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A NATURAL GENERALISATION OF BALANCED TABLEAUX

FRANÇOIS VIARD

ABSTRACT. We introduce the notion of “type” that allows us to define new families of tableaux, which include both balanced and standard Young tableaux. We use these new objects to describe the set of reduced decompositions of any permutation. Moreover, we generalise the work of Edelman and Greene on balanced tableaux by giving among other things, a new proof of the fact that balanced tableaux and standard Young tableaux are equinumerous.

INTRODUCTION

The problem of enumerating reduced decompositions of a permutation was first studied by Stanley in [7]. By using symmetric function techniques, he showed that $|\text{Red}(\omega_0)|$, the number of reduced decomposition of the longest permutation ω_0 in the symmetric group S_n , is given by the number of standard Young tableaux of *staircase shape* $\lambda_n = (n - 1, n - 2, \dots, 1)$. Moreover, he conjectured that for an arbitrary permutation σ , $|\text{Red}(\sigma)| = \sum a_{\sigma,\lambda} f^\lambda$, where λ runs over the partitions of n , f^λ is the number of standard tableaux of shape λ , and $a_{\sigma,\lambda}$ are nonnegative integers.

In [3], Edelman and Greene gave a combinatorial proof of this conjecture by introducing an algorithm, now called the Edelman-Green insertion. This algorithm has been recently investigated by Billey and Pawlowski [1], who highlighted the links between values of the coefficients $a_{\sigma,\lambda}$ and some patterns which are avoided by σ . Another proof of the conjecture is based on the Lascoux-Schutzenberger tree [5]. The link between these two different approaches has been recently studied by Hamaker and Young [4], using the very elegant work of Little [6].

In another part of the same paper [3], Edelman and Greene introduced a new family of combinatorial objects called *balanced tableaux*, in an attempt to prove bijectively Stanley’s result for the longest permutation. They proved that balanced and standard Young tableaux of same shape are equinumerous. Their proof is quite involved and a direct bijective one is still missing.

The goal of this paper is to generalize Edelman-Greene work on balanced tableaux in different directions. In order to do this, we deal with a bigger class of *tableaux* which are not required to be standard or balanced. To each tableau T of shape S , where S is any finite subset of $\mathbb{N} \times \mathbb{N}$, we associate a combinatorial object \mathcal{T} called the *type of T* . This allows us to split that set of tableaux into different classes: two tableaux will be in the same class if and only if they have the same type. Let us denote $\text{Tab}(\mathcal{T})$ the set of tableaux of type \mathcal{T} . Both sets of balanced and standard

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Young tableaux of a given shape λ can be seen as special classes of our classification. We construct all the tableaux of a fixed type thanks to an iterative process. In particular, we can use this procedure to easily construct all balanced tableaux of a given shape.

In [2] an explicit bijection between $\text{Red}(\omega_0)$ and the balanced tableaux of staircase shape is given. We generalise this result in (Theorem 2.10) as follows.

Theorem A. *For each $\sigma \in S_n$, there exists a bijection between $\text{Red}(\sigma)$ and $\text{Tab}(\mathcal{T}_\sigma)$, where \mathcal{T}_σ is a special type associated to σ .*

This result gives a new combinatorial way of obtaining all the reduced decompositions of any fixed permutation. In Section 3, we focus on vexillary permutations, namely 2143 avoiding permutations. It is well-known [7], that each vexillary permutation σ is associated with a partition $\lambda(\sigma)$, and that the number of reduced decompositions of σ is $f^{\lambda(\sigma)}$. In Section 3.2, we introduce a combinatorial process on types, that together with Theorem A, leads us to our second main result (Theorem 3.18), which can be written as follows.

Theorem B. *For each vexillary permutation σ , there exists a type \mathcal{T}_σ^E , such that $|\text{Tab}(\mathcal{T}_\sigma^E)| = f^{\lambda(\sigma)}$. Moreover, all tableaux in $\text{Tab}(\mathcal{T}_\sigma^E)$ have shape $\lambda(\sigma)$.*

From this, a new and more accessible proof of the Edelman-Greene result on the equality between the number of balanced and standard Young tableaux, will arise (Corollary 3.22).

Finally, consequences of our work allow us to answer to a problem raised by Edelman and Green. More precisely, in [3] the authors enumerate in some cases the number of balanced tableaux of shape λ having the integer n in a given fixed position, and ask for a general answer to this problem. In Section 3.6, we deal with a more general situation where the integers $n, n-1, \dots, n-k$ are in given fixed positions. This leads to the following result.

Theorem C. *Set $\lambda \vdash n$. Set A the number of balanced tableaux of shape λ with the integers $n, n-1, \dots, n-k$ in given fixed positions. Then there exists a permutation $\omega \in S_m$ such that $A = |\text{Red}(\omega)|$.*

1. TYPE OF A TABLEAU

1.1. Partitions. We begin with some standard notation. A *partition* λ of a non-negative integer $n \in \mathbb{N}$, is a nonincreasing sequence of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \dots$ such that $\sum \lambda_i = n$. The integers $\lambda_i \neq 0$ are called *parts* of the partition λ .

The *Ferrers diagram* of λ is a finite collection of boxes, or cells, arranged in left-justified rows of lengths given by the parts of λ . By flipping this diagram over its main diagonal we obtain the diagram of the *conjugate partition* of λ denoted λ' . We usually identify a partition with its Ferrers diagram.

1.2. Tableaux. In this paper we deal with Ferrers diagrams, but more generally with *diagrams*, namely finite subsets of $\mathbb{N} \times \mathbb{N}$, without any constrain. So let $S \subset \mathbb{N} \times \mathbb{N}$ such that $|S| = n$ (where $|S|$ denote the cardinal of S). A *tableau* T of

shape S is a bijective filling of S (seen as a set of boxes) with entries in $\{1, 2, \dots, n\}$. Given a tableau we denote its shape by $\text{Sh}(T)$. The set of all tableaux of shape S will be denoted by Tab_S . If we require $\text{Sh}(T)$ to be a partition λ , then T will be what is usually called a *Young tableau*. Moreover, if we consider tableaux satisfying the conditions that the filling is

- (1) increasing from left to right across each row;
- (2) increasing down each column;

we obtain the set of *standard Young tableaux* of shape λ , denoted by $\text{SYT}(\lambda)$, whose cardinality $f^\lambda = |\text{SYT}(\lambda)|$ is given by the well-known hook-length formula.

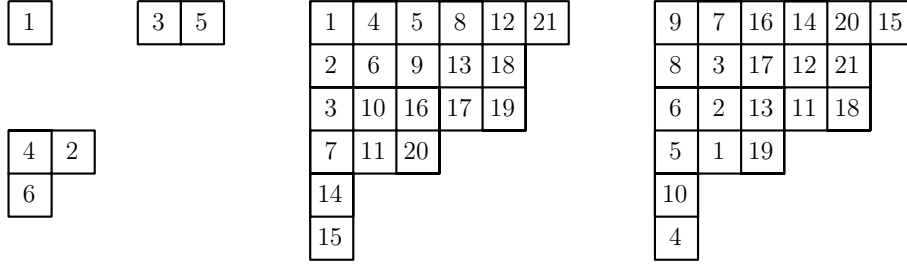


FIGURE 1. A tableau, a standard tableau, and a balanced tableau.

1.3. Type of a tableau.

Definition 1.1. Fix a diagram S . Any box $c = (a, b) \in S$ determines a *hook* $H_c(S)$, which consists of that box and all boxes below in the same column (the *leg* $L_{a,b}(S)$), and strictly on its right in the same row (the *arm* $A_{a,b}(S)$). More precisely

$$L_{a,b}(S) = \{ (k, b) \mid k \geq a, (k, b) \in S \}, \quad A_{a,b}(S) = \{ (a, k) \mid k > b, (a, k) \in S \}, \quad (1)$$

$$H_{a,b}(S) = A_{a,b} \uplus L_{a,b}. \quad (2)$$

We will denote by $l_{a,b}(S)$, $a_{a,b}(S)$, and $h_{a,b}(S)$ their cardinalities.

Definition 1.2. Set S a diagram and $T = (t_c)_{c \in S} \in \text{Tab}_S$. The *type* of T is the filling $\mathcal{T} = (\theta_c)_{c \in S}$ of S defined by $\theta_c = |\{z \in H_c(S) \mid t_z > t_c\}|$. We note by $\text{Tab}_S(\mathcal{T})$ the set of all tableaux of shape S whose type is \mathcal{T} . When there is not any ambiguity we will simply denote this set by $\text{Tab}(\mathcal{T})$.

More generally, we can define a type without using a tableau as follows.

Definition 1.3. Let S be a diagram and $\mathcal{T} = (\theta_c)_{c \in S}$ a filling of S with $\theta_c \in \mathbb{N}$. \mathcal{T} is called a *type of shape* S if and only if for all $c \in S$, $0 \leq \theta_c \leq h_c(S) - 1$. We denote $\text{Sh}(\mathcal{T})$ the shape of a type. The set of all types of shape S is denoted by $\text{Type}(S)$. When S is the Ferrers diagram of a partition λ , we will denote this set by $\text{Type}(\lambda)$.

Remark 1.4. By definition, if $T \in \text{Tab}_S$ for a given diagram S , then the type of T is a type (according to Definition 1.3). Furthermore, it is easy to see that this definition generalises both the definitions of *standard Young tableaux* and *balanced tableaux*

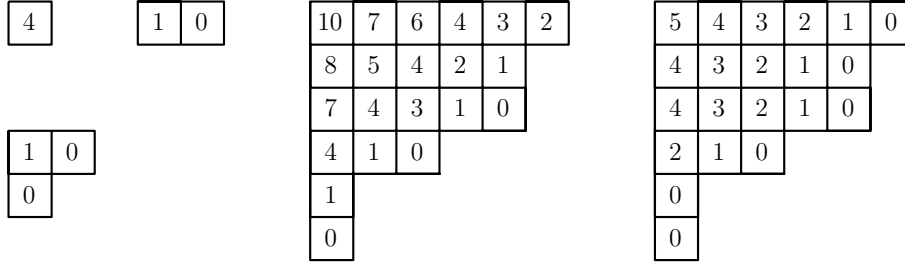


FIGURE 2. The types of the tableaux in Figure 1.

[3]. Set $\mathcal{S}_\lambda = (s_c) \in \text{Type}(\lambda)$ where $s_c = h_c(\lambda) - 1$, then $\text{Tab}_\lambda(\mathcal{S}_\lambda) = \text{SYT}(\lambda)$. In the same way, set $\mathcal{B}_\lambda = (\beta_c) \in \text{Type}(\lambda)$ with $\beta_c = a_c(\lambda)$, then $\text{Tab}_\lambda(\mathcal{B}_\lambda) = \text{Bal}(\lambda)$ the set of *balanced tableaux* of shape λ . An example of a balanced type is given at the right of Figure 2.

Now we recall one of the main result of ([3], Theorem 2.2) which will be generalised by Theorem 3.18 of this paper.

Theorem 1.5 (Edelman-Greene). *For all $\lambda \vdash n$, then $|\text{SYT}(\lambda)| = |\text{Bal}(\lambda)|$.*

1.4. Tableaux of fixed type. In this section, we introduce the definition of a *filling sequence* which will allowed us to prove that for each $\mathcal{T} \in \text{Type}(S)$, $\text{Tab}_S(\mathcal{T})$ is not empty.

Definition 1.6. Set S a diagram and $\mathcal{T} = (\theta_c)_{c \in S} \in \text{Type}(S)$. A box $c \in S$ is called *erasable in \mathcal{T}* if and only if $\theta_c = 0$ and for all $z \in S$ such that $c \in H_z(S)$, $\theta_z \neq 0$.

Lemma 1.7. *Set S a diagram and $\mathcal{T} = (\theta_c)_{c \in S} \in \text{Type}(S)$. Then there exists $c \in S$ which is erasable.*

Proof. There exists $z \in S$ such that $|H_z(S)| = 1$ because S is finite. Then we have that $0 \leq \theta_z \leq 1 - 1 = 0$. So $\theta_z = 0$. Then, because S is finite, we have that at least one of the boxes which contain a 0 is erasable, and this concludes the proof. \square

Definition 1.8 (Filling sequence). Set S a diagram, $n = |S|$, and $\mathcal{T} = (\theta_c)_{c \in S} \in \text{Type}(S)$. Set $L = [c_1, \dots, c_n]$ a sequence such that $S = \{c_1, \dots, c_n\}$. We say that L is a *filling sequence* of \mathcal{T} if and only if there exists a sequence $(\mathcal{P}^0, \mathcal{P}^1, \dots, \mathcal{P}^n)$ such that for all $1 \leq i \leq n$, $\mathcal{P}^i = (\theta_c^i) \in \text{Type}(S \setminus \{c_1, \dots, c_i\})$, that can be constructed recursively as follows:

- (1) $\mathcal{P}^0 = \mathcal{T}$;
- (2) c_{k+1} is erasable in \mathcal{P}^k , and \mathcal{P}^{k+1} can be obtained from \mathcal{P}^k in this way: for all $c \in \text{Sh}(\mathcal{P}^{k+1})$, if $c_{k+1} \in H_c(\text{Sh}(\mathcal{P}^k))$, then $\theta_c^{k+1} = \theta_c^k - 1$, and for all the other $c \in \text{Sh}(\mathcal{P}^{k+1})$, $\theta_c^{k+1} = \theta_c^k$.

The set of all the filling sequences of \mathcal{T} is denoted $\text{Fil}(\mathcal{T})$.

At first glance, this definition appears to be quite technical. Indeed it is quite easy to construct **all** filling sequences thanks to the next recursive procedure.

Recursive process 1.9. We start with the type \mathcal{T} and we set $\mathcal{P}^0 = \mathcal{T}$. In the first step we localise all the boxes $c \in S$ which are erasable in \mathcal{T} (circled boxes in

Example 1.10). In the second step we chose one of those boxes c_1 (we can chose anyone of them). In the last step, we construct the type \mathcal{P}^1 from \mathcal{P}^0 by first decreasing by one all the θ_z^0 , where $z \in S$ and $c_1 \in H_z(S)$, letting the others entries unchanged, and to finish by deleting the box c_1 . Then we repeat those steps until we obtain the type \mathcal{P}^n which is empty. At the end, the sequence $L = [c_1, \dots, c_n]$ is a filling sequence by construction.

Example 1.10. We consider $\lambda = (2, 2) \vdash 4$ and $\mathcal{T} \in \text{Type}(\lambda)$ which is represented in the top left of Figure 3. Then we perform Process 1.9 on \mathcal{T} in order to construct a filling sequence of \mathcal{T} . All the erasable boxes of \mathcal{T} are circled for the first iteration of the procedure, and this has been omitted in the others.

$\mathcal{T} = \mathcal{P}^0$			\mathcal{P}^1		\mathcal{P}^2
	Step 1	Step 2	Step 3		
		$c_1 = (1, 2)$		$c_2 = (1, 1)$	
	\mathcal{P}^3	\mathcal{P}^4	The sequence $L = [c_1, c_2, c_3, c_4]$ is a filling sequence of \mathcal{T}		
		\emptyset			
$c_3 = (2, 1)$	$c_4 = (2, 2)$				

FIGURE 3.

Remark 1.11. Thanks to Lemma 1.7, this process can always be performed. As a consequence, $\text{Fil}(\mathcal{T}) \neq \emptyset$.

Now we focus on the connection there exists between $\text{Fil}(\mathcal{T})$ and $\text{Tab}(\mathcal{T})$.

Definition 1.12. Set $\mathcal{T} \in \text{Type}(S)$, and $L = [c_1, \dots, c_n] \in \text{Fil}(\mathcal{T})$. We define a tableau $T_L = (t_c)_{c \in S} \in \text{Tab}_S$ by setting $t_{c_k} = n + 1 - k$.

Lemma 1.13. For any filling sequence L of $\mathcal{T} \in \text{Type}(S)$, $T_L \in \text{Tab}(\mathcal{T})$.

Proof. Let $L = [c_1, \dots, c_n]$ be a filling sequence of $\mathcal{T} = (\theta_c)$, set $T_L = (t_c)$, $\mathcal{T}' = (\theta'_c)$ the type of T_L and $1 \leq k \leq n$. By construction $t_{c_k} = n + 1 - k$ and $\theta_{c_k}^k = 0$. For each $c \in S$, we define the following set

$$U_{c,k} = H_c(S) \cap \{c_1, \dots, c_{k-1}\}. \tag{3}$$

We use the following convention: if $k = 1$, $U_{c,k} = \emptyset$. By construction of Process 1.9, $\theta_{c_k} = \theta_{c_k}^0 = |U_{c_k,k}|$. Moreover, by definition of T_L , if $z \in H_{c_k}(S)$ then $t_z > t_{c_k}$ if and only if $z \in U_{c_k,k}$, so by definition $\theta'_{c_k} = |U_{c_k,k}|$. Then \mathcal{T}' is equal to \mathcal{T} and the lemma is proved. \square

With Remark 1.11 we have the immediate corollary.

Corollary 1.14. *For any type \mathcal{T} , $\text{Tab}(\mathcal{T}) \neq \emptyset$.*

Theorem 1.15. *Set $\mathcal{T} \in \text{Type}(S)$ for a given diagram S . The application $L \mapsto T_L$ from $\text{Fil}(\mathcal{T})$ to $\text{Tab}(\mathcal{T})$ is a bijection.*

Proof. By definition and Lemma 1.13, this application is clearly injective. Now, we will prove recursively on $n = |S|$ that this application is surjective. If $|S| = 1$, then the property is true since $|\text{Fil}(\mathcal{T})| = |\text{Tab}(\mathcal{T})| = 1$.

Now we suppose that the property has been proved for all the diagrams of cardinality n . Set S a diagram such that $|S| = n + 1$, $\mathcal{T} = (\theta_c)_{c \in S} \in \text{Type}(S)$, and $T \in \text{Tab}_S(\mathcal{T})$. We define the sequence $L = [c_1, \dots, c_{n+1}]$ such that for all k , $t_{c_k} = n + 2 - k$. Hence proving the property is equivalent to show that $L \in \text{Fil}(\mathcal{T})$. To do this, we use the Process 1.9: by definition $t_{c_1} = n + 1$, hence for all $z \in H_{c_1}(S)$, $t_z < n + 1 = t_{c_1}$, so $\theta_{c_1} = 0$. Moreover, for all z such that $c_1 \in H_z(S)$ then $t_z < t_{c_1}$, so $\theta_z \neq 0$. So c_1 is erasable, and it can be picked at the first iteration of Process 1.9. Now we consider the type \mathcal{P}^1 obtained after we picked c_1 . By definition, $\mathcal{P}^1 \in \text{Type}(S \setminus \{c_1\})$. We also consider the tableau $T' = (t_c)_{c \in S \setminus \{c_1\}}$ obtained from T just by removing the box c_1 . Then, by construction we have that $T' \in \text{Tab}_{S \setminus \{c_1\}}(\mathcal{P}^1)$. Hence by induction, we have that the sequence $[c_2, \dots, c_{n+1}] \in \text{Fil}(\mathcal{P}^1)$, and by construction of Process 1.9 $[c_1, \dots, c_{n+1}] \in \text{Fil}(\mathcal{T})$, and this concludes the proof. \square

We give an example to visualise a dynamic construction of the tableau T_L associated with a filling sequence L .

Example 1.16. Consider the type \mathcal{T} on the top-left of Figure 4, and the filling sequence $L = [(1, 3); (1, 1); (1, 2); (2, 2); (2, 1)]$. In the top line is represented the sequence $(\mathcal{P}^0, \dots, \mathcal{P}^5)$ associated to L . The corresponding tableau T_L is depicted in the bottom-right of the figure.

$\mathcal{T} = \mathcal{P}^0$	\mathcal{P}^1	\mathcal{P}^2	\mathcal{P}^3	\mathcal{P}^4	\mathcal{P}^5																																				
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FIGURE 4.

Remark 1.17. We have some basic properties. Obviously, if $\lambda = (n)$ or $\lambda = (1^n)$ then for any $\mathcal{T} \in \text{Type}(\lambda)$, there exists a unique tableau of type \mathcal{T} . This is because at each iteration of the Process 1.9, there is only one $(i, j) \in \lambda$ which is erasable. This fact leads us to a first interesting property.

Proposition 1.18. *Let k and p be two integers. Set $\lambda = (k, 1^p)$ and \mathcal{T} a type of shape λ . Then there exists f^λ tableaux of type \mathcal{T} .*

Proof. Set $\mathcal{T} = (\theta_c) \in \text{Type}((k, 1^p))$. Set L a filling of \mathcal{T} . To begin, we remark that in the tableau $T_L = (t_c)$ we have $t_{1,1} = n - \theta_{1,1}$. So if we set $S = \lambda - \{(1, 1)\}$

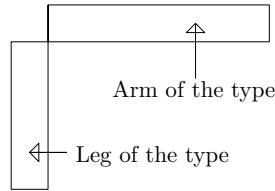


FIGURE 5.

(see Figure 5) and we consider the type $\mathcal{T}' = (\theta_c)_{c \in S}$, then we have $|\text{Tab}_\lambda(\mathcal{T})| = |\text{Tab}_S(\mathcal{T}')|$.

Now the result follows from Remark 1.17. In fact at each iteration of Process 1.9, the only thing we have to choose is an element in the leg or in the arm of S . This is independent of the choice of \mathcal{T} , so the result is proved. \square

This first result might lead us to think that there should be a simple way to calculate the number of tableaux of a given type, such as a general “hook-length formula”, but a quick verification shows that the situation seems to be considerably more complicated. For example the number of tableaux of type

0	0	0
2	0	
0		

is 11 which do not divide $6!$. Nevertheless, we have a sort of probabilist result.

Theorem 1.19. *If we consider the set of all types of shape λ and we choose uniformly a type \mathcal{T} , then the expected value for $|\text{Tab}_\lambda(\mathcal{T})|$ is f^λ .*

Proof. Set $H_\lambda = \prod_{(a,b) \in \lambda} h_{a,b}(\lambda)$. It is obvious that the number of type of shape λ is precisely H_λ . Then because of the uniform choice, the probability for a type \mathcal{T} to be chosen is exactly $\frac{1}{H_\lambda}$. Then by definition, the expected value is $\frac{\sum_{\mathcal{T} \in \text{Type}(\lambda)} |\text{Tab}_\mathcal{T}(\lambda)|}{H_\lambda}$, and the numerator is trivially equal to $n!$, so the result follows. \square

Now, a few questions arise.

Question 1.20. Is it possible to find a general formula to compute $|\text{Tab}(\mathcal{T})|$ for any given type \mathcal{T} ?

Question 1.21. Set $\lambda \vdash n$. Can we find a subset $\Lambda \subset \text{Type}(\lambda)$ such that $|\text{Tab}_\lambda(\mathcal{T})| = f^\lambda$ for any $\mathcal{T} \in \Lambda$, as the standard and the balanced type?

The first question is still open, but in Section 3 we will give a positive answer to the second one.

2. TYPES AND REDUCED DECOMPOSITIONS

In [3], Edelman and Greene exhibit a bijection between balanced tableaux of staircase shape λ_n and $\text{Red}(\omega_0)$. In this section we will rephrase this result in our terminology and show how it can be generalized to each permutation. A new combinatorial interpretation of the set of reduced decompositions of any permutation will arise.

2.1. Definition and Theorems. It is well-known that the symmetric group S_n is generated by the *elementary transpositions* $s_i = (i, i + 1)$, $1 \leq i \leq n - 1$. For any $\sigma \in S_n$, there exists a minimal integer k such that $\sigma = s_{i_1} \cdots s_{i_k}$ called the *length* of σ , and denoted $\ell(\sigma)$. Such a product is called a *reduced decomposition* of σ . The set of all the reduced decompositions of σ is denoted $\text{Red}(\sigma)$. Usually we denote a permutation $\sigma \in S_n$ in its *window notation* $\sigma = [\sigma(1), \dots, \sigma(n)]$. It is well-known that $\ell(\sigma)$ is equal to the number of inversions of σ , namely the cardinality of the set

$$\text{Inv}(\sigma) = \{(p, q) \in \mathbb{N}^2 \mid p < q \text{ and } \sigma^{-1}(p) > \sigma^{-1}(q)\}. \quad (4)$$

We have the following classical result: for σ and ω two permutations, $\sigma = \omega$ if and only if $\text{Inv}(\sigma) = \text{Inv}(\omega)$. Now we will define the (*right*) *weak Bruhat* order. Set σ and ω two permutations, we said that $\sigma \leq_R \omega$ if and only if there exists k such that $\omega = \sigma s_{i_1} \cdots s_{i_k}$ and $\ell(\omega) = \ell(\sigma) + k$.

For any undefined notations we refer to the book [2]

Remark 2.1. In all this section, it will be convenient to change the coordinates of the cells of the diagram of $\lambda_n : (a, b)$ will denote the cell previously denoted by $(a, n + 1 - b)$ (see Figure 6 for an illustration on λ_7).

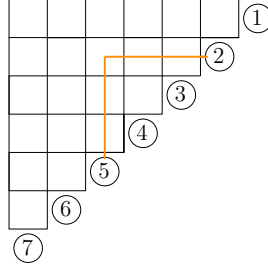


FIGURE 6. The cell previously denoted by $(2, 3)$ is now $(2, 5)$.

Recall the definition of the *balanced type* $\mathcal{B}_{\lambda_n} = (\beta_c)_{c \in \lambda_n} \in \text{Type}(\lambda_n)$ given in Remark 1.4: for all $c \in \lambda_n$, $\beta_c = a_c(\lambda_n)$ (Definition 1.1). With the coordinates of Remark 2.1 we have that for each $c = (a, b) \in \lambda_n$,

$$\beta_c = |\{q \in \mathbb{N} \mid a < q < b\}| = b - a - 1. \quad (5)$$

The next is a well-known result of Edelman and Greene ([3], Theorem 4.2).

Theorem 2.2. *There exists a bijection between $\text{Red}(\omega_0)$ and $\text{Tab}(\mathcal{B}_{\lambda_n})$.*

Hence with Theorem 1.12 we have this immediate corollary.

Corollary 2.3. *There exists a bijection between $\text{Red}(\omega_0)$ and $\text{Fil}(\mathcal{B}_{\lambda_n})$.*

Remark 2.4. Since we will use this bijection later on, we give a brief sketch of its construction. Set $N = \binom{n}{2}$, and

$$\text{Id} = \sigma_0 \leq_R \sigma_1 \leq_R \dots \leq_R \sigma_N = \omega_0$$

a maximal chain in the weak order of S_n (which corresponds to a unique reduced decomposition of ω_0). Set $c_i = (a_i, b_i)$ the unique element in $\text{Inv}(\sigma_i) \setminus \text{Inv}(\sigma_{i-1})$ and set $T = (t_c)_{c \in \lambda_n} \in \text{Tab}_{\lambda_n}$ with $t_{c_i} = N + 1 - i$. Set $a_i < q < b_i$ if it exists. Since a_i and b_i are adjacent in σ_{i-1} (i.e. there exists $1 \leq k \leq n - 1$ such that $\sigma_{i-1}(k) = a_i$

and $\sigma_{i-1}(k+1) = b_i$, there exists a unique $j < i$ such that either $c_j = (q, b_i)$ or $c_j = (a_i, q)$. This is enough to show that $|\{z \in H_{c_i}(S) \mid t_z > t_{c_i}\}| = |\{q \in \mathbb{N} \mid a < q < b\}| = \beta_{(a_i, b_i)}$ (thanks to Equation 5), so T is balanced. Then $L = [c_1, \dots, c_N]$ is a filling sequence of \mathcal{B}_{λ_n} and it obviously defines an injection from $\text{Red}(\omega_0)$ to $\text{Fil}(\mathcal{B}_{\lambda_n})$.

The reverse application is constructed in this way: set $L = [c_1, \dots, c_N]$ a filling sequence of \mathcal{B}_{λ_n} and $c_i = (a_i, b_i)$. We define a sequence of permutations $(\sigma_i)_{0 \leq i \leq N}$ as follows: $\sigma_0 = Id$, and for all $0 \leq i \leq N-1$, σ_{i+1} is obtained from σ_i by exchanging the positions of a_{i+1} and b_{i+1} in the window notation of σ_i (see Figure 7). Edelman and Greene showed in [3] that for all $0 \leq i \leq N-1$, σ_{i+1} covers σ_i in the weak order (more precisely, they showed that a_{i+1} and b_{i+1} are adjacent in σ_i). Hence the sequence $(\sigma_i)_{0 \leq i \leq N}$ defines a maximal chain in the weak order, which corresponds to a unique reduced decomposition of ω_0 .

Example 2.5. We illustrate how the bijection turns $L = [(1, 2), (1, 3), (2, 3)] \in \text{Fil}(\mathcal{B}_{\lambda_3})$ into a reduced decomposition of $\omega_0 \in S_3$.

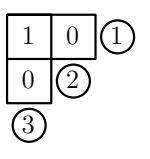
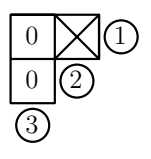
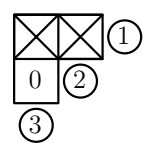
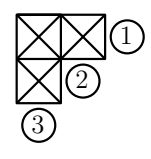
			
	$c_1 = (1, 2)$	$c_2 = (1, 3)$	$c_3 = (2, 3)$
$\sigma_0 = [1, 2, 3]$	$\sigma_1 = [2, 1, 3]$ $= \sigma_0 s_1$	$\sigma_2 = [2, 3, 1]$ $= \sigma_1 s_2$	$\sigma_3 = [3, 2, 1]$ $= \sigma_2 s_1$
<p>The element of $\text{Red}(\omega_0)$ corresponding to $[c_1, c_2, c_3] \in \text{Fil}(\mathcal{B}_{\lambda_3})$ is $s_1 s_2 s_1$.</p>			

FIGURE 7.

By using the maximality of ω_0 for the weak order and the previous result, we can in fact generalize Corollary 2.3 to all permutations. To do so, we introduce the next definition.

Definition 2.6. Set S a diagram contained in λ_n , and $\mathcal{B}_{\lambda_n} = (\beta_c)_{c \in \lambda_n}$. Set $\mathcal{A}_S = (\beta_c)_{c \in S}$, we call \mathcal{A}_S a *subtype* of \mathcal{B}_{λ_n} if and only if there exists $L = [c_1, c_2, \dots, c_N] \in \text{Fil}(\mathcal{B}_{\lambda_n})$ such that $S = \{c_1, \dots, c_{|S|}\}$. The set of all the subtypes of \mathcal{B}_{λ_n} is denoted $\text{Sub}(\mathcal{B}_{\lambda_n})$.

By definition, if $\mathcal{A} \in \text{Sub}(\mathcal{B}_{\lambda_n})$, then \mathcal{A} is a type.

Definition 2.7. For any $\sigma \in S_n$, we set $\mathcal{T}_\sigma = (\beta_c)_{c \in \text{Inv}(\sigma)}$ where $\text{Inv}(\sigma)$ is seen as a subset of λ_n with the new convention for the coordinates.

Lemma 2.8. For any $\sigma \in S_n$, \mathcal{T}_σ is a subtype of \mathcal{B}_{λ_n} .

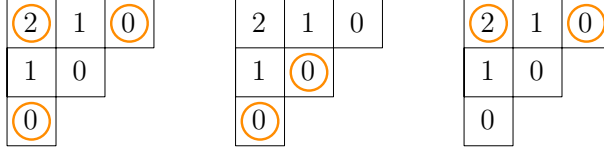


FIGURE 8. The first subset defines a subtype, the two others do not.

Proof. Set $\sigma \in S_n$ and $s_{i_1} \cdots s_{i_{\ell(\sigma)}} \in \text{Red}(\sigma)$. Then by the maximality of ω_0 , we can complete this reduced decomposition into a reduced decomposition $s_{i_1} \cdots s_{i_{\ell(\sigma)}} \cdots s_{i_N}$ of ω_0 . By Corollary 2.3 this decomposition correspond to a filling sequence $L = [c_1, \dots, c_N]$ of \mathcal{B}_{λ_n} , and by Remark 2.4 we have that $\{c_1, c_2, \dots, c_{\ell(\sigma)}\} = \text{Inv}(\sigma)$. This concludes the proof. \square

Proposition 2.9. *The application $\phi(\sigma) = \mathcal{T}_\sigma$ is a bijection between S_n and $\text{Sub}(\mathcal{B}_{\lambda_n})$*

Proof. By Lemma 2.8 and Definition 2.6, ϕ is an application from S_n to $\text{Sub}(\mathcal{B}_{\lambda_n})$. Set σ and ω two permutations. Since $\sigma = \omega$ if and only if $\text{Inv}(\sigma) = \text{Inv}(\omega)$, then ϕ is injective.

Set $\mathcal{A} \in \text{Sub}(\mathcal{B}_{\lambda_n})$, we have that $\text{Fil}(\mathcal{A}) \neq \emptyset$. Set $L \in \text{Fil}(\mathcal{A})$, we complete it into a filling sequence L' of \mathcal{B}_{λ_n} using Process 1.9. By Corollary 2.3, $L' = [c_1, \dots, c_N]$ corresponds to a reduced decomposition $s_{i_1} \cdots s_{i_N}$ of ω_0 . Set $\sigma_{\mathcal{A}} = s_{i_1} \cdots s_{i_{\ell(\sigma)}}$, then by Remark 2.4, $\{c_1, \dots, c_{|\mathcal{A}|}\} = \text{Sh}(\mathcal{A}) = \text{Inv}(\sigma_{\mathcal{A}})$. Hence $\phi(\sigma_{\mathcal{A}}) = \mathcal{A}$, and ϕ is surjective. This concludes the proof. \square

And now we can formulate the claimed generalisation.

Theorem 2.10. *For all $\sigma \in S_n$ there exists a bijection ϕ_σ between $\text{Fil}(\mathcal{T}_\sigma)$ and $\text{Red}(\sigma)$.*

Proof. Set $\sigma \in S_n$ and consider \mathcal{T}_σ . Consider a filling sequence $L \in \text{Fil}(\mathcal{T}_\sigma)$, and complete it into a filling sequence L' of \mathcal{B}_{λ_n} , which give rise to a reduced decomposition $s_{i_1} \cdots s_{i_{\ell(\sigma)}} \cdots s_{i_N}$ of ω_0 . By definition $\text{Inv}(\sigma) = \text{Sh}(\mathcal{T}_\sigma)$, so $s_{i_1} \cdots s_{i_{\ell(\sigma)}} = \sigma$. Moreover, since $s_{i_1} \cdots s_{i_N}$ is reduced, $s_{i_1} \cdots s_{i_{\ell(\sigma)}}$ is a reduced decomposition of σ . In this way we associated to each element of $\text{Fil}(\mathcal{T}_\sigma)$ a reduced decomposition of σ . Let us denote this map by ϕ_σ . By Corollary 2.3, ϕ_σ is injective.

Now consider $s_{j_1} \cdots s_{j_{\ell(\sigma)}} \in \text{Red}(\sigma)$. By maximality of ω_0 , we complete it into a reduced decomposition $s_{j_1} \cdots s_{j_{\ell(\sigma)}} \cdots s_{j_N} \in \text{Red}(\omega_0)$. By Corollary 2.3 it corresponds to $U = [z_1, \dots, z_n] \in \text{Fil}(\mathcal{B}_{\lambda_n})$, and by Remark 2.4, $\text{Inv}(\sigma) = \{z_1, \dots, z_{\ell(\sigma)}\}$, hence $[z_1, \dots, z_{\ell(\sigma)}] \in \text{Fil}(\mathcal{T}_\sigma)$, so $\phi_\sigma([z_1, \dots, z_{\ell(\sigma)}]) = s_{j_1} \cdots s_{j_{\ell(\sigma)}}$ and the application is surjective. This concludes the proof. \square

Example 2.11. In Figure 9, we show how we construct the reduced decomposition of $\sigma = [3, 1, 4, 2]$ associated to $[(2, 3), (1, 3), (2, 4)] \in \text{Fil}(\mathcal{T}_\sigma)$.

This bijection gives an easy way of obtaining all the reduced decompositions of a permutation using Process 1.9. An example of such a use of this bijection is given in Figure 10. Now, we give a combinatorial characterisation of the subtypes of \mathcal{B}_{λ_n} .

Proposition 2.12. *Set S a subdiagram of λ_n . $\mathcal{A} = (\beta_c)_{c \in S}$ is a subtype of \mathcal{B}_{λ_n} if and only if:*

- (1) for all $c \in S$, $\beta_c \leq |H_c(S)| - 1$,

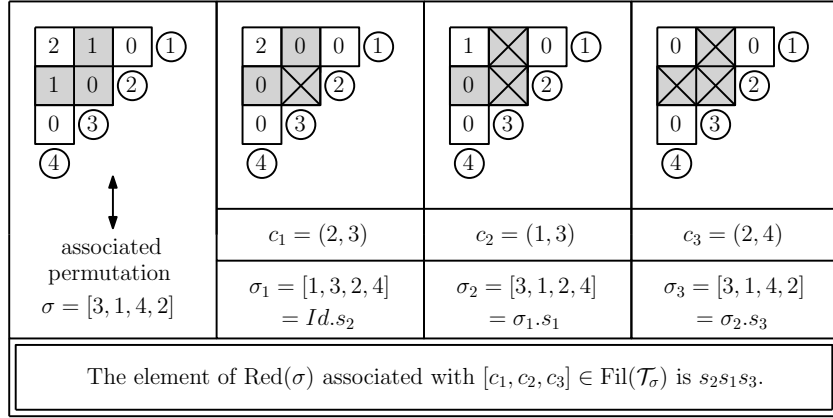


FIGURE 9.

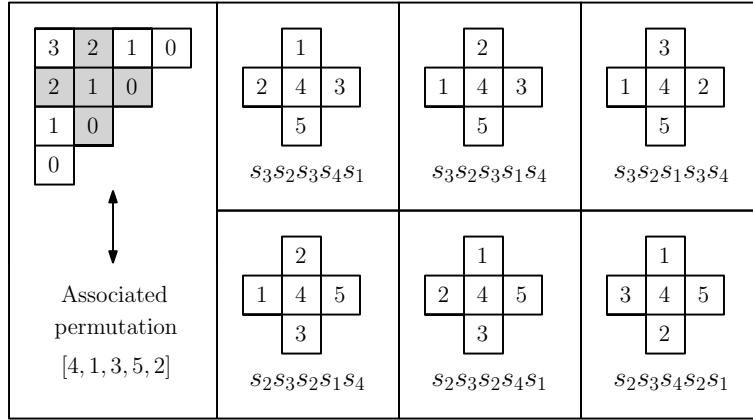


FIGURE 10.

(2) for all $c \in \lambda_n \setminus S$, $\beta_c \geq |S \cap H_c(\lambda_n)|$.

Proof. Those conditions are clearly necessary. We will show recursively that they are sufficient. By condition (1), there exists $c \in S$ such that $\beta_c = 0$. Moreover by condition (2), for all $x \in \lambda_n \setminus S$, if $S \cap H_c(\lambda_n) \neq \emptyset$ then $\beta_x \neq 0$. Then there exists $c_1 \in S$ erasable, and L a filling sequence of \mathcal{B}_{λ_n} which starts with c_1 . Set $[\mathcal{P}^0, \mathcal{P}^1, \dots, \mathcal{P}^n]$ the sequence of types associated with L and $S' = S \setminus \{c_1\}$. Then by definition of Process 1.9,

- (3) for all $c \in S'$, $\beta_c^1 \leq |H_c(S')| - 1$,
- (4) $c \in (\lambda_n \setminus \{c_1\}) \setminus S'$, $\beta_c^1 \geq |S' \cap H_c(\lambda_n \setminus \{c_1\})|$,

and by induction, the proposition is proved (λ_n has just been replaced by $\lambda_n \setminus \{c_1\}$). □

Proposition 2.12 allows to define the notion of subtype for any given type, but we will not talk about that in this paper.

3. VEXILLARY PERMUTATIONS AND BALANCED TABLEAUX

We have seen that some types are related with the theory of reduced decompositions in symmetric group. There exists a class of permutations called vexillary permutations which are strongly linked with the theory of standard Young tableaux. In this section we will show that a careful study of those permutations will give rise to a generalisation and a new proof of the fact that standard Young tableaux and balanced tableaux are equinumerous.

First, we give an important result of Stanley which will be central in the next sections.

Theorem 3.1 (Stanley, [7]). *Set $\sigma \in S_n$. For $i \in \{1, \dots, n\}$, we define:*

- (1) $d_i(\sigma) = |\{j > i \mid \sigma(j) < \sigma(i)\}|$,
- (2) $g_i(\sigma) = |\{j < i \mid \sigma(j) > \sigma(i)\}|$.

Set $\mu(\sigma)$ (resp. $\lambda(\sigma)$) the partition obtained by rearranging the sequence $(d_i(\sigma))$ (resp. $(g_i(\sigma))$) in a nonincreasing order. If $\lambda(\sigma) = \mu(\sigma)'$, then $\text{red}(\sigma) = f^{\lambda(\sigma)}$.

Definition 3.2. A permutation σ such that $\lambda(\sigma) = \mu(\sigma)'$ is called a vexillary permutation. In [7] it is shown that vexillary permutations are the 2143-avoiding permutations.

The two previous sequences have a further combinatorial interpretation when we use the convention of Remark 2.1 for the coordinates of the cells of λ_n .

Lemma 3.3. *Set $\sigma \in S_n$, then we have that $c_i(\mathcal{T}_\sigma) = d_i(\sigma)$ and $l_i(\mathcal{T}_\sigma) = g_i(\sigma)$ where*

$$c_i(\mathcal{T}_\sigma) = |\{j \mid (j, i) \in \text{Inv}(\sigma)\}| \text{ and } l_i(\mathcal{T}_\sigma) = |\{j \mid (i, j) \in \text{Inv}(\sigma)\}|, \quad (6)$$

called respectively the column and line sequences of \mathcal{T}_σ . Note that $\text{Inv}(\sigma) = \text{Sh}(\mathcal{T}_\sigma)$.

3.1. Stacking process. Set $\sigma \in S_n$, here we give a combinatorial interpretation of $\lambda(\sigma)$ and $\mu(\sigma)$ which will be useful in Section 3.3. We begin by putting the diagram $\text{Inv}(\sigma)$ in a grid such as in Figure 11.

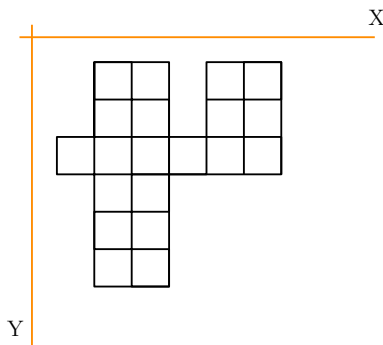


FIGURE 11. Diagram associated with $\sigma = [4, 8, 9, 5, 7, 6, 1, 3, 2] \in S_9$

Then we push all the boxes against the Y -axes. To finish, we push all the boxes against the X -axes, by obtaining a Ferrers diagram (Figure 12).

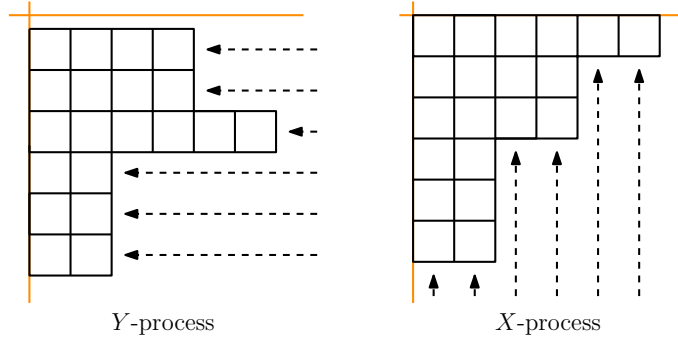


FIGURE 12.

Remark 3.4. The partition obtained after this YX -process is $\lambda(\sigma)$. In fact by Lemma 3.3, the number of boxes in the row i is precisely equal to $g_i(\sigma)$. Moreover after we packed all the boxes against the Y -axes, these rows of boxes are left-justified. So when we push everything on the X -axes, we are just rearranging those rows in a nonincreasing order. As a consequence, the diagram we obtain is precisely $\lambda(\sigma)$. Similarly, if we do the same thing but by first stacking on the X and then on the Y -axes (XY -process), then the partition we obtain is precisely $\mu(\sigma)'$.

Corollary 3.5. *As a consequence, a permutation σ is vexillary if and only if the two partitions obtained with these two combinatorial processes are the same.*

3.2. A transformation on type. At this point, we have two different descriptions of the set of reduced decompositions of any vexillary permutation $\sigma \in S_n$ in terms of tableaux. On the one hand, from Theorem 3.1 we have

$$|\text{Red}(\sigma)| = |\text{SYT}(\lambda(\sigma))|, \tag{7}$$

and on the other hand, from Theorem 2.10 we have

$$|\text{Red}(\sigma)| = |\text{Tab}(\mathcal{T}_\sigma)|. \tag{8}$$

Notice that in general, $\text{Sh}(\mathcal{T}_\sigma) \neq \lambda(\sigma)$. Hence it is not clear that those two sets of tableaux are linked. So it seems natural to ask for a generalisation of (8) where the shape $\lambda(\sigma)$ of (7) appears clearly. Indeed we can answer positively to that question, thanks to the transformation on types that we describe below.

Remark 3.6. From now on, we go back to the usual coordinates for the boxes of a diagram.

Definition 3.7. Let S be a diagram and “ a ” (resp. “ b ”) a row (resp. a column) of S . We define $S \downarrow_a$ (resp. \vec{S}^b) the diagram obtained by exchanging rows a and $a + 1$ (resp. columns b and $b + 1$) of S .

In the same way, we can exchange rows and columns in tableaux.

Definition 3.8. Let T be a tableau of shape S and a (resp. b) a row (resp. a column) of S . We define $T \downarrow_a$ (resp. \vec{T}^b) the tableau of shape $S \downarrow_a$ (resp. \vec{S}^b) obtained from T by exchanging rows a and $a + 1$ (resp. columns b and $b + 1$).

In general, if we consider $T \in \text{Tab}(\mathcal{T})$ and “ a ” a row of T , then the type \mathcal{T}' of $T \downarrow_a$ heavily depends on the tableau T and not only on \mathcal{T} . In the sequel, we focus on a special case where \mathcal{T}' is totally determined by \mathcal{T} .

Definition 3.9. Set S a diagram and $\mathcal{T} = (\theta_c)_{c \in S} \in \text{Type}(S)$. Let “ a ” (resp. “ b ”) be a row (resp. a column) of \mathcal{T} . This row (resp. column) will be called *dominant* if

- (D) for all $(a, y) \in S$ (resp. $(x, b) \in S$), $(a + 1, y) \in S$ (resp. $(x, b + 1) \in S$) and $\theta_{a,y} > \theta_{a+1,y}$ (resp. $\theta_{x,b} > \theta_{x,b+1}$).

Definition 3.10. Set \mathcal{T} a type of shape S and “ a ” (resp. “ b ”) a dominant row (resp. column) of \mathcal{T} . We define the type $\mathcal{T} \downarrow_a$ (resp. $\overrightarrow{\mathcal{T}}^b$) of shape $S \downarrow_a$ (resp. \overrightarrow{S}^b) obtained from \mathcal{T} by first decreasing by one all the integers in the row a (resp. column b) of \mathcal{T} , then by exchanging rows a and $a + 1$ (resp. columns b and $b + 1$) of \mathcal{T} , and keeping all other entries unchanged (see Figure 13).

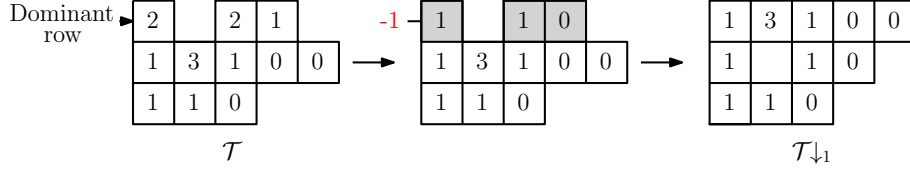


FIGURE 13.

Lemma 3.11. Set $\mathcal{T} = (\theta_{x,y}) \in \text{Type}(S)$, and consider a dominant row “ a ” of \mathcal{T} . Then for any $T = (t_c) \in \text{Tab}(\mathcal{T})$, we have $t_{a,y} < t_{a+1,y}$ for all $(a, y) \in S$.

Proof. Consider $T \in \text{Tab}(\mathcal{T})$. By Theorem 1.15 there exists $L = [(a_k, b_k)]_{1 \leq k \leq |S|} \in \text{Fil}(\mathcal{T})$ and $(\mathcal{P}^j = (\theta_{a,b}^j)_j)$ its associated sequence of types such that $T_L = T$. By definition of Process 1.9, to prove the lemma it is enough to show the following property:

- (Q) for all $0 \leq k \leq n$, and $(a, y) \in \text{Sh}(\mathcal{P}^k)$ such that $(a + 1, y) \in \text{Sh}(\mathcal{P}^k)$, then $\theta_{a,y}^k > \theta_{a+1,y}^k$.

So let c_1 be the biggest integer such that $(a, c_1) \in S$. Then we have that $H_{a,c_1}(S) \setminus \{(a, c_1)\} \subseteq H_{a+1,c_1}(S)$ and $\theta_{a,c_1}^0 > \theta_{a+1,c_1}^0$ because the line a is dominant. Set k such that $(a + 1, c_1) \in \text{Sh}(\mathcal{P}^k)$, then $\theta_{a,c_1}^k > \theta_{a+1,c_1}^k$ (such an integer exists, at least $k = 0$ works). Now let see why the property is still true for $k + 1$ by focusing on (a_k, b_k) . We have two main cases :

- (1) If $(a_k, b_k) = (a + 1, c_1)$, then $(a + 1, c_1) \notin \text{Sh}(\mathcal{P}^{k+1})$.
- (2) If $(a_k, b_k) \neq (a + 1, c_1)$, then $\theta_{a,c_1}^k > \theta_{a+1,c_1}^k \geq 0$. There are two possibilities:
 - $(a_k, b_k) \notin H_{a,c_1}(S) \setminus \{(a, c_1)\}$, then $\theta_{a,c_1}^{k+1} = \theta_{a,c_1}^k > \theta_{a+1,c_1}^k \geq \theta_{a+1,c_1}^{k+1}$.
 - $(a_k, b_k) \in H_{a,c_1}(S) \setminus \{(a, c_1)\}$, then $\theta_{a,c_1}^{k+1} = \theta_{a,c_1}^k - 1 > \theta_{a+1,c_1}^k - 1 = \theta_{a+1,c_1}^{k+1}$.

Finally, we proved by induction that the property (Q) holds for the column c_1 for any k . Now let c_2 be the biggest integer such that $c_2 < c_1$ and such that $(a, c_2) \in S$. Then $L_{a,c_2}(S) \setminus \{(a, c_2)\} \subseteq H_{a+1,c_2}(S)$ and $A_{a,c_2}(S) = \{(a, c_1)\}$. As before we will prove recursively that for all $0 \leq k \leq n$, the property (Q) holds for the column c_2 . As before we focus on (a_k, b_k) , once again we divide our study into several cases as for c_1 . The only new situation occurs when $(a_k, b_k) = (a, c_1)$. Based on our previous analysis about column c_1 , we know there exists $q < k$ such that

$(a_q, b_q) = (a + 1, c_1)$. Then for all $q + 1 \leq s \leq k$ we have that $\theta_{a,c_2}^s - \theta_{a+1,c_2}^s \geq 2$, so $\theta_{a,c_2}^{k+1} - \theta_{a+1,c_2}^{k+1} \geq 1$ and the property is proved. With similar arguments, we show that the property (Q) holds for all the columns, and this concludes the proof. \square

With this lemma we have the following property.

Proposition 3.12 (Exchange property). *Set \mathcal{T} a type and “ a ” (resp. b) a dominant row (resp. column) of \mathcal{T} . Then the operation \downarrow_a (resp. \rightarrow^b) is a bijection between $\text{Tab}(\mathcal{T})$ and $\text{Tab}(\mathcal{T}\downarrow_a)$ (resp. $\text{Tab}(\overrightarrow{\mathcal{T}}^b)$).*

Proof. We consider the tableau $T\downarrow_a = T' = (t'_{x,y}) \in \text{Tab}_{S\downarrow_a}(\mathcal{T}')$ where $\mathcal{T}' = (\theta'_{x,y})$. Let (x, y) be a cell of $S\downarrow_a$, and $H_{x,y}(T) = \{t_{a,b} \mid (a, b) \in H_{x,y}(\text{Sh}(T))\}$. If $x \neq a$ and $a + 1$, $H_{x,y}(T) = H_{x,y}(T')$, and $\theta'_{x,y} = \theta_{x,y}$. If $x = a$, then $H_{a,y}(T') = H_{a+1,y}(T) \cup \{t_{a,y}\}$ and by Lemma 3.11, $t'_{a,y} = t_{a+1,y} > t_{a,y}$ so $\theta'_{a,y} = \theta_{a+1,y}$. To finish if $x = a + 1$, then $\theta'_{a+1,y} = \theta_{a,y} - 1$ because $H_{a+1,y}(T') = H_{a,y}(T) \setminus \{t_{a+1,y}\}$. Then $\mathcal{T}' = \mathcal{T}\downarrow_a$, so \downarrow_a send an element of $\text{Tab}_S(T)$ to an element of $\text{Tab}_{S\downarrow_a}(\mathcal{T}\downarrow_a)$. Similar arguments show that \downarrow_a also sends an element of $\text{Tab}_{S\downarrow_a}(\mathcal{T}\downarrow_a)$ to an element of $\text{Tab}_S(T)$, hence \downarrow_a is an involution, and it concludes the proof of the line-exchange property. The proof of the same property for columns is similar. \square

We finish this section with a useful definition.

Definition 3.13. Set \mathcal{T} a type of shape S . The row (resp. column) a (resp. b) is called *dethroned* if: for all $(a, y) \in S$ (resp. $(x, b) \in S$), then $(a - 1, y) \in S$ (resp. $(x, b - 1) \in S$), and $\theta_{a-1,y} \leq \theta_{a,y}$ (resp. $\theta_{x,b-1} \leq \theta_{x,b}$).

Remark 3.14. Obviously, if a is a dominant row of \mathcal{T} , then $a + 1$ is a dethroned line of $\mathcal{T}\downarrow_a$ and viceversa. The same holds for dominant columns.

3.3. The exchange algorithm. In order to reach the goal described in the introduction of Section 3.2, we define an algorithm on \mathcal{T}_σ , whose definition will be motivated in Lemma 3.16 and Proposition 3.17. For the reader convenience we show how this algorithm works for the vexillary permutation $\sigma = [7, 8, 4, 5, 1, 2, 6, 9, 3] \in S_9$.

Definition 3.15 (The exchange algorithm). First, we erase all the boxes of \mathcal{B}_{λ_n} that do not belong to \mathcal{T}_σ . At this step, some lines and columns are possibly empty, so we remove them (Figure 14). We denote \mathcal{T}_σ^0 the type obtained. We note r (resp. s)

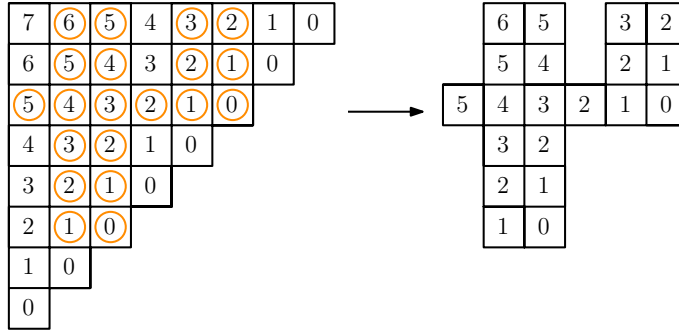


FIGURE 14.

the number of lines (resp. columns) of \mathcal{T}_σ^0 . As in Lemma 3.3, we define the column

and the line sequences $(c_i^0)_{1 \leq i \leq r}$ and $(l_i^0)_{1 \leq i \leq s}$ of \mathcal{T}_σ^0 . Now we create a sequence of types $(\mathcal{T}_\sigma^k)_{0 \leq k \leq r+s}$ such that $\text{Sh}(\mathcal{T}_\sigma^{r+s}) = \lambda(\sigma)$ and $|\text{Tab}(\mathcal{T}_\sigma^k)| = |\text{Tab}(\mathcal{T}_\sigma^{k+1})|$ with the following algorithm.

- (1) Let i_1 be the smallest integer such that $l_j^0 \leq l_{i_1}^0$ for all $1 \leq j \leq r$.
- (2) Exchange row i_1 with row $i_1 - 1$ using Proposition 3.12, then exchange the new row $i_1 - 1$ with the row $i_1 - 2$, and so on until i_1 has been exchanged with all the rows above it. Call \mathcal{T}_σ^1 the type obtained after this process, and denote $(l_i^1)_{1 \leq i \leq r}$ its line sequence (its column sequence is the same of \mathcal{T}_σ^0).
- (3) Define i_2 the smallest integer such that $l_j^1 \leq l_{i_2}^1$ for all $2 \leq j \leq r$.
- (4) Exchange the row i_2 with all the rows above it except the first. Call \mathcal{T}_σ^2 the type obtained and $(l_i^2)_{1 \leq i \leq r}$ its line sequence.
- (5) Continue this procedure till a type \mathcal{T}_σ^r having a nonincreasing line sequence is obtained. After that apply the same procedure to the columns, and at the end obtain a type \mathcal{T}_σ^{r+s} such that both line and column sequences are nonincreasing.
- (6) Denote $\mathcal{T}_\sigma^E = \mathcal{T}_\sigma^{r+s}$.

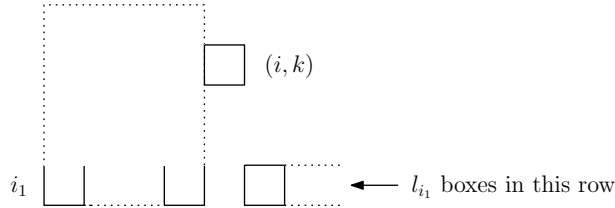
4	3	1	0	1	0
5	4	2	1		
4	3	1	0		
3	2				
2	1				
1	0				

FIGURE 15. This is the type \mathcal{T}_σ^E obtained from $\sigma = [4, 8, 9, 5, 7, 6, 1, 3, 2]$

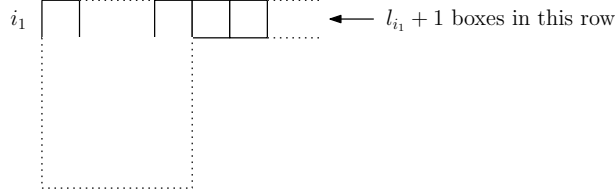
To prove that this algorithm can always be performed, we need an intermediate lemma.

Lemma 3.16. *Set $\sigma \in S_n$ a vexillary permutation. Set l_i (resp. c_i) the line sequence (resp. the column sequence) of $\text{Sh}(\mathcal{T}_\sigma)$ (see (6)). For all i and j in \mathbb{N} such that $l_j \leq l_i$ (resp. $c_j \leq c_i$), if $(i, a) \in \text{Sh}(\mathcal{T}_\sigma)$ (resp. $(a, i) \in \text{Sh}(\mathcal{T}_\sigma)$) then $(j, a) \in \text{Sh}(\mathcal{T}_\sigma)$ (resp. $(a, j) \in \text{Sh}(\mathcal{T}_\sigma)$).*

Proof. We prove this lemma for lines only (the proof for columns is similar) by induction using the stacking process (Section 3.1). Set i_1, \dots, i_n such that the sequence $(l_{i_1}, \dots, l_{i_n})$ is nonincreasing. We focus on i_1 , and consider the corresponding line in $S = \text{Sh}(\mathcal{T}_\sigma)$. Now set $i \neq i_1$ and k such that $(i, k) \in S$, then we will show that $(i_1, k) \in S$. To prove it, we assume that $(i_1, k) \notin S$.



So, if we push the boxes against the X -axes, then in the first row there must be strictly more than l_{i_1} boxes.



So this is again the case when we stack all the boxes on the Y -axes. If we reverse the order of the stacking process and we first pack the boxes against the Y -axes and after we stack the boxes on the X -axes, because of the maximality of l_{i_1} , the first row contains only l_{i_1} boxes. But σ is vexillary, so this is a contradiction, and the lemma is proved for the row i_1 .

Suppose the lemma is proved for rows $i_1, \dots, i_k, k \in \mathbb{N}$. Set $\lambda(\sigma) = (\lambda_1, \dots, \lambda_m)$. By induction, if we delete rows i_1, \dots, i_k in S , and then perform the XY or YX stacking process on the obtained diagram, we found the partition $(\lambda_{k+1}, \dots, \lambda_m)$ in both cases. With a similar argument as for the case i_1 , we prove that the lemma holds for row i_{k+1} , and this concludes the proof. \square

Proposition 3.17. *Set σ a vexillary permutation, then the exchange algorithm can always be performed.*

Proof. First we focus on the row $i_1 - 1$: since the integers in \mathcal{T}_σ^0 are strictly increasing from bottom to top, by Lemma 3.16 we have that the row $i_1 - 1$ satisfies condition (D), hence it is dominant. Then we can use Proposition 3.12 to exchange the row i_1 with the row $i_1 - 1$. We denote \mathcal{T}' the type obtained after this exchange. With a similar argument we can show that row $i_1 - 2$ of \mathcal{T}' is dominant, so we can exchange rows $i_1 - 2$ and $i_1 - 1$ in \mathcal{T}' , and so on. At the end, we exchanged the row i_1 of \mathcal{T} with all the rows above it. Moreover, since the integers in \mathcal{T}_σ^0 are strictly increasing from bottom to top, by Proposition 3.12 the integers in the rows from 2 to r in \mathcal{T}_σ^1 are again strictly increasing from bottom to top. Then by Lemma 3.16, we prove that we can use Proposition 3.12 to move the row i_2 to the second position. And so on.

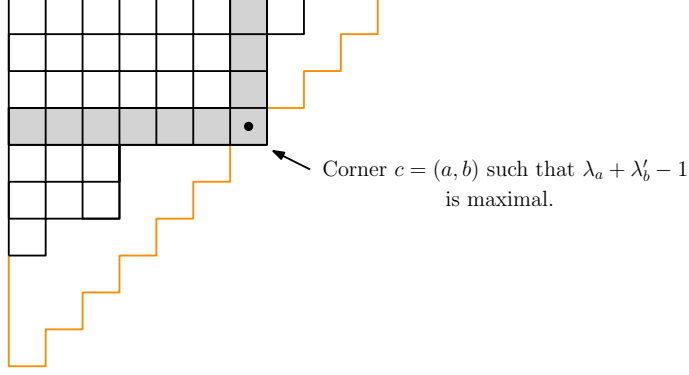
We remark that the integers in each row of \mathcal{T}_σ^r are increasing from right to left. Then with exactly the same arguments, we show that the process over the columns can be performed. \square

At this point, we have a type \mathcal{T}_σ^E such that both column and line sequences are non-increasing. It is easy to see by using the stacking process and Lemma 3.16 that \mathcal{T}_σ^E is of shape $\lambda(\sigma)$. As a consequence of Proposition 3.12, Lemma 3.3, and Theorem 3.1 we have the expected theorem.

Theorem 3.18. *Set $\sigma \in S_n$ vexillary, then the type \mathcal{T}_σ^E is of shape $\lambda(\sigma)$ and $|\text{Tab}_\lambda(\mathcal{T}_\sigma^E)| = |\text{Tab}(\mathcal{T}_\sigma)| = f^{\lambda(\sigma)}$.*

3.4. Link with balanced tableaux. Now we use the theory developed in the previous section to give an alternative proof of the fact that balanced tableaux and standard Young tableaux of same shape are equinumerous. Consider the Ferrers diagram of λ . A *corner* of λ is a cell c of its diagram such that there is not any cell below and on the right of c . Now consider a corner $c = (a, b)$ of λ such that the

integer $k = \lambda_a + \lambda'_b - 1$ is maximal. Then we can place λ in the staircase partition λ_{k+1} as depicted in the following figure.



Then we look at the corners (u, v) of λ which are on the diagonal border of the staircase partition. For each such corner set $R_{(u,v)} = \{(x, y) \in \lambda \mid x \leq u, y \leq v\}$, and consider the union R of all these $R_{(u,v)}$. Then we let each other connected parts of λ fall in the staircase tableau as depicted in Figure 16. Then we repeat the same procedure for each connected part of the obtained diagram, until this is possible. At the end, we obtain a subdiagram S of λ_{k+1} .

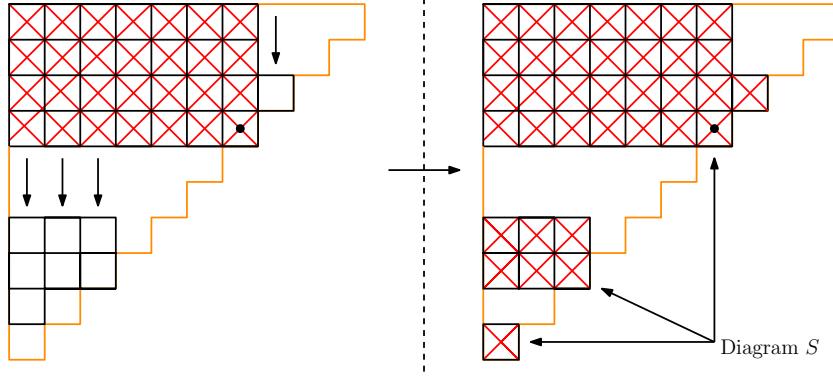


FIGURE 16.

Definition 3.19. Set S the diagram obtained with the procedure depicted above. We define $\mathcal{F} = (\beta_c)_{c \in S}$.

Lemma 3.20. \mathcal{F} is a subtype of $\mathcal{B}_{\lambda_{k+1}}$. Moreover, if we denote $\sigma_\lambda \in S_{k+1}$ the permutation such that $\mathcal{F} = \mathcal{T}_{\sigma_\lambda}$, then σ_λ is vexillary of shape λ .

Proof. To prove this Lemma, we use Proposition 2.12. Consider $z = (x, y) \in \lambda_{k+1}$;

- if $z \notin S$, then the only elements in $H_z(\lambda_{k+1}) \cap S$ have to be strictly below z . Moreover, by definition β_z is precisely equal to the number of boxes strictly below z . Hence $\beta_z \geq |H_z(\lambda_{k+1}) \cap S|$;
- if $z = (x, y) \in S$, by construction of S there exists $z' = (x', y') \in S$ such that: $x' \geq x$, $y' \geq y$, $\beta_{z'} = 0$, and for all $x \leq u \leq x'$ and $y \leq v \leq y'$ then $(u, v) \in S$. Then $\beta_z = (x' - x) + (y' - y) \leq |H_z(S)| - 1$.

Hence, $\mathcal{F} \in \text{Sub}(\mathcal{B}_{\lambda_{k+1}})$. By Proposition 2.9 there exists σ_λ such that $\mathcal{T}_{\sigma_\lambda} = \mathcal{F}$. Moreover, if we perform the stacking process on $\text{Inv}(\sigma_\lambda) = S$, it is easy to see that both XY and YX processes end with the partition λ . Hence by Corollary 3.5, σ_λ is vexillary of shape λ , and this concludes the proof. \square

Definition 3.21. Set $\lambda = (\lambda_1, \dots, \lambda_k)$ a partition of n . We said that two elements (a, b) and (c, d) in the Ferrers diagram of λ are in the same *block* if and only if $\lambda_a = \lambda_c$.

Now we focus on $\mathcal{T}_{\sigma_\lambda}^E = (\theta_c)_{c \in \lambda}$. We first remark that if c is a corner of λ , then $\theta_c = 0$. Moreover by construction, the column sequence of S is nonincreasing, hence when we perform the exchange algorithm on σ_λ , it does not exchange any columns. Hence, we have that:

(A) for all $(x, y) \in \lambda$, if $(x, y - 1) \in \lambda$ then $\theta_{(x, y-1)} = \theta_{(x, y)} + 1$.

More technical now: set $j \in \mathbb{N}$ and $i < j$ maximal such that $l_i(S) = l_j(S)$. Then by construction of S we have that for all k such that $i < k < j$, then $l_k(S) > l_i(S)$ (in particular the line k is not empty). Then when we perform the exchange algorithm, we exchange row j with each rows k where $i < k < j$. Moreover, each time we do that, we decrease by one the integers in row j . Hence, since there is not any empty rows between row i and row j in S , we have that:

(B) for all $(x, y) \in \lambda$, if $(x - 1, y) \in \lambda$ and is in the same block as (x, y) , then $\theta_{(x-1, y)} = \theta_{(x, y)} + 1$.

From (A) and (B) we deduce that $\mathcal{T}_{\sigma_\lambda}^E$ is of the form depicted in Figure 17.

7	6	5	4	3	2	1	0
8	7	6	5	4	3	2	
7	6	5	4	3	2	1	
6	5	4	3	2	1	0	
3	2	1					
2	1	0					
0							

FIGURE 17.

Definition 3.21 splits $\mathcal{T}_{\sigma_\lambda}^E$ into several blocks. We can use Proposition 3.12 on each block in this way: the bottom row is translated to the top, then the new bottom row is translated to the second position, and so on (see Figure 18). And at

8	7	6	5	4	3	2	↻	6	5	4	3	2	1	0
7	6	5	4	3	2	1		6	5	4	3	2	1	0
6	5	4	3	2	1	0		6	5	4	3	2	1	0

FIGURE 18.

the end, the type we obtained is effectively \mathcal{B}_λ , and this proves the following result.

Corollary 3.22. *For all $\lambda \vdash n$ we have that*

$$|\text{Bal}(\lambda)| = |\text{Tab}(\mathcal{B}_\lambda)| = |\text{Tab}(\mathcal{T}_{\sigma_\lambda}^E)| = f^\lambda = |\text{SYT}(\lambda)|.$$

At this point, one could expect that there exists a vexillary permutation σ_{eq} such that we directly have $\mathcal{T}_{\sigma_{eq}}^E = \mathcal{B}_\lambda$. Unfortunately, we can show that such a permutation does not exist in general.

We conclude this section with a natural question: given two vexillary permutations σ and ω , when do we have $\mathcal{T}_\sigma^E = \mathcal{T}_\omega^E$? We will answer to this question in the following section.

3.5. An equivalence relation between vexillary permutations. Now we will exhibit an equivalence relation \sim_v on the set of vexillary permutations with the property that for any two vexillary permutations $\sigma \in S_n$ and $\omega \in S_m$, then $\mathcal{T}_\sigma^E = \mathcal{T}_\omega^E$ if and only if $\sigma \sim_v \omega$.

Set $\sigma \in S_n$, and $p \leq n$ the lowest integer such that $\sigma(p) \neq p$ and $q \leq n$ the biggest integer such that $\sigma(q) \neq q$. Then we define,

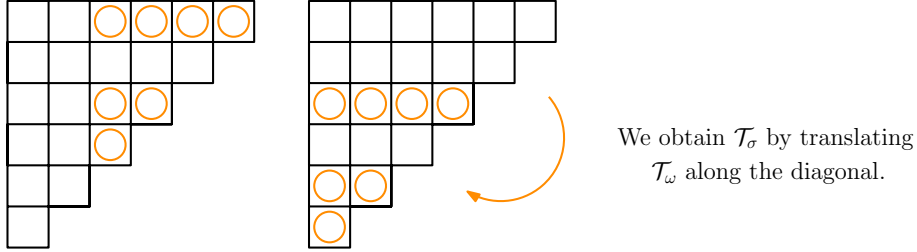
$$\bar{\sigma} = [\sigma(p) - (p - 1) ; \sigma(p + 1) - (p - 1) ; \dots ; \sigma(q) - (p - 1)].$$

Note that $\bar{\sigma}$ is an element of $S_{n+1-p-q}$ because of the choice of p and q .

Definition 3.23. We say that $\sigma \sim_v \omega$ if and only if $\bar{\sigma} = \bar{\omega}$.

Theorem 3.24. *Set σ and ω two vexillary permutations. Then $\mathcal{T}_\sigma^E = \mathcal{T}_\omega^E$ if and only if $\sigma \sim_v \omega$.*

Proof. We have a combinatorial interpretation for the relation \sim_v . Set $\sigma \in S_n$ and $\omega \in S_m$ such as $\sigma \sim_v \omega$. We can assume n bigger than m . Then we can see ω as a permutation of S_n by adding $n - m$ fix points at the end of ω . Set p_σ (resp. p_ω) the smallest integer such that $\sigma(p_\sigma) \neq p_\sigma$ (resp. $\omega(p_\omega) \neq p_\omega$). By definition $\text{Inv}(\bar{\sigma}) = \text{Inv}(\bar{\omega})$, moreover $\text{Inv}(\sigma) = \{(x + (p_\sigma - 1), y + (p_\sigma - 1)) \mid (x, y) \in \text{Inv}(\bar{\sigma})\}$. So if we look at \mathcal{T}_σ and \mathcal{T}_ω , we have the situation described in the following figure.



Then it is clear that we have $\mathcal{T}_\sigma^E = \mathcal{T}_\omega^E$ by construction.

To prove the opposite direction, we will construct the inverse procedure of the Exchange algorithm (Definition 3.15). This algorithm is divided in two parts, one for the columns, and another for the lines. We will show how it works on columns (on lines is similar). Set $\sigma \in S_n$ vexillary.

- (1) Start with \mathcal{T}_σ^E : set r its number of lines and s its number of columns.
- (2) Scan the columns of \mathcal{T}_σ^E from right to left starting from column s . When one meet a column k which is dethroned (see Definition 11) then use the Proposition 3.12 on k and $k + 1$. Then scan again the columns from right

to left, starting from column s and look for a column which is dethroned in order to use Proposition 3.12 again, and so on.

(3) The algorithm stops when there is not any dethroned column left.

Recall that $\mathcal{T}_\sigma^E = \mathcal{T}_\sigma^{r+s}$. By Definition of the Exchange Algorithm, there exist $1 \leq u_1 \leq s$ and $k_1 \in \mathbb{N}$ (which can be 0) such that: if we exchange the column u_1 of $\mathcal{T}_\sigma^{r+s-1}$ with the k_1 columns on its left using Property 3.12, then the type we obtain is \mathcal{T}_σ^{r+s} . Note that for any $u_1 - k_1 < i < j \leq s$, the columns i and j of \mathcal{T}_σ^{r+s} have not been exchanged at any step of the exchange algorithm. Since the integers in each row of \mathcal{T}_σ are increasing from right to left, we have that this property is still true for columns $u_1 - k_1 + 1$ to s in \mathcal{T}_σ^{r+s} . This implies that the rightmost dethroned column of \mathcal{T}_σ^{r+s} is the column $u_1 - k_1 + 1$. After k_1 iteration of the algorithm described above, the obtained type is $\mathcal{T}_\sigma^{r+s-1}$. With similar arguments, we show that the type we obtain when this algorithm stops is \mathcal{T}_σ^r . Then we perform the same process on the lines of \mathcal{T}_σ^r , and at the end we obtain \mathcal{T}_σ^0 . Now let us see how we can turn this type into \mathcal{T}_σ .

In the Exchange algorithm we removed empty lines and empty columns from \mathcal{T}_σ in order to obtain $\mathcal{T}_\sigma^0 = (\theta_{i,j})$, so now we have to put them back. Set (L_i) the line sequence of \mathcal{T}_σ^0 and i_{max} an integer such that $L_{i_{max}}$ is maximal in the line sequence. Then let s be the biggest integer such that $(i_{max}, s) \in \text{Sh}(\mathcal{T}_\sigma^0)$. Then for all $1 \leq j \leq s$, we put $\theta_{i_{max},j} - \theta_{i_{max},j+1} - 1$ empty columns between the column j and the column $j + 1$. We do the same thing with lines. We call $\bar{\mathcal{T}} = (\bar{\theta}_c)$ the obtained type.

By hypothesis, $\bar{\mathcal{T}} = \mathcal{T}_\sigma$. Set $\omega \in S_m$ such that $\mathcal{T}_\omega = \bar{\mathcal{T}}$. We can suppose that $m = n$ by adding some fix points at the beginning of σ and ω . Then $\text{Inv}(\bar{\sigma}) = \text{Inv}(\bar{\omega})$, hence with a similar argument as in the first paragraph of this proof, $\omega \sim_v \sigma$. This concludes the proof. \square

4. ON A QUESTION RAISED BY EDELMAN AND GREENE

In this last section, we use our previous work to give an answer to a question raised in the [3]. Set $\lambda \vdash n$, it is obvious that

$$f^\lambda = \sum_{\lambda^-} f^{\lambda^-}, \tag{9}$$

where the sum runs over all the partition λ^- obtained by removing a corner from λ . Since it holds for standard tableaux, it also holds for balanced tableaux. In [3] was raised the question to give a direct proof of this result. We can answer this problem by considering a more general question. Set $\sigma \in S_n$ a vexillary permutation of shape λ and associated type \mathcal{T}_σ^E . Set z_1, \dots, z_k a sequence of boxes of λ , we want to enumerate the tableaux $T = (t_c)_{c \in \lambda} \in \text{Tab}(\mathcal{T}_\sigma^E)$ such that for all $1 \leq j \leq k$, $t_{z_j} = n + 1 - j$. Equivalently, this consists in enumerating the filling sequences $[c_1, \dots, c_n]$ of \mathcal{T}_σ^E such that for all $1 \leq j \leq k$, $c_j = z_j$. In the sequel, we will give a complete answer to this question using the theory of reduced decompositions of the symmetric group. To do so, we introduce the following definition.

Definition 4.1. Set $S \subset S' \subseteq \lambda_n$ such that both $\mathcal{T}_S = (\beta_c)_{c \in S}$ and $\mathcal{T}_{S'} = (\beta_c)_{c \in S'}$ are in $\text{Sub}(\mathcal{B}_{\lambda_n})$. Define $\mathcal{T}_{S'/S} = (\theta_c)_{c \in S' \setminus S}$, where $\theta_c = \beta_c - |H_c(\lambda_n) \cap S|$. Then $\mathcal{T}_{S'/S}$ is a type, and we say that $\mathcal{T}_{S'/S}$ is a S -subtype of \mathcal{B}_{λ_n} .

Then we have the following property.

Proposition 4.2. *Set $S \subset S' \subseteq \lambda_n$ such that \mathcal{T}_S and $\mathcal{T}_{S'} \in \text{Sub}(\mathcal{B}_{\lambda_n})$. Set σ and $\omega \in S_n$ such that $\mathcal{T}_S = \mathcal{T}_\sigma$ and $\mathcal{T}_{S'} = \mathcal{T}_\omega$, then $|\text{Fil}(\mathcal{T}_{S'/S})| = |\text{Red}(\sigma^{-1}\omega)|$.*

Proof. Set $k = |S' \setminus S|$, $V = [z_1, \dots, z_k] \in \text{Fil}(\mathcal{T}_{S'/S})$, and $U = [c_1, \dots, c_{|S|}] \in \text{Fil}(\mathcal{T}_S)$. Then $W = [c_1, \dots, c_{|S|}, z_1, \dots, z_k] \in \text{Fil}(\mathcal{T}_{S'})$. By Theorem 2.10, W corresponds to $s_{i_1} \cdots s_{i_{\ell(W)}} \in \text{Red}(\omega)$. Moreover, by construction of W we have that $s_{i_1} \cdots s_{i_{|S|}} \in \text{Red}(\sigma)$. Hence we have that $s_{i_{|S|+1}} \cdots s_{i_{\ell(\omega)}} \in \text{Red}(\sigma^{-1}\omega)$. In this way, we associate to each element of $\text{Fil}(\mathcal{T}_{S'/S})$ an element of $\text{Red}(\sigma^{-1}\omega)$, and as in the proof of Theorem 2.10, this defines a bijection. \square

This result implies the following theorem which answers to the question raised at the beginning of this section.

Theorem 4.3. *Set $\sigma \in S_m$ a vexillary permutation of shape $\lambda \vdash n$, \mathcal{T}_σ^E its associated type, and $U = [z_1, \dots, z_k]$ a sequence of boxes in the Ferrers diagram of λ . Then if we consider the number of filling sequences $L = [c_1, \dots, c_n]$ of \mathcal{T}_σ^E such that $L_k = [c_1, \dots, c_k] = U$, we have the two following cases :*

- (1) *If U can be completed into a filling of \mathcal{T}_σ^E , then there exists ω a permutation such that this number is equal to $|\text{Red}(\omega)|$.*
- (2) *If not, this number is 0.*

Proof. By construction, the exchange algorithm defines a bijection ψ from the diagram λ to $\text{Inv}(\sigma) \subseteq \lambda_m$ such that: $[c_1, \dots, c_n] \in \text{Fil}(\mathcal{T}_\sigma^E)$ if and only if $[\psi(c_1), \dots, \psi(c_n)] \in \text{Fil}(\mathcal{T}_\sigma)$. Set $S = \{\psi(z_1), \dots, \psi(z_k)\}$, then there is two possibilities:

- (1) if there exists $\tau \in S_m$ such that $\mathcal{T}_\tau = \mathcal{T}_S$, then by Proposition 4.2 the number we are looking for is $|\text{Red}(\tau^{-1}\sigma)|$;
- (2) if not, this number is 0.

This concludes the proof. \square

This theorem can also be applied to balanced tableaux. Moreover, this result is not just theoretical : the permutation ω can be computed by using the Exchange algorithm 3.15 and Proposition 4.2. Since the algorithm is not easy to manipulate, we cannot provide a systematic description of the associate permutation. However we have this nice combinatorial result.

Theorem 4.4. *Set σ , \mathcal{T}_σ^E and $U = [z_1, \dots, z_k]$ as in the previous theorem. We consider the diagram $S = \lambda(\sigma) \setminus \{z_1, \dots, z_k\}$. We perform the stacking process on S , and we set $\lambda(S)$ (resp. $\mu(S)'$) the partition obtained after the XY (resp. YX) process. If U can be completed into a filling of \mathcal{T}_σ^E and $\lambda(S) = \mu(S)'$, then we have that*

$$|\{L = [c_1, \dots, c_n] \in \text{Fil}(\mathcal{T}_\sigma^E) \mid [c_1, \dots, c_k] = U\}| = f^{\lambda(S)}.$$

As a corollary of Theorem 4.3, if U cannot be completed into a filling of \mathcal{T}_σ^E then this number is 0. Moreover, as before this theorem is still true with the balanced type. This theorem is an immediate consequence of Theorem 3.1 together with Proposition 4.6, which is given after an intermediate lemma.

Lemma 4.5. *Set $S \subset S' \subseteq \lambda_n$ such that \mathcal{T}_S and $\mathcal{T}_{S'} \in \text{Sub}(\mathcal{B}_{\lambda_n})$. Set σ and $\omega \in S_n$ such that $\mathcal{T}_S = \mathcal{T}_\sigma$ and $\mathcal{T}_{S'} = \mathcal{T}_\omega$. Then $\text{Inv}(\sigma^{-1}\omega) = \sigma^{-1}(S' \setminus S)$ where $\sigma^{-1}(S' \setminus S) = \{(\sigma^{-1}(i), \sigma^{-1}(j)) \mid (i, j) \in (S' \setminus S)\}$.*

Proof. We have that $\omega = [\omega(1), \omega(2), \dots, \omega(n)]$, then

$$\sigma^{-1}\omega = [\sigma^{-1}(\omega(1)), \dots, \sigma^{-1}(\omega(n))].$$

Since $S \subseteq S'$, by Proposition 2.9 we have that $\text{Inv}(\sigma) \subseteq \text{Inv}(\omega)$. Now let (i, j) be an inversion of ω , then $\omega = [\omega(1), \dots, j, \dots, i, \dots, \omega(n)]$. We have the two following cases.

(1) If $(i, j) \in \text{Inv}(\sigma)$, then by definition we have that $\sigma^{-1}(i) > \sigma^{-1}(j)$. But $\sigma^{-1}\omega = [\sigma^{-1}(\omega(1)), \dots, \sigma^{-1}(j), \dots, \sigma^{-1}(i), \dots, \sigma^{-1}(\omega(n))]$, so

$$(\sigma^{-1}(i), \sigma^{-1}(j)) \notin \text{Inv}(\sigma^{-1}\omega).$$

(2) If $(i, j) \notin \text{Inv}(\sigma)$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$, so $(\sigma^{-1}(i), \sigma^{-1}(j)) \in \text{Inv}(\sigma^{-1}\omega)$.

Hence we have that $\sigma^{-1}(\text{Inv}(\omega) \setminus \text{Inv}(\sigma)) \subseteq \text{Inv}(\sigma^{-1}\omega)$. Moreover it is easy to see that $\ell(\sigma^{-1}\omega) = \ell(\omega) - \ell(\sigma)$, so $\sigma^{-1}(\text{Inv}(\omega) \setminus \text{Inv}(\sigma)) = \text{Inv}(\sigma^{-1}\omega)$. But $S' \setminus S = \text{Inv}(\omega) \setminus \text{Inv}(\sigma)$, so the lemma is proved. \square

Proposition 4.6. *Set S, S', σ and ω as defined in Lemma 4.5. Then with convention of Remark 2.1, we have that for all i , $c_{\sigma(i)}(S' \setminus S) = d_i(\sigma^{-1}\omega)$ and $l_{\sigma(i)}(S' \setminus S) = g_i(\sigma^{-1}\omega)$.*

Proof. Set $1 \leq i \leq n$, we have that $d_i(\sigma^{-1}\omega) = |\{(k, i) \mid (k, i) \in \text{Inv}(\sigma^{-1}\omega)\}|$ and $c_i(S' \setminus S) = |\{(k, i) \mid (k, i) \in (S' \setminus S)\}|$. With the previous lemma, we have that

$$\sigma^{-1}.\{(k, \sigma(i)) \mid (k, \sigma(i)) \in (S' \setminus S)\} = \{(j, i) \mid (j, i) \in \text{Inv}(\sigma^{-1}\omega)\}$$

so $d_i(\sigma^{-1}\omega) = c_{\sigma(i)}(S' \setminus S)$, and we prove the other equality with a similar argument. \square

Now we give a direct proof of Equation (9). First we need a definition.

Definition 4.7. Set $\lambda \vdash n$ seen as a diagram. Set B a block of λ , then there exists i minimal such that $(i, \lambda_i) \in B$. The box (i, λ_i) is called the upper right corner of the block B .

Set T a balanced tableau of shape $\lambda \vdash n$. By definition of Process 1.9, we have that the integers n appears in the upper right corner of a block (Figure 19).

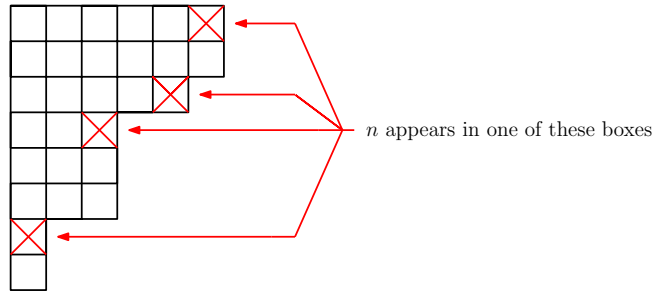


FIGURE 19.

Then by Theorem 4.4 we directly have the next result.

Proposition 4.8. *Set B a block of $\lambda \vdash n$ and c the upper right corner of B . Then the number of balanced tableaux of shape λ such that n appears in c is equal to the number of standard tableaux of shape λ^- , where $\lambda^- \vdash n - 1$ is obtained from λ by suppressing the corner of the block B .*

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