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Abstract

Let $(\xi_i)_{i\in\mathbb{Z}}$ be a stationary Harris recurrent geometrically ergodic Markov chain on a countably generated state space $(E,\mathcal{B})$. Let f be a bounded and measurable function from E into $\mathbb{R}$ satisfying the condition $\mathbb{E}(f(\xi_0))=0$. In this paper, we obtain the almost sure strong approximation of the partial sums $S_n(f)=\sum_{i=1}^n f(\xi_i)$ by the partial sums of a sequence of independent and identically distributed Gaussian random variables with the optimal rate $O(\log n)$.

1 Introduction and main result

This paper focuses on a Komlós-Major-Tusnády type strong approximation for additive functionals of Markov chains. We first recall the famous Komlós-Major-Tusnády theorem (1975 and 1976): let $(X_i)_{i>0}$ be a sequence of independent and identically distributed (iid) centered real-valued random variables with a finite moment generating function in a neighborhood of 0. Set $\sigma^2=\mathrm{Var}\,X_1$ and $S_n=X_1+X_2+\cdots+X_n$. Then one can construct a standard Brownian motion $(B_t)_{t\geq 0}$ in such a way that

$$\mathbb{P}\Big(\sup_{k\leq n}|S_k-\sigma B_k|\geq x+c\log n\Big)\leq a\exp(-bx), \tag{1.1}$$

where a, b and c are positive constants depending only on the law of X_1. From this result, the almost sure approximation of the partial sum process by a Brownian motion holds with the rate $O(\log n)$. It comes from the Erdös-Rényi law that this result is unimprovable. This result has been later extended to the multivariate case by Einmahl (1989), who obtained the rate $O((\log n)^2)$ in the almost sure approximation of partial sums of random vectors with finite moment generating function in a neighborhood of 0 by Gaussian partial sums. Next Zaitsev (1998) removed the extra logarithmic factor and obtained (1.1) in the case of random vectors. We refer to Götze and Zaitsev (2009) for a detailed review of the results on this subject.

We now come to the framework of this paper. Let (ξ_n) be an irreducible and aperiodic Harris recurrent Markov chain on a countably generated measurable state space $(E,\mathcal{B})$. We will consider only chains which are positive recurrent and π will exclusively denote the (unique) invariant probability measure of (ξ_n). In that case the transition probability $P(x,.)$ of the Markov chain satisfies the following minorization condition: there exists some positive integer

m, some measurable function h with values in $[0,1]$ with $\pi(h) > 0$, and some probability measure ν on E, such that

$$P^m(x, A) \geq h(x)\nu(A)\,. \tag{1.2}$$

In order to avoid additional difficulties, we will assume throughout the paper that the above condition holds true with $m = 1$. Let then $Q(x, \cdot)$ be the sub-stochastic kernel defined by

$$Q = P - h \otimes \nu\,. \tag{1.3}$$

Under assumption (1.2), proceeding as in Nummelin (1984), we can define an extended chain $(\bar{\xi}_n, U_n)$ in $E \times [0,1]$ as follows. At time 0, U_0 is independent of $\bar{\xi}_0$ and has the uniform distribution over $[0,1]$; for any nonnegative integer n,

$$\mathbb{P}(\bar{\xi}_{n+1} \in A \mid \bar{\xi}_n = x, U_n = y) = \mathbf{1}_{y \leq h(x)}\nu(A) + \mathbf{1}_{y > h(x)}\frac{Q(x, A)}{1 - h(x)} := \bar{P}((x, y), A) \tag{1.4}$$

and U_{n+1} is independent of $(\bar{\xi}_{n+1}, \bar{\xi}_n, U_n)$ and has the uniform distribution over $[0,1]$. Then the kernel $\tilde{P}$ of the extended chain is equal to $\bar{P} \otimes \lambda$ (here λ denotes the Lebesgue measure on $[0,1]$). This extended chain is also an irreducible and aperiodic Harris recurrent chain, with unique invariant probability measure $\pi \otimes \lambda$. It can easily be seen that $(\bar{\xi}_n)$ is an homogenous Markov chain with transition probability $P(x, .)$. Define now the set C in $E \times [0,1]$ by

$$C = \{(x, y) \in E \times [0,1] \;\text{ such that } y \leq h(x)\}. \tag{1.5}$$

For any (x, y) in C, $\mathbb{P}(\bar{\xi}_{n+1} \in A \mid \bar{\xi}_n = x, U_n = y) = \nu(A)$. Since $\pi \otimes \lambda(C) = \pi(h) > 0$, the set C is an atom of the extended chain, and it can be proven that this atom is recurrent.

Everywhere in the paper, we shall use the following notations: $\mathbb{P}_\pi$ (respectively $\mathbb{P}_C$) will denote the probability measure on the underlying space such that $\bar{\xi}_0 \sim \pi$ (resp. $(\bar{\xi}_0, U_0) \in C$), and $\mathbb{E}_\pi(\cdot)$ will denote the $\mathbb{P}_\pi$-expectation (resp. $\mathbb{E}_C(\cdot)$ the $\mathbb{P}_C$-expectation).

Define now the stopping times $(T_k)_{k \geq 0}$ by

$$T_0 = \inf\{n \geq 1 : U_n \leq h(\bar{\xi}_n)\} \;\text{ and }\; T_k = \inf\{n > T_{k-1} : U_n \leq h(\bar{\xi}_n)\} \text{ for } k \geq 1\,, \tag{1.6}$$

and the return times $(\tau_k)_{k>0}$ by

$$\tau_k = T_k - T_{k-1}\,. \tag{1.7}$$

Then T_0 is almost surely finite and the return times τ_k are iid and integrable. Moreover, from the strong Markov property, it is well known that the random vectors $(\bar{\xi}_{T_k+1}, \ldots, \bar{\xi}_{T_{k+1}})$ $(k \geq 0)$ are identically distributed and independent. Their common law is the law of $(\bar{\xi}_1, \ldots, \bar{\xi}_{T_0})$ under the probability $\mathbb{P}_C$. Let then

$$S_n(f) = \sum_{k=1}^{n} f(\bar{\xi}_k)\,. \tag{1.8}$$

From the above property, for any measurable function f from E into $\mathbb{R}$, the random vectors $(\tau_k, S_{T_k}(f) - S_{T_{k-1}}(f))_{k>0}$ are independent and identically distributed. This fact was used in Csáki and Csörgö (1995) to get strong approximation results for the partial sums $S_n(f)$ under moment assumptions on the return times τ_k. Let us recall their result. Assume that the chain satisfies (1.2) with $m = 1$. If the random variables $S_{T_k}(|f|) - S_{T_{k-1}}(|f|)$ have a finite moment of order p for some p in $]2, 4]$ and if the return times τ_k satisfy $\mathbb{E}(\tau_k^{p/2}) < \infty$, then one can construct a standard Wiener process $(W_t)_{t \geq 0}$ such that

$$S_n(f) - n\pi(f) - \sigma(f)\, W_n = O(a_n) \text{ a.s., with } \sigma^2(f) = \lim_n \frac{1}{n} \operatorname{Var} S_n(f) \text{ and } a_n = n^{1/p} \log n\,. \tag{1.9}$$

Note that the above result holds true for any bounded function f only if the return times have a finite moment of order p. The proof of Csáki and Csörgö (1995) is based on the regeneration

properties of the chain, on the Skorohod embedding and on an application of the results of Komlós, Major and Tusnády (1975) to the partial sums of the iid random variables $S_{T_{k+1}}(f) - S_{T_k}(f)$, $k > 0$. Since the moments of the return times essentially play the same role as the moments of the random variables in the case of iid random variables, it seems clear that such a result is optimal, up to a possible power of $\log n$. However this result has not been extended to the case $p > 4$. By contrast the strong approximation of the renewal process associated to the chain holds with the optimal rate $O(n^{1/p})$ if $\mathbb{E}(\tau_1^p) < \infty$, for any $p > 2$. Furthermore, if the chain is geometrically ergodic, then the strong approximation of the renewal process holds with the rate $O(\log n)$ (see Corollaries 3.1 and 4.2 in Csörgö, Horváth, and Steinebach (1987) for these results).

We now recall some possible methods to get strong approximation results. Some of these methods are based on the ergodicity properties of the Markov chain. For positive measures μ and ν, let $\|\mu - \nu\|$ denote the total variation of $\mu - \nu$. Set

$$\beta_n = \int_E \|P^n(x, .) - \pi\| d\pi(x)\,. \tag{1.10}$$

The coefficients β_n are called absolute regularity (or β-mixing) coefficients of the chain. Then, as proved by Bolthausen (1980 and 1982), for any $p > 1$,

$$\mathbb{E}_C(T_0^p) < \infty \text{ if and only if } \sum_{n>0} n^{p-2}\beta_n < \infty\,. \tag{1.11}$$

The second part of (1.11) is also called a weak dependence condition. Under a mixing condition which is more restrictive than (1.11) in the context of Markov chains, Shao and Lu (1987) obtained (1.9) with the rate $a_n = O(n^{1/p}(\log n)^c)$ for some $c > 1$ for p in $]2, 4]$. Their proof was based on the so-called Skorohod embedding. Recently, using a direct method based on constructions via quantile transformations, as in Major (1976), Merlevède and Rio (2012) improved the results of Shao and Lu (1987). For p in $]2, 3[$, they obtained (1.9) under the ergodicity condition (1.11) with the better rate $a_n = n^{1/p}(\log n)^{(p-2)/(2p)}$. The results of Merlevède and Rio (2012) involve more general weak dependence coefficients than the coefficients β_n, so that their result applies also to non irreducible Markov chains and to some dynamical systems. In the context of dynamical systems, Gouëzel (2010) used spectral methods to construct coupling with independent random variables and applied then strong approximation results for partial sums of independent random vectors to get rates of the order of $n^{1/p}$ for p in $]2, 4[$ in (1.9). The techniques used in these papers are suitable for Markov chains or non trivial dynamical systems, including the Liverani-Saussol-Vaienti map. Nevertheless the applied tools limit the accuracy to the rate $O(n^{1/4})$.

Recently, for stationary processes that are functions of iid innovations, Berkes, Liu and Wu (2014) obtained (1.9) with the rate $O(n^{1/p})$ for any $p > 2$ provided that the inovations have finite moments of order p and the process has a fast enough arithmetically decay of some coupling coefficients. Moreover they give some application to nonlinear time series (see Example 2.2). However their condition (2.15) is too restrictive (even for functional autoregressive processes) and they do not give estimates of their coupling coefficients for more general Markov chains.

In this paper we are interested in general Harris recurrent Markov chains. Our aim is to obtain the optimal rate $O(\log n)$. Recall that, in the dependent case the rate $o(n^{1/p})$ has never been surpassed. In order to get better rates of approximation, we will assume thoughout the paper that the Markov chain is geometrically ergodic, which means that (see Theorem 2.1 in Nummelin and Tuominen (1982))

$$\beta_n = O(\rho^n) \text{ for some real } \rho \text{ with } 0 < \rho < 1, \tag{1.12}$$

where β_n is defined in (1.10). Note now that $\mathbb{P}(\tau_1 > n) = \mathbb{P}_C(T_0 > n) = \nu Q^n(1)$ and in addition $\mathbb{P}_\pi(T_0 > n) = \pi Q^{n+1}(1) + \nu Q^n(1)\pi(h)$ where Q is defined by (1.3). Therefore, condition (1.12)

together with Corollary 2.4 and Lemma 2.8 in Nummelin and Tuominen (1982) imply that both $\mathbb{P}(\tau_1 > n)$ and $\mathbb{P}_\pi(T_0 > n)$ decrease exponentially fast. Hence, if (1.12) holds there exists a positive real δ such that

$$\mathbb{E}\big(e^{t\tau_1}\big) < \infty \text{ and } \mathbb{E}_\pi\big(e^{tT_0}\big) < \infty \text{ for any } |t| \le \delta\,. \tag{1.13}$$

We will use this fact together with a strategy inherited from the papers of Bolthausen (1980 and 1982) to get the optimal rates of strong approximation in that case: we will apply a strong approximation result of Zaitsev (1998) to the multidimensional partial sum process $(T_n - T_0, S_{T_n}(f) - S_{T_0}(f))$ rather than the initial theorems of Komlós, Major and Tusnády (1975 and 1976). This method enables us to get the optimal rate of convergence. Let us now give our main result.

Theorem 1.1. *Let (ξ_n) be a stationary, irreducible and aperiodic Harris positive recurrent Markov chain on E, with invariant probability measure π. Assume that the chain satisfies* (1.2) *with $m = 1$ and the geometric ergodicity condition* (1.12). *Let g be any bounded measurable function from $E \times [0,1]$ to $\mathbb{R}$ such that $\pi \otimes \lambda(g) = 0$ and let $\bar{S}_n(g) = \sum_{k=1}^n g(\bar{\xi}_k, U_k)$. Let $\tilde{P} = \bar{P} \otimes \lambda$. If*

$$\sigma^2(g) = \pi \otimes \lambda(g^2) + 2\sum_{n>0} \pi \otimes \lambda(g\tilde{P}^n g) > 0\,,$$

then there exists a standard Wiener process $(W_t)_{t\ge 0}$ and positive constants a, b and c depending on g and on the transition probability $P(x,\cdot)$ such that, for any positive real x and any integer $n \ge 2$,

$$\mathbb{P}_\pi\Big(\sup_{k\le n} \big|\bar{S}_k(g) - \sigma(g)W_k\big| \ge c\log n + x\Big) \le a\exp(-bx)\,. \tag{1.14}$$

We now give in a separate corollary the application of this result to additive functionals of the initial chain. The proof, being immediate, will be omitted.

Corollary 1.1. *Let (ξ_n) be a stationary, irreducible and aperiodic Harris positive recurrent Markov chain on E, with invariant probability measure π. Assume that the chain satisfies* (1.2) *with $m = 1$ and the geometric ergodicity condition* (1.12). *Let f be any bounded measurable function from E to $\mathbb{R}$ such that $\pi(f) = 0$ and let $S_n(f) = \sum_{k=1}^n f(\xi_k)$. If*

$$\sigma^2(f) = \pi(f^2) + 2\sum_{n>0} \pi(fP^n f) > 0\,,$$

then there exists a standard Wiener process $(W_t)_{t\ge 0}$ and positive constants a, b and c depending on f and on the transition probability $P(x,\cdot)$ such that, for any positive real x and any integer $n \ge 2$,

$$\mathbb{P}_\pi\Big(\sup_{k\le n} \big|S_k(f) - \sigma(f)W_k\big| \ge c\log n + x\Big) \le a\exp(-bx)\,. \tag{1.15}$$

Remark 1.1. *Corollary 1.1 may be generalized to the nonstationary case. Let μ be any law on E such that*

$$\int_E \|P^n(x,.) - \pi\| d\mu(x) = O(r^n) \quad \textit{for some } r < 1.$$

Corollary 2.4 and Lemma 2.8 in Nummelin and Tuominen (1982) ensure that $\mathbb{P}_\mu(T_0 > n)$ decreases exponentially fast. Consequently the proof of Theorem 1.1 extends to the Markov chain (ξ_n) with transition probability P and initial law μ without modification.

2 Proof of Theorem 1.1

Before proving our main result, we give an idea of the proof. The constants v, $\tilde{v}$, λ and γ appearing below will be specified in Subsection 2.3. For any $i \geq 1$, let

$$X_i = \sum_{\ell=T_{i-1}+1}^{T_i} g(\bar{\xi}_\ell, U_\ell)\,.$$

The random variables $(X_i, \tau_i)_{i>0}$ are independent and identically distributed. Let then α be the unique real such that $\mathrm{Cov}(X_k - \alpha\tau_k, \tau_k) = 0$. Applying the multidimensional extension of the results of Komlós Major and Tusnády (1976), which is due to Zaitsev (1998), we obtain that there exist two independent standard Brownian motions $(B_t)_t$ and $(\tilde{B}_t)_t$ such that

$$\bar{S}_{T_n}(g) - \alpha(T_n - n\mathbb{E}(\tau_1)) - vB_n = O(\log n) \text{ a.s. and } \; T_n - n\mathbb{E}(\tau_1) - \tilde{v}\tilde{B}_n = O(\log n) \text{ a.s.}$$

Next, using the Komlós-Major-Tusnády strong approximation theorem, one can construct a Poisson process N with parameter λ from $\tilde{B}$ in such a way that

$$n\mathbb{E}(\tau_1) + \tilde{v}\tilde{B}_n - \gamma N(n) = O(\log n) \text{ a.s.}$$

For this Poisson process,

$$\bar{S}_{\gamma N(n)}(g) - \alpha\gamma N(n) + \alpha n\mathbb{E}(\tau_1) - vB_n = O(\log n) \text{ a.s.}$$

The processes $(B_t)_t$ and $(N_t)_t$ appearing here are independent. From the above result one can deduce that

$$\bar{S}_n(g) = vB_{N^{-1}(n/\gamma)} + \alpha n - \alpha\mathbb{E}(\tau_1)N^{-1}(n/\gamma) + O(\log n) \text{ a.s.} \tag{2.1}$$

If $v = 0$, which corresponds to the case of renewal processes, then

$$\bar{S}_n(g) = \alpha n - \alpha\mathbb{E}(\tau_1)N^{-1}(n/\gamma) + O(\log n) \text{ a.s.}$$

Up to some multiplicative constant, the process on right hand is a partial sum process associated to iid random variables with exponential law. Hence, using the Komlós-Major-Tusnády strong approximation theorem again, one can construct a Brownian motion $\widetilde{W}$ such that

$$\alpha n - \alpha\mathbb{E}(\tau_1)N^{-1}(n/\gamma) = \widetilde{W}_n + O(\log n) \text{ a.s.}\,, \tag{2.2}$$

which leads to the expected result. Notice that the Brownian motion $\widetilde{W}$ depends only on the Poisson process N and on some auxiliary atomless random variables independent of the σ-field generated by the processes B and N.

If $v \neq 0$ and $\alpha = 0$, (2.1) ensures that

$$\bar{S}_n(g) = vB_{N^{-1}(n/\gamma)} + O(\log n) \text{ a.s.}$$

As noted by Csörgö, Deheuvels and Horváth (1987), since the renewal process of the Poisson process is the partial sum process associated to independent random variables with exponential law, the above compound process is a partial sum process associated to iid random variables with a finite Laplace transform, and consequently, one can construct a Brownian motion W such that

$$B_{N^{-1}(n/\gamma)} - W_n = O(\log n) \text{ a.s.}\,,$$

which leads to the expected result. However the Brownian motion W depends on N. It follows that, in the case $\alpha \neq 0$ and $v \neq 0$, the so constructed processes W and $\widetilde{W}$ are not independent.

Then the construction of Csörgö, Deheuvels and Horváth (1987) cannot be used to prove our theorem.

In order to perform the exact rate in the case $\alpha \neq 0$, it will be necessary to construct a Brownian motion W^* independent of N in such a way that

$$B_n - W^*_{\gamma N(n)} = O(\log n) \text{ a.s.} \tag{2.3}$$

Since W^* is independent of N, it will also be independent of $\widetilde{W}$. Then, using (2.1) and (2.2), we will get that

$$\bar{S}_n(g) = W^*_n + \widetilde{W}_n + O(\log n) \text{ a.s.}$$

which will imply our strong approximation theorem. The proof of (2.3) will be done in Subsection 2.2. Then, starting from this fundamental result, we will prove the main theorem.

2.1 Some technical lemmas

Lemma below follows from the classical Cramér-Chernoff calculation (see also, for instance, Lemma 1 in Bretagnolle and Massart (1989)).

Lemma 2.1. *Let Z be a real-valued random variable with Poisson distribution of parameter m. Then, for any positive x and any sign ε, we have*

$$\mathbb{P}\big(\varepsilon(Z-m) > x\big) \leq \exp\big(-mh(\varepsilon x/m)\big)\,.$$

where

$$h(t) = (1+t)\log(1+t) - t \quad \text{for } t > -1 \text{ and } \quad h(t) = +\infty \quad \text{for } t \leq -1. \tag{2.4}$$

Next lemma follows once again from the classical Cramér-Chernoff calculation together with the Doob maximal inequality.

Lemma 2.2. *Let $(N(t) : t \geq 0)$ be a real-valued homogeneous Poisson process of parameter m. Then, for any positive reals x and s, we have*

$$\mathbb{P}\big(\sup_{t\leq s} |N(t) - tm| > x\big) \leq \exp\big(-msh(x/(ms))\big) + \exp\big(-msh(-x/(ms))\big)\,.$$

where $h(\cdot)$ is defined by (2.4).

Lemma 2.3 below is due to Tusnády in his Phd-thesis (see Bretagnolle and Massart (1989) for a complete proof of it).

Lemma 2.3. *Let ξ be a random variable with law $\mathcal{N}(0,1)$, Φ its distribution function and Φ_m the distribution function of a Binomial law $\mathcal{B}(m,1/2)$. Let $B_m = 2\Phi_m^{-1}(\Phi(\xi)) - m$ where Φ_m^{-1} is the generalized inverse of Φ_m. Then the following inequality holds:*

$$|B_m| \leq 2 + |\xi|\sqrt{m}\,.$$

2.2 A fundamental lemma

The main new tool for proving Theorem 1.1 is the lemma below.

Lemma 2.4. *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on the line and $\{N(t) : t \geq 0\}$ be a Poisson process with parameter $\lambda > 0$, independent of $(B_t)_{t\geq 0}$. Then one can construct a standard Brownian process $(W_t)_{t\geq 0}$ independent of the Poisson process $N(\cdot)$ and such that, for any positive integer $n \geq 2$ and any positive real x,*

$$\mathbb{P}\Big(\sup_{k\leq n} \big|B_k - \frac{1}{\sqrt{\lambda}} W_{N(k)}\big| \geq C\log n + x\Big) \leq A\exp(-Bx)\,,$$

where A, B and C are positive constants depending only on λ. Furthermore $(W_t)_{t\geq 0}$ may be constructed from the processes $(B_t)_{t\geq 0}$, $N(\cdot)$ and some auxiliary atomless random variable δ independent of the σ-field generated by the processes $(B_t)_{t\geq 0}$ and $N(\cdot)$.

Proof. For $j \in \mathbb{Z}$ and $k \in \mathbb{N}$, let

$$\tilde{e}_{j,k} = 2^{-j/2}\big(\mathbf{1}_{]k2^j,(k+\frac{1}{2})2^j]} - \mathbf{1}_{](k+\frac{1}{2})2^j,(k+1)2^j]}\big)\,,$$

and

$$Y_{j,k} = \int_0^\infty \tilde{e}_{j,k}(t)dB(t) = 2^{-j/2}\big(2B_{(k+\frac{1}{2})2^j} - B_{k2^j} - B_{(k+1)2^j}\big)\,.$$

Note that $(\tilde{e}_{j,k})_{j\in\mathbb{Z},k\geq 0}$ is a total orthonormal system of $\ell^2(\mathbb{R})$. Hence for any $t \in \mathbb{R}^+$, B_t can be written as

$$B_t = \sum_{j\in\mathbb{Z}}\sum_{k\geq 0}\Big(\int_0^t \tilde{e}_{j,k}(t)dt\Big)Y_{j,k}\,. \tag{2.5}$$

For any $j \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $N(k2^j) < N((k+\frac{1}{2})2^j) < N((k+1)2^j)$, let

$$\tilde{f}_{j,k} = c_{j,k}^{-1/2}\big(b_{j,k}\mathbf{1}_{]N(k2^j),N((k+\frac{1}{2})2^j)]} - a_{j,k}\mathbf{1}_{]N((k+\frac{1}{2})2^j),N((k+1)2^j)]}\big)\,,$$

where

$$a_{j,k} = N((k+\frac{1}{2})2^j) - N(k2^j)\,,\; b_{j,k} = N((k+1)2^j) - N((k+\frac{1}{2})2^j)\,,$$

and

$$c_{j,k} = a_{j,k}b_{j,k}(a_{j,k}+b_{j,k})\,.$$

For $j \in \mathbb{Z}$, let $E_j = \{k \in \mathbb{N} : N(k2^j) < N((k+\frac{1}{2})2^j) < N((k+1)2^j)\}$, and notice that $(\tilde{f}_{j,k})_{j\in\mathbb{Z},k\in E_j}$ is an orthonormal system whose closure contains the vectors $\mathbf{1}_{]0,N(t)]}$ for $t \in \mathbb{R}^+$ and then the vectors $\mathbf{1}_{]0,\ell]}$ for $\ell \in \mathbb{N}^*$. With the convention $\tilde{f}_{j,k} = 0$ if $k \notin E_j$, we then set

$$W_\ell = \sum_{j\in\mathbb{Z}}\sum_{k\geq 0}\Big(\int_0^\ell \tilde{f}_{j,k}(t)dt\Big)Y_{j,k} \text{ for any } \ell \in \mathbb{N}^* \text{ and } W_0 = 0\,. \tag{2.6}$$

Since conditionally to $N(\cdot)$, $(\tilde{f}_{j,k})_{j\in\mathbb{Z},k\in E_j}$ is an orthonormal system and $(Y_{j,k})$ is a sequence of iid standard Gaussian random variables, independent of $N(\cdot)$, one can easily check that, conditionally to $N(\cdot)$, $(W_\ell)_{\ell\geq 0}$ is a Gaussian sequence such that $\mathrm{Cov}(W_\ell, W_m) = \ell \wedge m$. Therefore this Gaussian sequence is independent of the Poisson process $N(\cdot)$. By the Skorohod embedding theorem, there exists a standard Wiener process $(W_t)_t$ which coincides with the Gaussian sequence (W_ℓ) at integer values. Furthermore this Wiener process can be constructed from the Gaussian sequence and an auxiliary atomless random variable δ independent of the σ-field generated by the processes $(B_t)_{t\geq 0}$ and $N(\cdot)$.

Let c_1 and c_2 be two positive reals such that

$$c_1 \geq \tilde{c}_1 := \max\Big(8 + \frac{1765(2+\sqrt{2})^2(1+\sqrt{2})^2}{\lambda}, \frac{1656}{\lambda^2(\log 2)^2}\Big)\,, \tag{2.7}$$

and

$$c_2 \geq \tilde{c}_2 := \frac{1765(2+\sqrt{2})^2}{(\sqrt{2}-1)^2}\,. \tag{2.8}$$

Let n_0 be the smallest integer such that $n_0 \geq c_1$ and

$$n_0 - c_1\log(n_0) - c_2 \geq 0\,. \tag{2.9}$$

The lemma will be proven if we can show that there exist positive constants a and b depending only on λ, such that for any $n \geq \max(2^5, n_0)$,

$$\mathbb{P}\big(\sup_{k\leq n}|B_k - \lambda^{-1/2}W_{N(k)}| \geq 3c_1\log n + 3c_2x\big) \leq ae^{-bx}\,. \tag{2.10}$$

Indeed, for any integer n in $[2, \max(2^5, n_0)]$, it can be easily shown that the conclusion of the lemma holds. From now on, n is a positive integer such that $n \geq \max(2^5, n_0)$. To prove (2.10), we first define j_0 as the smallest integer such that

$$2^{j_0} \geq c_1 \log n + c_2 x \,, \tag{2.11}$$

where c_1 and c_2 are positive reals satisfying (2.7) and (2.8) respectively. Now, let K be the integer such that $2^{K-1} + 1 < n \leq 2^K$. Notice that

$$\begin{aligned}\mathbb{P}\big(\sup_{k\leq n} |B_k - \lambda^{-1/2} W_{N(k)}| \geq 3c_1 \log n + 3c_2 x\big) &\leq \mathbb{P}\Big(\sup_{1\leq \ell \leq 2^{K-j_0}} \big|B_{\ell 2^{j_0}} - \lambda^{-1/2} W_{N(\ell 2^{j_0})}\big| \geq c_1 \log n + c_2 x\Big) \\ &\quad + \mathbb{P}\big(\sup_{0\leq \ell \leq 2^{K-j_0}-1} \sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |B_k - B_{\ell 2^{j_0}}| \geq c_1 \log n + c_2 x\big) \\ &\quad + \mathbb{P}\big(\sup_{0\leq \ell \leq 2^{K-j_0}-1} \sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |W_{N(k)} - W_{N(\ell 2^{j_0})}| \geq \lambda^{1/2}(c_1 \log n + c_2 x)\big)\,. \end{aligned} \tag{2.12}$$

But, by Lévy's inequality,

$$\begin{aligned}&\mathbb{P}\big(\sup_{0\leq \ell \leq 2^{K-j_0}-1} \sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |B_k - B_{\ell 2^{j_0}}| \geq c_1 \log n + c_2 x\big) \\ &\leq \sum_{\ell=0}^{2^{K-j_0}-1} \mathbb{P}\big(\sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |B_k - B_{\ell 2^{j_0}}| \geq c_1 \log n + c_2 x\big) \\ &\leq 2^{K-j_0+1} \mathbb{P}\big(|B_{2^{j_0}}| \geq c_1 \log n + c_2 x\big) \leq \frac{2^{K-j_0+2}}{\sqrt{2\pi}} \exp(-2^{-(j_0+1)}(c_1 \log n + c_2 x)^2)\,. \end{aligned}$$

Using the definition (2.11) of j_0, it follows that

$$\begin{aligned}&\mathbb{P}\big(\sup_{0\leq \ell \leq 2^{K-j_0}-1} \sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |B_k - B_{\ell 2^{j_0}}| \geq c_1 \log n + c_2 x\big) \\ &\qquad\qquad \leq \frac{2^3}{\sqrt{2}\pi} \times \frac{n^{1-c_1/4}}{(c_1 \log n + c_2 x)^{3/2}} \exp(-c_2 x/4)\,. \end{aligned} \tag{2.13}$$

Next,

$$\begin{aligned}&\mathbb{P}\big(\sup_{0\leq \ell \leq 2^{K-j_0}-1} \sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |W_{N(k)} - W_{N(\ell 2^{j_0})}| \geq \lambda^{1/2}(c_1 \log n + c_2 x)\big) \\ &\leq \sum_{\ell=0}^{2^{K-j_0}-1} \mathbb{P}\big(\sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |W_{N(k)} - W_{N(\ell 2^{j_0})}| \geq \lambda^{1/2}(c_1 \log n + c_2 x)\big) \\ &\leq 2^{K-j_0} \mathbb{P}\big(\sup_{0<t\leq \lambda 2^{j_0+1}} |W_t| \geq \lambda^{1/2}(c_1 \log n + c_2 x)\big) \\ &\qquad + \sum_{\ell=0}^{2^{K-j_0}-1} \mathbb{P}\big(\sup_{\ell 2^{j_0} < k \leq (\ell+1)2^{j_0}} |N(k) - N(\ell 2^{j_0})| \geq \lambda 2^{j_0+1}\big)\,. \end{aligned}$$

Using once again Lévy's inequality and the definition (2.11) of j_0, we get

$$\begin{aligned}&2^{K-j_0} \mathbb{P}\big(\sup_{0<t\leq \lambda 2^{j_0+1}} |W_t| \geq \lambda^{1/2}(c_1 \log n + c_2 x)\big) \\ &\qquad\qquad \leq 2^{K-j_0+1} \frac{2^{1+j_0/2}}{\sqrt{\pi}(c_1 \log n + c_2 x)} \exp\Big(-\frac{(c_1 \log n + c_2 x)^2}{2^{j_0+2}}\Big) \\ &\qquad\qquad \leq \frac{2^3}{\sqrt{\pi}} \times \frac{n^{1-c_1/8}}{(c_1 \log n + c_2 x)^{3/2}} \exp(-c_2 x/8)\,. \end{aligned}$$

On another hand, by Lemma 2.2,

$$\mathbb{P}\big(\sup_{1\le k\le 2^{j_0}} N(k)\ge \lambda 2^{j_0+1}\big)\le \mathbb{P}\big(\sup_{1\le k\le 2^{j_0}} |N(k)-k\lambda|\ge \lambda 2^{j_0}\big)\le \exp\big(-\lambda 2^{j_0}h(1)\big)\,.$$

Hence, by using (2.11),

$$2^{K-j_0}\mathbb{P}\big(\sup_{1\le k\le 2^{j_0}} N(k)\ge \lambda 2^{j_0+1}\big)\le 2\frac{n^{1-\lambda c_1/3}}{c_1\log n+c_2x}\exp(-\lambda c_2x/3)\,.$$

So, overall,

$$\begin{aligned}\mathbb{P}\big(\sup_{0\le \ell\le 2^{K-j_0}-1}\ \sup_{\ell 2^{j_0}<k\le(\ell+1)2^{j_0}} |W_{N(k)}-W_{N(\ell 2^{j_0})}|\ge \lambda^{1/2}(c_1\log n+c_2x)\big)\\ \le \frac{2^3}{\sqrt{\pi}}\times\frac{n^{1-c_1/8}}{c_1\log n+c_2x}\exp(-c_2x/8)+2\frac{n^{1-\lambda c_1/3}}{c_1\log n+c_2x}\exp(-\lambda c_2x/3)\,. \quad (2.14)\end{aligned}$$

Therefore, starting from (2.12) and considering the upper bounds (2.13) and (2.14), we derive that to prove the lemma, it suffices to show that there exist positive constants A_1 and B_1 depending only on λ, such that for any $n\ge\max(2^5,n_0)$,

$$\mathbb{P}\Big(\sup_{1\le\ell\le 2^{K-j_0}}\big|B_{\ell 2^{j_0}}-\lambda^{-1/2}W_{N(\ell 2^{j_0})}\big|\ge c_1\log n+c_2x\Big)\le A_1\exp(-B_1x)\,. \quad (2.15)$$

In the rest of the proof, we shall prove the inequality above.

Taking into account (2.5) and (2.6), we first write that, for any $\ell\in\mathbb{N}^*$,

$$B_{\ell 2^{j_0}}-\lambda^{-1/2}W_{N(\ell 2^{j_0})}=\sum_{j\ge j_0}\sum_{k\ge 0}\Big(\int_0^{\ell 2^{j_0}}\tilde e_{j,k}(t)dt-\lambda^{-1/2}\int_0^{N(\ell 2^{j_0})}\tilde f_{j,k}(t)dt\Big)Y_{j,k}\,.$$

Notice that if $\ell 2^{j_0}\notin]k2^j,(k+1)2^j[$ then

$$\int_0^{\ell 2^{j_0}}\tilde e_{j,k}(t)dt=\int_0^{N(\ell 2^{j_0})}\tilde f_{j,k}(t)dt=0\,.$$

Therefore setting

$$\ell_j=[\ell 2^{j_0-j}]\,,$$

we get

$$B_{\ell 2^{j_0}}-\lambda^{-1/2}W_{N(\ell 2^{j_0})}=\sum_{j\ge j_0}\Big(\int_0^{\ell 2^{j_0}}\tilde e_{j,\ell_j}(t)dt-\lambda^{-1/2}\int_0^{N(\ell 2^{j_0})}\tilde f_{j,\ell_j}(t)dt\Big)Y_{j,k}\,.$$

Setting

$$t_j=\frac{\ell 2^{j_0}-\ell_j 2^j}{2^j}\,,$$

this leads to

$$B_{\ell 2^{j_0}}-\lambda^{-1/2}W_{N(\ell 2^{j_0})}=\sum_{j\ge j_0}\widetilde U_{j,k}Y_{j,k}+\sum_{j\ge j_0}\widetilde V_{j,k}Y_{j,k}\,,$$

where $\widetilde U_{j,k}=U_{j,\ell_j}\mathbf{1}_{t_j\in]0,1/2]}$ with

$$U_{j,\ell_j}=2^{j/2}t_j-\lambda^{-1/2}\frac{b_{j,\ell_j}}{\sqrt{c_{j,\ell_j}}}\big(N(\ell 2^{j_0})-N(\ell_j 2^j)\big) \quad (2.16)$$

and $\widetilde{V}_{j,k} = V_{j,\ell_j} \mathbf{1}_{t_j \in]1/2,1[}$ with

$$V_{j,\ell_j} = 2^{j/2}(1-t_j) - \lambda^{-1/2} \frac{a_{j,\ell_j} b_{j,\ell_j}}{\sqrt{c_{j,\ell_j}}} + \lambda^{-1/2} \frac{a_{j,\ell_j}}{\sqrt{c_{j,\ell_j}}} \big(N(\ell 2^{j_0}) - N((\ell_j + \frac{1}{2})2^j) \big) \,. \tag{2.17}$$

It follows that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{1 \le \ell \le 2^{K-j_0}} \big| B_{\ell 2^{j_0}} - \lambda^{-1/2} W_{N(\ell 2^{j_0})} \big| \ge c_1 \log n + c_2 x \Big) \\
&\qquad \le \sum_{k=j_0}^{K} \mathbb{P}\Big(\sup_{\ell \,:\, 2^k \le \ell 2^{j_0} \le 2^{k+1}-1} \big| B_{\ell 2^{j_0}} - \lambda^{-1/2} W_{N(\ell 2^{j_0})} \big| \ge c_1 \log n + c_2 x \Big) \\
&\qquad \le \sum_{k=j_0}^{K} \sum_{\ell = 2^{k-j_0}}^{2^{k+1-j_0}-1} \mathbb{P}\Big(\Big| \sum_{j \ge j_0} (\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}) Y_{j,\ell_j} \Big| \ge c_1 \log n + c_2 x \Big) \,.
\end{aligned}$$

Recall now that $(Y_{j,k})_{j>0,k\ge 1}$ is a sequence of standard centered Gaussian random variables that are mutually independent. In addition this sequence is independent of $(N(t), t \ge 0)$. Therefore,

$$\begin{aligned}
&\mathbb{P}\Big(\Big| \sum_{j \ge j_0} (\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}) Y_{j,\ell_j} \Big| \ge c_1 \log n + c_2 x \Big) \\
&\le \mathbb{P}\Big(\Big| \sum_{j \ge j_0} (\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}) Y_{j,\ell_j} \Big| \ge c_1 \log n + c_2 x, \sum_{j \ge j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j} \big)^2 < c_1 \log n + c_2 x \Big) \\
&\qquad + \mathbb{P}\Big(\sum_{j \ge j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j} \big)^2 \ge c_1 \log n + c_2 x \Big) \\
&\le \frac{2(2\pi)^{-1/2}}{\sqrt{c_1 \log n + c_2 x}} e^{-(c_1 \log n + c_2 x)/2} + \mathbb{P}\Big(\sum_{j \ge j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j} \big)^2 \ge c_1 \log n + c_2 x \Big) \,.
\end{aligned}$$

So, overall, by using (2.11) and the fact that $2^K \le 2n$,

$$\begin{aligned}
\mathbb{P}\Big(\sup_{1 \le \ell \le 2^{K-j_0}} \big| B_{\ell 2^{j_0}} - \lambda^{-1/2} W_{N(\ell 2^{j_0})} \big| \ge c_1 \log n + c_2 x \Big) &\le \frac{8(2\pi)^{-1/2} n^{1-c_1/2}}{(c_1 \log n + c_2 x)^{3/2}} e^{-c_2 x/2} \\
&\quad + \sum_{k=j_0}^{K} \sum_{\ell = 2^{k-j_0}}^{2^{k+1-j_0}-1} \mathbb{P}\Big(\sum_{j \ge j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j} \big)^2 \ge c_1 \log n + c_2 x \Big) \,.
\end{aligned} \tag{2.18}$$

Let now

$$\Theta_{\ell,j_0} = \Theta_{a,\ell,j_0} \cap \Theta_{b,\ell,j_0} \,, \tag{2.19}$$

where

$$\Theta_{a,\ell,j_0} = \{ a_{j,\ell_j} \le \frac{3}{2}(\lambda 2^{j-1}) \text{ for all } j \ge j_0 \} \cap \{ a_{j,\ell_j} \ge \frac{1}{2}(\lambda 2^{j-1}) \text{ for all } j \ge j_0 \} \,,$$

and

$$\Theta_{b,\ell,j_0} = \{ b_{j,\ell_j} \le \frac{3}{2}(\lambda 2^{j-1}) \text{ for all } j \ge j_0 \} \cap \{ b_{j,\ell_j} \ge \frac{1}{2}(\lambda 2^{j-1}) \text{ for all } j \ge j_0 \} \,.$$

We have

$$\mathbb{P}(\Theta^c_{a,\ell,j_0}) \le \sum_{j \ge j_0} \mathbb{P}\big(a_{j,\ell_j} > \frac{3}{2}(\lambda 2^{j-1}) \big) + \sum_{j \ge j_0} \mathbb{P}\big(a_{j,\ell_j} < \frac{1}{2}(\lambda 2^{j-1}) \big) \,.$$

Hence, by Lemma 2.1,

$$\mathbb{P}(\Theta^c_{a,\ell,j_0}) \le \sum_{j \ge j_0} \Big(\exp\big(-\lambda 2^{j-1} h(2^{-1}) \big) + \exp\big(-\lambda 2^{j-1} h(-2^{-1}) \big) \Big) \,.$$

Therefore,

$$\mathbb{P}(\Theta^c_{a,\ell,j_0}) \le 2\sum_{j\ge j_0} \exp\big(-\lambda 2^j/20\big) \le \frac{80}{\lambda 2^{j_0}} \exp\big(-\lambda 2^{j_0}/40\big)\,.$$

A similar bound is valid for $\mathbb{P}(\Theta^c_{b,\ell,j_0})$. Hence by (2.11),

$$\sum_{k=j_0}^{K} \sum_{\ell=2^{k-j_0}}^{2^{k+1-j_0}-1} \mathbb{P}(\Theta^c_{\ell,j_0}) \le \frac{5\times 2^7}{\lambda(c_1\log n + c_2 x)^2} n^{1-\lambda c_1/40} \exp(-\lambda c_2 x/40)\,. \tag{2.20}$$

Starting from (2.18) and taking into account the upper bound (2.20), we infer that to prove (2.15) and then the lemma, it suffices to show that there exist two positive constants A_2 and B_2 such that, for any $n \ge \max(2^5, n_0)$,

$$\sum_{k=j_0}^{K} \sum_{\ell=2^{k-j_0}}^{2^{k+1-j_0}-1} \mathbb{P}\Big(\sum_{j\ge j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}\big)^2 \ge c_1\log n + c_2 x, \Theta_{\ell,j_0}\Big) \le A_2\exp(-B_2 x)\,, \tag{2.21}$$

for any $c_1 \ge \tilde{c}_1$ and any $c_2 \ge \tilde{c}_2$ where $\tilde{c}_1$ and $\tilde{c}_2$ are defined in (2.7) and (2.8) respectively.

To prove the inequality (2.21), we first notice that, by definition of $\widetilde{U}_{j,\ell_j}$ and $\widetilde{V}_{j,\ell_j}$,

$$\sum_{j\ge j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}\big)^2 = \sum_{j\ge j_0} \widetilde{U}^2_{j,\ell_j} + \sum_{j\ge j_0} \widetilde{V}^2_{j,\ell_j}$$

and that if $k \in \{j_0,\dots,K\}$ with $\ell \in [2^{k-j_0}, 2^{k+1-j_0}[\cap\mathbb{N}$, then $\ell_j = 0$ for any $j \ge k+1$, and $t_j \le 1/2$ for any $j \ge k+2$. Therefore,

$$\sum_{j\ge j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}\big)^2 = \sum_{j=j_0}^{2(k+4)} \big(\widetilde{U}^2_{j,\ell_j} + \widetilde{V}^2_{j,\ell_j}\big) + \sum_{j\ge 2(k+4)} U^2_{j,0}\,. \tag{2.22}$$

In the rest of the proof, if it is not specified, k and ℓ are two integers such that $k \in \{j_0,\dots,K\}$ and $\ell \in [2^{k-j_0}, 2^{k+1-j_0}[$. On the set Θ_{ℓ,j_0},

$$|U_{j,0}| \le \frac{\ell 2^{j_0}}{2^{j/2}} + 3\sqrt{2}\,2^{-j/2}\frac{N(\ell 2^{j_0})}{\lambda} \le \frac{2^{k+1}}{2^{j/2}} + 3\sqrt{2}\,2^{-j/2}\frac{N(2^{k+1})}{\lambda}\,,$$

leading to

$$\sum_{j\ge 2(k+4)} U^2_{j,0} \le \frac{1}{2^4} + 9\frac{N^2(2^{k+1})}{\lambda^2 2^{2k+5}} \le \frac{1}{2^4} + \frac{N^2(2^{k+1})}{\lambda^2 2^{2k+1}}\,.$$

Hence, for any $y > 2^{-4}$,

$$\mathbb{P}\Big(\sum_{j\ge 2(k+4)} U^2_{j,0} \ge y, \Theta_{\ell,j_0}\Big) \le \mathbb{P}\Big(\frac{N(2^{k+1})}{\lambda 2^{k+1}} \ge \sqrt{(y-2^{-4})/2}\Big)\,.$$

Next, if $\sqrt{(y-2^{-4})/2} \ge 3/2$, by Lemma 2.1,

$$\begin{aligned}\mathbb{P}\Big(\frac{N(2^{k+1})}{\lambda 2^{k+1}} \ge \sqrt{(y-2^{-4})/2}\Big) &\le \mathbb{P}\Big(N(2^{k+1}) - \lambda 2^{k+1} \ge \lambda 2^k\Big)\\ &\le \exp(-\lambda 2^{k+1} h(1/2)) \le \exp(-\lambda 2^k/20)\,,\end{aligned}$$

and taking into account (2.11),

$$\sum_{k=j_0}^{K} \sum_{\ell=2^{k-j_0}}^{2^{k+1-j_0}-1} \exp(-\lambda 2^k/20) \le \frac{5\times 2^4}{\lambda(c_1\log n + c_2 x)^2} n^{1-\lambda c_1/40} \exp(-\lambda c_2 x/40)\,.$$

So, overall, starting from (2.22) and taking into account the considerations above, we get, for any $n \geq 10$,

$$\sum_{k=j_0}^{K} \sum_{\ell=2^{k-j_0}}^{2^{k+1-j_0}-1} \mathbb{P}\Big(\sum_{j\geq j_0} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}\big)^2 \geq c_1 \log n + c_2 x, \Theta_{\ell,j_0}\Big)$$
$$\leq \sum_{k=j_0}^{K} \sum_{\ell=2^{k-j_0}}^{2^{k+1-j_0}-1} \mathbb{P}\Big(\sum_{j=j_0}^{2(k+4)} \big(\widetilde{U}_{j,\ell_j} + \widetilde{V}_{j,\ell_j}\big)^2 \geq (c_1-2) \log n + c_2 x, \Theta_{\ell,j_0}\Big)$$
$$+ \frac{5 \times 2^4}{\lambda (c_1 \log n + c_2 x)^2} n^{1-\lambda c_1/40} \exp(-\lambda c_2 x/40)\,. \quad (2.23)$$

We prove now that, for any $n \geq 2^5$, and any $c_1 \geq \tilde{c}_1$ where $\tilde{c}_1$ is defined in (2.7),

$$\sum_{k=j_0}^{K} \sum_{\ell=2^{k-j_0}}^{2^{k+1-j_0}-1} \mathbb{P}\Big(\sum_{j=j_0}^{2(k+4)} \big(\widetilde{U}^2_{j,\ell_j} + \widetilde{V}^2_{j,\ell_j}\big) \geq (c_1-2) \log n + c_2 x, \Theta_{\ell,j_0}\Big)$$
$$\leq \frac{4}{c_1 \log n + c_2 x} \exp\Big(- \frac{\lambda(\sqrt{2}-1)^2 c_2 x}{1765(2+\sqrt{2})^2}\Big)\,. \quad (2.24)$$

Clearly taking into account the restriction on c_1 and the fact that $c_2 \geq \frac{1765(2+\sqrt{2})^2}{(\sqrt{2}-1)^2}$, the inequality (2.21), and then the lemma, will follow from (2.23) and (2.24). To prove (2.24), we first write the following decomposition

$$\frac{b_{j,\ell_j}}{\sqrt{c_{j,\ell_j}}} = \frac{1}{2a_{j,\ell_j}} \sqrt{\frac{4a_{j,\ell_j} b_{j,\ell_j}}{a_{j,\ell_j}+b_{j,\ell_j}}} = \frac{1}{2a_{j,\ell_j}} \sqrt{a_{j,\ell_j} + b_{j,\ell_j} - \frac{(a_{j,\ell_j}-b_{j,\ell_j})^2}{a_{j,\ell_j}+b_{j,\ell_j}}}\,.$$

Therefore

$$\frac{1}{2a_{j,\ell_j}} \sqrt{a_{j,\ell_j}+b_{j,\ell_j}} - \frac{1}{2a_{j,\ell_j}} \frac{|a_{j,\ell_j}-b_{j,\ell_j}|}{\sqrt{a_{j,\ell_j}+b_{j,\ell_j}}} \leq \frac{b_{j,\ell_j}}{\sqrt{c_{j,\ell_j}}} \leq \frac{1}{2a_{j,\ell_j}} \sqrt{a_{j,\ell_j}+b_{j,\ell_j}}\,. \quad (2.25)$$

Set, for any $j > 0$ and $k \geq 0$,

$$\Pi_{j,k} = N((k+1)2^j) - N(k2^j)\,. \quad (2.26)$$

Recalling the definition (2.16) of $U_{j,k}$ and noticing that $a_{j,\ell_j} + b_{j,\ell_j} = \Pi_{j,\ell_j}$, we then get

$$|U_{j,\ell_j}| \leq t_j |\lambda^{-1/2} \Pi^{1/2}_{j,\ell_j} - 2^{j/2}|$$
$$+ \lambda^{-1/2} \Pi^{1/2}_{j,\ell_j} \Big| \frac{1}{2a_{j,\ell_j}} \big(N(\ell 2^{j_0}) - N(\ell_j 2^j)\big) - t_j \Big| + \frac{|a_{j,\ell_j}-b_{j,\ell_j}|}{\lambda^{1/2}\Pi^{1/2}_{j,\ell_j}} \frac{N(\ell 2^{j_0}) - N(\ell_j 2^j)}{2a_{j,\ell_j}}\,.$$

Whence, using the fact that, for $t_j \in]0,1/2]$, $N(\ell 2^{j_0}) - N(\ell_j 2^j) \leq a_{j,\ell_j}$, we infer that

$$|\widetilde{U}_{j,\ell_j}| \leq |\lambda^{-1/2}\Pi^{1/2}_{j,\ell_j} - 2^{j/2}| \mathbf{1}_{t_j \in]0,1/2]} + (2\lambda^{1/2}\Pi^{1/2}_{j,\ell_j})^{-1} |a_{j,\ell_j} - b_{j,\ell_j}| \mathbf{1}_{t_j \in]0,1/2]}$$
$$+ 2^{j/2} \Big| \frac{1}{2a_{j,\ell_j}} \big(N(\ell 2^{j_0}) - N(\ell_j 2^j)\big) - t_j \Big| \mathbf{1}_{t_j \in]0,1/2]}\,.$$

Moreover using the fact that, on the set Θ_{ℓ,j_0}, $a_{j,\ell_j} \geq \lambda 2^{j-2}$ and $\Pi_{j,\ell_j} = a_{j,\ell_j} + b_{j,\ell_j} \geq \lambda 2^{j-1}$, we get that, on the set Θ_{ℓ,j_0},

$$|\widetilde{U}_{j,\ell_j}| \leq |\lambda^{-1/2}\Pi^{1/2}_{j,\ell_j} - 2^{j/2}| \mathbf{1}_{t_j \in]0,1/2]} + 2^{-(j+1)/2} \lambda^{-1} |a_{j,\ell_j} - b_{j,\ell_j}| \mathbf{1}_{t_j \in]0,1/2]}$$
$$+ 2^{1-j/2} \lambda^{-1} \big| N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} \big| \mathbf{1}_{t_j \in]0,1/2]}\,. \quad (2.27)$$

On another hand, permuting the roles of a_{j,ℓ_j} and of b_{j,ℓ_j} in (2.25), we get

$$\frac{1}{2b_{j,\ell_j}}\sqrt{\Pi_{j,\ell_j}} - \frac{1}{2b_{j,\ell_j}}\frac{|b_{j,\ell_j}-a_{j,\ell_j}|}{\sqrt{\Pi_{j,\ell_j}}} \leq \frac{a_{j,\ell_j}}{\sqrt{c_{j,\ell_j}}} \leq \frac{1}{2b_{j,\ell_j}}\sqrt{\Pi_{j,\ell_j}}\,,$$

where we recall that $\Pi_{j,\ell_j} = a_{j,\ell_j} + b_{j,\ell_j}$. Since

$$V_{j,\ell_j} = 2^{j/2}(1-t_j) - \frac{a_{j,\ell_j}}{\sqrt{c_{j,\ell_j}}}\lambda^{-1/2}\big(N((\ell_j+1)2^j) - N(\ell 2^{j_0})\big)\,,$$

it follows that for any $t_j \in]1/2,1[$,

$$|V_{j,\ell_j}| \leq (1-t_j)|\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}| + \lambda^{-1/2}\Pi_{j,\ell_j}^{1/2}\Big|\frac{1}{2b_{j,\ell_j}}\big(N((\ell_j+1)2^j) - N(\ell 2^{j_0})\big) - (1-t_j)\Big| \\ + \frac{|a_{j,\ell_j}-b_{j,\ell_j}|}{\lambda^{1/2}\Pi_{j,\ell_j}^{1/2}}\frac{N((\ell_j+1)2^j) - N(\ell 2^{j_0})}{2b_{j,\ell_j}}\,.$$

Whence, using the fact that, for $t_j \in]1/2,1[$, $N((\ell_j+1)2^j) - N(\ell 2^{j_0}) \leq b_{j,\ell_j}$, we infer that

$$|\widetilde{V}_{j,\ell_j}| \leq |\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|\mathbf{1}_{t_j\in]1/2,1[} + \frac{|a_{j,\ell_j}-b_{j,\ell_j}|}{2\lambda^{1/2}\Pi_{j,\ell_j}^{1/2}}\mathbf{1}_{t_j\in]1/2,1[} \\ + 2^{j/2}\Big|\frac{1}{2b_{j,\ell_j}}\big(N((\ell_j+1)2^j) - N(\ell 2^{j_0})\big) - (1-t_j)\Big|\mathbf{1}_{t_j\in]1/2,1[}\,.$$

Since, on the set Θ_{ℓ,j_0}, $b_{j,\ell_j} \geq \lambda 2^{j-2}$ and $\Pi_{j,\ell_j} \geq \lambda 2^{j-1}$, we get that, on the set Θ_{ℓ,j_0},

$$|\widetilde{V}_{j,\ell_j}| \leq |\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|\mathbf{1}_{t_j\in]1/2,1[} + 2^{-(j+1)/2}\lambda^{-1}|a_{j,\ell_j}-b_{j,\ell_j}|\mathbf{1}_{t_j\in]1/2,1[} \\ + 2^{1-j/2}\lambda^{-1}\big|N((\ell_j+1)2^j) - N(\ell 2^{j_0})\big) - 2(1-t_j)b_{j,\ell_j}\big|\mathbf{1}_{t_j\in]1/2,1[}\,.$$

Notice now that

$$N((\ell_j+1)2^j) - N(\ell 2^{j_0})\big) - 2(1-t_j)b_{j,\ell_j} \\ = N((\ell_j+1)2^j) - N(\ell_j 2^j) + 2t_j(b_{j,\ell_j}-a_{j,\ell_j}) - 2b_{j,\ell_j} - \Big(N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j}\Big) \\ = (2t_j-1)(b_{j,\ell_j}-a_{j,\ell_j}) - \big(N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j}\big)\,.$$

So, overall, on the set Θ_{ℓ,j_0},

$$|\widetilde{V}_{j,\ell_j}| \leq |\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|\mathbf{1}_{t_j\in]1/2,1[} + 2^{-(j+1)/2}(2^{3/2}+1)\lambda^{-1}|a_{j,\ell_j}-b_{j,\ell_j}|\mathbf{1}_{t_j\in]1/2,1[} \\ + 2^{1-j/2}\lambda^{-1}\big|N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j}\big|\mathbf{1}_{t_j\in]1/2,1[}\,. \quad (2.28)$$

Taking into account (2.27) and (2.28), it follows that, on the set Θ_{ℓ,j_0},

$$\widetilde{U}_{j,\ell_j}^2 + \widetilde{V}_{j,\ell_j}^2 \leq 3|\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|^2 + 24\times\lambda^{-2}\times 2^{-j}|a_{j,\ell_j}-b_{j,\ell_j}|^2 \\ + 12\times\lambda^{-2}\times 2^{-j}\big|N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j}\big|^2\,.$$

Therefore, on the set Θ_{ℓ,j_0},

$$\sum_{j=j_0}^{2(k+4)}\big(\widetilde{U}_{j,\ell_j}^2 + \widetilde{V}_{j,\ell_j}^2\big) \leq 3\sum_{j=j_0}^{2(k+4)}|\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|^2 + 24\times\lambda^{-2}\sum_{j=j_0}^{2(k+4)}2^{-j}|a_{j,\ell_j}-b_{j,\ell_j}|^2 \\ + 12\times\lambda^{-2}\sum_{j=j_0+1}^{2(k+4)}2^{-j}\big|N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j}\big|^2\,, \quad (2.29)$$

the last sum starting at $j = j_0 + 1$ since $\ell_{j_0} 2^{j_0} = \ell 2^{j_0}$ and $t_{j_0} = 0$.

To handle the terms in the inequality above we shall introduce the following double indexed sequence $(\xi_{j,k})_{j>0,k\geq 0}$ of Gaussian random variables. Let Φ be the distribution function of a standard real-valued Gaussian random variable and Φ_n be the distribution function of the Binomial law $\mathcal{B}(n, 1/2)$. Let $(\delta_{j,k})_{j>0,k\geq 0}$ be a sequence of iid random variables with uniform law on $[0, 1]$, independent of the Poisson process $N(\cdot)$. For any $j \in \mathbb{N}^*$ and $k \in \mathbb{N}$, let

$$\xi_{j,k} = \Phi^{-1}\Big(\Phi_{\Pi_{j,k}}(\Pi_{j-1,2k} - 0) + \delta_{j,k}\big(\Phi_{\Pi_{j,k}}(\Pi_{j-1,2k}) - \Phi_{\Pi_{j,k}}(\Pi_{j-1,2k} - 0)\big)\Big)\,, \tag{2.30}$$

where we recall that the $\Pi_{j,k}$'s have been defined in (2.26). Note that, conditionnally to the sigma algebra, say $\mathcal{F}_j$, generated by the random variables $\{\Pi_{j,k} : k \geq 0\}$ and $\{\delta_{i,k} : i < j, k \geq 0\}$ the random variables $(\xi_{j,k})_{k\geq 0}$ are independent with law $\mathcal{N}(0, 1)$. By recurrence, it follows that for any positive integer m_0, $(\xi_{j,k})_{j\leq m_0,k\geq 0}$ is a sequence of independent random variables with law $\mathcal{N}(0, 1)$, and therefore $(\xi_{j,k})_{j>0,k\geq 0}$ is a sequence of iid standard real-valued Gaussian random variables. Moreover according to Lemma 2.3,

$$\big|\Pi_{j-1,2k} - \frac{1}{2}\Pi_{j,k}\big| \leq 1 + \frac{1}{2}\Pi_{j,k}^{1/2}|\xi_{j,k}|\,. \tag{2.31}$$

Since $\lim_{m\to\infty} 2^{-m}\Pi_{m,\ell_m} = \lambda$ almost surely, we have

$$\sum_{j=j_0}^{2(k+4)} |\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|^2 = \lambda^{-1}\sum_{j=j_0}^{2(k+4)}\Big(\sum_{m\geq j}\big(2^{\frac{j-m}{2}}\Pi_{m,\ell_m}^{1/2} - 2^{\frac{j-m-1}{2}}\Pi_{m+1,\ell_{m+1}}^{1/2}\big)\Big)^2.$$

But

$$\Pi_{m,\ell_m}^{1/2} - 2^{-1/2}\Pi_{m+1,\ell_{m+1}}^{1/2} = \frac{\Pi_{m,\ell_m} - 2^{-1}\Pi_{m+1,\ell_{m+1}}}{\Pi_{m,\ell_m}^{1/2} + 2^{-1/2}\Pi_{m+1,\ell_{m+1}}^{1/2}}\,. \tag{2.32}$$

Notice now that $\ell_{m+1} = [\ell_m/2]$. Therefore, setting

$$\widetilde{\Pi}_{j,k} = \Pi_{j-1,2k} - \frac{1}{2}\Pi_{j,k}\,, \tag{2.33}$$

we have

$$\Pi_{m,\ell_m} - 2^{-1}\Pi_{m+1,\ell_{m+1}} = (-1)^{\ell_m}\widetilde{\Pi}_{m+1,\ell_{m+1}}\,. \tag{2.34}$$

In addition, recall that on the set Θ_{ℓ,j_0}, $\Pi_{j,\ell_j} = a_{j,\ell_j} + b_{j,\ell_j} \geq \lambda 2^{j-1}$. Hence, starting from (2.32) and using (2.34) and (2.31), we get that, on the set Θ_{ℓ,j_0},

$$\big|\Pi_{m,\ell_m}^{1/2} - 2^{-1/2}\Pi_{m+1,\ell_{m+1}}^{1/2}\big| \leq \lambda^{-1/2}2^{-(m-1)/2} + \frac{1}{\sqrt{2}}|\xi_{m+1,\ell_{m+1}}|\,.$$

Whence, on the set Θ_{ℓ,j_0},

$$\begin{aligned}
\sum_{j=j_0}^{2(k+4)} |\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|^2 &\leq \lambda^{-1}\sum_{j=j_0}^{2(k+4)}\Big(\sum_{m\geq j} 2^{\frac{j-m}{2}}\big(\lambda^{-1/2}2^{-(m-1)/2} + \frac{1}{\sqrt{2}}|\xi_{m+1,\ell_{m+1}}|\big)\Big)^2 \\
&= \lambda^{-1}\sum_{j=j_0}^{2(k+4)}\Big(\frac{2\sqrt{2}}{\sqrt{\lambda}}2^{-j/2} + \frac{1}{\sqrt{2}}\sum_{m\geq j} 2^{\frac{j-m}{2}}|\xi_{m+1,\ell_{m+1}}|\Big)^2 \\
&\leq \lambda^{-2}\times 2^{-j_0+5} + \lambda^{-1}\sum_{j=j_0}^{2(k+4)}\sum_{i\geq j} 2^{\frac{j-i}{2}}\sum_{m\geq j} 2^{\frac{j-m}{2}}\xi_{m+1,\ell_{m+1}}^2 \\
&\leq \lambda^{-2}\times 2^{-j_0+5} + \lambda^{-1}(2+\sqrt{2})\sum_{j=j_0}^{2(k+4)}\sum_{m\geq j} 2^{\frac{j-m}{2}}\xi_{m+1,\ell_{m+1}}^2\,,
\end{aligned}$$

Therefore, on the set Θ_{ℓ,j_0},

$$\sum_{j=j_0}^{2(k+4)} |\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|^2 \leq \lambda^{-2} \times 2^{-j_0+5} + \lambda^{-1}(2+\sqrt{2}) \sum_{m=j_0}^{2(k+4)} \sum_{j=j_0}^{m} 2^{\frac{j-m}{2}} \xi_{m+1,\ell_{m+1}}^2 + \lambda^{-1}(2+\sqrt{2}) \sum_{m\geq 2k+9} \sum_{j=j_0}^{2(k+4)} 2^{\frac{j-m}{2}} \xi_{m+1,\ell_{m+1}}^2 .$$

This leads, by taking into account (2.11), to

$$\sum_{j=j_0}^{2(k+4)} |\lambda^{-1/2}\Pi_{j,\ell_j}^{1/2} - 2^{j/2}|^2 \mathbf{1}_{\Theta_{\ell,j_0}} \leq \frac{2^5}{\lambda^2(c_1 \log n + c_2 x)} + \lambda^{-1}(2+\sqrt{2})^2 \sum_{m=j_0}^{2(k+4)} \xi_{m+1,\ell_{m+1}}^2 + \lambda^{-1}(2+\sqrt{2})^2 \sum_{m\geq 2k+9} \frac{2^{k+4}}{2^{m/2}} \xi_{m+1,\ell_{m+1}}^2 . \quad (2.35)$$

On another hand,

$$|a_{j,\ell_j} - b_{j,\ell_j}|^2 = |2a_{j,\ell_j} - \Pi_{j,\ell_j}|^2 = 4\big|\Pi_{j-1,2\ell_j} - \frac{1}{2}\Pi_{j,\ell_j}\big|^2 .$$

Therefore by using (2.31) and the fact that on the set Θ_{ℓ,j_0}, $\Pi_{j,\ell_j} \leq 3\lambda \times 2^{j-1}$, we derive

$$\mathbf{1}_{\Theta_{\ell,j_0}} \sum_{j=j_0}^{2(k+4)} 2^{-j} |a_{j,\ell_j} - b_{j,\ell_j}|^2 \leq 4 \sum_{j=j_0}^{2(k+4)} \big(2^{1-j} + 3\lambda \times 2^{-2} \xi_{j,\ell_j}^2\big) ,$$

leading, by taking into account (2.11), to

$$\mathbf{1}_{\Theta_{\ell,j_0}} \sum_{j=j_0}^{2(k+4)} 2^{-j} |a_{j,\ell_j} - b_{j,\ell_j}|^2 \leq \frac{16}{c_1 \log n + c_2 x} + 3\lambda \sum_{j=j_0}^{2(k+4)} \xi_{j,\ell_j}^2 . \quad (2.36)$$

To handle now the last term in the right-hand side of (2.29), we first note that $\ell 2^{j_0} = \ell_{j_0} 2^{j_0}$, and write the following decomposition

$$N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} = \sum_{m=j_0+1}^{j} \big(N(\ell_{m-1}2^{m-1}) - N(\ell_m 2^m) - (\ell_{m-1}2^{m-1} - \ell_m 2^m)\frac{a_{j,\ell_j}}{2^{j-1}}\big) .$$

Since $\ell_m = [\ell_{m-1}/2]$, $\ell_{m-1}2^{m-1} \neq \ell_m 2^m$ only if $\ell_{m-1} = 2\ell_m + 1$ and in this case $\ell_{m-1}2^{m-1} - \ell_m 2^m = 2^{m-1}$. Therefore, using the notation (2.26) and that $a_{j,\ell_j} = \Pi_{j-1,2\ell_j}$, we have

$$\begin{aligned} N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} &= \sum_{m=j_0+1}^{j} \mathbf{1}_{\ell_{m-1}2^{m-1} \neq \ell_m 2^m} \big(N((2\ell_m+1)2^{m-1}) - N(\ell_m 2^m) - 2^{m-j} a_{j,\ell_j}\big) \\ &= \sum_{m=j_0+1}^{j} \mathbf{1}_{\ell_{m-1}2^{m-1} \neq \ell_m 2^m} \Big(\Pi_{m-1,2\ell_m} - \frac{1}{2}\Pi_{m,\ell_m} + \sum_{u=m}^{j-1} \frac{1}{2^{u+1-m}} \big(\Pi_{u,\ell_u} - \frac{1}{2}\Pi_{u+1,\ell_{u+1}}\big) \\ &\qquad + \frac{1}{2^{j-m}} \big(\frac{1}{2}\Pi_{j,\ell_j} - \Pi_{j-1,2\ell_j}\big)\Big) . \end{aligned}$$

Using the notation (2.33) and the relation (2.34), we then derive

$$N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} \\ = \sum_{m=j_0+1}^{j} \mathbf{1}_{\ell_{m-1}2^{m-1} \neq \ell_m 2^m} \Big(\widetilde{\Pi}_{m,\ell_m} + \sum_{u=m}^{j-1} \frac{(-1)^{\ell_u}}{2^{u+1-m}} \widetilde{\Pi}_{u+1,\ell_{u+1}} - \frac{1}{2^{j-m}} \widetilde{\Pi}_{j,\ell_j} \Big) .$$

Therefore by (2.31),

$$\big| N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} \big| \\ \leq \sum_{m=j_0+1}^{j} \mathbf{1}_{\ell_{m-1}2^{m-1} \neq \ell_m 2^m} \Big(\sum_{u=m}^{j} \frac{1}{2^{u-m}} \big(1 + \frac{1}{2} \Pi_{u,\ell_u}^{1/2} |\xi_{u,\ell_u}|\big) + \frac{1}{2^{j-m}} \big(1 + \frac{1}{2} \Pi_{j,\ell_j}^{1/2} |\xi_{j,\ell_j}|\big) \Big) \\ \leq 2j + \sum_{m=j_0+1}^{j} \sum_{u=m}^{j} \frac{1}{2^{u-m}} \Pi_{u,\ell_u}^{1/2} |\xi_{u,\ell_u}| .$$

It follows that

$$\mathbf{1}_{\Theta_{\ell,j_0}} \sum_{j=j_0+1}^{2(k+4)} 2^{-j} \big| N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} \big|^2 \\ \leq 8 \sum_{j=j_0+1}^{2(k+4)} \frac{j^2}{2^j} + 3\lambda \sum_{j=j_0+1}^{2(k+4)} \frac{1}{2^j} \Big(\sum_{m=j_0+1}^{j} \sum_{u=m}^{j} \frac{2^m}{2^{u/2}} |\xi_{u,\ell_u}| \Big)^2 ,$$

since on the set Θ_{ℓ,j_0}, $\Pi_{u,\ell_u} = a_{u,\ell_u} + b_{u,\ell_u} \leq 3\lambda \times 2^{u-1}$. Hence by taking into account (2.11) and by using Cauchy-Schwarz's inequality,

$$\mathbf{1}_{\Theta_{\ell,j_0}} \sum_{j=j_0+1}^{2(k+4)} 2^{-j} \big| N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} \big|^2 \\ \leq \frac{32(k+4)^2}{c_1 \log n + c_2 x} + 12\lambda \sum_{j=j_0+1}^{2(k+4)} \frac{1}{2^j} \Big(\sum_{u=j_0+1}^{j} 2^{u/2} |\xi_{u,\ell_u}| \Big)^2 \\ = \frac{32(k+4)^2}{c_1 \log n + c_2 x} + 12\lambda \sum_{j=j_0+1}^{2(k+4)} \frac{1}{2^j} \sum_{i=j_0+1}^{j} 2^{i/2} \sum_{u=j_0+1}^{j} 2^{u/2} \xi_{u,\ell_u}^2 \\ \leq \frac{32(k+4)^2}{c_1 \log n + c_2 x} + 12(2+\sqrt{2})\lambda \sum_{j=j_0+1}^{2(k+4)} \frac{1}{2^{j/2}} \sum_{u=j_0+1}^{j} 2^{u/2} \xi_{u,\ell_u}^2 .$$

Therefore

$$\mathbf{1}_{\Theta_{\ell,j_0}} \sum_{j=j_0+1}^{2(k+4)} 2^{-j} \big| N(\ell 2^{j_0}) - N(\ell_j 2^j) - 2t_j a_{j,\ell_j} \big|^2 \leq \frac{32(k+4)^2}{c_1 \log n + c_2 x} + 12(2+\sqrt{2})^2 \lambda \sum_{u=j_0+1}^{2(k+4)} \xi_{u,\ell_u}^2 . \tag{2.37}$$

Hence starting from (2.29) and considering the upper bounds (2.35), (2.36) and (2.37), we derive

$$\mathbb{P}\Big(\sum_{j=j_0}^{2(k+4)} \big(\widetilde{U}_{j,\ell_j}^2 + \widetilde{V}_{j,\ell_j}^2 \big) \geq (c_1 - 2) \log n + c_2 x, \Theta_{\ell,j_0} \Big) \\ \leq \mathbb{P}\Big(A_4 \sum_{m=j_0-1}^{2(k+4)} \xi_{m+1,\ell_{m+1}}^2 + A_5 2^k \sum_{m \geq 2k+9} 2^{-m/2} \xi_{m+1,\ell_{m+1}}^2 \geq (c_1 - 2) \log n + c_2 x - A_3 \Big) .$$

where

$$A_3 = \frac{3\times 2^5 + 16\times 24}{\lambda^2(c_1\log n + c_2 x)} + \frac{32\times 12(k+4)^2}{\lambda^2(c_1\log n + c_2 x)}\,,$$

$$A_4 = 3\lambda^{-1}(24 + (2+\sqrt{2})^2 + (12)^2\times(2+\sqrt{2})^2) \text{ and } A_5 = 3\lambda^{-1}\times 2^4(2+\sqrt{2})^2\,.$$

Recall that $k \le K$. Hence $k < 1 + (\log n)/(\log 2)$. Therefore if $n \ge 2^5$, we get

$$(k+4)^2 < \frac{4}{(\log 2)^2}(\log n)^2\,. \tag{2.38}$$

Whence, if $n \ge 2^5$,

$$A_3 \le \frac{(30+32\times 12)(k+4)^2}{\lambda^2(c_1\log n + c_2 x)} \le \frac{1656\times(\log n)^2}{\lambda^2(\log 2)^2(c_1\log n + c_2 x)} \le \frac{1656\times(\log n)}{\lambda^2 c_1(\log 2)^2}\,.$$

Therefore if $n \ge 2^5$ and we take c_1 such that

$$c_1 \ge \max\big(3, \frac{1656}{\lambda^2(\log 2)^2}\big)\,, \tag{2.39}$$

we get $A_3 \le \log n$ implying that

$$\begin{aligned}
&\mathbb{P}\Big(\sum_{j=j_0-1}^{2(k+4)} (\widetilde{U}^2_{j,\ell_j} + \widetilde{V}^2_{j,\ell_j}) \ge (c_1-2)\log n + c_2 x, \Theta_{\ell,j_0}\Big) \\
&\quad\le \mathbb{P}\Big(A_4 \sum_{m=j_0-1}^{2(k+4)} \xi^2_{m+1,\ell_{m+1}} + A_5 2^k \sum_{m\ge 2k+9} 2^{-m/2}\xi^2_{m+1,\ell_{m+1}} \ge (c_1-3)\log n + c_2 x\Big)\,.
\end{aligned} \tag{2.40}$$

Notice now that for any $0 < t < (2A_4)^{-1}$,

$$\mathbb{E}\exp\Big(A_4 t \sum_{m=j_0-1}^{2(k+4)} \xi^2_{m+1,\ell_{m+1}}\Big) = \Big(\frac{1}{1-2tA_4}\Big)^{k+4-(j_0-2)/2} \le \exp\Big(\frac{2tA_4(k+4)}{1-2tA_4}\Big)\,,$$

where, for the last inequality, we have used that $-\log(1-u) \le \frac{u}{1-u}$ for any $u \in [0,1[$ and that $j_0 \ge 2$ since $c_1 \ge 3$ and $n \ge 4$. On another hand, for any $0 < t < (2^{-7/2}A_5)^{-1}$,

$$\mathbb{E}\exp\Big(A_5 t \sum_{m\ge 2k+9} 2^{k-m/2}\xi^2_{m+1,\ell_{m+1}}\Big) = \prod_{m\ge 2k+9}\Big(\frac{1}{1-2tA_5\times 2^{k-m/2}}\Big)^{1/2}\,.$$

Using once again the fact that $-\log(1-u) \le \frac{u}{1-u}$ for any $u \in [0,1[$, we get that

$$\sum_{m\ge 2k+9}\log\Big(\frac{1}{1-2tA_5\times 2^{k-m/2}}\Big) \le \sum_{m\ge 2k+9}\frac{2tA_5\times 2^{k-m/2}}{1-2tA_5\times 2^{k-m/2}} \le \frac{tA_5(\sqrt{2}+1)\times 2^{-3}}{1-2^{-7/2}tA_5}\,.$$

Since $2^{-7/2}A_5 \le 2A_4$, it follows that, for any $0 < t < (2A_4)^{-1}$,

$$\mathbb{E}\exp\Big(A_5 t \sum_{m\ge 2k+9} 2^{k-m/2}\xi^2_{m+1,\ell_{m+1}}\Big) \le \exp\Big(\frac{tA_5(\sqrt{2}+1)\times 2^{-4}}{1-2tA_4}\Big)\,.$$

So, overall, for any $0 < t < (2A_4)^{-1}$,

$$\mathbb{E}\exp\Big(tA_4 \sum_{m=j_0-1}^{2(k+4)} \xi^2_{m+1,\ell_{m+1}} + A_5 t 2^k \sum_{m\ge 2k+9} 2^{-m/2}\xi^2_{m+1,\ell_{m+1}}\Big) \le \exp\Big(\frac{t(2A_4k + A_6)}{1-2tA_4}\Big)\,,$$

where

$$A_6 = 8A_4 + A_5(\sqrt{2}+1) \times 2^{-4}\,.$$

Therefore, starting from (2.40), we get, if $n \geq 2^5$ and c_1 satisfies (2.39),

$$\mathbb{P}\Big(\sum_{j=j_0}^{2(k+4)} \big(\widetilde{U}_{j,\ell_j}^2 + \widetilde{V}_{j,\ell_j}^2\big) \geq (c_1-2)\log n + c_2 x, \Theta_{\ell,j_0}\Big)$$
$$\leq \inf_{0<t<(2A_4)^{-1}} \exp\Big(-t((c_1-3)\log n + c_2 x) + \frac{t(2A_4 k + A_6)}{1-2tA_4}\Big)\,.$$

Hence, if $(c_1-3)\log n > 2A_4 k + A_6$,

$$\mathbb{P}\Big(\sum_{j=j_0}^{2(k+4)} \big(\widetilde{U}_{j,\ell_j}^2 + \widetilde{V}_{j,\ell_j}^2\big) \geq (c_1-2)\log n + c_2 x, \Theta_{\ell,j_0}\Big)$$
$$\leq \exp\Big(-\frac{(\sqrt{(c_1-3)\log n + c_2 x} - \sqrt{2A_4 k + A_6}\,)^2}{2A_4}\Big)\,.$$

Let

$$A_7 = 3 \times 148 \times \lambda^{-1}(2+\sqrt{2})^2\,.$$

Notice that $A_4 \leq A_7$ and $A_6 \leq 9A_7$. Therefore $2A_4 k + A_6 \leq 9A_7(k+4)/4$. Hence, if $n \geq 2^5$, and if $c_1 \geq 3 + 9A_7/(\log 2)$, taking into account (2.38), we get

$$2A_4 k + A_6 \leq \frac{9A_7}{2(\log 2)}(\log n) \leq \frac{(c_1-3)}{2}(\log n)\,.$$

So, overall, if

$$c_1 \geq \max\Big(3 + \frac{3996(2+\sqrt{2})^2}{\lambda(\log 2)}, \frac{1656}{\lambda^2(\log 2)^2}\Big)\,,$$

then for any $n \geq 2^5$,

$$\mathbb{P}\Big(\sum_{j=j_0}^{2(k+4)} \big(\widetilde{U}_{j,\ell_j}^2 + \widetilde{V}_{j,\ell_j}^2\big) \geq (c_1-2)\log n + c_2 x, \Theta_{\ell,j_0}\Big)$$
$$\leq \exp\Big(-\frac{(\sqrt{2}-1)^2\big((c_1-3)\log n + c_2 x\big)}{4A_4}\Big)\,.$$

This last inequality leads to (2.24) as soon as $c_1 \geq \tilde{c}_1$ where $\tilde{c}_1$ is defined in (2.7), taking into account that $4A_4 \leq 1765 \times \lambda^{-1}(2+\sqrt{2})^2$ and 2^{j_0} satisfies (2.11). This ends the proof of the lemma.

2.3 Proof of Theorem 1.1

Notice first that it suffices to prove the result for any positive real x such that $x \leq 2n\|g\|_\infty$. Indeed since $|\bar{S}_k(g)| \leq k\|g\|_\infty$ for any positive integer k, it follows, by Lévy's inequality, that for any standard Wiener process $(W_t)_{t\geq 0}$ and any real $x > 2n\|g\|_\infty$,

$$\mathbb{P}\Big(\sup_{k\leq n} \big|\bar{S}_k(g) - \sigma(g)W_k\big| \geq c\log n + x\Big) \leq 2\mathbb{P}\Big(|\sigma(g)W_n| \geq x/2\Big)$$
$$\leq \frac{4\sqrt{2}}{\sqrt{\pi}} \times \frac{\sigma(g)\sqrt{n}}{x} \exp\Big(-\frac{x^2}{8\sigma^2(g)n}\Big) \leq \frac{2\sqrt{2}\sigma(g)}{\|g\|_\infty\sqrt{\pi}} n^{-1/2} \exp\Big(-\frac{x\|g\|_\infty}{4\sigma^2(g)}\Big)\,.$$

Therefore, to prove the theorem, it suffices to show that there exists a standard Wiener process $(W_t)_{t\geq 0}$ such that (1.15) holds for any positive real x satisfying $x \leq 2n\|g\|_\infty$. From now on, x will be a positive real satisfying the latter condition.

For any $i \in \mathbb{N}^*$, let

$$X_i = \sum_{\ell=T_{i-1}+1}^{T_i} g(\bar{\xi}_\ell, U_\ell)\,.$$

With this notation $\sum_{i=1}^k X_i = \bar{S}_{T_k}(g) - \bar{S}_{T_0}(g)$. Let τ_k be defined by (1.7). Notice that $(\tau_k, X_k)_{k\geq 1}$ forms a sequence of iid random vectors. In addition for any k, $\mathbb{E}(X_k) = 0$ since $\pi \otimes \lambda(g) = 0$. We can assume without loss of generality that $\mathrm{Var}(\tau_1) > 0$. Indeed if $\mathrm{Var}(\tau_1) = 0$ then τ_1 is almost surely equal to some positive integer d. Then $\tau_i = d$ almost surely for any positive integer i, which implies that $T_k = kd + T_0$ almost surely. The result follows then easily from the Komlós-Major-Tusnády theorem applied to the above sequence $(X_i)_{i>0}$ and the fact that T_0 has a finite Laplace transform in a neighborhood of 0.

We now assume that $\mathrm{Var}(\tau_1) \neq 0$. Let

$$\alpha = \frac{\mathrm{Cov}(\tau_1, X_1)}{\mathrm{Var}(\tau_1)}\,. \tag{2.41}$$

It follows that $(\tau_k, X_k - \alpha(\tau_k - \mathbb{E}(\tau_k)))_{k\geq 1}$ is a sequence of iid random vectors such that, for any $k \in \mathbb{N}^*$,

$$\mathrm{Cov}(\tau_k, X_k - \alpha(\tau_k - \mathbb{E}(\tau_k))) = 0\,.$$

Let

$$v^2 = \mathrm{Var}\big(X_1 - \alpha(\tau_1 - \mathbb{E}(\tau_1))\big)\,. \tag{2.42}$$

As it was recalled in the introduction, under the condition (1.12), the return times τ_k have finite Laplace transform on some neighborhood of 0 (see (1.13)). Since g is assumed to be bounded, the random variables $X_k - \alpha(\tau_k - \mathbb{E}(\tau_k))$ also a finite Laplace transform on some neighborhood of 0. More precisely, by (1.13),

$$\mathbb{E}\big(e^{t(X_1 - \alpha(\tau_1 - \mathbb{E}(\tau_1)))}\big) \leq e^{t\alpha\mathbb{E}(\tau_1)}\mathbb{E}\big(e^{t(\|g\|_\infty + |\alpha|)\tau_1}\big) < \infty \ \text{ for any } |t| \leq \delta(\|g\|_\infty + |\alpha|)^{-1}.$$

Taking into account all the considerations above mentioned, we can apply Theorem 1.3 in Zaitsev (1998) to the multivariate sequence of iid random variables $(\tau_k, X_k - \alpha(\tau_k - \mathbb{E}(\tau_k)))_{k>0}$ to conclude that there exists a sequence $(Y_i, Z_i)_{i\geq 1}$ of independent random variables in $\mathbb{R}^2$ such that $(Y_i)_{i\geq 1}$ is independent of $(Z_i)_{i\geq 1}$,

$$\mathcal{L}(Y_i) = \mathcal{N}(0, v^2)\ ,\ \mathcal{L}(Z_i) = \mathcal{N}(0, \mathrm{Var}(\tau_1))\,,$$

and satisfying, for some positive constants C_1, A_1 and B_1 depending on g and on the transition probability $P(x, \cdot)$, the following inequalities: for any integer $n \geq 2$,

$$\mathbb{P}\Big(\sup_{k\leq n} \big|\bar{S}_{T_k}(g) - \bar{S}_{T_0}(g) - \alpha(T_k - T_0 - k\mathbb{E}(\tau_1)) - \sum_{i=1}^k Y_i\big| \geq C_1 \log n + x\Big) \leq A_1 \exp(-B_1 x)\,, \tag{2.43}$$

and

$$\mathbb{P}\Big(\sup_{k\leq n} \big|T_k - T_0 - k\mathbb{E}(\tau_1) - \sum_{i=1}^k Z_i\big| \geq C_1 \log n + x\Big) \leq A_1 \exp(-B_1 x)\,. \tag{2.44}$$

Using the Skorohod embedding theorem, we can then construct two independent standard Wiener processes $(B_t)_{t\geq 0}$ and $(\tilde{B}_t)_{t\geq 0}$ such that for any positive integer k,

$$vB_k = \sum_{i=1}^k Y_i \ \text{ and } \ \sqrt{\mathrm{Var}(\tau_1)}\tilde{B}_k = \sum_{i=1}^k Z_i\,.$$

In addition, according to Theorem 1(ii) in Komlós, Major and Tusnády (1975), there exists a Poisson process $(N(t), t \geq 0)$ with parameter λ defined by

$$\lambda = \frac{(\mathbb{E}(\tau_1))^2}{\mathrm{Var}(\tau_1)} \tag{2.45}$$

such that, setting

$$\gamma = \frac{\mathrm{Var}(\tau_1)}{\mathbb{E}(\tau_1)}\,, \tag{2.46}$$

the following inequality holds: for any integer $n \geq 2$,

$$\mathbb{P}\Big(\sup_{k\leq n} \big|\gamma N(k) - k\mathbb{E}(\tau_1) - \sqrt{\mathrm{Var}(\tau_1)}\tilde{B}_k\big| \geq C_2 \log n + x\Big) \leq A_2 \exp(-B_2 x)\,, \tag{2.47}$$

where C_2, A_2 and B_2 are positive constants depending on λ. According to the dyadic construction of Komlós, Major and Tusnády (1975), this Poisson process may be defined from $(\tilde{B}_t)_{t\geq 0}$ in a deterministic way. Therefore $N(\cdot)$ is independent of $(B_t)_{t\geq 0}$. Notice that (2.44) together with (2.47) imply that

$$\mathbb{P}\Big(\sup_{k\leq n} \big|\gamma N(k) - (T_k - T_0)\big| \geq C_3 \log n + 2x\Big) \leq A_3 \exp(-B_3 x)\,, \tag{2.48}$$

where $C_3 = C_1 + C_2$, $A_3 = A_1 + A_2$ and $B_3 = B_1 \wedge B_2$. Actually, as we shall see, the above upper bound also implies that, for any $n \geq 2$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\leq n} \big|\gamma N(t) - T_{[t]}\big| \geq \big(C_3 + \frac{3\lambda\gamma}{\log 2}\big)\log n + 4x\Big)& \\ \leq A_3 \exp(-B_3 x) + \exp\Big(-\frac{x\log 3}{2\gamma}\Big) + \mathbb{E}_\pi\big(e^{\delta T_0}\big)\exp(-\delta x)\,,&\end{aligned} \tag{2.49}$$

where δ is defined in (1.13). Therefore, for any $n \geq 2$,

$$\mathbb{P}\Big(\sup_{t\leq n} \big|\gamma N(t) - T_{[t]}\big| \geq C_4 \log n + 4x\Big) \leq A_4 \exp(-B_4 x)\,, \tag{2.50}$$

where

$$C_4 = C_3 + \frac{3\lambda\gamma}{\log 2}\ ,\ A_4 = 1 + A_3 + \mathbb{E}_\pi\big(e^{\delta T_0}\big)\ \text{ and }\ B_4 = \min\Big(B_3, \delta, \frac{\log 3}{2\gamma}\Big)\,.$$

Let us prove (2.49). By using (1.13) and (2.48), we have

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,n]} \big|\gamma N(t) - T_{[t]}\big| \geq \big(C_3 + \frac{3\lambda\gamma}{\log 2}\big)\log n + 4x\Big)& \\ \leq \mathbb{E}_\pi\big(e^{\delta T_0}\big)e^{-\delta x} + \mathbb{P}\Big(\sup_{1\leq k\leq n} \big|\gamma N(k) - (T_k - T_0)\big| \geq C_3 \log n + 2x\Big)& \\ + \mathbb{P}\Big(\sup_{1\leq k\leq n}\sup_{k-1<t\leq k} \gamma(N(t) - N(k-1)) \geq \frac{3\lambda\gamma}{\log 2}\log n + x\Big)& \\ \leq \mathbb{E}_\pi\big(e^{\delta T_0}\big)e^{-\delta x} + A_3\exp(-B_3 x) + \mathbb{P}\Big(\sup_{1\leq k\leq n}\sup_{k-1<t\leq k} \gamma(N(t) - \gamma N(k-1)) \geq \frac{3\lambda\gamma}{\log 2}\log n + x\Big)\,.&\end{aligned} \tag{2.51}$$

Now, for any $n \geq 2$, by using Lemma 2.2, we get

$$\begin{aligned}\mathbb{P}\Big(\sup_{1\leq k\leq n}\sup_{k-1<t\leq k}\gamma(N(t)-N(k-1)) \geq \frac{3\lambda\gamma}{\log 2}\log n + x\Big) &\leq n\mathbb{P}\Big(\sup_{0<t\leq 1}\gamma N(t) \geq \frac{3\lambda\gamma}{\log 2}\log n + x\Big)\\ &\leq n\mathbb{P}\Big(\sup_{0<t\leq 1}\gamma|N(t)-\lambda t| \geq \lambda\gamma\big(\frac{3}{\log 2}\log n - 1\big) + x\Big)\\ \leq n\mathbb{P}\Big(\sup_{0<t\leq 1}|N(t)-\lambda t| \geq \frac{2\lambda}{\log 2}\log n + x\gamma^{-1}\Big) &\leq n\exp\Big(-\lambda h\big(\frac{2}{\log 2}\log n + (\gamma\lambda)^{-1}x\big)\Big)\\ &\leq n\exp\Big(-\big(\frac{\lambda\log n}{\log 2} + \frac{\gamma^{-1}x}{2}\big)\log\big(1+\frac{2}{\log 2}\log n + (\gamma\lambda)^{-1}x\big)\Big),\end{aligned}$$

where for the last inequality, we have used that $h(x) \geq \frac{x}{2}\log(1+x)$. Hence, taking into account that $\lambda > 1$, we derive that, for any $n \geq 2$,

$$\mathbb{P}\Big(\sup_{1\leq k\leq n}\sup_{k-1<t\leq k}\gamma(N(t)-N(k-1)) \geq \frac{3\lambda\gamma}{\log 2}\log n + x\Big) \leq \exp\Big(-\frac{x\log 3}{2\gamma}\Big). \tag{2.52}$$

Starting from (2.51) and taking into account (2.52), (2.49) follows.

Note now that the random variables Γ_k defined by

$$\Gamma_0 = 0 \ \text{ and } \Gamma_k := N^{-1}(k) = \inf\{t > 0 \,:\, N(t) \geq k\} \text{ for } k \geq 1$$

are such that $(\Gamma_k - \Gamma_{k-1})_{k\geq 1}$ forms a sequence of iid random variables with exponential law of parameter λ. Therefore, according to Theorem 1(i) in Komlós, Major and Tusnády (1975), there exists a standard Wiener process $(\widetilde{W}_t)_{t\geq 0}$ such that, for any integer $n \geq 2$,

$$\mathbb{P}\Big(\sup_{k\leq n}\big|N^{-1}(k) - \frac{k}{\lambda} - \frac{1}{\lambda}\widetilde{W}_k\big| \geq C_5\log n + x\Big) \leq A_5\exp(-B_5 x). \tag{2.53}$$

where C_5, A_5 and B_5 are positive constants depending on λ. Notice that the so constructed Wiener process $\widetilde{W}$ depends only on the process N^{-1} and on some auxiliary atomless random variable U independent of the σ-field generated by the processes B, N and the auxiliary random variable δ of Lemma 2.4.

On another hand, since $(B_t)_{t\geq 0}$ is independent of $(N(t) : t \geq 0)$, according to Lemma 2.4, there exists a standard Brownian process $(W^*_t)_{t\geq 0}$ independent of the Poisson process $N(\cdot)$ and such that, for any integer $n \geq 2$,

$$\mathbb{P}\big(\sup_{k\leq n}|B_k - \frac{1}{\sqrt{\lambda}}W^*_{N(k)}| \geq C_6\log n + x\big) \leq A_6\exp(-B_6 x), \tag{2.54}$$

where C_6, A_6 and B_6 are positive constants depending on λ. Moreover $(W^*_t)_t$ is measurable with respect to the σ-field generated by the processes B, N and the auxiliary random variable δ of Lemma 2.4, which ensures that $(W^*_t)_t$ is independent of the σ-field generated by $N(\cdot)$ and U. Hence the Wiener processes $\widetilde{W}$ and W^* are independent.

In what follows we shall prove that (1.15) holds true with

$$W_t = \frac{1}{\sigma(g)}\Big(\frac{v}{\sqrt{\lambda}}W^*_{t/\gamma} - \alpha\frac{\mathbb{E}(\tau_1)}{\lambda}\widetilde{W}_{t/\gamma}\Big). \tag{2.55}$$

Recall here that $\sigma(g)$ is assumed to be positive. Notice that $(W_t)_{t\geq 0}$ defined by (2.55) is a standard Brownian motion. Indeed

$$\frac{v^2}{\gamma\lambda} + \alpha^2\frac{(\mathbb{E}(\tau_1))^2}{\gamma\lambda^2} = \frac{\mathrm{Var}(X_1)}{\mathbb{E}(\tau_1)} = \lim_{n\to\infty}\frac{\mathrm{Var}(\bar{S}_n(g))}{n} = \sigma^2(g).$$

The two latter inequalities follow from a well known fact concerning the asymptotic variance (see e.g. Nummelin (1984) or Chen (1999)).

Before proving that (1.15) holds true with $(W_t)_{t\geq 0}$ defined by (2.55), let us prove that, in addition to (2.54), we also have, for any integer $n \geq 2$,

$$\mathbb{P}\big(\sup_{0\leq t\leq n} |B_t - \frac{1}{\sqrt{\lambda}} W^*_{N(t)}| \geq C_7 \log n + 3x\big) \leq A_7 \exp(-B_7 x)\,, \tag{2.56}$$

where

$$C_7 = C_6 + (6 + 4\lambda^{-1})(\log 2)^{-1}\ ,\ A_7 = A_6 + 2 + \frac{1}{2^5\sqrt{\pi}}$$

and

$$B_7 = \min(1, B_6, (\lambda \log 2)/4)\,.$$

With this aim we first write the following decomposition:

$$\begin{aligned}
&\mathbb{P}\big(\sup_{0\leq t\leq n} |B_t - \frac{1}{\sqrt{\lambda}} W^*_{N(t)}| \geq C_7 \log n + 3x\big) = \mathbb{P}\big(\sup_{1\leq k\leq n} \sup_{k-1\leq t\leq k} |B_t - \frac{1}{\sqrt{\lambda}} W^*_{N(t)}| \geq C_7 \log n + 3x\big) \\
&\leq \mathbb{P}\big(\sup_{1\leq k\leq n} |B_k - \frac{1}{\sqrt{\lambda}} W^*_{N(k)}| \geq C_6 \log n + x\big) + \mathbb{P}\big(\sup_{1\leq k\leq n} \sup_{k-1\leq t\leq k} |B_t - B_{k-1}| \geq 2(\log 2)^{-1} \log n + x\big) \\
&\qquad + \mathbb{P}\big(\sup_{1\leq k\leq n} \sup_{k-1\leq t\leq k} |\frac{1}{\sqrt{\lambda}} W^*_{N(t)} - \frac{1}{\sqrt{\lambda}} W^*_{N(k-1)}| \geq (4 + 4\lambda^{-1})(\log 2)^{-1} \log n + x\big) \\
&:= I_1 + I_2 + I_3\,.
\end{aligned} \tag{2.57}$$

By Lévy's inequality, for any $n \geq 2$,

$$\begin{aligned}
I_2 &\leq \sum_{k=1}^{n} \mathbb{P}\big(\sup_{k-1\leq t\leq k} |B_t - B_{k-1}| \geq 2(\log 2)^{-1} \log n + x\big) \leq 2n\mathbb{P}\big(|B_1| \geq 2(\log 2)^{-1} \log n + x\big) \\
&\leq \frac{2\sqrt{2}n}{\sqrt{\pi}(2(\log 2)^{-1} \log n + x)} \exp(-2^{-1}(2(\log 2)^{-1} \log n + x)^2) \leq \exp(-2x)\,.
\end{aligned} \tag{2.58}$$

On another hand, for any $y > 0$,

$$\begin{aligned}
I_3 &\leq \sum_{k=1}^{n} \mathbb{P}\big(\sup_{k-1\leq t\leq k} \mathbb{P}\big|\frac{1}{\sqrt{\lambda}} W^*_{N(t)} - \frac{1}{\sqrt{\lambda}} W^*_{N(k-1)}| \geq (4 + 4\lambda^{-1})(\log 2)^{-1} \log n + x\big) \\
&\leq n\mathbb{P}\big(\sup_{0\leq t\leq y} \lambda^{-1/2}|W_t| \geq (4+4\lambda^{-1})(\log 2)^{-1} \log n + x\big) + \sum_{k=1}^{n} \mathbb{P}\big(\sup_{k-1\leq t\leq k} (N(t) - N(k-1)) \geq y\big)\,.
\end{aligned}$$

Using once again Lévy's inequality and taking $y = 2^{-1}\lambda((4 + 4\lambda^{-1})(\log 2)^{-1} \log n + x)$, we get, for any integer $n \geq 2$,

$$\begin{aligned}
&\mathbb{P}\big(\sup_{0\leq t\leq y} \lambda^{-1/2}|W_t| \geq (4 + 4\lambda^{-1})(\log 2)^{-1} \log n + x\big) \\
&\qquad \leq \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{(4 + 4\lambda^{-1})(\log 2)^{-1} \log n + x}} e^{-((4+4\lambda^{-1})(\log 2)^{-1} \log n + x)} \\
&\qquad \leq \frac{n^{-4/(\log 2)}}{\sqrt{\pi}} \exp(-x)\,.
\end{aligned}$$

On another hand, by Lemma 2.2, for any $y \geq 2\lambda$,

$$\begin{aligned}
\mathbb{P}\big(\sup_{k-1\leq t\leq k} (N(t) - N(k-1)) \geq y\big) &= \mathbb{P}\big(\sup_{0\leq t\leq 1} N(t) \geq y\big) \leq \mathbb{P}\big(\sup_{0\leq t\leq 1} |N(t) - \lambda t| \geq y - \lambda\big) \\
&\leq \exp(-\lambda h((y-\lambda)/\lambda)) \leq \exp\Big(- \frac{y-\lambda}{2} \log(y/\lambda)\Big) \leq \exp\Big(- \frac{(y-\lambda)\log 2}{2}\Big)\,.
\end{aligned}$$

Therefore, for $y = 2^{-1}\lambda((4+4\lambda^{-1})(\log 2)^{-1}\log n + x)$ and $n \geq 2$,

$$\mathbb{P}\big(\sup_{k-1\leq t\leq k}(N(t)-N(k-1)) \geq 2^{-1}\lambda((2+4\lambda^{-1})(\log 2)^{-1}\log n + x)\big) \leq n^{-1}\exp\Big(-\frac{x\lambda\log 2}{4}\Big).$$

So overall, for any $n \geq 2$,

$$I_3 \leq \frac{1}{2^5\sqrt{\pi}}\exp(-x) + \exp\Big(-\frac{x\lambda\log 2}{4}\Big). \tag{2.59}$$

Starting from (2.57) and considering the upper bounds (2.54), (2.58) and (2.59), (2.56) follows.

We turn now to the proof of (1.15) with $(W_t)_{t\geq 0}$ defined by (2.55). In the rest of the proof we shall show that, for any $n \geq 2$,

$$\mathbb{P}\big(\sup_{k\leq n}|\bar{S}_k - \sigma(g)W_k| \geq c\log n + dx\big) \leq A\exp(-Bx), \tag{2.60}$$

where

$$d := 3 + |\alpha|\mathbb{E}(\tau_1) + 4v + 5\|g\|_\infty + 5|\alpha|$$

$$A := A_1 + 2A_4 + A_5 + A_7 + \mathbb{E}_\pi(e^{\delta T_0}) + 2 + \sqrt{\frac{2}{\pi}}$$

$$B := \min\Big(B_1, B_4, B_5, B_7, 2, \delta, \frac{\lambda(1-\log 2)}{2\|g\|_\infty}, 2v^{-1}\sqrt{\lambda}, \frac{2\lambda}{\alpha\mathbb{E}(\tau_1)}\Big)$$

and

$$c := c_1 + \frac{2v}{\sqrt{\lambda}\log 2} + \frac{2\alpha\mathbb{E}(\tau_1)}{\lambda\log 2} + \frac{\|g\|_\infty(1+\gamma)}{\log 2} + |\alpha\mathbb{E}(\tau_1)|C_5\big(1 + \frac{|\log\gamma|}{\log 2}\big),$$

with

$$c_1 = 2vC_7 + 2C_4\|g\|_\infty + 2v(\log 2)^{-1} + 2C_1 + \frac{|\alpha|\mathbb{E}(\tau_1)}{\log 2} + 2|\alpha|C_4. \tag{2.61}$$

The reals α, v , λ, γ and δ involved in the definition of the constants above are defined in (2.41), (2.42), (2.45), (2.46) and (1.13) respectively, whereas the constants A_1, B_1, A_2, B_2, A_5, B_5, A_7 and B_7 have been defined previously all along the proof.

To prove (2.60), we recall the definition (2.55) of $(W_t)_{t\geq 0}$ and first write

$$\begin{aligned}
&\mathbb{P}\big(\sup_{k\leq n}|\bar{S}_k(g) - \sigma(g)W_k| \geq c\log n + dx\big)\\
&\leq \mathbb{P}\Big(\sup_{k\leq n}\big|\bar{S}_k(g) - \frac{v}{\sqrt{\lambda}}W^*_{[k/\gamma]} + \alpha\frac{\mathbb{E}(\tau_1)}{\lambda}\widetilde{W}_{[k/\gamma]}\big| \geq \Big(c - \frac{2v}{\sqrt{\lambda}\log 2} - \frac{2\alpha\mathbb{E}(\tau_1)}{\lambda\log 2}\Big)\log n + (d-2)x\Big)\\
&\qquad + \mathbb{P}\Big(\sup_{k\leq n}|W^*_{k/\gamma} - W^*_{[k/\gamma]}| \geq \frac{2}{\log 2}\log n + v^{-1}\sqrt{\lambda}x\Big)\\
&\qquad + \mathbb{P}\Big(\sup_{k\leq n}|\widetilde{W}_{k/\gamma} - \widetilde{W}_{[k/\gamma]}| \geq \frac{2}{\log 2}\log n + \frac{\lambda}{\alpha\mathbb{E}(\tau_1)}x\Big).
\end{aligned} \tag{2.62}$$

For any integer $n \geq 2$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{k\leq n}|W^*_{k/\gamma} - W^*_{[k/\gamma]}| \geq \frac{2}{\log 2}\log n + v^{-1}\sqrt{\lambda}x\Big)\\
&\qquad \leq \frac{n}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\Big(\frac{2}{\log 2}\log n + v^{-1}\sqrt{\lambda}x\Big)^2\Big) \leq \frac{1}{\sqrt{2\pi}}\exp\big(-2xv^{-1}\sqrt{\lambda}\big).
\end{aligned} \tag{2.63}$$

Similarly

$$\mathbb{P}\Big(\sup_{k\le n}\big|\widetilde{W}_{k/\gamma}-\widetilde{W}_{[k/\gamma]}\big|\ge\frac{2}{\log 2}\log n+\frac{\lambda}{\alpha\mathbb{E}(\tau_1)}x\Big)\le\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{2x\lambda}{\alpha\mathbb{E}(\tau_1)}\Big)\,. \tag{2.64}$$

On another hand, notice that

$$\sup_{k\le n}\big|\bar{S}_k(g)-\bar{S}_{[\gamma[k/\gamma]]}(g)\big|\le\|g\|_\infty\sup_{k\le n}\big|k-[\gamma[k/\gamma]]\big|\le\|g\|_\infty(1+\gamma)$$

which is less than $\frac{\|g\|_\infty(1+\gamma)}{\log 2}\log n$ for any integer $n\ge 2$. Therefore, since $N(N^{-1}(\ell))=\ell$ for any positive integer ℓ, we get that for any integer $n\ge 2$,

$$\mathbb{P}\Big(\sup_{k\le n}\big|\bar{S}_k(g)-\frac{v}{\sqrt{\lambda}}W^*_{[k/\gamma]}+\alpha\frac{\mathbb{E}(\tau_1)}{\lambda}\widetilde{W}_{[k/\gamma]}\big|\ge\Big(c-\frac{2v}{\sqrt{\lambda}\log 2}-\frac{2\alpha\mathbb{E}(\tau_1)}{\lambda\log 2}\Big)\log n+(d-2)x)\Big)$$
$$\le I_4+I_5\,, \tag{2.65}$$

where, by setting $k_\gamma=k/\gamma$,

$$I_4:=\mathbb{P}\Big(\sup_{\gamma\le k\le n}\big|N^{-1}([k_\gamma])-\frac{1}{\lambda}[k_\gamma]-\frac{1}{\lambda}\widetilde{W}_{[k_\gamma]}\big)\big|\ge\Big(1+\frac{|\log\gamma|}{\log 2}\Big)C_5\log n+x\Big)\,,$$

and

$$I_5:=\mathbb{P}\Big(\sup_{\gamma\le k\le n}\big|\bar{S}_{[\gamma N(N^{-1}([k_\gamma])]}(g)-\frac{v}{\sqrt{\lambda}}W^*_{N(N^{-1}([k_\gamma])}+\alpha\mathbb{E}(\tau_1)\big(N^{-1}([k_\gamma])-\frac{1}{\lambda}[k_\gamma]\big)\big|\ge c_1\log n+d_1x\Big)\,,$$

where $d_1=d-2-|\alpha|\mathbb{E}(\tau_1)$ and c_1 is defined in (2.61). Applying (2.53), we infer that, for any $n\ge 2$,

$$I_4\le A_5\exp(-B_5x)\,. \tag{2.66}$$

We handle now I_5. Note that

$$I_5\le\mathbb{P}\Big(\sup_{t\le N^{-1}([n/\gamma])}\big|\bar{S}_{[\gamma N(t)]}(g)-\frac{v}{\sqrt{\lambda}}W^*_{N(t)}+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\lambda}N(t)\big)\big|\ge c_1\log n+d_1x\Big)$$
$$\le\mathbb{P}\Big(\sup_{t\le 2n}\big|\bar{S}_{[\gamma N(t)]}(g)-\frac{v}{\sqrt{\lambda}}W^*_{N(t)}+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\lambda}N(t)\big)\big|\ge c_1\log n+d_1x\Big)+\mathbb{P}\Big(N^{-1}([n/\gamma])>2n\Big)\,. \tag{2.67}$$

If $n<\gamma$, then $N^{-1}([n/\gamma])=0$ and the second term in the right-hand side is zero. Assume now that $n\ge\gamma$. Since $N^{-1}([n/\gamma])$ has a Gamma distribution with parameters $[n/\gamma]$ and λ, we have

$$\mathbb{P}\Big(N^{-1}([n/\gamma])>2n\Big)=\frac{\lambda}{([n/\gamma]-1)!}\int_{2n}^{+\infty}(\lambda x)^{[n/\gamma]-1}e^{-\lambda x}dx$$
$$\le\lambda\times 2^{[n/\gamma]-1}\int_{2n}^{+\infty}e^{-\lambda x/2}dx=2^{[n/\gamma]}e^{-n\lambda}\,.$$

Therefore since $\lambda\gamma=\mathbb{E}(\tau_1)\ge 1$ and $x\le 2n\|g\|_\infty$,

$$\mathbb{P}\Big(N^{-1}([n/\gamma])>2n\Big)\le\exp\big(-n\lambda(1-\log 2)\big)\le\exp\Big(-x\frac{\lambda(1-\log 2)}{2\|g\|_\infty}\Big)\,. \tag{2.68}$$

Moreover, by using (2.56), we get that, for any integer $n\ge 2$,

$$\mathbb{P}\Big(\sup_{t\le 2n}\big|\bar{S}_{[\gamma N(t)]}(g)-\frac{v}{\sqrt{\lambda}}W^*_{N(t)}+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\lambda}N(t)\big)\big|\ge c_1\log n+d_1x\Big)\le A_7\exp(-B_7x)$$
$$+\mathbb{P}\Big(\sup_{t\le 2n}\big|\bar{S}_{[\gamma N(t)]}(g)-vB_t+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\lambda}N(t)\big)\big|\ge(c_1-2vC_7)\log n+(d_1-3v)x\Big)\,. \tag{2.69}$$

But

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\le 2n}\big|\bar S_{[\gamma N(t)]}(g)-vB_t+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\lambda}N(t)\big)\big|\ge (c_1-2vC_7)\log n+(d_1-3v)x\Big)\\
&\le \mathbb{P}\Big(\sup_{t\le 2n}\big|\bar S_{[\gamma N(t)]}(g)-\bar S_{T_{[t]}}(g)\big|\ge \|g\|_\infty(2C_4\log n+4x)\Big)\\
&\quad+\mathbb{P}\Big(\sup_{t\le 2n} v\big|B_t-B_{[t]}\big|\ge v(2(\log 2)^{-1}\log n+x)\Big)\\
&\quad+\mathbb{P}\Big(\sup_{t\le 2n}\big|\bar S_{T_{[t]}}(g)-vB_{[t]}+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\gamma\lambda}T_{[t]}\big)\big|\ge \big(2C_1+\frac{|\alpha|\mathbb{E}(\tau_1)}{\log 2}\big)\log n+(1+|\alpha|+\|g\|_\infty)x\Big)\\
&\quad+\mathbb{P}\Big(\sup_{t\le 2n}\big|\alpha\mathbb{E}(\tau_1)\big(\lambda^{-1}N(t)-(\lambda\gamma)^{-1}T_{[t]}\big)\big|\ge |\alpha|(2C_4\log n+4x)\Big)\\
&:= I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}\,. \qquad (2.70)
\end{aligned}$$

By (2.50), for any integer $n\ge 2$,

$$I_5^{(1)}\le \mathbb{P}\Big(\sup_{t\le 2n}\big|\gamma N(t)-T_{[t]}\big|\ge C_4\log(2n)+4x\Big)\le A_4\exp(-B_4x)\,. \qquad (2.71)$$

To handle $I_5^{(4)}$, we first notice that $\lambda\gamma=\mathbb{E}(\tau_1)$. Therefore, applying (2.50), we get that, for any integer $n\ge 2$,

$$I_5^{(4)}\le A_4\exp(-B_4x)\,. \qquad (2.72)$$

On another hand, by using Lévy's inequality as we did in (2.58), we infer that, for any $n\ge 2$,

$$I_5^{(2)}\le 2n\mathbb{P}\Big(\sup_{0\le t\le 1}\big|B_t-B_{[t]}\big|\ge 2(\log 2)^{-1}\log n+x\Big)\le\exp(-2x)\,. \qquad (2.73)$$

Let us now handle $I_5^{(3)}$. With this aim, taking into account that $\gamma\lambda=\mathbb{E}(\tau_1)$, we first write

$$\begin{aligned}
\sup_{t\le 2n}\big|\bar S_{T_{[t]}}(g)-vB_{[t]}+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\gamma\lambda}T_{[t]}\big)\big|&\le \sup_{k\le 2n}\big|\bar S_{T_k}-vB_k+\alpha\big(T_k-k\mathbb{E}(\tau_1)\big)\big|+|\alpha|\mathbb{E}(\tau_1)\\
&\le \sup_{k\le 2n}\big|\bar S_{T_k}(g)-\bar S_{T_0}(g)-vB_k+\alpha\big(T_k-T_0-k\mathbb{E}(\tau_1)\big)\big|+|\alpha|\mathbb{E}(\tau_1)+T_0(|\alpha|+\|g\|_\infty)\,.
\end{aligned}$$

Therefore, taking into account (2.43), we derive that, for any integer $n\ge 2$,

$$\begin{aligned}
I_5^{(3)}&\le \mathbb{P}_\pi\Big(\sup_{k\le 2n}\big|\bar S_{T_k}(g)-\bar S_{T_0}(g)-vB_k+\alpha\big(T_{[k]}-T_0-k\mathbb{E}(\tau_1)\big)\big|\ge 2C_1\log n+x\Big)\\
&\quad+\mathbb{P}\Big(|\alpha|\mathbb{E}(\tau_1)+T_0(|\alpha|+\|g\|_\infty)\ge\frac{|\alpha|\mathbb{E}(\tau_1)}{\log 2}\log n+(|\alpha|+\|g\|_\infty)x\Big)\\
&\le A_1\exp(-B_1x)+\mathbb{P}_\pi\Big(T_0\ge x\Big)\,.
\end{aligned}$$

Hence, for any $n\ge 2$,

$$I_5^{(3)}\le A_1\exp(-B_1x)+\mathbb{E}_\pi(e^{\delta T_0})\exp(-\delta x)\,, \qquad (2.74)$$

where δ is defined in (1.13). Starting from (2.70) and considering the upper bounds (2.71), (2.72), (2.73) and (2.74), it follows that, for any integer $n\ge 2$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\le 2n}\big|\bar S_{[\gamma N(t)]}-vB_t+\alpha\mathbb{E}(\tau_1)\big(t-\frac{1}{\lambda}N(t)\big)\big|\ge (c_1-2vC_7)\log n+(d_1-3v)x\Big)\\
&\qquad\le A_1\exp(-B_1x)+2A_4\exp(-B_4x)+\exp(-2x)+\mathbb{E}_\pi(e^{\delta T_0})\exp(-\delta x)\,. \qquad (2.75)
\end{aligned}$$

Starting from (2.67) and considering the upper bounds (2.68), (2.69) and (2.75), we then get that, for any integer $n \geq 2$,

$$I_5 \leq A_1 \exp(-B_1 x) + 2A_4 \exp(-B_4 x) + A_7 \exp(-B_7 x) \\ + \exp\Big(- x \frac{\lambda(1-\log 2)}{2\|g\|_\infty} \Big) + \exp(-2x) + \mathbb{E}_\pi(e^{\delta T_0}) \exp(-\delta x)\,. \quad (2.76)$$

Starting from (2.62) and considering the upper bounds (2.63), (2.64), (2.65), (2.66) and (2.76), the inequality (2.60) follows. This ends the proof of the theorem.

References

[1] Berkes, I., Liu, W. and Wu, W.B. (2014). Komlós-Major-Tusnády approximation under dependence. *Ann. Probab.* **42**, 794-817.

[2] Bolthausen, E. (1980). The Berry-Esseen theorem for functionals of discrete Markov chains. *Z. Wahrsch. Verw. Gebiete* **54**, 59-73.

[3] Bolthausen, E. (1982). The Berry-Esseen theorem for strongly mixing Harris recurrent Markov chains. *Z. Wahrsch. Verw. Gebiete* **60**, 283-289.

[4] Bretagnolle, J. and Massart, P. (1989). Hungarian constructions from the nonasymptotic viewpoint. *Ann. Probab.* **17**, 239-256.

[5] Chen, X. (1999). Limit theorems for functionals of ergodic Markov chains with general state space. *Mem. Amer. Math. Soc.* **139** no. 664.

[6] Csáki, E. and Csörgö, M. (1995). On additive functionals of Markov chains. *J. Theoret. Probab.* **8**, 905-919.

[7] Csörgö, M., Deheuvels, P. and Horváth, L. (1987). An approximation of stopped sums with applications in queueing theory. *Adv. in Appl. Probab.* **19**, 674-690.

[8] Csörgö, M., Horváth, L. and Steinebach, J. (1987). Invariance principles for renewal processes. *Ann. Probab.* **15**, 1441-1460.

[9] Einmahl, U. (1989). Extensions of results of Komlós, Major, and Tusnády to the multivariate case. *J. Multivariate Anal.* **28**, 20-68.

[10] Götze, F. and Zaitsev, A. Yu. (2009). Bounds for the rate of strong approximation in the multidimensional invariance principle. *Theory Probab. Appl.* **53**, 59-80

[11] Gouëzel, S. (2010). Almost sure invariance principle for dynamical systems by spectral methods, *Ann. Probab.* **38**, 1639-1671.

[12] Komlós, J., Major, P. and Tusnády, G. (1975). An approximation of partial sums of independent RV's, and the sample DF. I. *Z. Wahrsch. Verw. Gebiete* **32**, 111-131.

[13] Komlós, J., Major, P. and Tusnády, G. (1976). An approximation of partial sums of independent RV's, and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* **34**, 33-58.

[14] Major, P. (1976). The approximation of partial sums of independent RV's. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **35**, 213-220.

[15] Merlevède, F. and Rio, E. (2012). Strong approximation of partial sums under dependence conditions with application to dynamical systems. *Stochastic Process. Appl.* **122**, 386-417.

[16] Nummelin, E. (1984). *General irreducible Markov chains and nonnegative operators*. Cambridge Tracts in Mathematics, **83**. Cambridge University Press, Cambridge.

[17] Nummelin, E. and Tuominen, P. (1982). Geometric ergodicity of Harris recurrent Markov chains with applications to renewal theory. *Stochastic Process. Appl.* **12**, 187-202.

[18] Shao, Q.M. and Lu, C.R. (1987). Strong approximations for partial sums of weakly dependent random variables. *Sci. Sinica Ser. A* **30**, 575-587.

[19] Zaitsev, A. Yu. (1998). Multidimensional version of the results of Komlós, Major and Tusnády for vectors with finite exponential moments. *ESAIM Probab. Statist.* **2** 41-108