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BOUNDEDNESS OF THE DIFFERENTIATION OPERATOR IN MODEL SPACES AND APPLICATION TO PELLER TYPE INEQUALITIES

ANTON BARANOV AND RACHID ZAROUF

ABSTRACT. Given an inner function Θ in the unit disc \mathbb{D} , we study the boundedness of the differentiation operator which acts from the model subspace $K_\Theta = (\Theta H^2)^\perp$ of the Hardy space H^2 , equipped with the *BMOA*-norm, to some radial-weighted Bergman space. As an application, we generalize Peller's inequality for Besov norms of rational functions f of degree $n \geq 1$ having no poles in the closed unit disc $\overline{\mathbb{D}}$.

1. INTRODUCTION AND NOTATIONS

A well-known inequality by Peller (see inequality (2.1) below) majorizes a Besov norm of any rational function f of degree $n \geq 1$ having no poles in the closed unit disc $\overline{\mathbb{D}} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$ in terms of its *BMOA*-norm and its degree n . The original proof of Peller makes use of the theory of Hankel operators. One of the aims of this paper is to give a direct proof of this inequality and extend it to more general radial-weighted Bergman norms. Our proof combines integral representation for the derivative of f by using a model spaces approach, and the generalization of a theorem by Dyn'kin. The corresponding inequalities are obtained in terms of radial-weighted Bergman norms of the derivative of finite Blaschke products (of degree $n = \deg f$), instead of n itself. The finite Blaschke products in question has the same poles as f . The inequalities are sharp and attained by these Blaschke products. The study of radial-weighted Bergman norms of the derivatives of finite Blaschke products of degree n and their asymptotic as n tends to $+\infty$ may be of independent interest. A contribution to this topic, which we are going to exploit here, was given by Arazy, Fisher and Peetre.

Let \mathcal{P}_n be the space of complex analytic polynomials of degree at most n and let

$$\mathcal{R}_n^+ = \left\{ \frac{P}{Q} : P, Q \in \mathcal{P}_n, Q(\xi) \neq 0 \text{ } |\xi| \geq 1 \right\},$$

be the set of rational functions of degree at most n with poles outside of the closed unit disc $\overline{\mathbb{D}}$. In this paper, we consider the norm of a rational function $f \in \mathcal{R}_n^+$ in different function spaces which will consist of analytic functions in the open unit disc $\mathbb{D} = \{\xi : |\xi| < 1\}$.

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1.1. **Some Banach spaces of analytic functions.** We denote by $\mathcal{H}ol(\mathbb{D})$ the space of holomorphic functions in \mathbb{D} .

1.1.1. *The Besov spaces B_p , $1 < p \leq +\infty$.*

- A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the Besov space B_p , $1 < p \leq +\infty$, if and only if

$$\|f\|_{B_p} = |f(0)| + \|f\|_{B_p}^* < +\infty,$$

where $\|f\|_{B_p}^*$ is the seminorm defined by

$$\|f\|_{B_p}^* = \left(\int_{\mathbb{D}} (1 - |u|)^{p-2} |f'(u)|^p dA(u) \right)^{\frac{1}{p}},$$

A being the normalized area measure on \mathbb{D} .

- A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the Besov space B_1 if and only if

$$\|f\|_{B_1} = |f(0)| + |f'(0)| + \|f\|_{B_1}^* < +\infty,$$

where $\|f\|_{B_1}^*$ is the seminorm defined by

$$\|f\|_{B_1}^* = \int_{\mathbb{D}} |f''(u)| dA(u).$$

- A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the space B_∞ (known as the Bloch space) if and only if

$$\|f\|_{B_\infty} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| (1 - |z|) < \infty.$$

We refer to [Pee, Tri, BeLo] for general properties of Besov spaces.

1.1.2. *The radial-weighted Bergman spaces $A_p(w)$, $1 \leq p < \infty$.* The radial-weighted Bergman space $A_p(w)$, $1 \leq p < \infty$ (where "a" means analytic) is defined as:

$$A_p(w) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \|f\|_{A_p(w)}^p = \int_{\mathbb{D}} w(|u|) |f(u)|^p dA(u) < \infty \right\},$$

where the weight w satisfies $w \geq 0$ and $\int_0^1 w(r) dr < \infty$. The classical power weights $w(r) = w_\alpha(r) = (1 - r^2)^\alpha$, $\alpha > -1$, are of special interest; in this case we put $A_p(\alpha) = A_p(w_\alpha)$. We refer to [HKZ] for general properties of weighted Bergman spaces.

1.1.3. *The spaces $A_p^1(\alpha)$, $1 \leq p \leq +\infty$, $\alpha > -1$.* A function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the space $A_p^1(\alpha)$, $1 \leq p \leq +\infty$, $\alpha > -1$, if and only if

$$\|f\|_{A_p^1(\alpha)} = |f(0)| + \|f'\|_{A_p(\alpha)} < +\infty.$$

We also define the $A_p^1(\alpha)$ -seminorm by $\|f\|_{A_p^1(\alpha)}^* = \|f'\|_{A_p(\alpha)}$. Note that the B_p and $A_p^1(p-2)$ coincide for $1 < p \leq +\infty$.

1.1.4. *The space BMOA.* There are many ways to define *BMOA*; see [Gar, Chapter 6]. For the purposes of this paper we choose the following one: a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to the *BMOA* space (of analytic functions of bounded mean oscillation) if and only if

$$\|f\|_{BMOA} = \inf \|g\|_{L^\infty(\mathbb{T})} < +\infty,$$

where the infimum is taken over all $g \in L^\infty(\mathbb{T})$, $\mathbb{T} = \{\xi : |\xi| = 1\}$ being the unit circle, for which the representation

$$f(\xi) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(u)}{u - \xi} du, \quad |\xi| < 1,$$

holds. Recall that *BMOA* is the dual space of the Hardy space H^1 under the pairing

$$\langle f, g \rangle = \int_{\mathbb{T}} f(u) \overline{g(u)} du, \quad f \in H^1, g \in BMOA,$$

where this integral must be understood as the extension of the pairing acting on a dense subclass of H^1 , see [Bae, page 23].

1.2. Model spaces.

1.2.1. *General inner functions.* By H^p , $1 \leq p \leq \infty$, we denote the standard Hardy spaces (see [Gar, Nik]). Recall that H^2 is a reproducing kernel Hilbert space, with the kernel

$$k_\lambda(w) = \frac{1}{1 - \bar{\lambda}w}, \quad \lambda, w \in \mathbb{D},$$

known as the Szegő kernel (or the Cauchy kernel) associated with λ . Thus $\langle f, k_\lambda \rangle = f(\lambda)$ for all $f \in H^2$ and for all $\lambda \in \mathbb{D}$, where $\langle \cdot, \cdot \rangle$ means the scalar product on H^2 .

Let Θ be an *inner function*, i.e. $\Theta \in H^\infty$ and $|\Theta(\xi)| = 1$ a.e. $\xi \in \mathbb{T}$. We define the model subspace K_Θ of the Hardy space H^2 by

$$K_\Theta = (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$

By the famous theorem of Beurling, these and only these subspaces of H^2 are invariant with respect to the backward shift operator. We refer to [Nik] for the general theory of the spaces K_Θ and their numerous applications.

For any *inner function* Θ , the reproducing kernel of the model space K_Θ corresponding to a point $\xi \in \mathbb{D}$ is of the form

$$k_\lambda^\Theta(w) = \frac{1 - \overline{\Theta(\lambda)}\Theta(w)}{1 - \bar{\lambda}w}, \quad \lambda, w \in \mathbb{D},$$

that is $\langle f, k_\lambda^\Theta \rangle = f(\lambda)$ for all $f \in K_\Theta$ and for all $\lambda \in \mathbb{D}$.

1.2.2. *The case of finite Blaschke products.* From now on, for any $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$, we consider the finite Blaschke product

$$B_\sigma = \prod_{k=1}^n b_{\lambda_k},$$

where $b_\lambda(z) = \frac{\lambda-z}{1-\bar{\lambda}z}$, is the elementary Blaschke factor corresponding to $\lambda \in \mathbb{D}$. We suppose here that $\Theta = B_\sigma$ with

$$\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\} \in \mathbb{D}^n,$$

where every λ_s is repeated according to its multiplicity n_s , $\sum_{s=1}^t n_s = n$. Then, we have

$$K_{B_\sigma} = H^2 \ominus B_\sigma H^2 = \text{span} \left(k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq n_j - 1 \right),$$

where if $\lambda \neq 0$, $k_{\lambda, i} = \left(\frac{d}{d\lambda}\right)^i k_\lambda$ and $k_\lambda = \frac{1}{1-\bar{\lambda}z}$ is the standard Cauchy kernel at the point λ , whereas if $\lambda = 0$, $k_{0, i} = z^i$.

The subspace K_{B_σ} consists of rational functions of the form p/q , where $p \in \mathcal{P}_{n-1}$ and $q \in \mathcal{P}_n$, with the poles $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ of corresponding multiplicities (including possible poles at ∞). Thus, if $f \in \mathcal{R}_n^+$ and $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ are the poles of f (repeated according to multiplicities), then $f \in K_{zB_\sigma}$ with $\sigma = (\lambda_1, \dots, \lambda_n)$. Conversely, K_{B_σ} is obviously a subset of \mathcal{R}_n^+ for any $\sigma \in \mathbb{D}^n$.

From now on, for two positive functions a and b , we say that a is dominated by b , denoted by $a \lesssim b$, if there is a constant $c > 0$ such that $a \leq cb$; and we say that a and b are comparable, denoted by $a \asymp b$, if both $a \lesssim b$ and $b \lesssim a$.

2. MAIN RESULTS

2.1. Main ingredients. It has been proved in 1980 by Peller [Pel1] that

$$(2.1) \quad \|f\|_{B_p}^* \leq c_p n^{\frac{1}{p}} \|f\|_{BMOA},$$

for any $f \in \mathcal{R}_n^+$ and $1 \leq p \leq +\infty$, where c_p is a constant depending only on p . Later, this result was extended to the range $p > 0$ independently and with different proofs by Peller [Pel2] and Semmes [Sem]. The aim of the present article is:

- (1) study the boundedness of the differentiation operator from $(K_\Theta, \|\cdot\|_{BMOA})$ to $A_p(\alpha)$, $1 < p \leq +\infty$, $\alpha > -1$, and
- (2) generalize Peller's result (2.1) replacing the B_p -seminorm by the $A_p^1(\alpha)$ -one.

In both of these problems, we make use of a method based on two main ingredients:

- integral representation for the derivative of functions in K_Θ or in \mathcal{R}_n^+ and
- a generalization of a theorem by Dyn'kin, see Subsection 2.2.3.

As a one more ingredient (required in problem (2)) we use estimates of B_p -seminorms of finite Blaschke products by Arazy, Fischer and Peetre [AFP].

2.2. Main results. Let us consider the differentiation operator $Df = f'$ and the shift and the backward shift operators defined respectively by

$$(2.2) \quad Sf = zf, \quad S^*f = \frac{f - f(0)}{z},$$

for any $f \in \mathcal{H}ol(\mathbb{D})$. From now on, for any inner function Θ , we put

$$\tilde{\Theta} = z\Theta = S\Theta.$$

2.2.1. *Boundedness of the differentiation operator from $(K_\Theta, \|\cdot\|_{BMOA})$ to $A_p(\alpha)$.* First, note that the operator

$$D : BMOA \rightarrow A_p(\alpha)$$

is bounded when $p \geq 2$ and $\alpha \geq p - 1$, and also when $1 \leq p < 2$ and $\alpha > p - 1$, and, thus, for these values of parameters the embedding $BMOA \subset A_p^1(\alpha)$ is continuous. Indeed, for any f in $BMOA$ we have

$$\begin{aligned} \|f'\|_{A_p(\alpha)}^p &= \int_{\mathbb{D}} (1 - |u|)^{p-2} |f'(u)|^{p-2} (1 - |u|)^{\alpha-(p-2)} |f'(u)|^2 dA(u) \\ &\leq \|f\|_{B_\infty}^{p-2} \int_{\mathbb{D}} (1 - |u|)^{\alpha-(p-2)} |f'(u)|^2 dA(u) \\ &\leq \|f\|_{B_\infty}^{p-2} \int_{\mathbb{D}} (1 - |u|) |f'(u)|^2 dA(u), \end{aligned}$$

when $\alpha - (p - 2) \geq 1$. Here B_∞ is the Bloch space. Since $\int_{\mathbb{D}} (1 - |u|) |f'(u)|^2 dA(u) = \|f\|_{H^2}^2$, $\|f\|_{H^2} \lesssim \|f\|_{BMOA}$ and $\|f\|_{B_\infty} \lesssim \|f\|_{BMOA}$, we conclude that

$$\|f'\|_{A_p(\alpha)} \lesssim \|f\|_{BMOA}.$$

In the case $1 \leq p < 2$ and $\alpha > p - 1$ the boundedness of D from $(K_\Theta, \|\cdot\|_{BMOA})$ to $A_p(\alpha)$ is trivial. However, for $1 \leq p < 2$, the space $A_p^1(p - 1)$ does not contain even some functions from the disc-algebra. The reason for that is that if $f \in A_p^1(p - 1)$, $1 \leq p < 2$, then, by a result of S.A. Vinogradov [Vin, Lemma 1.6], $\sum_{n=0}^{\infty} |\hat{f}(2^n)|^p < \infty$, (where $\hat{f}(n)$ stands for the n^{th} Fourier coefficient of f). Thus, $BMOA \not\subset A_p^1(p - 1)$ when $1 \leq p < 2$.

Now, we consider an arbitrary *inner function* Θ . Our first main result gives necessary and sufficient conditions under which the differentiation operator

$$D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$$

is bounded. When this is the case, we estimate its norm in terms of $\|\Theta'\|_{A_p(\alpha)}$.

Theorem 2.1. *Let $1 < p \leq \infty$, $\alpha > -1$, and $p > 1 + \alpha$. We distinguish three cases:*

- (a) *If $\alpha > p - 1$, then $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded.*
- (b) *If $p - 2 < \alpha < p - 1$, then $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded if and only if $\Theta' \in A_1(\alpha - p + 1)$.*
- (c) *If $\alpha < p - 2$ and $p > 1$, then $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded if and only if Θ is a finite Blaschke product.*

In cases (b) and (c), we have

$$(2.3) \quad \|D\| \lesssim \|\Theta'\|_{A_p(\alpha)} \lesssim \|D\| + \text{const},$$

with constants depending on p and α only.

Remark. More precisely, we show that for $\alpha < p - 1$, D is bounded if and only if $\Theta' \in A_p(\alpha)$ and then use a theorem by Ahern [Ahe1] and its generalizations by Verbitsky [Ver] and Gluchoff [Glu]. The case $\alpha = p - 1$ and $1 \leq p < 2$ is still open.

It should be noted that the membership of Blaschke products in various function spaces is a well-studied topic. Besides the above-cited papers by Ahern, Gluchoff and Verbitsky

let us mention the Ahern–Clark papers [AC1, AC2] and recent works by D. Girela, J. Peláez, D. Vukotić and A. Aleman [GPV, AV].

2.2.2. *Generalization of Peller’s inequalities.* In the following theorem, we give a generalization of Peller’s inequality (2.1).

Theorem 2.2. *Let $f \in \mathcal{R}_n^+$, $\deg f = n$ and $\sigma \in \mathbb{D}^n$ be the set of its poles counting multiplicities (including poles at ∞). For any $\alpha > -1$, $1 < p \leq \infty$, and $p > 1 + \alpha$, we have*

$$(2.4) \quad \|f\|_{A_p^1(\alpha)}^* \leq K_{p,\alpha} \|f\|_{BMOA} \|\tilde{B}'_\sigma\|_{A_p(\alpha)},$$

where $K_{p,\alpha}^p = \frac{2^{\frac{1}{\alpha+1}}}{2^{\frac{1}{\alpha+1}-1}} \left(\frac{p}{p-1-\alpha}\right)^p 2^{p+1}$.

Remark. The inequality (2.4) is sharp and attained by $f = B_\sigma$. Indeed, we have

$$\|\tilde{B}'_\sigma\|_{A_p(\alpha)} \leq \|zB'_\sigma\|_{A_p(\alpha)} + \|B_\sigma\|_{A_p(\alpha)} \lesssim \|B'_\sigma\|_{A_p(\alpha)},$$

and $\|B_\sigma\|_{BMOA} = 1$.

Remark. Now, taking $\alpha = p - 2$, we have

$$\|f'\|_{A_p(\alpha)} = \|f\|_{B_p}^*, \quad \|\tilde{B}'_\sigma\|_{A_p(\alpha)} = \|\tilde{B}_\sigma\|_{B_p}^*.$$

To deduce Peller’s inequalities (2.1) it remains to apply the following theorem by Arazy, Fischer and Peetre [AFP]: if $1 \leq p \leq \infty$, then there exist absolute positive constants m_p and M_p such that

$$(2.5) \quad m_p n^{\frac{1}{p}} \leq \|B\|_{B_p}^* \leq M_p n^{\frac{1}{p}}.$$

Indeed, we obtain for $1 < p \leq \infty$,

$$\|f\|_{B_p}^* \leq K_{p,p-2}^{\frac{1}{p}} M_p \|f\|_{BMOA} (n+1)^{\frac{1}{p}} \lesssim n^{\frac{1}{p}} \|f\|_{BMOA},$$

which is Peller’s estimate (2.1) for $1 < p \leq \infty$. The case $p = 1$ requires a special treatment which will be developed later because of the particular definition of B_1 .

2.2.3. *Generalization of a theorem by Dyn’kin.* Dyn’kin proved that [Dyn, Theorem 3.2] that

$$(2.6) \quad \int_{\mathbb{D}} \left(\frac{1 - |B(u)|^2}{1 - |u|^2} \right)^2 dA(u) \leq 8(n+1),$$

for any finite Blaschke product B of degree n .

From now on, for any inner function Θ , and for any $\alpha > -1$, $p > 1$, we put

$$(2.7) \quad I_{p,\alpha}(\Theta) = \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\frac{1 - |\Theta(u)|^2}{1 - |u|^2} \right)^p dA(u).$$

Dyn’kin’s Theorem can be stated as follows: *for any finite Blaschke product B of degree n , we have*

$$I_{2,0}(B) \leq 8(n+1).$$

Here, we generalize this result to the case $\alpha > -1$, $p > 1$, and $p > 1 + \alpha$.

In the proof of Theorem 2.2, we will need the following generalization of Dyn'kin's result.

Theorem 2.3. *We suppose that $p > 1$, $\alpha > -1$ and $p > 1 + \alpha$. Then,*

$$\|\Theta'\|_{A_p(\alpha)}^p \leq I_{p,\alpha}(\Theta) \leq K_{p,\alpha} \|\Theta'\|_{A_p(\alpha)}^p,$$

where $K_{p,\alpha}$ is the same constant as in Theorem 2.2.

The paper is organized as follows. We first focus in Section 3 on the generalization of Dyn'kin's result. In Section 4 Theorem 2.1 is proved, while Section 5 is devoted to the proof of Peller type inequalities (Theorem 2.2). Finally, in Section 6 we discuss some estimates of radial-weighted Bergman norms of Blaschke products.

3. GENERALIZATION OF DYN'KIN'S THEOREM

The aim of this Section is to prove Theorem 2.3. The lower bound is trivially obtained as an application of the well-known Schwarz-Pick inequality applied to Θ . The proof of the upper bound will be done step by step, using lemmas which are stated below. The main ideas come from [Dyn, Theorem 3.2]. In this Section, Θ is any inner function.

Lemma 3.1. *For $p > 1$, $\alpha > -1$ and $p > 1 + \alpha$, we have*

$$I_{p,\alpha}(\Theta) \leq 2^p \int_0^1 \int_0^{2\pi} (1-r)^\alpha \left(\frac{1}{1-r} \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \frac{d\theta}{\pi}.$$

Proof. Writing the integral $I_{p,\alpha}(\Theta)$ in polar coordinates, and using the fact that

$$1 - |\Theta(u)|^2 \leq 2(1 - |\Theta(u)|),$$

we obtain

$$\begin{aligned} I_{p,\alpha}(\Theta) &\leq 2^p \int_0^1 r(1-r^2)^{\alpha-p} \left(\int_0^{2\pi} (1 - |\Theta(re^{i\theta})|)^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 r(1-r^2)^{\alpha-p} \left(\int_0^{2\pi} |\Theta(e^{i\theta}) - \Theta(re^{i\theta})|^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 r(1-r^2)^{\alpha-p} \left(\int_0^{2\pi} \left(\int_r^1 |\Theta'(se^{i\theta})| ds \right)^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 \int_0^{2\pi} (1-r)^\alpha \frac{1}{(1-r)^p} \left(\int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \frac{d\theta}{\pi}, \end{aligned}$$

which completes the proof of the lemma. □

We recall now a general version of Hardy's inequality, see [HLP, page 245], which after change of variables gives (as in [Ahe2, Lemma 7]):

Lemma 3.2. *If $h : (0, 1) \rightarrow [0, +\infty)$, $p > 1$, $\alpha > -1$ and $p > 1 + \alpha$,*

$$\int_0^1 (1-r)^\alpha \left(\frac{1}{1-r} \int_r^1 h(s) ds \right)^p dr \leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^1 (1-r)^\alpha h(r)^p dr.$$

Corollary 3.3. *We suppose that $p > 1$, $\alpha > -1$ and $p > 1 + \alpha$. Then,*

$$I_{p,\alpha}(\Theta) \leq C_{p,\alpha} \int_0^{2\pi} \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi},$$

where $C_{p,\alpha} = \left(\frac{p}{p-1-\alpha}\right)^p 2^p$.

Proof. Combining estimates in Lemma 3.1 and Lemma 3.2 (setting $h(s) = h_\theta(s) = |\Theta'(se^{i\theta})|$, for any fixed $\theta \in (0, 2\pi)$), we obtain

$$\int_0^1 (1-r)^\alpha \left(\frac{1}{1-r} \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr.$$

Thus,

$$I_{p,\alpha}(\Theta) \leq \left(\frac{p}{p-1-\alpha} \right)^p 2^p \int_0^{2\pi} \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi}$$

which completes the proof. \square

Lemma 3.4. *Let any nonzero weight w satisfying $w \geq 0$ and $\int_0^1 w(r) dr < \infty$. Let $\beta = \beta_w \in (0, 1)$ such that $\int_0^1 w(r) dr = 2 \int_0^\beta w(r) dr$. Then, for $f \in A_p(w)$, $1 \leq p < \infty$,*

$$\begin{aligned} \|f\|_{A_p(w)}^p &\leq \int_0^{2\pi} w(r) \int_0^1 |f(re^{i\theta})|^p dr \frac{d\theta}{\pi} \\ &\leq \frac{2}{\beta} \int_\beta^1 r w(r) \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi} \right) dr \leq \frac{2}{\beta} \|f\|_{A_p(w)}^p. \end{aligned}$$

Proof. The proof follows easily from the fact that for any f in $\mathcal{Hol}(\mathbb{D})$, the function

$$r \mapsto \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi},$$

is nondecreasing on $[0, 1]$. \square

We are now ready to prove Theorem 2.3.

Proof. We first prove (2.4). Applying the previous Lemma 3.4 with $f = \Theta'$ and $w(r) = (1-r^2)^\alpha$, $\alpha > -1$, we obtain, going back to the above Corollary that

$$I_{p,\alpha}(\Theta) \leq C_{p,\alpha} \int_0^{2\pi} \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} \leq \frac{2}{\beta} C_{p,\alpha} \|\Theta'\|_{A_p(\alpha)}^p,$$

where $C_{p,\alpha} = \left(\frac{p}{p-1-\alpha}\right)^p 2^p$, and $\beta = \beta_\alpha$ satisfies the condition

$$\int_\beta^1 w(r) dr = \int_0^\beta w(r) dr.$$

By a direct computation, we see that $\beta = \beta_\alpha$ is given by the equation $\frac{1-(1-\beta)^{\alpha+1}}{1+\alpha} = \frac{(1-\beta)^{\alpha+1}}{1+\alpha}$, which is equivalent to

$$(3.1) \quad \beta = \beta_\alpha = 1 - \frac{1}{2^{\frac{1}{\alpha+1}}}.$$

\square

4. PROOF OF THEOREM 2.1

4.1. Integral representation for the derivative of functions in K_Θ . An important ingredient of our proof is the following simple and well-known integral representation for the derivative of a function from a model space.

Lemma 4.1. *Let Θ be an inner function and $f \in K_\Theta$. We have*

$$f'(u) = \left\langle f, z (k_u^\Theta)^2 \right\rangle,$$

for any $u \in \mathbb{D}$.

Proof. For a fixed $u \in \mathbb{D}$, we have

$$f'(u) = \left\langle f, \frac{z}{(1 - \bar{u}z)^2} \right\rangle = \left\langle f, z (k_u^\Theta)^2 \right\rangle.$$

Here the first equality is the standard Cauchy formula, while the second follows from the fact that $z(1 - \bar{u}z)^{-2} - z(k_u^\Theta(z))^2 \in \Theta H^2$ and $f \perp z\Theta H^2$. \square

4.2. Proof of the right-hand side inequality in (2.3). Now, we state and prove the following proposition in which the quantity $I_{p,\alpha}(\Theta)$ is involved.

Proposition 4.2. *Let $\alpha > -1$, and $1 < p \leq \infty$, Θ be an inner function and $f \in K_\Theta$. We have*

$$\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA} (I_{p,\alpha}(\Theta))^{\frac{1}{p}}.$$

Proof. We use the integral representation of a f from Lemma 4.1:

$$f'(u) = \left\langle f, z (k_u^\Theta)^2 \right\rangle = \int_{\mathbb{T}} f(\tau) \tau \overline{(k_u^\Theta(\tau))^2} dm(\tau),$$

for any $u \in \mathbb{D}$, and thus

$$\begin{aligned} \|f'\|_{A_p(\alpha)}^p &= \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left| \int_{\mathbb{T}} f(\tau) \tau \overline{(k_u^\Theta(\tau))^2} dm(\tau) \right|^p dA(u) \\ &\leq \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left| \int_{\mathbb{T}} \tau \overline{(k_u^\Theta(\tau))^2} dm(\tau) \right|^p dA(u) \\ &\leq \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\int_{\mathbb{T}} |k_u^\Theta(\tau)|^2 dm(\tau) \right)^p dA(u) \\ &= \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\frac{1 - |\Theta(u)|^2}{1 - |u|^2} \right)^p dA(u), \end{aligned}$$

whih completes the proof. \square

4.3. Proof of the left-hand side inequality in (2.3).

Proof. We prove the left-hand side inequality implicitly stated in (2.3) from Theorem 2.1. We consider the test function

$$f = S^* \Theta = \frac{\Theta - \Theta(0)}{z},$$

where S^* is the backward shift operator (2.2). It is well-known that f belongs to K_Θ and easy to check that $\|f\|_{BMOA} \leq 2$ and thus,

$$\|D\|_{(K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)(w_\alpha)} \geq \frac{\|f'\|_{A_p(\alpha)}}{2}.$$

Now,

$$\|f'\|_{A_p(\alpha)}^p \geq \int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} |f'(r\xi)|^p dm(\xi) dr,$$

where β_α is given by (3.1) and thus,

$$\begin{aligned} \|f'\|_{A_p(\alpha)} &\geq \left(\int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p dm(\xi) dr \right)^{\frac{1}{p}} + \\ &\quad - \left(\int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi) - \Theta(0)}{r^2\xi^2} \right|^p dm(\xi) dr \right)^{\frac{1}{p}}. \end{aligned}$$

On one hand we have

$$\int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p dm(\xi) dr \geq \int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} |\Theta'(r\xi)|^p dm(\xi) dr,$$

and applying Lemma 3.4 with $w = w_\alpha$ (and $\beta = \beta_\alpha$) we obtain

$$\begin{aligned} \int_{\beta_\alpha}^1 r w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p dm(\xi) dr &\geq \frac{\beta_\alpha}{2} \int_0^{2\pi} \int_0^1 w_\alpha(r) |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} \\ &\geq \frac{\beta_\alpha}{2} \int_0^{2\pi} \int_0^1 w_\alpha(r) |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} = \frac{\beta_\alpha}{2} \|\Theta'\|_{A_p(\alpha)}^p. \end{aligned}$$

On the other hand,

$$\int_{\beta_\alpha}^1 r w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi) - \Theta(0)}{r^2\xi^2} \right|^p dm(\xi) dr \leq 2^p \int_{\beta_\alpha}^1 \frac{w_\alpha(r)}{r} dr \leq \frac{2^p}{\beta_\alpha} \int_{\beta_\alpha}^1 w_\alpha(r) dr.$$

Finally, we conclude that

$$\|f'\|_{A_p(\alpha)} \geq \frac{\beta_\alpha^{1/p}}{2^{1/p}} \|\Theta'\|_{A_p(\alpha)} - \frac{2}{\beta_\alpha^{1/p}} \left(\int_{\beta_\alpha}^1 w_\alpha(r) dr \right)^{\frac{1}{p}}.$$

□

4.4. Proof of Theorem 2.1. To be complete the proof of Theorem 2.1, we need to recall the following theorem proved by Ahern [Ahe1] for the case $1 \leq p \leq 2$ and generalized by Verbitsky [Ver] and Gluchoff [Glu] to the range $1 \leq p < \infty$. This theorem characterizes inner functions Θ which derivative belong to $A_p(\alpha)$.

Theorem. *Let Θ be an inner function, $1 \leq p < \infty$, and $\alpha > -1$.*

- (a) *If $\alpha > p - 1$, then $\Theta' \in A_p(\alpha)$.*
- (b) *If $p - 2 < \alpha < p - 1$, then $\Theta' \in A_p(\alpha)$ if and only if $\Theta' \in A_p(\alpha - p + 1)$.*
- (c) *If $\alpha < p - 2$ and $p > 1$, then $\Theta' \in A_p(\alpha)$ if and only if Θ is a finite Blaschke product.*

Let us prove Theorem 2.1.

Proof. We start with the proof of (a). Going back to Proposition 4.2, we have that

$$\|f'\|_{A_p(\alpha)}^p \leq \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\frac{1 - |\Theta(u)|^2}{1 - |u|^2} \right)^p dA(u),$$

and bounding roughly (from above) $1 - |\Theta(u)|^2$ by 2, we obtain that for $\alpha > p - 1$,

$$\|f'\|_{A_p(\alpha)}^p \leq \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|)^{\alpha-p} dA(u) \lesssim \|f\|_{BMOA}^p,$$

with a constant depending only on p and α . \square

In order to prove (b) and (c), we first remark that for $\alpha < p - 1$, it follows from (2.3) that $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$ is bounded if and only if $\Theta' \in A_p(\alpha)$. A direct application of the above Ahern–Verbitsky–Gluchoff theorem completes the proof.

5. PROOF OF PELLER TYPE INEQUALITIES

In this section we prove Theorem 2.2. From now on the inner function Θ is a finite Blaschke product. Recall that if $f \in \mathcal{R}_n^+$ and $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$ are the poles of f (repeated according to multiplicities), then $f \in K_{zB_\sigma}$ with $\sigma = (\lambda_1, \dots, \lambda_n)$.

Proof. Let $f \in \mathcal{R}_n^+$; there exists $\sigma \in \mathbb{D}^n$ such that $f \in K_{\tilde{B}_\sigma}$, $\tilde{B}_\sigma = zB_\sigma$. Then, by Proposition 4.2 we have

$$\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA} \left(I_{p,\alpha}(\tilde{B}_\sigma) \right)^{\frac{1}{p}},$$

for any $\alpha > -1$, and $1 < p \leq \infty$. Now applying Theorem 2.3, we complete the proof. \square

Remark. In Subsection 2.2.2, we have shown how to deduce Peller's inequality (2.1) from the result of Arazy–Fischer–Peetre (2.5). Let us show that for $p \geq 2$, one can give a very simple proof which uses only Proposition 4.2 and Dyn'kin's estimate $I_{2,0}(\tilde{B}_\sigma) \leq 8(n+2)$, where $n = \deg B_\sigma$. Indeed, in this case, we have

$$\begin{aligned} I_{p,p-2}(\tilde{B}_\sigma) &= \int_{\mathbb{D}} (1 - |u|^2)^{p-2} \left(\frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^{p-2+2} dA(u) \\ &= \int_{\mathbb{D}} \left(1 - |\tilde{B}_\sigma(u)|^2 \right)^{p-2} \left(\frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^2 dA(u) \\ &\leq I_{2,0}(\tilde{B}_\sigma) \leq 8(n+2). \end{aligned}$$

Thus, applying Proposition 4.2 with $\alpha = p - 2$ we obtain

$$\|f\|_{B_p}^p = \|f\|_{A_p^1(p-2)}^p \leq \|f\|_{BMOA}^p \left(I_{2,0}(\tilde{B}_\sigma) \right)^{\frac{1}{p}},$$

which completes the proof.

6. RADIAL-WEIGHTED BERGMAN NORMS OF THE DERIVATIVE OF FINITE BLASCHKE PRODUCTS AND APPLICATIONS

Again, let $n \geq 1$, $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ and B_σ be the finite Blaschke product corresponding to σ . For any $1 \leq p < \infty$ and $\alpha > -1$, we set

$$\varphi_n(p, \alpha) = \sup \left\{ \|B'_\sigma\|_{A_p(\alpha)} : \sigma \in \mathbb{D}^n \right\}.$$

An application of [AFP] gives

$$(6.1) \quad \varphi_n(p, p-2) \asymp n^{1/p}.$$

We have seen above (see Subsection 2.2.2) how (6.1) implies Peller's inequality (2.1). Thus, it could be of interest to find a more general estimate (for other values of α and p) of $\varphi_n(p, \alpha)$. Notice that for each fixed p , the function $\alpha \mapsto \varphi_n(p, \alpha)$ is decreasing and there exists a critical $\alpha_p \geq -1$:

$$\alpha_p = \inf \left\{ \alpha < -1 : \sup_n \varphi_n(p, \alpha) < \infty \right\}.$$

As a consequence, the asymptotic behavior of $\varphi_n(p, \alpha)$ as n tends to infinity (for fixed values of α and p) can exist only if $\alpha \geq \alpha_p$. In this notation we can rewrite Theorem 2.2 as

$$\|f\|_{A_p^+(\alpha)}^* \lesssim \varphi_n(p, \alpha) \|f\|_{BMOA},$$

for any $f \in \mathcal{R}_n^+$, $\alpha > -1$, and $1 < p \leq \infty$ such that $p > 1 + \alpha$.

Proposition 6.1. *For any $p \geq 1$, $\alpha_p = p - 1$.*

Proof. By the Schwarz–Pick lemma, we have that for any $\sigma \in \mathbb{D}^n$,

$$\|B'_\sigma\|_{A_p(\alpha)}^p \leq I_{p,\alpha}(B_\sigma),$$

and for any $\alpha > p - 1$,

$$I_{p,\alpha}(B_\sigma) \leq \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left(\frac{1 - |B_\sigma(u)|^2}{1 - |u|^2} \right)^p dA(u) \lesssim \int_{\mathbb{D}} \frac{1}{(1 - |u|)^{p-\alpha}} dA(u) < \infty,$$

and thus $\alpha_p \leq p - 1$ for each p .

Next we show that $\alpha_p \geq p - 1$. Let us consider the set $\sigma = (0, \dots, 0) \in \mathbb{D}^n$, for which $B_\sigma(z) = z^n$. In this case, we have

$$\|B'_\sigma\|_{A_p(\alpha)}^p = \|nz^{n-1}\|_{A_p(\alpha)}^p = n^p \int_0^1 r (1 - r^2)^\alpha \int_{\mathbb{T}} |r\xi|^{p(n-1)} dm(\xi) dr,$$

which gives

$$\|B'_\sigma\|_{A_p(\alpha)}^p = n^p \int_0^1 (1 - r^2)^\alpha r^{p(n-1)+1} dr,$$

and

$$\beta(pn - p + 2, \alpha + 1) \leq \frac{\|B'_\sigma\|_{A_p(\alpha)}^p}{n^p} \leq 2^\alpha \beta(pn - p + 2, \alpha + 1),$$

where β stands for the *Beta function* $\beta(x, y) = \int_0^1 r^{x-1}(1-r)^{y-1}dr$. Let $\alpha = p - 1 - \varepsilon$, $\varepsilon > 0$. Then by the standard Γ -function asymptotics, we obtain

$$\|B'_\sigma\|_{A_p(\alpha)}^p \geq \Gamma(\alpha + 1)n^p \frac{\Gamma(pn - p + 2)}{\Gamma(pn + \alpha - p + 3)} \sim_{n \rightarrow \infty} \Gamma(\alpha + 1)n^p (pn)^{\varepsilon - p},$$

whence $\sup_n \varphi_p(\alpha, n) = \infty$. □

REFERENCES

- [Ahe1] P.R. Ahern, *The mean modulus and derivative of an inner function*, Indiana University Math. J. **28** (1979), 311–347.
- [Ahe2] P.R. Ahern, *The Poisson integral of a singular measure*, Can. J. Math. **25** (1983), 4, 735–749.
- [AC1] P.R. Ahern, D.N. Clark, *On inner functions with H^p derivative*, Michigan Math. J. **21** (1974), 115–127.
- [AC2] P.R. Ahern, D.N. Clark, *On inner functions with B^p derivative*, Michigan Math. J. **23** (1976), 2, 107–118.
- [AV] A. Aleman, D. Vukotić, *On Blaschke products with derivatives in Bergman spaces with normal weights*, J. Math. Anal. Appl. **361** (2010), 2, 492–505.
- [AFP] J. Arazy, S.D. Fisher, J. Peetre, *Besov norms of rational functions*, Lecture Notes Math. Vol. 1302, Springer-Verlag, Berlin, 1988, 125–129.
- [Bae] A. Baernstein II, *Analytic functions of bounded mean oscillation*, in Aspects of Contemporary Complex Analysis, Academic Press, London, 1980, 3–36.
- [BaZa] A. Baranov, R. Zarouf, *A model spaces approach to some classical inequalities for rational functions*, J. Math. Anal. Appl. **418** (2014), 1, 121–141.
- [BeLo] J. Bergh, J. Lofstrom, *Interpolation Spaces*, Springer-Verlag, Berlin, 1976.
- [Dol] E.P. Dolzhenko, *Rational approximations and boundary properties of analytic functions*, Mat. Sb. **69(111)** (1966), 4, 497–524.
- [Dyn] E.M. Dyn'kin, *Rational functions in Bergman spaces*, in: Complex analysis, operators, and related topics, Oper. Theory Adv. Appl., Vol. 113, Birkhäuser, Basel, 2000, 77–94.
- [GPV] D. Girela, J.A. Peláez, D. Vukotić, *Integrability of the derivative of a Blaschke product*, Proc. Edinb. Math. Soc. **50** (2007), 3, 673–687.
- [Glu] A. Gluchoff, *On inner functions with derivatives in Bergman spaces*, Illinois J. Math. **31** (1987), 518–528.
- [Gar] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
- [HLP] G. Hardy, J. Littlewood, G. Polya, *Inequalities*, Cambridge, 1934.
- [HKZ] H. Hedenmalm, B. Korenblum, K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, Vol. 199, Springer-Verlag, New York, 2000.
- [Nik] N. Nikolski, *Operators, Function, and Systems: an Easy Reading*, Vol.1, Amer. Math. Soc. Monographs and Surveys, 2002.
- [Pel1] V.V. Peller, *Hankel operators of class S_p and their applications (rational approximation, Gaussian processes, the problem of majorization of operators)*, Mat. Sb. **113(155)** (1980), 4, 538–581; English transl. in: Math. USSR-Sb. **41** (1982), 443–479.
- [Pel2] V.V. Peller, *Description of Hankel operators of the class S_p for $p > 0$, investigation of the rate of rational approximation and other applications*, Mat. Sb. **122(164)** (1983), 4, 481–510; English transl. in: Math. USSR- Sb. **50** (1985), 465–494.
- [Pee] J. Peetre, *New thoughts on Besov spaces*, Duke Univ. Math. Ser., No. 1, Math. Dept., Duke Univ., Durham, NC, 1976
- [Sem] S. Semmes, *Trace ideal criteria for Hankel operators, and applications to Besov spaces*, Integr. Equat. Oper. Theory **7** (1984), 2, 241–281.
- [Tri] H. Triebel, *Spaces of Besov-Hardy-Sobolev Type*, Teubner Verlag, Leipzig, 1978.
- [Ver] J.E. Verbitsky, *Inner functions, Besov spaces, and multipliers*, Soviet Math Doklady **29** (1984), AMS Translation.

- [Vin] S.A. Vinogradov, *Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces*, Zap. Nauchn. Sem. POMI **222** (1995), 45–77; English transl. in: J. Math. Sci. (New York) **87** (1997), 5, 3806–3827.
- [Zhu] K. Zhu, *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 139. Marcel Dekker, Inc., New York, 1990.

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