electric field which in turn makes the configuration unstable. On the other hand, going to anti-de Sitter space changes the stability condition: taking the different normalizability condition into account, one finds a mode of mass m is unstable if and only if

$$m^2 < -\frac{1}{4} \quad (61)$$

for a unit radius anti-de Sitter space, instead of $m^2 < 0$ in asymptotically Minkowski space. So, modes with wave-vectors close to zero, which have a very small negative mass [9] are not unstable. This is just what is found in a Bose-Einstein condensate with a roton-maxon structure when the velocity is between the Landau and sound velocities. We conjecture a deep analogy exists between these models, with the relations:

$$\begin{aligned} \text{Minkowski space} &\leftrightarrow \text{no roton-maxon} \\ \text{Anti-de Sitter space} &\leftrightarrow \text{yes roton-maxon} \\ \text{electric field } E &\leftrightarrow v - v_L \text{ (anti-de Sitter) or } v - c \text{ (Minkowski)} \end{aligned}$$

From this we expect the Anti-de Sitter-black hole geometry alone is not enough to get a symmetry-breaking phase. This is actually what is found in [8]: when the charge of the black hole goes to zero (so that the electric field at the horizon vanishes), the homogeneous phase is stable at any temperature.

A more detailed analysis is required to find whether this analogy goes beyond a few qualitative similarities. In particular, it would be interesting to study the stability of black holes in asymptotically Minkowski space with the Chern-Simons term, as well as extremal black holes (which have a simpler geometry) in asymptotically anti-de Sitter space. In [9] the non-linear solutions are found assuming the symmetry-breaking phase has a well-defined wave-vector, just as in [11, 1] for the Bose-Einstein case. Given the results presented in subsection 2.5, one is tempted to check this hypothesis, which should break down if our conjecture is correct. We plan to investigate this in a near future.

4.2 Appendix B: One-dimensional Dipolar Gas

This subsection is devoted to a particular model for which the dispersion relation has the right roton-maxon structure. The aim is to show explicitly that such a dispersion relation occurs in a physical model. It might provide a way to test experimentally the above results, although there may be important practical difficulties.

We shall here consider a quasi-one-dimensional Bose-Einstein condensate subject to an anisotropic harmonic trap with a frequency in two dimensions (let us call them x and y) much larger than in the third one z . Strictly speaking, Bose-Einstein condensation does not occur in one-dimensional uniform gases in the thermodynamical limit. Indeed it is well-known that one-dimensional statistical systems with only short-range interactions (which is assumed here) have no phase transition at

finite temperature. This is because entropic effects, which are logarithmic in the number of atoms N , dominate over energetic ones, independent of N . However, condensates can appear provided a suitable confining potential is added [14]. Since link with actual experiments is not the primary aim of this internship, we shall simply assume we work at a sufficiently low temperature so that thermal effects are negligible despite the weakness of the confining potential in the z direction. Despite this rather unrealistic assumption, the calculation below shows the roton-maxon structure can appear in a simple model.

Let ω be the trapping frequency in the two directions x , y , and ω_z the frequency in the z direction. We assume $\omega \gg \omega_z$, and $\omega \gg k_B T$. The wave-function of each atom can then be written as:

$$\psi(\vec{r}) = \psi_0(x) \psi_0(y) \phi(z), \quad (62)$$

where ψ_0 is the wave-function of the ground-state of a harmonic oscillator of frequency ω . Since virtually all the atoms are in their ground state in the x and y directions, we shall “only quantize the z -dependence of the field”, *ie* we write

$$\psi(\vec{r}) = \frac{m\omega}{\pi\hbar} \exp\left(-\frac{m\omega\rho^2}{2\hbar}\right) \phi(z) \quad (63)$$

where $\rho = x^2 + y^2$ and ϕ a one-dimensional quantum field:

$$\phi(z) = \sum_k \phi_k(z) a_k^{(z)}. \quad (64)$$

Here ϕ_k are the eigenfunctions of the Hamiltonian for a one-dimensional harmonic oscillator of frequency ω_z with annihilation operators $a_k^{(z)}$.

We consider a Hamiltonian of the form:

$$\hat{H} = \int d^3r \frac{\hbar^2}{2m} \vec{\nabla} \psi^\dagger \cdot \vec{\nabla} \psi + \hat{H}_{int}, \quad (65)$$

where the interaction Hamiltonian is given by

$$\hat{H}_{int} = \frac{1}{2} \int d^3r d^3r' V(\vec{r} - \vec{r}') \psi^\dagger(\vec{r}) \psi^\dagger(\vec{r}') \psi(\vec{r}) \psi(\vec{r}'). \quad (66)$$

g is an even function (for the moment unspecified). Integration over x and y gives an effective two-body potential whose Fourier transform is

$$\tilde{g}(k) = \frac{1}{2} \frac{m\omega}{\pi\hbar} \int d^2\rho dz \exp\left(-\frac{m\omega}{\hbar} \frac{\rho^2}{2}\right) V(\vec{\rho} + z\vec{e}_z) \cos(kz), \quad (67)$$

where $\vec{e}_z$ is a unit vector in the z direction and $\vec{\rho}$ is in the (x, y) plane. We assume the atoms interact via a dipole-dipole term, plus some local interactions which will contribute by a constant to $\tilde{g}$, g_0 . If we further assume all the dipoles $\vec{d}$ are parallel to each others and orthogonal to the (Oz) direction (see Fig. 9), V has the form:

$$V(\vec{\rho}, z) = \frac{d^2}{(\rho^2 + z^2)^{3/2}} - \frac{3d^2\rho^2}{(\rho^2 + z^2)^{5/2}} \quad (68)$$

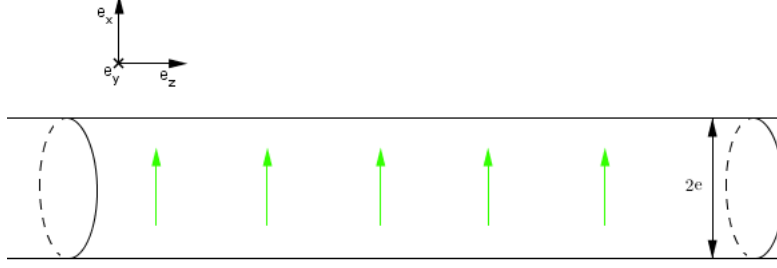


Figure 9: Schematic diagram of the setup. The Bose-Einstein condensate is approximately cylindrical, with a diameter $2e = \sqrt{\hbar/(m\omega)}$. The atoms have a magnetic or electric dipole (green arrows), oriented along the x direction by a magnetic or electric field.

After a few lines of algebra, we get:

$$\tilde{g}(k) - g_0 \approx k^2 d^2 \log \left(\frac{ke}{\alpha} \right) \quad (69)$$

where α is a quantity of order 1 and $e = \sqrt{\frac{\hbar}{m\omega}}$. More precisely, we find

$$\alpha = ke \exp \left(-\frac{1}{k^2 e^2} \int_0^\infty y dy \int_{-\infty}^{+\infty} dx \exp \left(-\frac{y^2}{2} \right) \frac{\cos(kex) - 1}{(y^2 + x^2)^{3/2}} \right) \quad (70)$$

In the limit $ke \rightarrow 0$, $\alpha \approx 1.746$. It can be treated as a constant for $ke \ll 1$.

One can check the corresponding dispersion relation is linear in k for $k \approx 0$, has a unique inflexion point, and is proportional to k^2 in the large k limit, so that our above analysis applies. The stability condition (44) takes a rather simple form in terms of the Landau velocity v_L , the speed of sound c and the critical wave-vector k_c :

$$\frac{v_L}{c} \leq \sqrt{1 + \frac{1}{2 \log \left(\frac{ek_c}{\alpha} \right)}} \quad (71)$$

This can be rewritten in terms of microscopic quantities. Imposing also that the spectrum of perturbations be real (which is implicitly assumed in (71) since v_L is not defined when the spectrum has complex-frequency modes), we get:

$$1 \leq g_0 \frac{2e^2}{\alpha^2 d^2} \exp \left(\frac{\hbar^2}{2m\rho_0 d^2} + 1 \right) < 1 + \frac{\hbar^2}{2m\rho_0 d^2} \quad (72)$$

4.3 Appendix C: Landau Horizon and Hawking Radiation

Here we argue there is no Hawking Radiation (HR) in the presence of the undulation below the sound velocity. More precisely, HR can be seen as the mechanism which feeds the unstable modes

and makes the undulation grow. At the end of the process, the “renormalised” spectrum of perturbations is positive and HR stops, hence the saturation of the undulation. We first briefly review why HR appears before explaining what prevents it in the true ground state.

We consider a stationary one-dimensional Bose-Einstein condensate with an inhomogeneous velocity. More specifically, we assume the velocity v goes from $v_- < v_L$ at $x \rightarrow -\infty$ to v_+ such that $c > v_+ > v_L$ at $x \rightarrow +\infty$. The crucial point lies in the mode expansion of the perturbations ϕ .

In a homogeneous condensate, the wave-vector k is a good quantum number and ϕ can be expanded as:

$$\phi = \int_{-\infty}^{+\infty} dk \left[a_k \phi_k + a_k^\dagger \varphi_k^* \right] \quad (73)$$

where a_k and $a_k^\dagger$ are annihilation and creation operators for the mode k , respectively, and ϕ_k , φ_k are the corresponding normalized eigenfunctions. Creation and annihilation operators satisfy:

$$\begin{aligned} [a_k, a_{k'}] &= [a_k^\dagger, a_{k'}^\dagger] = 0 \\ [a_k, a_{k'}^\dagger] &= \delta(k - k') \end{aligned} \quad (74)$$

Remark that the presence of two terms in (73) is related to the two branches of the dispersion relation.

In the presence of inhomogeneities, k is not a good quantum number anymore, *ie* plane waves in e^{ikx} are no longer eigenmodes of the Hamiltonian. But the frequency ω can be used instead and (73) must be written as an integral over ω . As a first step, let us do this for a homogeneous condensate. As one can expect from the above remark, it proceeds differently depending on the number of modes for a given ω . If $v < v_L$, there are only two modes for each frequency and the field may be written as:

$$\phi(t, x) = \int_0^\infty d\omega \left[e^{-i\omega t} \hat{\phi}_\omega(x) + e^{i\omega t} \hat{\phi}_\omega(x)^\dagger \right], \quad (75)$$

where $\hat{\phi}_\omega$ and $\hat{\varphi}_\omega$ contain respectively the positive- and negative-frequency modes (see Fig. 11). For $v < v_L$, they can be written as

$$\begin{aligned} \hat{\phi}_\omega(x) &= a_\omega^u \phi_\omega^u(x) + a_\omega^v \phi_\omega^v(x) \\ \hat{\varphi}_\omega(x) &= a_\omega^u \varphi_\omega^u(x) + a_\omega^v \varphi_\omega^v(x), \end{aligned} \quad (76)$$

where u stands for a left-moving mode and v for a right-moving one. We work with a dispersion relations for which there are four solutions in some interval $[\omega_1, \omega_2]$ and only two outside it³. Generalization to more complicated situations is straightforward. Between ω_1 and ω_2 , the two new negative-frequency modes are naturally seen as the complex conjugates of two positive-frequency modes (see Fig. 12) and we can write

$$\begin{aligned} \hat{\phi}_\omega(x) &= a_\omega^u \phi_\omega^u(x) + a_\omega^v \phi_\omega^v(x) + (a_{-\omega,1}^v)^\dagger (\phi_{-\omega,1}^v(x))^* + (a_{-\omega,2}^v)^\dagger (\phi_{-\omega,2}^v(x))^* \\ \hat{\varphi}_\omega(x) &= a_\omega^u \varphi_\omega^u(x) + a_\omega^v \varphi_\omega^v(x) + (a_{-\omega,1}^v)^\dagger (\varphi_{-\omega,1}^v(x))^* + (a_{-\omega,2}^v)^\dagger (\varphi_{-\omega,2}^v(x))^* \end{aligned} \quad (77)$$

³For the dispersion relation shown on figures 10,11,12, there are additional modes for some values of ω . Including them would only make the equations unnecessarily complicated, while the conclusion would be unaffected.

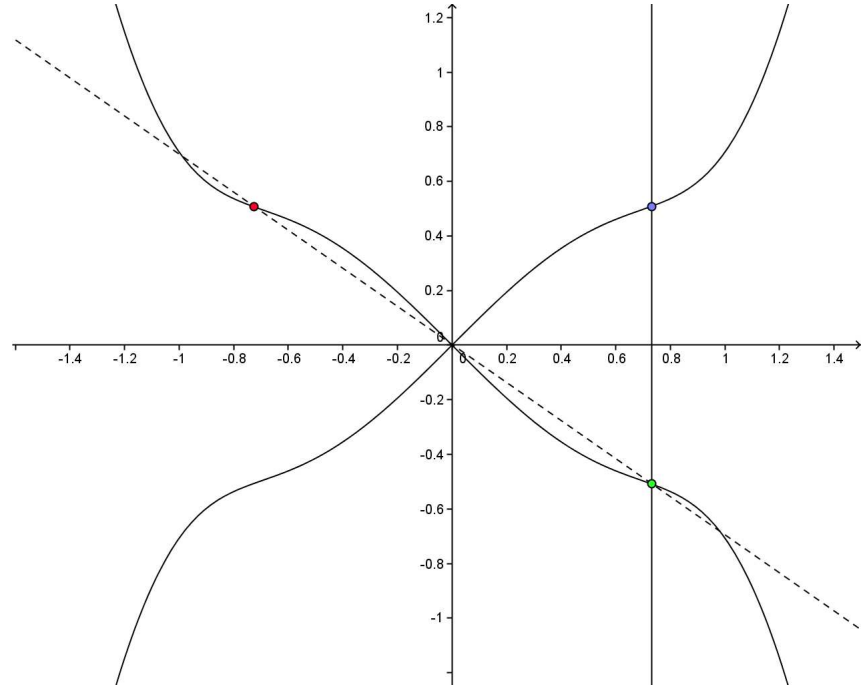


Figure 10: Dispersion relation in the homogeneous phase. The frequency in the fluid frame Ω is plotted as a function of the wave-vector k . Two modes (blue and green dots) correspond to the same k . The negative-frequency mode (green dot) is the complex conjugate of a positive-frequency one (red dot), its symmetric with respect to the origin. The plain line is a line of constant k . The dashed line passes through the origin.

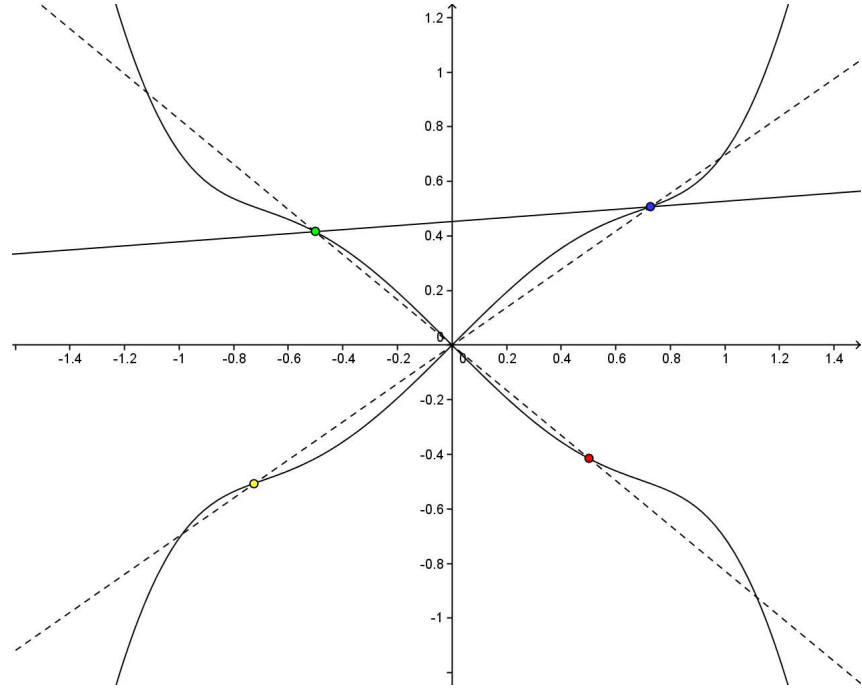


Figure 11: Same as Fig. 10 at fixed frequency ω in the frame where the undulation is stationary. The u and v positive-frequency modes (blue and green dots, respectively) and their negative-frequency counterparts (yellow and red dots, respectively) are represented. The plain line is the locus of constant ω . A symmetry with respect to the origin exchanges the blue dot with the yellow one, and the green dot with the red one.

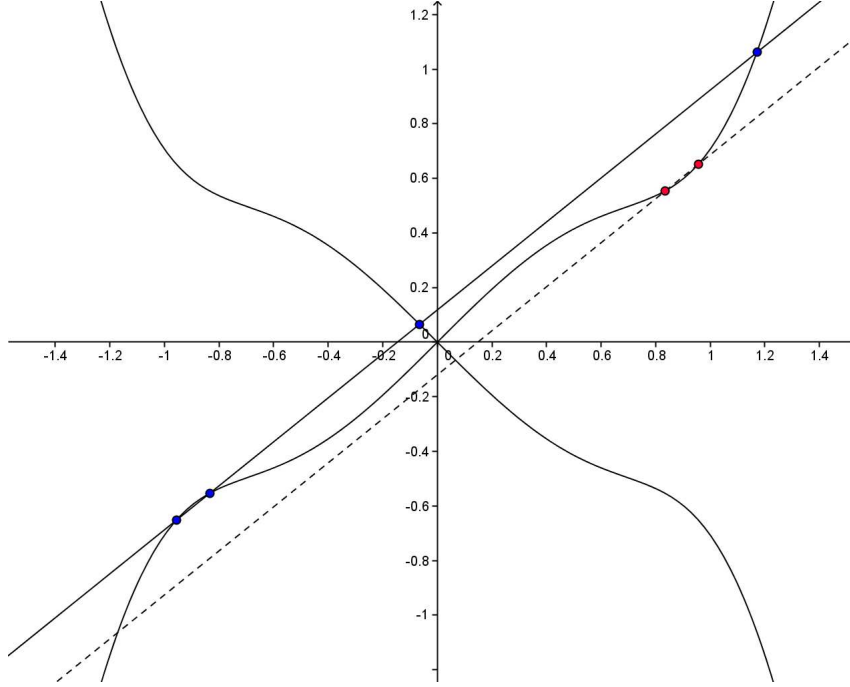


Figure 12: Dispersion relation in the homogeneous phase for $v > v_L$. The frequency in the fluid frame Ω is plotted as a function of the wave-vector k . The plain line is the locus of constant frequency ω in the frame where the undulation is stationary. The dashed line corresponds to $-\omega$ (here the dashed line does not pass through the origin). The four modes of frequency ω are represented as blue dots. Red dots represent the complex conjugates of the two new modes, absent when $v < v_L$.

These two new modes have a negative energy, as can be seen from the exchange of ϕ_ω and φ_ω with respect to the previous ones, and the fact that the scalar product used to normalize the modes is antisymmetric in (ϕ, φ) .

How do we connect these two pictures? If there is a Landau horizon, we must ask how many globally normalizable⁴ modes exist. Assume for definiteness that the equation on k at fixed ω is a polynomial of degree d with real coefficients (in our case $d = 6$). We also assume it has n_1 real solutions on one side of the horizon, in the asymptotic region where v becomes constant, and n_2 on the other side (in the above configuration $n_1 = 2$ and $n_2 = 4$). We ask how many modes are asymptotically bounded on both sides of the horizon. Here the number of solutions takes into account possible multiple roots, so that the equation $x^2 = 0$, for instance, is said to have two real solutions. On one side, we have n_1 modes with a real wave-vector which oscillate as $x \rightarrow -\infty$, and $d - n_1$ complex ones. Each complex root is related to another one by complex conjugation. This gives $\frac{d-n_1}{2}$ exponentially decaying modes and $\frac{d-n_1}{2}$ exponentially growing ones at $x \rightarrow -\infty$. So, $\frac{d+n_1}{2}$ modes are asymptotically bounded at $x \rightarrow -\infty$ (note that $d - n_1$ is necessarily even). Since there are $\frac{d-n_2}{2}$ exponentially growing modes at $x \rightarrow +\infty$ by the same argument, we have $\frac{d-n_2}{2}$, *a priori* independent, linear relations to be satisfied for a mode to be well-behaved as $x \rightarrow +\infty$, so that the number of normalizable modes is $\frac{n_1+n_2}{2}$. In our case, this gives 3 globally well-behaved

⁴More precisely, we consider modes with which we can form normalizable wave-packets.

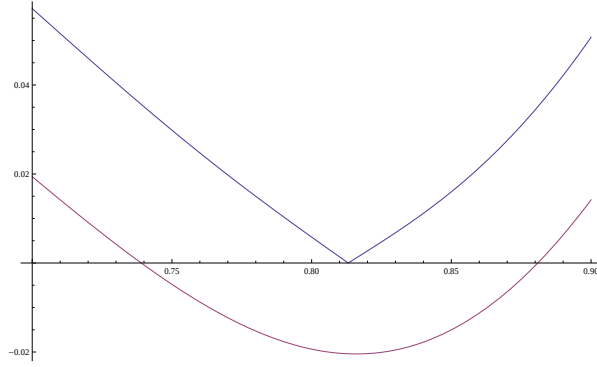


Figure 13: Spectrum of perturbations on top of the homogeneous state (purple curve) and of the true ground state for $v > v_L$ (blue curve) in the laboratory frame. Adding the undulation makes the spectrum go up by just the right amount to erase the additional solutions leading to HR.

modes. We can thus write

$$\begin{aligned}\hat{\phi}_\omega(x) &= a_\omega^u \phi_\omega^u(x) + a_\omega^v \phi_\omega^v(x) + (a_{-\omega}^v)^\dagger (\phi_{-\omega}^v(x))^* \\ \hat{\varphi}_\omega(x) &= a_\omega^u \varphi_\omega^u(x) + a_\omega^v \varphi_\omega^v(x) + (a_{-\omega}^v)^\dagger (\varphi_{-\omega}^v(x))^*\end{aligned}\quad (78)$$

The Hamiltonian takes the form:

$$H = \int_{\mathbb{R}-[\omega_1, \omega_2]} d\omega \, \hbar \omega \left(a_\omega^{u\dagger} a_\omega^u + a_\omega^{v\dagger} a_\omega^v \right) + \int_{[\omega_1, \omega_2]} d\omega \, \hbar \omega \left(a_\omega^{u\dagger} a_\omega^u + a_\omega^{v\dagger} a_\omega^v - a_{-\omega}^{v\dagger} a_{-\omega}^v \right). \quad (79)$$

To make a long discussion short, Hawking Radiation takes place when positive- and negative-energy modes are spontaneously created (because the in- and out-modes are not the same, so that the in-vacuum, for instance, contains out-excitations). This of course requires these modes to exist, and in particular only the range $[\omega_1, \omega_2]$ contributes to HR. The same mechanism leads to HR in analogue black holes, for which $v_+ > c$. The only difference with respect to Landau horizons is that $\omega_1 = 0$ if $v_+ > c$.

This was for the “false vacuum” without undulation. As we have seen, the spectrum of perturbations on top of the undulation has no negative-energy modes (see Fig. 13). This means HR can not occur, since the “additional modes” for $v > v_L$ disappear. In some sense, this was expected. Indeed, the very reason why the undulation is stable is that it changes the spectrum of perturbations by just the right amount to get rid of all the modes of negative energy in the fluid frame, which were our additional modes. As is common in non-linear physics, the saturation occurs when the undulation is large enough to neutralize the initial instability.

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