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DIFFERENTIATION OPERATOR FROM MODEL SPACES TO BERGMAN SPACES AND PELLER TYPE INEQUALITIES

ANTON BARANOV AND RACHID ZAROUF

ABSTRACT. Given an inner function Θ in the unit disc \mathbb{D} , we study the boundedness of the differentiation operator which acts from the model subspace $K_\Theta = (\Theta H^2)^\perp$ of the Hardy space H^2 , equipped with the *BMOA*-norm, to some radial-weighted Bergman space. As an application, we generalize Peller's inequality for Besov norms of rational functions f of degree $n \geq 1$ having no poles in the closed unit disc $\overline{\mathbb{D}}$.

1. INTRODUCTION AND NOTATIONS

A well-known inequality by Vladimir Peller (see inequality (2.1) below) majorizes a Besov norm of any rational function f of degree $n \geq 1$ having no poles in the closed unit disc $\overline{\mathbb{D}} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$ in terms of its *BMOA*-norm and its degree n . The original proof of Peller is based on his description of Hankel operators in the Schatten classes. One of the aims of this paper is to give a short and direct proof of this inequality and extend it to more general radial-weighted Bergman norms. Our proof combines integral representation for the derivative of f (which come from the theory of model spaces) and the generalization of a theorem by E.M. Dyn'kin. The corresponding inequalities are obtained in terms of radial-weighted Bergman norms of the derivative of finite Blaschke products (of degree $n = \deg f$), instead of n itself. The finite Blaschke products in question have the same poles as f . The study of radial-weighted Bergman norms of the derivatives of finite Blaschke products of degree n and their asymptotic as n tends to $+\infty$ is of independent interest. A contribution to this topic, which we are going to exploit here, was given by J. Arazy, S.D. Fisher and J. Peetre.

Let \mathcal{P}_n be the space of complex analytic polynomials of degree at most n and let

$$\mathcal{R}_n^+ = \left\{ \frac{P}{Q} : P, Q \in \mathcal{P}_n, Q(\xi) \neq 0 \text{ for } |\xi| \leq 1 \right\}$$

be the set of rational functions of degree at most n with poles outside of the closed unit disc $\overline{\mathbb{D}}$. In this paper, we consider the norm of a rational function $f \in \mathcal{R}_n^+$ in different spaces of analytic functions in the open unit disc $\mathbb{D} = \{\xi : |\xi| < 1\}$.

1.1. Some Banach spaces of analytic functions. We denote by $\mathcal{H}ol(\mathbb{D})$ the space of all holomorphic functions in \mathbb{D} .

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1.1.1. *The Besov spaces B_p .* A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the Besov space B_p , $1 < p < \infty$, if and only if

$$\|f\|_{B_p} = |f(0)| + \|f\|_{B_p}^* < +\infty,$$

where $\|f\|_{B_p}^*$ is the seminorm defined by

$$\|f\|_{B_p}^* = \left(\int_{\mathbb{D}} (1 - |u|^2)^{p-2} |f'(u)|^p dA(u) \right)^{\frac{1}{p}},$$

A being the normalized area measure on \mathbb{D} .

For the case $0 < p \leq 1$ the definition of the Besov norm requires a modification:

$$\|f\|_{B_p} = \sum_{j=0}^{k-1} |f^{(j)}(0)| + \|f\|_{B_p}^*, \quad \|f\|_{B_p}^* = \left(\int_{\mathbb{D}} |f^{(k)}(u)|^p (1 - |u|^2)^{pk-2} dA(u) \right)^{\frac{1}{p}},$$

where k is the smallest positive integer such that $pk > 1$. We refer to [Pee, Tri, BeLo] for general properties of Besov spaces.

A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the space B_∞ (known as the Bloch space) if and only if $\|f\|_{B_\infty} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| (1 - |z|) < \infty$.

1.1.2. *The radial-weighted Bergman spaces $A_p(w)$.* The radial-weighted Bergman space $A_p(w)$, $1 \leq p < \infty$, is defined as:

$$A_p(w) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \|f\|_{A_p(w)}^p = \int_{\mathbb{D}} w(|u|) |f(u)|^p dA(u) < \infty \right\},$$

where the weight w satisfies $w \geq 0$ and $\int_0^1 w(r) dr < \infty$. The classical power weights $w(r) = w_\alpha(r) = (1 - r^2)^\alpha$, $\alpha > -1$, are of special interest; in this case we put $A_p(\alpha) = A_p(w_\alpha)$. We refer to [HKZ] for general properties of weighted Bergman spaces.

1.1.3. *The spaces $A_p^1(\alpha)$.* A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the space $A_p^1(\alpha)$, $1 \leq p \leq +\infty$, $\alpha > -1$, if and only if

$$\|f\|_{A_p^1(\alpha)} = |f(0)| + \|f'\|_{A_p(\alpha)} < +\infty.$$

We also define the $A_p^1(\alpha)$ -seminorm by $\|f\|_{A_p^1(\alpha)}^* = \|f'\|_{A_p(\alpha)}$. Note that the spaces B_p and $A_p^1(p-2)$ coincide for $1 < p < +\infty$.

1.1.4. *The space $BMOA$.* There are many ways to define $BMOA$; see [Gar, Chapter 6]. For the purposes of this paper we choose the following one: a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the $BMOA$ space (of analytic functions of bounded mean oscillation) if and only if

$$\|f\|_{BMOA} = \inf \|g\|_{L^\infty(\mathbb{T})} < +\infty,$$

where the infimum is taken over all $g \in L^\infty(\mathbb{T})$, $\mathbb{T} = \{\xi : |\xi| = 1\}$ being the unit circle, for which the representation

$$f(\xi) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(u)}{u - \xi} du, \quad |\xi| < 1,$$

holds. Recall that $BMOA$ is the dual space of the Hardy space H^1 under the pairing

$$\langle f, g \rangle = \int_{\mathbb{T}} f(u) \overline{g(u)} du, \quad f \in H^1, g \in BMOA,$$

where this integral must be understood as the extension of the pairing acting on a dense subclass of H^1 , see [Bae, p. 23].

1.2. Model spaces.

1.2.1. *General inner functions.* By H^p , $1 \leq p \leq \infty$, we denote the standard Hardy spaces (see [Gar, Nik]). Recall that H^2 is a reproducing kernel Hilbert space, with the kernel

$$k_\lambda(w) = \frac{1}{1 - \bar{\lambda}w}, \quad \lambda, w \in \mathbb{D},$$

known as the Szegő kernel (or the Cauchy kernel) associated with λ . Thus $\langle f, k_\lambda \rangle = f(\lambda)$ for all $f \in H^2$ and for all $\lambda \in \mathbb{D}$, where $\langle \cdot, \cdot \rangle$ is the scalar product on H^2 .

Let Θ be an *inner function*, i.e., $\Theta \in H^\infty$ and $|\Theta(\xi)| = 1$ a.e. $\xi \in \mathbb{T}$. We define the model subspace K_Θ of the Hardy space H^2 by

$$K_\Theta = (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$

By the famous theorem of Beurling, these and only these subspaces of H^2 are invariant with respect to the backward shift operator. We refer to [Nik] for the general theory of the spaces K_Θ and their numerous applications.

For any *inner function* Θ , the reproducing kernel of the model space K_Θ corresponding to a point $\xi \in \mathbb{D}$ is of the form

$$k_\lambda^\Theta(w) = \frac{1 - \overline{\Theta(\lambda)}\Theta(w)}{1 - \bar{\lambda}w}, \quad \lambda, w \in \mathbb{D},$$

that is $\langle f, k_\lambda^\Theta \rangle = f(\lambda)$ for all $f \in K_\Theta$ and for all $\lambda \in \mathbb{D}$.

1.2.2. *The case of finite Blaschke products.* From now on, for any $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$, we consider the finite Blaschke product

$$B_\sigma = \prod_{k=1}^n b_{\lambda_k},$$

where $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$, is the elementary Blaschke factor corresponding to $\lambda \in \mathbb{D}$. It is well known that if

$$\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\} \in \mathbb{D}^n,$$

where every λ_s is repeated according to its multiplicity n_s , $\sum_{s=1}^t n_s = n$, then

$$K_{B_\sigma} = H^2 \ominus B_\sigma H^2 = \overline{\text{span}}\{k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq n_j - 1\},$$

where for $\lambda \neq 0$, $k_{\lambda, i} = \left(\frac{d}{d\lambda}\right)^i k_\lambda$ and $k_\lambda = \frac{1}{1 - \bar{\lambda}z}$ is the standard Cauchy kernel at the point λ , whereas $k_{0, i} = z^i$. Thus the subspace K_{B_σ} consists of rational functions of the form p/q , where $p \in \mathcal{P}_{n-1}$ and $q \in \mathcal{P}_n$, with the poles $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ of corresponding multiplicities (including possible poles at ∞). Hence, if $f \in \mathcal{R}_n^+$ and $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ are the poles of f (repeated according to multiplicities), then $f \in K_{zB_\sigma}$ with $\sigma = (\lambda_1, \dots, \lambda_n)$.

From now on, for two positive functions a and b , we say that a is dominated by b , denoted by $a \lesssim b$, if there is a constant $C > 0$ such that $a \leq Cb$; we say that a and b are comparable, denoted by $a \asymp b$, if both $a \lesssim b$ and $b \lesssim a$.

2. MAIN RESULTS

2.1. Main ingredients. In 1980 V. Peller proved in his seminal paper [Pel1] that

$$(2.1) \quad \|f\|_{B_p} \leq c_p n^{\frac{1}{p}} \|f\|_{BMOA}$$

for any $f \in \mathcal{R}_n^+$ and $1 \leq p \leq +\infty$, where c_p is a constant depending only on p . Later, this result was extended to the range $p > 0$ independently and with different proofs by Peller [Pel2], S. Semmes [Sem] and also by A. Pekarskii [Pek1] who found a proof which does not use the theory of Hankel operators (see also [Pek2]).

The aim of the present article is:

- (1) study the boundedness of the differentiation operator from $(K_\Theta, \|\cdot\|_{BMOA})$ to $A_p(\alpha)$, $1 < p < +\infty$, $\alpha > -1$, and
- (2) generalize Peller's result (2.1) replacing the B_p -seminorm by the $A_p^1(\alpha)$ -one.

In both of these problems, we make use of a method based on two main ingredients:

- integral representation for the derivative of functions in K_Θ or in \mathcal{R}_n^+ , and
- a generalization of a theorem by E.M. Dyn'kin, see Subsection 2.2.3.

One more tool (that we will need in problem (2)) is the estimate of B_p -seminorms of finite Blaschke products by Arazy, Fischer and Peetre [AFP].

2.2. Main results. Let us consider the differentiation operator $Df = f'$ and the shift and the backward shift operators defined respectively by

$$(2.2) \quad Sf = zf, \quad S^*f = \frac{f - f(0)}{z},$$

for any $f \in \mathcal{H}ol(\mathbb{D})$. From now on, for any inner function Θ , we put

$$\tilde{\Theta} = z\Theta = S\Theta.$$

2.2.1. Boundedness of the differentiation operator from $(K_\Theta, \|\cdot\|_{BMOA})$ to $A_p(\alpha)$. Let us first discuss the boundedness of the operator D from $BMOA$ to $A_p(\alpha)$. The following (essentially well-known) proposition gives necessary and sufficient conditions on p and α so that a continuous embedding $BMOA \subset A_p^1(\alpha)$ hold.

Proposition 2.1. *Let $\alpha > -1$ and $1 \leq p < \infty$. Then $BMOA \subset A_p^1(\alpha)$ if and only if either $\alpha > p - 1$ or $\alpha = p - 1$ and $p \geq 2$.*

Now, we consider an arbitrary *inner function* Θ . Our first main result gives necessary and sufficient conditions under which the differentiation operator

$$D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$$

is bounded. When this is the case, we estimate its norm in terms of $\|\Theta'\|_{A_p(\alpha)}$.

Theorem 2.2. *Let $1 < p < \infty$ and $\alpha > -1$. Then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded if and only if $\Theta' \in A_p(\alpha)$.*

Moreover, one can distinguish three cases:

(a) *If $\alpha > p - 1$ or $\alpha = p - 1$, and $p \geq 2$ then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded.*

(b) *If $p - 2 < \alpha < p - 1$, then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded if and only if $\Theta' \in A_1(\alpha - p + 1)$.*

(c) *If $\alpha \leq p - 2$, then the operator $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded if and only if Θ is a finite Blaschke product.*

In cases (b) and (c), we have

$$(2.3) \quad \|D\| \lesssim \|\Theta'\|_{A_p(\alpha)} \lesssim \|D\| + \text{const},$$

with constants depending on p and α only.

Remark. 1. In the cases (b) and (c), to show that their conditions are equivalent to the inclusion $\Theta' \in A_p(\alpha)$ we use a theorem by P.R. Ahern [Ahe1] and its generalizations by I.E. Verbitsky [Ver] and A. Gluchoff [Glu]. We do not know whether the inclusions $\Theta' \in A_p(\alpha)$ and $\Theta' \in A_1(\alpha - p + 1)$ are equivalent for $\alpha = p - 1$ and $1 < p < 2$.

2. The membership of Blaschke products in various function spaces is a well-studied topic. Besides the above-cited papers by Ahern, Gluchoff and Verbitsky, let us mention the papers by Ahern and D.N. Clark [AC1, AC2] and recent works by D. Girela, J. Peláez, D. Vukotić, and A. Aleman [GPV, AV].

2.2.2. *Generalization of Peller's inequalities.* In the following theorem, we give a generalization of Peller's inequality (2.1).

Theorem 2.3. *Let $f \in \mathcal{R}_n^+$, $\deg f = n$ and $\sigma \in \mathbb{D}^n$ be the set of its poles counting multiplicities (including poles at ∞). For any $\alpha > -1$, $1 < p < \infty$, and $p > 1 + \alpha$, we have*

$$(2.4) \quad \|f\|_{A_p^1(\alpha)}^* \leq K_{p,\alpha} \|f\|_{BMOA} \|B'_\sigma\|_{A_p(\alpha)},$$

where $K_{p,\alpha}^p = \frac{2^{\frac{1}{\alpha+1}}}{2^{\frac{1}{\alpha+1}-1}} \left(\frac{p}{p-1-\alpha}\right)^p 2^{p+1}$.

Remark. The inequality (2.4) is sharp up to a constant in the following sense: for $f = B_\sigma$ we, obviously, have $\|f\|_{A_p^1(\alpha)}^* = \|f\|_{BMOA} \|B'_\sigma\|_{A_p(\alpha)}$ (note that $\|B_\sigma\|_{BMOA} = 1$).

Let us show how Peller's inequality (2.1) for $1 < p < \infty$ follows from Theorem 2.3. For $\alpha = p - 2$, we have

$$\|f'\|_{A_p(\alpha)} = \|f\|_{B_p}^*, \quad \|B'_\sigma\|_{A_p(\alpha)} = \|B_\sigma\|_{B_p}^*.$$

To deduce Peller's inequalities it remains to apply the following theorem by Arazy, Fischer and Peetre [AFP]: if $1 \leq p \leq \infty$, then there exist absolute positive constants m_p and M_p such that

$$(2.5) \quad m_p n^{\frac{1}{p}} \leq \|B\|_{B_p}^* \leq M_p n^{\frac{1}{p}}.$$

for any Blaschke product of degree n . Then we obtain for $1 < p < \infty$,

$$\|f\|_{B_p}^* \leq K_{p,p-2}^{\frac{1}{p}} M_p \|f\|_{BMOA} (n+1)^{\frac{1}{p}} \lesssim n^{\frac{1}{p}} \|f\|_{BMOA}.$$

To make the expositions self-contained, we give in Section 5 a very simple proof of the upper estimate in (2.5) (which is slightly different from the proof by D. Marshall presented in [AFP]).

The method of integral representations for higher order derivatives in model spaces allows to prove Peller's inequalities also for $0 < p \leq 1$. In Section 6 we present the proof for the case $p > \frac{1}{2}$.

2.2.3. *Generalization of a theorem by Dyn'kin.* E.M. Dyn'kin proved in [Dyn, Theorem 3.2] that

$$(2.6) \quad \int_{\mathbb{D}} \left(\frac{1 - |B(u)|^2}{1 - |u|^2} \right)^2 dA(u) \leq 8(n+1),$$

for any finite Blaschke product B of degree n .

From now on, for any inner function Θ and for any $\alpha > -1$, $p > 1$, we put

$$(2.7) \quad I_{p,\alpha}(\Theta) = \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\frac{1 - |\Theta(u)|^2}{1 - |u|^2} \right)^p dA(u).$$

Dyn'kin's Theorem can be stated as follows: *for any finite Blaschke product B of degree n , we have*

$$I_{2,0}(B) \leq 8(n+1).$$

Here, we generalize this result to the case $\alpha > -1$, $p > 1$ and $p > 1 + \alpha$. This generalization is the key step of the proof of Theorem 2.3.

Theorem 2.4. *Let $1 < p < \infty$, $\alpha > -1$ and $p > 1 + \alpha$. Then,*

$$\|\Theta'\|_{A_p(\alpha)}^p \leq I_{p,\alpha}(\Theta) \leq K_{p,\alpha} \|\Theta'\|_{A_p(\alpha)}^p,$$

where $K_{p,\alpha}$ is the same constant as in Theorem 2.3.

The paper is organized as follows. We first focus in Section 3 on the generalization of Dyn'kin's result. In Section 4, Proposition 2.1 and Theorem 2.2 are proved, while Section 5 is devoted to the proof of Peller type inequalities (Theorem 2.3). The case $\frac{1}{2} < p \leq 1$ in Peller's inequality is considered in Section 6. In Section 7, we discuss some estimates of radial-weighted Bergman norms of Blaschke products. Finally, in Section 8 we discuss some related inequalities by Dolzhenko for which we give a very simple proof for the case $1 \leq p \leq 2$ based on Dyn'kin's estimate and suggest a way to extend these inequalities to the range $p > 2$.

3. GENERALIZATION OF DYN'KIN'S THEOREM

The aim of this Section is to prove Theorem 2.4. The lower bound follows trivially from the Schwarz–Pick inequality applied to Θ . The main ideas for the proof of the upper bound come from [Dyn, Theorem 3.2]. In this Section, Θ is an arbitrary inner function.

Lemma 3.1. *For $p > 1$, $\alpha > -1$ and $p > 1 + \alpha$, we have*

$$I_{p,\alpha}(\Theta) \leq 2^p \int_0^{2\pi} \int_0^1 (1-r)^\alpha \left(\frac{1}{1-r} \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \frac{d\theta}{\pi}.$$

Proof. Writing the integral $I_{p,\alpha}(\Theta)$ in polar coordinates, and using the fact that

$$1 - |\Theta(u)|^2 \leq 2(1 - |\Theta(u)|),$$

we obtain

$$\begin{aligned} I_{p,\alpha}(\Theta) &\leq 2^p \int_0^1 r(1-r^2)^{\alpha-p} \left(\int_0^{2\pi} (1 - |\Theta(re^{i\theta})|)^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 r(1-r)^{\alpha-p} \left(\int_0^{2\pi} |\Theta(e^{i\theta}) - \Theta(re^{i\theta})|^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 r(1-r)^{\alpha-p} \left(\int_0^{2\pi} \left(\int_r^1 |\Theta'(se^{i\theta})| ds \right)^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^{2\pi} \int_0^1 (1-r)^\alpha \frac{1}{(1-r)^p} \left(\int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \frac{d\theta}{\pi}, \end{aligned}$$

which completes the proof of the lemma. \square

We recall now a general version of the Hardy inequality, see [HLP, page 245], which after change of variables gives (as in [Ahe2, Lemma 7]):

Lemma 3.2. *If $h : (0, 1) \rightarrow [0, +\infty)$, $p > 1$, $\alpha > -1$ and $p > 1 + \alpha$, then*

$$\int_0^1 (1-r)^\alpha \left(\frac{1}{1-r} \int_r^1 h(s) ds \right)^p dr \leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^1 (1-r)^\alpha h(r)^p dr.$$

Corollary 3.3. *Let $p > 1$, $\alpha > -1$ and $p > 1 + \alpha$. Then,*

$$I_{p,\alpha}(\Theta) \leq C_{p,\alpha} \int_0^{2\pi} \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi},$$

where $C_{p,\alpha} = \left(\frac{p}{p-1-\alpha} \right)^p 2^p$.

Proof. Combining estimates in Lemma 3.1 and Lemma 3.2 (setting $h(s) = h_\theta(s) = |\Theta'(se^{i\theta})|$, for any fixed $\theta \in [0, 2\pi)$), we obtain

$$\int_0^1 (1-r)^\alpha \left(\frac{1}{1-r} \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr.$$

Thus,

$$I_{p,\alpha}(\Theta) \leq \left(\frac{p}{p-1-\alpha} \right)^p 2^p \int_0^{2\pi} \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi}$$

which completes the proof. \square

Lemma 3.4. *Let any nonzero weight w satisfying $w \geq 0$ and $\int_0^1 w(r) dr < \infty$. Let $\beta = \beta_w \in (0, 1)$ such that $\int_0^1 w(r) dr = 2 \int_0^\beta w(r) dr$. Then, for $f \in A_p(w)$, $1 \leq p < \infty$,*

$$\begin{aligned} \|f\|_{A_p(w)}^p &\leq \int_0^{2\pi} w(r) \int_0^1 |f(re^{i\theta})|^p dr \frac{d\theta}{\pi} \\ &\leq \frac{2}{\beta} \int_\beta^1 rw(r) \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi} \right) dr \leq \frac{2}{\beta} \|f\|_{A_p(w)}^p. \end{aligned}$$

Proof. The proof follows easily from the fact that for any f in $\mathcal{H}ol(\mathbb{D})$, the function

$$r \mapsto \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi},$$

is nondecreasing on $[0, 1]$. □

We are now ready to prove Theorem 2.4.

Proof. We first prove (2.4). Applying Lemma 3.4 with $f = \Theta'$ and $w(r) = (1 - r^2)^\alpha$, $\alpha > -1$, and Corollary 3.3 we obtain that

$$I_{p,\alpha}(\Theta) \leq C_{p,\alpha} \int_0^{2\pi} \int_0^1 (1 - r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} \leq \frac{2}{\beta} C_{p,\alpha} \|\Theta'\|_{A_p(\alpha)}^p,$$

where $C_{p,\alpha} = \left(\frac{p}{p-1-\alpha}\right)^p 2^p$, and $\beta = \beta_\alpha$ satisfies the condition

$$\int_\beta^1 w(r) dr = \int_0^\beta w(r) dr.$$

By a direct computation, we see that $\beta = \beta_\alpha$ is given by the equation $\frac{1-(1-\beta)^{\alpha+1}}{1+\alpha} = \frac{(1-\beta)^{\alpha+1}}{1+\alpha}$, which is equivalent to

$$(3.1) \quad \beta = \beta_\alpha = 1 - \frac{1}{2^{\frac{1}{\alpha+1}}}.$$

□

4. PROOF OF PROPOSITION 2.1 AND THEOREM 2.2

4.1. Proof of Proposition 2.1. The statement for $\alpha > p - 1$ is trivial. Indeed, by the standard Cauchy formula,

$$f'(u) = \left\langle f, \frac{z}{(1 - \bar{u}z)^2} \right\rangle, \quad u \in \mathbb{D},$$

and thus, bounding $|f'(u)|$ from above by $\|f\|_{BMOA} \left\| \frac{z}{(1 - \bar{u}z)^2} \right\|_{H^1} = (1 - |u|^2)^{-1} \|f\|_{BMOA}$, we get

$$\|f'\|_{A_p(\alpha)}^p \lesssim \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|)^{\alpha-p} dA(u) \lesssim \|f\|_{BMOA}^p.$$

For $p \geq 2$ and $\alpha = p - 1$ we have

$$\begin{aligned} \|f'\|_{A_p(\alpha)}^p &= \int_{\mathbb{D}} (1 - |u|^2) |f'(u)|^p dA(u) \\ &\leq \|f\|_{B_\infty}^{p-2} \int_{\mathbb{D}} (1 - |u|^2) |f'(u)|^2 dA(u), \end{aligned}$$

where $\|f\|_{B_\infty}$ is the norm of f in the Bloch space. Since $\int_{\mathbb{D}} (1 - |u|^2) |f'(u)|^2 dA(u) \leq \|f\|_{H^2}^2$, $\|f\|_{H^2} \lesssim \|f\|_{BMOA}$ and $\|f\|_{B_\infty} \lesssim \|f\|_{BMOA}$, we conclude that

$$(4.1) \quad \|f'\|_{A_p(\alpha)} \lesssim \|f\|_{BMOA}.$$

Now we turn to the necessity of the restrictions on p and α for the estimate (4.1). If $\alpha < p - 1$ then it is well known that there exist interpolating Blaschke products B such that $B' \notin A_p(\alpha)$ (see, e.g., [Glu, Theorem 6], where an explicit criterion for the inclusion is given in terms of the zeros of B). Finally, by a result of S.A. Vinogradov [Vin, Lemma 1.6], if $f \in A_p^1(p - 1)$, $1 \leq p < 2$, then, $\sum_{n=0}^{\infty} |\hat{f}(2^n)|^p < \infty$ (where $\hat{f}(n)$ stands for the n^{th} Taylor coefficient of f). Hence, $A_p^1(p - 1)$ does not contain even some functions from the disc algebra, and so $BMOA \not\subseteq A_p^1(p - 1)$ when $1 \leq p < 2$. \square

4.2. Integral representation for the derivative of functions in K_Θ . An important ingredient of our proof is the following simple and well-known integral representation for the derivative of a function from a model space.

Lemma 4.1. *Let Θ be an inner function, $f \in K_\Theta$, $n \in \mathbb{N}$. We have*

$$f^{(n)}(u) = \left\langle f, z^n (k_u^\Theta)^{n+1} \right\rangle,$$

for any $u \in \mathbb{D}$.

Proof. For a fixed $u \in \mathbb{D}$, we have

$$f^{(n)}(u) = \left\langle f, \frac{z^n}{(1 - \bar{u}z)^{n+1}} \right\rangle = \left\langle f, z^n (k_u^\Theta)^{n+1} \right\rangle.$$

Here the first equality is the standard Cauchy formula, while the second follows from the fact that $z^n(1 - \bar{u}z)^{-n-1} - z^n(k_u^\Theta(z))^{n+1} \in \Theta H^2$ and $f \perp \Theta H^2$. \square

4.3. Proof of the left-hand side inequality in (2.3). Sufficiency of the condition $\Theta' \in A_p(\alpha)$ in Theorem 2.2 and the left-hand side inequality in (2.3) follow immediately from Theorem 2.4 and the following proposition.

Proposition 4.2. *Let $\alpha > -1$ and $1 < p < \infty$, let Θ be an inner function and $f \in K_\Theta$. Then we have*

$$\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA} (I_{p,\alpha}(\Theta))^{\frac{1}{p}}.$$

Proof. We use the integral representation of f' from Lemma 4.1:

$$f'(u) = \left\langle f, z (k_u^\Theta)^2 \right\rangle = \int_{\mathbb{T}} f(\tau) \overline{\tau (k_u^\Theta(\tau))^2} dm(\tau),$$

for any $u \in \mathbb{D}$, and thus

$$\begin{aligned} \|f'\|_{A_p(\alpha)}^p &= \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left| \int_{\mathbb{T}} f(\tau) \overline{\tau (k_u^\Theta(\tau))^2} dm(\tau) \right|^p dA(u) \\ &\leq \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\int_{\mathbb{T}} |k_u^\Theta(\tau)|^2 dm(\tau) \right)^p dA(u) \\ &= \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\frac{1 - |\Theta(u)|^2}{1 - |u|^2} \right)^p dA(u), \end{aligned}$$

which completes the proof. \square

It remains to combine Proposition 4.2 with Theorem 2.4 to complete the proof of the left-hand side inequality in (2.3).

4.4. Proof of the right-hand side inequality in (2.3). To prove the necessity of the inclusion $\Theta' \in A_p(\alpha)$ and the left-hand side inequality in (2.3), consider the test function

$$f = S^* \Theta = \frac{\Theta - \Theta(0)}{z},$$

where S^* is the backward shift operator (2.2). It is well-known that f belongs to K_Θ and easy to check that $\|f\|_{BMOA} \leq 2$, whence

$$(4.2) \quad \|D\|_{(K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)} \geq \frac{\|f'\|_{A_p(\alpha)}}{2}.$$

Now,

$$\|f'\|_{A_p(\alpha)}^p \geq \int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} |f'(r\xi)|^p dm(\xi) dr,$$

where β_α is given by (3.1) and thus,

$$\begin{aligned} \|f'\|_{A_p(\alpha)} &\geq \left(\int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p dm(\xi) dr \right)^{\frac{1}{p}} \\ &\quad - \left(\int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi) - \Theta(0)}{r^2 \xi^2} \right|^p dm(\xi) dr \right)^{\frac{1}{p}}. \end{aligned}$$

On one hand, applying Lemma 3.4 with $w = w_\alpha$ and $\beta = \beta_\alpha$ we obtain

$$\begin{aligned} \int_{\beta_\alpha}^1 r w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p dm(\xi) dr &\geq \int_0^{2\pi} \int_{\beta_\alpha}^1 r w_\alpha(r) |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} \\ &\geq \frac{\beta_\alpha}{2} \int_0^{2\pi} \int_0^1 r w_\alpha(r) |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} = \frac{\beta_\alpha}{2} \|\Theta'\|_{A_p(\alpha)}^p. \end{aligned}$$

On the other hand, since $\|f\|_{H^\infty} \leq 2$, we have

$$\int_{\beta_\alpha}^1 r w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi) - \Theta(0)}{r^2 \xi^2} \right|^p dm(\xi) dr \leq 2^p \int_{\beta_\alpha}^1 \frac{w_\alpha(r)}{r^{p-1}} dr \leq \frac{2^p}{\beta_\alpha^{p-1}} \int_{\beta_\alpha}^1 w_\alpha(r) dr.$$

Finally, we conclude that

$$\|f'\|_{A_p(\alpha)} \geq \left(\frac{\beta_\alpha}{2}\right)^{\frac{1}{p}} \|\Theta'\|_{A_p(\alpha)} - 2\beta_\alpha^{\frac{1}{p}-1} \left(\int_{\beta_\alpha}^1 w_\alpha(r) dr\right)^{\frac{1}{p}},$$

which, combined with (4.2), gives us the right-hand side inequality in (2.3). \square

4.5. Proof of Theorem 2.2. To complete the proof of Theorem 2.2, we need to recall the following theorem proved by Ahern [Ahe1] for the case $1 \leq p \leq 2$ and generalized by Verbitsky [Ver] and Gluchoff [Glu] to the range $1 \leq p < \infty$. This theorem characterizes inner functions Θ whose derivative belong to $A_p(\alpha)$.

Theorem. ([Glu]) *Let Θ be an inner function, $1 \leq p < \infty$, and $\alpha > -1$.*

- (i) *If $\alpha > p - 1$, then $\Theta' \in A_p(\alpha)$.*
- (ii) *If $p - 2 < \alpha < p - 1$, then $\Theta' \in A_p(\alpha)$ if and only if $\Theta \in A_1(\alpha - p + 1)$.*
- (iii) *If $\alpha < p - 2$ and $p > 1$, then $\Theta' \in A_p(\alpha)$ if and only if Θ is a finite Blaschke product.*

Proof of Theorem 2.2. Statement (a) is contained in Proposition 2.1. In order to prove (b) and (c) of Theorem 2.2, we first remark that for $\alpha < p - 1$, it follows from (2.3) that $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded if and only if $\Theta' \in A_p(\alpha)$. A direct application of the above Ahern–Verbitsky–Gluchoff theorem completes the proof for $\alpha > p - 2$. The case $\alpha = p - 2$ follows from the Arazy–Fisher–Peetre inequality (2.5). \square

5. PROOF OF PELLER TYPE INEQUALITIES

In this section we prove Theorem 2.3. From now on the inner function Θ is a finite Blaschke product. Recall that if $f \in \mathcal{R}_n^+$ and $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ are the poles of f (repeated according to multiplicities), then $f \in K_{zB_\sigma}$ with $\sigma = (\lambda_1, \dots, \lambda_n)$.

We start with the proof of the upper bound in the Arazy–Fisher–Peetre inequality (2.5).

Lemma 5.1. *Let B be a finite Blaschke product with the zeros $\{z_j\}_{j=1}^n$. Then*

$$|B''(u)| \leq \sum_{j=1}^n \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^3} + \left(\frac{1 - |B(u)|}{1 - |u|}\right)^2, \quad u \in \mathbb{D}.$$

Proof. Let $B = \prod_{j=1}^n b_{z_j}$, where $b_\lambda = \frac{|\lambda|}{\lambda} \cdot \frac{\lambda - z}{1 - \bar{\lambda}z}$. Then it is easy to see that

$$(5.1) \quad |B''(u)| \leq \sum_{j=1}^n \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^3} \left| \frac{B(u)}{b_{z_j}(u)} \right| + 2 \sum_{1 \leq j < k \leq n} \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^2} \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \left| \frac{B(u)}{b_{z_j}(u)b_{z_k}(u)} \right|,$$

To estimate the second sum in (5.1), first we note that $\frac{1 - |\lambda|^2}{2|1 - \bar{\lambda}u|^2} \leq \frac{1 - |b_\lambda(u)|}{1 - |u|} \leq \frac{2(1 - |\lambda|^2)}{|1 - \bar{\lambda}u|^2}$. Let us introduce the notations $B_j = \prod_{l=1}^{j-1} b_{z_l}$ (assuming $B_1 \equiv 1$) and $\widehat{B}_k = \prod_{l=k}^n b_{z_l}$. Then

$$(5.2) \quad \frac{1 - |B(u)|}{1 - |u|} \asymp \sum_{j=1}^n |B_j(u)| \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^2} \asymp \sum_{k=1}^n |\widehat{B}_{k+1}(u)| \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

It follows from the estimate $|B/(b_{z_j}b_{z_k})| \leq |B_j\widehat{B}_{k+1}|$ and (5.2) that

$$\begin{aligned} \sum_{1 \leq j < k \leq n} \frac{1 - |z_j|^2}{|1 - \overline{z_j}u|^2} \frac{1 - |z_k|^2}{|1 - \overline{z_k}u|^2} \left| \frac{B(u)}{b_{z_j}(u)b_{z_k}(u)} \right| &\leq \left(\sum_{j=1}^n |B_j(u)| \frac{1 - |z_j|^2}{|1 - \overline{z_j}u|^2} \right) \times \\ &\times \left(\sum_{k=1}^n |\widehat{B}_{k+1}(u)| \frac{1 - |z_k|^2}{|1 - \overline{z_k}u|^2} \right) \lesssim \left(\frac{1 - |B(u)|}{1 - |u|} \right)^2. \end{aligned}$$

□

Using Lemma 5.1, we first obtain the Arazy–Fisher–Peetre inequality for $p = 1$:

$$\|B\|_{B_1} \lesssim \int_{\mathbb{D}} |B''(u)| dA(u) \lesssim \sum_{j=1}^n \int_{\mathbb{D}} \frac{1 - |z_j|^2}{|1 - \overline{z_j}u|^3} dA(u) + I_{2,0}(B) \lesssim n.$$

We used Dyn’kin’s inequality (2.6) and the fact that, by the well-known estimates (see [HKZ, Theorem 1.7]), each integral in the above sum does not exceed some absolute constant, which does not depend on z_j . Finally, for $1 < p < \infty$, we have

$$\begin{aligned} \|B\|_{B_p}^{*p} &\asymp \int_{\mathbb{D}} |B''(u)|^p (1 - |u|^2)^{2p-2} dA(u) \\ &\leq \left(\sup_{u \in \mathbb{D}} |B''(u)| (1 - |u|^2)^2 \right)^{p-1} \int_{\mathbb{D}} |B''(u)| dA(u) \lesssim n, \end{aligned}$$

since $\sup_{u \in \mathbb{D}} |f''(u)| (1 - |u|^2)^2 \leq 2\|f\|_{H^\infty}$.

Proof of Theorem 2.3. Let $f \in \mathcal{R}_n^+$; there exists $\sigma \in \mathbb{D}^n$ such that $f \in K_{\widetilde{B}_\sigma}$, $\widetilde{B}_\sigma = zB_\sigma$. Then, by Proposition 4.2 we have

$$\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA} \left(I_{p,\alpha}(\widetilde{B}_\sigma) \right)^{\frac{1}{p}}$$

for any $\alpha > -1$ and $1 < p < \infty$. Now applying Theorem 2.4, we obtain $\|f\|_{A_p^1(\alpha)}^* \leq K_{p,\alpha} \|f\|_{BMOA} \|B'_\sigma\|_{A_p(\alpha)}$. Finally, note that

$$\|\widetilde{B}'_\sigma\|_{A_p(\alpha)} \leq \|zB'_\sigma\|_{A_p(\alpha)} + \|B_\sigma\|_{A_p(\alpha)} \lesssim \|B'_\sigma\|_{A_p(\alpha)}.$$

□

Remark. In Subsection 2.2.2, we have shown how to deduce Peller’s inequality (2.1) from Theorem 2.3 and the result of Arazy–Fischer–Peetre (2.5). Let us show that for $p \geq 2$ one can give a very simple proof which uses only Proposition 4.2 and Dyn’kin’s estimate $I_{2,0}(\widetilde{B}_\sigma) \leq 8(n+2)$, where $n = \deg B_\sigma$. Indeed, in this case, we have

$$\begin{aligned} I_{p,p-2}(\widetilde{B}_\sigma) &= \int_{\mathbb{D}} (1 - |u|^2)^{p-2} \left(\frac{1 - |\widetilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^{p-2+2} dA(u) \\ &= \int_{\mathbb{D}} \left(1 - |\widetilde{B}_\sigma(u)|^2 \right)^{p-2} \left(\frac{1 - |\widetilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^2 dA(u) \leq I_{2,0}(\widetilde{B}_\sigma). \end{aligned}$$

It remains to apply Proposition 4.2 with $\alpha = p - 2$.

6. AN ELEMENTARY PROOF OF PELLER'S INEQUALITY FOR $p > \frac{1}{2}$

In this section we prove the inequality

$$(6.1) \quad \|f\|_{B_p} \leq cn^{\frac{1}{p}} \|f\|_{BMOA}$$

for $1 \geq p > \frac{1}{2}$ using the integral representations of the derivatives in model spaces. It is well known and easy to see that, for $p > \frac{1}{2}$,

$$\|f\|_{B_p}^p \asymp |f(0)|^p + |f'(0)|^p + |f''(0)|^p + \int_{\mathbb{D}} |f'''(u)|^p (1 - |u|^2)^{3p-2} dA(u).$$

Thus in what follows it is the last integral (which we denote $\|f\|_{B_p}^{**}$) that we will estimate.

Let Θ be an inner function and let $f \in K_{\Theta}$. Then, by Lemma 4.1, $|f'''(u)| = |\langle f, z^3(k_u^{\Theta})^4 \rangle| \leq \|k_u^{\Theta}\|_4^4 \|f\|_{BMOA}$, and so

$$\|f\|_{B_p}^{**} \leq \|f\|_{BMOA}^p \int_{\mathbb{D}} \|k_u^{\Theta}\|_4^{4p} (1 - |u|^2)^{3p-2} dA(u).$$

Lemma 6.1. *For any $u \in \mathbb{D}$,*

$$\|k_u^{\Theta}\|_4^4 = \frac{(1 + |u|^2)(1 - |\Theta(u)|^4)}{(1 - |u|^2)^3} - \frac{4\operatorname{Re}(u\Theta'(u)\overline{\Theta(u)})}{(1 - |u|^2)^2}.$$

Proof. The lemma follows from straightforward computations based on the formula $f'(u) = \langle f, z(1 - \bar{u}z)^{-2} \rangle$. We omit the details. \square

We continue to estimate $\|f\|_{B_p}^{**}$. Since $1 + |u|^2 \leq 2$ and $1 - |\Theta(u)|^4 \leq 2(1 - |\Theta(u)|^2)$, we have

$$\|k_u^{\Theta}\|_4^4 \leq \frac{4(1 - |\Theta(u)|^2)}{(1 - |u|^2)^3} - \frac{4\operatorname{Re}(u\Theta'(u)\overline{\Theta(u)})}{(1 - |u|^2)^2}.$$

From now on assume that Θ is a finite Blaschke product $B = \prod_{k=1}^n b_{z_k}$. Then

$$\begin{aligned} uB'(u)\overline{B(u)} &= u|B(u)|^2 \frac{B'(u)}{B(u)} = u|B(u)|^2 \sum_{k=1}^n \left(\frac{1}{u - z_k} + \frac{\bar{z}_k}{1 - \bar{z}_k u} \right) \\ &= |B(u)|^2 \sum_{k=1}^n \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} + |B(u)|^2 \sum_{k=1}^n \frac{z_k(1 - |z_k|^2)(1 - |u|^2)}{|1 - \bar{z}_k u|^2(u - z_k)}. \end{aligned}$$

Denote the last term by $S_1(u)$. Since $|B(u)| \leq |b_{z_k}(u)|$, we have

$$|S_1(u)| \leq \sum_{k=1}^n \frac{(1 - |u|^2)(1 - |z_k|^2)}{|1 - \bar{z}_k u|^3},$$

whence (recall that $p \leq 1$)

$$\int_{\mathbb{D}} \frac{|S_1(u)|^p}{(1 - |u|^2)^p} (1 - |u|^2)^{3p-2} dA(u) \leq \sum_{k=1}^n \int_{\mathbb{D}} \frac{(1 - |z_k|^2)^p}{|1 - \bar{z}_k u|^{3p}} (1 - |u|^2)^{3p-2} dA(u) \lesssim n,$$

since, by [HKZ, Theorem 1.7], each integral in the above sum does not exceed some constant depending only on p , but not on z_k .

Thus, to prove (6.1), it remains to estimate the weighted area integral of the difference

$$S_2(u) = \frac{1 - |B(u)|^2}{(1 - |u|^2)^2} - \frac{|B(u)|^2}{1 - |u|^2} \sum_{k=1}^n \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

We use again the notations $B_k = \prod_{l=1}^{k-1} b_{z_l}$ (assuming $B_1 \equiv 1$) and $\widehat{B}_k = \prod_{l=k}^n b_{z_l}$. It is easy to see that

$$(6.2) \quad \frac{1 - |B(u)|^2}{1 - |u|^2} = \sum_{k=1}^n |B_k(u)|^2 \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

Hence,

$$\begin{aligned} S_2(u) &= \sum_{k=1}^n |B_k(u)|^2 \cdot \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \cdot \frac{1 - |\widehat{B}_k(u)|^2}{1 - |u|^2} \\ &= \sum_{k=1}^n \sum_{l=k}^n \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \cdot \frac{1 - |z_l|^2}{|1 - \bar{z}_l u|^2} \cdot |B_l(u)|^2. \end{aligned}$$

Note that, by a formula analogous to (6.2), but without squares,

$$\frac{1 - |B(u)|}{1 - |u|} = \sum_{k=1}^n |B_k(u)| \frac{1 - |b_{z_k}(u)|}{1 - |u|} \geq \frac{1}{2} \sum_{k=1}^n |B_k(u)| \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

Hence,

$$4 \left(\frac{1 - |B(u)|}{1 - |u|} \right)^2 = \sum_{k=1}^n \sum_{l=1}^n |B_k(u)| \cdot |B_l(u)| \cdot \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \cdot \frac{1 - |z_l|^2}{|1 - \bar{z}_l u|^2}.$$

Denote the last double sum by $S_3(u)$. Since $|B_k B_l| \geq |B_l|^2$, $l \geq k$, we see that $S_2(u) \leq S_3(u)$. Now we have

$$\begin{aligned} \int_{\mathbb{D}} \frac{|S_2(u)|^p}{(1 - |u|^2)^p} (1 - |u|^2)^{3p-2} dA(u) &\leq 4^p \int_{\mathbb{D}} \left(\frac{1 - |B(u)|}{1 - |u|} \right)^{2p} (1 - |u|^2)^{2p-2} dA(u) \\ &\lesssim I_{2p, 2p-2}(B) \lesssim \|B'\|_{B_{2p}}^{2p} \lesssim n. \end{aligned}$$

Here we used Theorem 2.4 to estimate $I_{2p, 2p-2}(B)$ (recall that $2p > 1$) and the Arazy–Fisher–Peetre inequality (2.5). \square

7. RADIAL-WEIGHTED BERGMAN NORMS OF THE DERIVATIVE OF FINITE BLASCHKE PRODUCTS

Again, let $n \geq 1$, $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ and let B_σ be the finite Blaschke product corresponding to σ . For any $1 < p < \infty$ and $\alpha > -1$, we set

$$\varphi_n(p, \alpha) = \sup \left\{ \|B'_\sigma\|_{A_p(\alpha)} : \sigma \in \mathbb{D}^n \right\}.$$

Note that for any $n \geq 1$, $\varphi_n(p, \alpha) = \varphi_1(p, \alpha) = \infty$ if and only if $\alpha < p - 2$. Indeed, if $\alpha \geq p - 2$, then $\varphi_n(p, \alpha) \leq \varphi_n(p, p - 2) \asymp n^{\frac{1}{p}}$ by the Arazy–Fisher–Peetre inequality (2.5). For $\alpha < p - 2$, consider the test function b_r , $r \in (0, 1)$. It is easily seen (see, e.g., [HKZ, Theorem 1.7]) that $\|b'_r\|_{A_p(\alpha)} \rightarrow \infty$ as $r \rightarrow 1^-$.

We have seen in Subsection 2.2.2 how the estimate $\varphi_n(p, p-2) \asymp n^{\frac{1}{p}}$ implies Peller's inequality (2.1). It could be of interest to find a more general estimate (for other values of α and p) of $\varphi_n(p, \alpha)$. Notice that for each fixed p , the function $\alpha \mapsto \varphi_n(p, \alpha)$ is decreasing and there exists the second critical value $\alpha_p \geq -1$,

$$\alpha_p = \inf \left\{ \alpha > -1 : \sup_n \varphi_n(p, \alpha) < \infty \right\}.$$

The sequence $\{\varphi_n(p, \alpha)\}_{n \geq 1}$ may be unbounded and, thus, have a nontrivial asymptotics if and only if $p-2 \leq \alpha \leq \alpha_p$. In this notation we can rewrite Theorem 2.3 as

$$\|f\|_{A_p^1(\alpha)}^* \lesssim \varphi_n(p, \alpha) \|f\|_{BMOA},$$

for any $f \in \mathcal{R}_n^+$, $1 < p < \infty$, $p-2 \leq \alpha \leq \alpha_p$.

We will show now that $\alpha_p = p-1$, and so $p-1$ is the second critical value of α , as is expected from Theorem 2.2.

Proposition 7.1. *For any $p > 1$, $\alpha_p = p-1$.*

Proof. By the Schwarz–Pick lemma, we have that for any $\sigma \in \mathbb{D}^n$,

$$\|B'_\sigma\|_{A_p(\alpha)}^p \leq I_{p,\alpha}(B_\sigma),$$

and for any $\alpha > p-1$,

$$I_{p,\alpha}(B_\sigma) \leq \int_{\mathbb{D}} (1-|u|^2)^\alpha \left(\frac{1-|B_\sigma(u)|^2}{1-|u|^2} \right)^p dA(u) \lesssim \int_{\mathbb{D}} \frac{1}{(1-|u|)^{p-\alpha}} dA(u) < \infty,$$

and thus $\alpha_p \leq p-1$ for each p .

Next we show that $\alpha_p \geq p-1$. Let us consider the set $\sigma = (0, \dots, 0) \in \mathbb{D}^n$, for which $B_\sigma(z) = z^n$. In this case, we have

$$\|B'_\sigma\|_{A_p(\alpha)}^p = \|nz^{n-1}\|_{A_p(\alpha)}^p = n^p \int_0^1 r(1-r^2)^\alpha \int_{\mathbb{T}} |r\xi|^{p(n-1)} dm(\xi) dr,$$

which gives

$$\|B'_\sigma\|_{A_p(\alpha)}^p = n^p \int_0^1 (1-r^2)^\alpha r^{p(n-1)+1} dr,$$

and

$$\beta(pn-p+2, \alpha+1) \leq \frac{\|B'_\sigma\|_{A_p(\alpha)}^p}{n^p} \leq 2^\alpha \beta(pn-p+2, \alpha+1),$$

where β stands for the *Beta function* $\beta(x, y) = \int_0^1 r^{x-1}(1-r)^{y-1} dr$. Let $\alpha = p-1-\varepsilon$, $\varepsilon > 0$. Then by the standard Γ -function asymptotics, we obtain

$$\|B'_\sigma\|_{A_p(\alpha)}^p \geq \Gamma(\alpha+1)n^p \frac{\Gamma(pn-p+2)}{\Gamma(pn+\alpha-p+3)} \sim_{n \rightarrow \infty} \Gamma(\alpha+1)n^p (pn)^{\varepsilon-p},$$

whence $\sup_n \varphi_p(\alpha, n) = \infty$. □

8. REMARKS ON DOLZHENKO'S INEQUALITIES

8.1. Proof of Dolzhenko's inequalities for $1 \leq p \leq 2$. In [Dol, Theorem 2.2] E.P. Dolzhenko proved that for any $f \in \mathcal{R}_n^+$,

$$(8.1) \quad \|f'\|_{A_p} \lesssim \begin{cases} n^{1-\frac{1}{p}} \|f\|_{H^\infty}, & 1 < p \leq 2, \\ \log n \|f\|_{H^\infty}, & p = 1, \end{cases}$$

where the constants involved in \lesssim may depend on p only. Let us show that these inequalities (and even with $BMOA$ -norm in place of H^∞ -norm) are direct corollaries of Proposition 4.2 and the following simple lemma.

Lemma 8.1. *For any Blaschke product B of degree n we have*

$$(8.2) \quad I_{p,0}(B) = \int_{\mathbb{D}} \left(\frac{1 - |B(u)|^2}{1 - |u|^2} \right)^p dA(u) \lesssim \begin{cases} n^{p-1}, & 1 < p \leq 2, \\ \log n, & p = 1. \end{cases}$$

Proof. Clearly, the integral over the disc $\{|z| \leq 1 - \frac{1}{n}\}$ has the required estimate. The estimate over the annulus $\{1 - \frac{1}{n} \leq |z| < 1\}$ follows from the result of Dyn'kin ($I_{2,0}(B) \lesssim n$) and the Hölder inequality. Indeed, for $1 \leq p < 2$,

$$\int_{\{1-\frac{1}{n} \leq |z| < 1\}} \left(\frac{1 - |B(u)|^2}{1 - |u|^2} \right)^p dA(u) \leq (I_{2,0}(B))^{\frac{p}{2}} \left(\pi \left(1 - \left(1 - \frac{1}{n}\right)^2\right) \right)^{1-\frac{p}{2}} \lesssim n^{p-1}.$$

□

Now inequality (8.1) follows from (8.2) and from the inequality $\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA(I_{p,\alpha}(B_\sigma))}^{\frac{1}{p}}$ which holds for any function $f \in K_{B_\sigma}$ (see Proposition 4.2). It should be mentioned, however, that Dolzhenko proves his inequalities for more general domains than the unit disc.

8.2. An extension of Dolzhenko's inequalities to the range $p > 2$. The case $p > 2$ is also treated by Dolzhenko (see the last inequality in [Dol, Theorem 2.2]), but the corresponding analog is of somewhat different nature. As the example $f(z) = (1 - rz)^{-1}$ with $r \rightarrow 1-$ shows, there exist no estimate of $\|f'\|_{A_p}$ in terms of $\|f\|_{BMOA}$ and $n = \deg f$.

Here we obtain another extension of Dolzhenko's result for $p > 2$.

Theorem 8.2. *Let $2 < p \leq \infty$, let $f \in \mathcal{R}_n^+$, $n \geq 1$, and let $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ be its poles (repeated according to multiplicities). We have*

$$(8.3) \quad \|f'\|_{A_p} \lesssim n^{\frac{1}{p}} \left(\sum_{k=1}^n \frac{1 + |\lambda_k|}{1 - |\lambda_k|} \right)^{1-\frac{2}{p}} \|f\|_{BMOA}.$$

Moreover, the inequality (8.3) is asymptotically sharp in the following sense: for any $r \in (0, 1)$ there exists $g \in \mathcal{R}_n^+$ having $\frac{1}{r}$ as a pole of multiplicity n such that

$$(8.4) \quad \|g'\|_{A_p} \gtrsim n^{1-\frac{1}{p}} \left(\frac{1}{1-r} \right)^{1-\frac{2}{p}} \|g\|_{BMOA}.$$

Proof. We first prove (8.3). Set $\sigma = (\lambda_1, \dots, \lambda_n)$, so that $f \in K_{\tilde{B}_\sigma}$. By Proposition 4.2 and Dyn'kin's inequality (2.6),

$$\|f'\|_{A_p}^p \leq \|f\|_{BMOA}^p \int_{\mathbb{D}} \left(\frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^p dA(u) \lesssim n \|f\|_{BMOA} \sup_{u \in \mathbb{D}} \left(\frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^{p-2}.$$

By (6.2), we have

$$\frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \leq 1 + \sum_{k=1}^n \frac{1 - |\lambda_k|^2}{|1 - \overline{\lambda_k}u|^2} \lesssim \sum_{k=1}^n \frac{1 + |\lambda_k|}{1 - |\lambda_k|}.$$

This proves (8.3). Now we prove (8.4). Take $g = b_{-r}^n(u)$, $r \in (0, 1)$, then

$$\|g'\|_{A_p}^p = n^p \int_{\mathbb{D}} |b'_{-r}(u)|^2 |b'_{-r}(u)|^{p-2} |b_{-r}(u)|^{p(n-1)} dA(u).$$

Taking $v = b_{-r}(u)$ as the new variable and using the fact that $u = b_{-r}(v)$, we get

$$\|g'\|_{A_p}^p = n^p \int_{\mathbb{D}} |b'_{-r}(b_{-r}(v))|^{p-2} |v|^{p(n-1)} dA(v).$$

Since $b'_{-r} \circ b_{-r}(v) = -\frac{(1+rv)^2}{1-r^2}$, we obtain

$$\|g'\|_{A_p}^p = \frac{n^p}{(1-r^2)^{p-2}} \int_{\mathbb{D}} |1+rv|^{2(p-2)} |v|^{p(n-1)} dA(v).$$

Supposing that $p \geq 2$, we have

$$\int_{\mathbb{D}} |1+rv|^{2(p-2)} |v|^{p(n-1)} dA(v) \asymp \int_{\mathbb{D}} |v|^{p(n-1)} dA(v) = \frac{2}{pn - p + 2},$$

whence

$$\|g'\|_{A_p}^p \asymp \frac{n^{p-1}}{(1-r)^{p-2}}.$$

Since $\|g\|_{BMOA} = 1$ this completes the proof (8.4). \square

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