



**HAL**  
open science

# On the infinite-dimensional representation of stochastic controlled systems with delayed control in the diffusion term

Giorgio Fabbri, Salvatore Federico

► **To cite this version:**

Giorgio Fabbri, Salvatore Federico. On the infinite-dimensional representation of stochastic controlled systems with delayed control in the diffusion term. *Mathematical Economics Letters*, 2014, 2 (3/4), pp.33-43. 10.1515/mel-2014-0011 . hal-01038088

**HAL Id: hal-01038088**

**<https://hal.science/hal-01038088>**

Submitted on 23 Jul 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# On the infinite-dimensional representation of stochastic controlled systems with delayed control in the diffusion term

G. Fabbri<sup>1</sup>      S. Federico<sup>2</sup>

July 23, 2014

## Abstract

In the deterministic context a series of well established results allow to reformulate delay differential equations (DDEs) as evolution equations in infinite dimensional spaces. Several models in the theoretical economic literature have been studied using this reformulation. On the other hand, in the stochastic case only few results of this kind are available and only for specific problems.

The contribution of the present letter is to present a way to reformulate in infinite dimension a prototype controlled stochastic DDE, where the control variable appears delayed in the diffusion term. As application, we present a model for quadratic risk minimization hedging of European options with execution delay and a time-to-build model with shock.

Some comments concerning the possible employment of the dynamic programming after the reformulation in infinite dimension conclude the letter.

*Keywords:* Stochastic delay differential equations, Evolution equations in Hilbert space, Dynamic programming.

*JEL Classification:* C61, G11, O40.

*AMS 2010 Classification:* 39A50, 34K30, 47D06, 60H99, 49L20

## 1 Introduction

The setting of optimal control problems with delays in the state and/or the control variable has been employed in the last decades to model a wide variety of economic and financial phenomena where the dynamics of the system has a natural memory of the past. We may quote, among others: the problem of growth models with time-to-build in production (see [1, 2, 4]) or investment (see [19, 22]) or with delays in the learning-by-doing process (see [7]); vintage capital models (see [6, 16]); advertising models (see [15, 17, 20, 21]).

Considering for simplicity the case of one-dimensional linear and autonomous state equation, the mathematical problem can be formulated in a quite wide generality as follows. Given  $d > 0$  and two Borel measures  $\mu, \nu$  on the interval  $[-d, 0]$ , consider the linear functionals, defined in the space of continuous functions from  $[-d, 0]$  to  $\mathbb{R}$ ,

$$\mathcal{L}f := \int_{-d}^0 f \, d\mu, \quad \mathcal{M}f := \int_{-d}^0 f \, d\nu.$$

---

<sup>1</sup>Giorgio Fabbri, EPEE, Université d'Evry-Val-d'Essonne (TEPP, FR-CNRS 3126), Département d'Economie, E-mail: [giorgio.fabbri@univ-evry.fr](mailto:giorgio.fabbri@univ-evry.fr). The research of G. Fabbri has been developed in the framework of the center of excellence LABEX MME-DII (ANR-11-LABX-0023-01).

<sup>2</sup>Salvatore Federico, Dipartimento di Economia, Management e Metodi Quantitativi, Univeristà di Milano, Italy. E-mail: [salvatore.federico@unimi.it](mailto:salvatore.federico@unimi.it).

Then we can consider the controlled differential equation

$$x'(t) = \mathcal{L}x_t + \mathcal{M}u_t, \quad (1)$$

where  $x(\cdot)$  is the state variable,  $u(\cdot)$  is the control of the system belonging to a suitable class of functions, and where  $x_t$  and  $u_t$  denote, respectively, the trajectory of the state and of the control in the time interval  $[t-d, t]$ , i.e.  $x_t := x(t+\xi)|_{\xi \in [-d, 0]}$  and  $u_t := u(t+\xi)|_{\xi \in [-d, 0]}$ . Obviously, to give a precise sense to (1), one needs, beyond completing it with initial conditions involving the past of the control and the state variable, also to impose some conditions to the operators  $\mathcal{L}$  and  $\mathcal{M}$  ensuring that (1) has a unique solution for each control  $u(\cdot)$  in a suitable class of functions. We do not discuss that in detail and refer instead to Chapter 3, Part II of [5]. We only notice that the nature of (1) – as soon as  $\mathcal{L}$  and  $\mathcal{M}$  are not trivial, i.e.  $\mu$  and  $\nu$  are not both (multiple of) Dirac measures concentrated at 0 – is basically infinite dimensional: to have an intuition of that, one can just think that (1) invokes, to state the dynamics, the past of the state and of the control variable, i.e. data which belong to (infinite dimensional) functional spaces.

Now, given a dynamics in the form (1), one can consider the problem of optimizing a payoff functional of the form

$$\int_0^T g(t, x(t), u(t)) dt, \quad T \in [0, +\infty], \quad (2)$$

where  $g$  is a measurable function. We notice that one can consider also the case of a function  $g$  depending on  $x_t$  and  $u_t$  too, and the problem of optimizing (2) would be basically infinite dimensional even in the case of no delay in the state equation (1).

Given the observation of the structural infinite dimensional nature of the problem we have done, one can understand the reason for which the mathematical literature has proposed infinite-dimensional formulations of the problem above to “absorb” (and remove) the delay, paying the price of a passage to the infinite dimensional setting. This is the first step to do if we want to try to employ the dynamic programming tools to tackle these problems. We observe that, when the delay is only in the state variable, i.e.  $\nu$  is a multiple of  $\delta_{\{0\}}$ , the infinite dimensional representation is straightforward, just considering as infinite dimensional space the couple  $(x(t), x_t)$  (see [8, 9, 11, 12]). On the other hand, when the control enters into the dynamics with delay, i.e.  $\nu$  is not a multiple of  $\delta_{\{0\}}$ , then two approaches can be employed:

1. Consider as infinite dimensional state the triple  $(x(t), x_t, u_t)$  (or the couple  $(x(t), u_t)$  if there is delay no delay in the state, i.e.  $\mu$  is a multiple of  $\delta_{\{0\}}$ ). This leads to an infinite-dimensional control problem with boundary control, i.e. with unbounded control operator (see [18]).
2. Consider as infinite dimensional state a kind of “minimal summary” combining together through a linear transformation leading to the construction of the so called *structural state*, the current state  $x(t)$ , the past state  $x_t$  (if the delay in the state is present in the equation, i.e.  $\mu$  is not a multiple of  $\delta_{\{0\}}$ ) and the past of the control  $u_t$  (see [23]). With respect to the latter one, this approach has the advantage that the infinite dimensional control problem has not a always the complication of becoming a boundary control problem, see e.g. [14, 23].

A systematic treatment of the methods described above can be found in Chapter 3, Part II of [5]. The aim of this letter is to try to understand how the above method can be extended to the stochastic case. We have to mention that attempts in this direction have already been done by some authors. For instance, [17] consider the case of a stochastic system with delays in state and control, but where the noise is purely additive. This allows to generalize in a straightforward way the method described in point 2 above. On the other hand, [13] consider the case when there is delay in the state both in the drift and the diffusion, but no delay in the control variable. Other cases seem not treated in the literature. A challenging case seems to be the case when the control variable appears delayed in the diffusion term. Several

interesting aspects of the problem could be already discussed dealing with the prototype equation

$$dX(t) = u(t-d) dW(t), \quad (3)$$

which may work as a guide to investigate the possibilities and the limits of an infinite-dimensional representation along the direction described above for deterministic systems. In fact we will deal with a slightly more general equation that can be specified to obtain the state equations of the two motivating examples we will introduce below.

The main result is provided by Theorem 3.4 that allow to connect such controlled DDE with a suitable controlled stochastic evolution equation (without delay) in an infinite dimensional space. In Section 4 we will comment on the possibility of exploiting this result by means of a dynamic programming approach to the resulting infinite dimensional control problem. Now we provide two concrete examples to motivate our mathematical problem.

### Hedging of European options with execution delay

Let us consider a complete filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  on which a Brownian motion  $(W(t))_{t \geq 0}$  is defined. We consider a standard Black-Scholes financial market, composed by a riskless asset with dynamics

$$dB(t) = rB(t) dt \quad (4)$$

and a risky asset with dynamics

$$\begin{cases} dP(t) = rP(t) dt + \sigma P(t)(\lambda dt + dW(t)), \\ P(0) = p > 0. \end{cases} \quad (5)$$

Above  $r, \lambda, \sigma$  are positive constants representing the riskless spot rate, the risk premium and the volatility of the risky asset, respectively. Now, let  $d > 0$  be a fixed execution delay time: at time  $t \geq 0$  the investor chooses, on the basis of the information  $\mathcal{F}_t$ , to allocate the amount of money  $u(t) \geq 0$  of its portfolio in the risky asset. This is the control process, which is an  $\mathbb{F}$ -adapted process. However, due to the execution delay this order will be executed at time  $t + d$  **when the price of the risky asset has changed** (see [3] for the definition of this problem in a stochastic impulse control framework). Letting  $(X(t))_{t \geq 0}$  be the  $\mathbb{F}$ -adapted process representing the value of the portfolio and assuming that the portfolio is self-financing, we get the dynamics

$$dX(t) = (rX(t) + \sigma \lambda u(t-d)) dt + \sigma u(t-d) dW(t), \quad t \geq 0. \quad (6)$$

We notice that (6) is a controlled stochastic differential equation with delay in the control variable. As a delay equation, it requires, other than the specification of the initial state  $X_0$ , also the specification of part of the past of the control variable as initial data. More precisely (6) requires the specification of  $u$  in the time interval  $[-d, 0)$ , so that the complete state equation looks like

$$\begin{cases} dX(t) = (rX(t) + \sigma \lambda u(t-d)) dt + \sigma u(t-d) dW(t), & t \geq 0, \\ X(0) = x_I; \quad u(s) = u_I(s), & s \in [-d, 0). \end{cases} \quad (7)$$

Within the setting above, we can consider the problem of quadratic risk minimization in hedging a European option  $g(P(T))$ , where  $T > 0$  is the exercise date:

$$\inf_{u(\cdot)} \mathbb{E} \left[ (g(P(T)) - X(T))^2 \right]. \quad (8)$$

**Remark 1.1** Since in (7) the control variable appears only evaluated at time  $t - d$  and in the functional (8), the optimization problem (7)-(8) can be actually seen as a stochastic optimal control problem under partial information (in this case just a delayed information, simply setting  $\tilde{u}(t) = u(t - d)$ ). This kind of problems can be approached also by means of the Malliavin Calculus as done, e.g., in [10].  $\square$

### Time-to-build with shock

In the same stochastic framework defined in the previous example, let us consider a state process  $K(t)$  representing the capital stock of a certain enterprise at time  $t$  and a control process  $i(t) \geq 0$  representing the investment undertaken at time  $t$  to increase  $K$ . Assume that there is a time lag (time-to-build)  $d > 0$  between the time when the investment is undertaken and the time when it becomes productive, and that there exists some randomness affecting the achievement of the investment, so that the dynamics of  $K$  can be described by the stochastic equation

$$dK(t) = i(t-d)(dt + \sigma dW(t)), \quad (9)$$

where  $\sigma > 0$  is a volatility constant measuring the uncertainty of achievement of the investment plans. The goal is to maximize the expected integral of the discounted future profit flow in the form

$$\sup_{i(\cdot)} \mathbb{E} \left[ \int_0^\infty e^{-pt} (pF(K(t)) - C(i(t))) dt \right], \quad \sigma > 0,$$

where  $F : \mathbb{R} \rightarrow \mathbb{R}$  is a production function,  $p > 0$  is the price of the produced good, and  $C : \mathbb{R}^+ \rightarrow \mathbb{R}$  is a cost function.

**Remark 1.2** In this case the approach suggested in Remark 1.1 does not apply.  $\square$

### Structure of the letter

In this letter we will deal with the state equation (7) that, varying the parameters, gives as a particular cases (3) and (9). The letter is organized as follows. In Section 2 we present some preliminary facts and results, in Section 3 we see how we can rewrite (3) as an evolution stochastic equation in a suitable Hilbert space and in Section 4 we conclude with some remarks on the use of dynamic programming in the described setting.

## 2 Preliminary facts and results

In this section we reformulate the problem in infinite dimension. More precisely we reformulate the problem in the space

$$H := \mathbb{R} \times L^2([-d, 0]; \mathbb{R}).$$

An element  $\psi$  in  $H$  is then a couple  $(\psi_0, \psi_1)$ , with  $\psi_0 \in \mathbb{R}$  and  $\psi_1 \in L^2([-d, 0]; \mathbb{R})$ .  $H$  is a Hilbert space when endowed with the inner product

$$\langle \phi, \psi \rangle = \langle (\phi_0, \phi_1), (\psi_0, \psi_1) \rangle := \phi_0 \psi_0 + \langle \phi_1, \psi_1 \rangle_{L^2([-d, 0]; \mathbb{R})}.$$

The symbol  $\|\cdot\|$  will denote the norm induced by the inner product  $\langle \cdot, \cdot \rangle$  on  $H$ . In the following by  $M'$  we will denote the topological dual of a Banach space  $(M, \|\cdot\|_M)$ . When differently not specified, we shall consider the spaces endowed with their natural norms which make them Banach space, suppressing the norm in the notations.

On  $H$ , we consider the unbounded linear operator

$$\begin{aligned} A : D(A) \subset H &\longrightarrow H, \\ \psi = (\psi_0, \psi_1) &\longmapsto (r\psi_0, \mathcal{D}\psi_1), \end{aligned} \quad (10)$$

where

$$D(A) := \{(\psi_0, \psi_1) \in H \mid \psi_1 \in W^{1,2}([-d, 0]; \mathbb{R}), \psi_0 = \psi_1(0)\} \subset H,$$

and  $\mathcal{D}$  is the (closed) derivative operator on the Sobolev space  $W^{1,2}([-d, 0]; \mathbb{R})$ . It is worth recalling here that by well known Sobolev's embedding theorems, the space  $W^{1,2}([-d, 0]; \mathbb{R})$  is continuously embedded into the space of continuous function from  $[-d, 0]$  into  $\mathbb{R}$ , so given an

element of  $W^{1,2}([-d, 0]; \mathbb{R})$  there is a unique continuous representative of it. In the following we will always refer to this continuous representative when pointwise definitions are invoked.

The space  $D(A)$  is a Banach space when endowed with the graph norm

$$\|\psi\|_{D(A)} = \|\psi\| + \|A\psi\|, \quad \psi \in D(A).$$

Identifying  $H$  with its topological dual we have the inclusions

$$D(A) \subseteq H = H' \subseteq D(A)'$$

The operator  $A$  is the generator of a  $C_0$ -semigroup on  $H$  - see e.g. Theorem 4.3, page 254 of [5] - that we denote by  $e^{tA}$ . The semigroup  $e^{tA}$  acts as follows<sup>1</sup> on  $\psi = (\psi_0, \psi_1) \in H$ :

$$e^{tA}\psi = \left( e^{rt}\psi_0, \mathbf{1}_{[-d, (-t) \vee (-d)]}\psi_1(t + \cdot) + \mathbf{1}_{[(-t) \vee (-d), 0]}e^{r(t+\cdot)}\psi_0 \right). \quad (11)$$

The adjoint operator  $A^*$  is also a closed unbounded operator defined on a domain  $D(A^*) \subset H$  generating a  $C_0$ -semigroup  $e^{tA^*}$  on  $H$  that we denote by  $e^{tA^*}$ . The operator  $A^*$  and its domain can be characterized explicitly as follows (see e.g. Theorem 4.6, page 269 of [5]):

$$\begin{aligned} A^* : D(A^*) \subset H &\longrightarrow H, \\ \psi = (\psi_0, \psi_1) &\longmapsto (r\psi_0 + \psi_1(0), -\mathcal{D}\psi_1), \end{aligned} \quad (12)$$

where

$$D(A^*) = \{(\psi_0, \psi_1) \in H \mid \psi_1 \in W^{1,2}([-d, 0]; \mathbb{R}), \psi_1(-d) = 0\} \subset H,$$

**Lemma 2.1** *For  $\phi \in H$ , the adjoint semigroup  $e^{tA^*}$  acts as follows on a element  $\phi = (\phi_0, \phi_1) \in H$ :*

$$e^{tA^*}\phi = \left( e^{rt}\phi_0 + e^{rt} \int_{-d}^0 e^{rs} \mathbf{1}_{[(-t) \vee (-d), 0]}(s) \phi_1(s) ds, \phi_1(\cdot - t) \mathbf{1}_{[-d+t \wedge 0, 0]}(\cdot) \right). \quad (13)$$

*Proof.* Given  $\phi$  and  $\psi$  in  $H$ , we have

$$\begin{aligned} &\langle e^{tA}\psi, \phi \rangle \\ &= e^{rt}\psi_0\phi_0 + \int_{-d}^0 \left( \mathbf{1}_{[-d, (-t) \vee (-d)]}(s) \psi_1(t+s) + \mathbf{1}_{[(-t) \vee (-d), 0]}(s) e^{r(t+s)} \psi_0 \right) \phi_1(s) ds \\ &= \psi_0 e^{rt}\phi_0 + \int_{-d}^0 \psi_1(s) \phi_1(s-t) \mathbf{1}_{[(-d+t) \wedge 0, 0]}(s) ds + \psi_0 e^{rt} \int_{-d}^0 e^{rs} \mathbf{1}_{[(-t) \vee (-d), 0]}(s) \phi_1(s) ds \\ &= \psi_0 \left( e^{rt}\phi_0 + e^{rt} \int_{-d}^0 e^{rs} \mathbf{1}_{[(-t) \vee (-d), 0]}(s) \phi_1(s) ds \right) \\ &\quad + \int_{-d}^0 \psi_1(s) \phi_1(s-t) \mathbf{1}_{[(-d+t) \wedge 0, 0]}(s) ds \end{aligned} \quad (14)$$

so we have the claim.  $\square$

**Lemma 2.2** *The semigroup  $e^{tA^*}$  can be extended to a  $C_0$ -semigroup to the space  $D(A)'$  and the semigroup  $e^{tA}$  can be restricted to a  $C_0$ -semigroup on  $D(A)$ .*

*Proof.* The claims follow by the general Semigroup Theory, see e.g. [5] pages 202–204.  $\square$

<sup>1</sup>Hereafter, given  $f \in L^2([-d, 0]; \mathbb{R})$ , with a slight abuse of notation we consider  $f$  extended on  $[-d, +\infty)$  as

$$\xi \longmapsto \begin{cases} f(\xi), & \text{if } \xi \in [-d, 0], \\ 0, & \text{if } \xi > 0. \end{cases}$$

On the space  $D(A)$  we introduce the continuous linear functional

$$\begin{aligned} B : D(A) \subseteq H &\longrightarrow \mathbb{R}, \\ \psi = (\psi_0, \psi_1) &\longmapsto \psi_1(-d). \end{aligned} \quad (15)$$

The adjoint operator  $B^* : \mathbb{R} \rightarrow D(A)'$  can be identified with  $(0, \delta_{-d}) \in D(A)'$ , defined by

$$(0, \delta_{-d})\psi = \psi_1(-d), \quad \psi = (\psi_0, \psi_1) \in D(A),$$

where  $\delta_{-d}$  is the Dirac measure concentrated at  $-d$ : indeed

$$B^*a = a(0, \delta_{-d}) \in D(A)', \quad a \in \mathbb{R},$$

With an abuse of language we will confuse  $B^* : \mathbb{R} \rightarrow D(A)'$  and  $(0, \delta_{-d}) \in D(A)'$ .

Given  $\psi = (\psi_0, \psi_1) \in D(A)$ , we have

$$Be^{tA}\psi = \begin{cases} \psi_1(t-d), & t \in [0, d], \\ \psi_0 e^{r(t-d)}, & t > d. \end{cases} \quad (16)$$

By Lemma 2.2,  $e^{tA}B^*$  is again an element of  $D(A)'$ . In the following lemma we give an explicit expression for it.

**Lemma 2.3** *We have*

$$e^{tA}B^* = \left( e^{(t-d)r} \mathbf{1}_{[d, +\infty)}(t), \delta_{-d+t} \mathbf{1}_{[(-d+t) \wedge 0, 0]}(\cdot) \right), \quad (17)$$

*i.e.*

$$e^{tA}B^* = \begin{cases} (0, \delta_{-d+t}) & \text{if } t \in [0, d], \\ (e^{r-d}, 0) & \text{if } t \in [d, +\infty). \end{cases} \quad (18)$$

*Proof.* We consider  $\psi \in D(A)$  - which implies in particular, by Lemma 2.2,  $e^{tA}\psi \in D(A)$  for all  $t \geq 0$  - and we write

$$\begin{aligned} \left\langle \psi, e^{tA}B^* \right\rangle_{D(A) \times D(A)'} &= \left\langle e^{tA}\psi, B^* \right\rangle_{D(A) \times D(A)'} \\ &= \left\langle \left( e^{rt}\psi_0, \mathbf{1}_{[-d, (-t) \vee (-d)]}(s)\psi_1(t+s) + \mathbf{1}_{[(-t) \vee (-d), 0]}(s)e^{r(t+s)}\psi_0 \right), B^* \right\rangle_{D(A) \times D(A)'} \\ &= \left( \mathbf{1}_{[-d, (-t) \vee (-d)]}(-d)\psi_1(t-d) + \mathbf{1}_{[(-t) \vee (-d), 0]}(-d)e^{r(t-d)}\psi_0 \right). \end{aligned} \quad (19)$$

This yields the expression of  $e^{tA}B^*$  when acting on a generic element  $\psi$  of  $D(A)$ , yielding the expression of  $e^{tA}B^*$  described in the claim.  $\square$

### 3 Infinite dimensional representation of the delay differential equation

We want to find a way to reformulate (7) as a stochastic evolution equation in  $H$ . First we state existence and uniqueness of solutions for (7). In the following given a Banach space  $M$ , the symbols  $L_{loc}^2([0, +\infty); M)$  and  $L_{loc}^\infty([0, +\infty); M)$  will denote the space of  $M$ -valued functions which are square integrable over compact intervals and, respectively, the space of  $M$ -valued functions which are essentially bounded.

**Proposition 3.1** *Let  $(x_I, u_I) \in H$  and let  $u \in L_{loc}^2([0, +\infty); L^2(\Omega; \mathbb{R}))$  be  $\mathbb{F}$ -adapted. The SDE (7) has a unique (adapted) strong solution  $X \in L_{loc}^2([0, +\infty); L^2(\Omega; \mathbb{R}))$  and such solution admits a continuous version.*

*Proof.* This is a straightforward application of classical results from the theory of SDEs with measurable coefficients. See, e.g., Chapter 1 of [24].  $\square$

Given  $a \in \mathbb{R}$  and  $v \in L^2([-d, 0]; \mathbb{R})$ , we denote by  $\Gamma(a, v)$  the element of  $L^2(\Omega, \mathcal{F}_d, \mathbb{P}; D(A)')$  defined as

$$\Gamma(a, v) := \begin{pmatrix} a \\ 0 \end{pmatrix} + \int_{-d}^0 \sigma \lambda v(\tau) e^{-\tau A^*} B^* d\tau + \int_{-d}^0 \sigma v(\tau) e^{-\tau A^*} B^* dW(\tau + d),$$

where  $\begin{pmatrix} a \\ 0 \end{pmatrix} \in H \subset D(A)'$ . The expression of  $\Gamma(a, v)$  provides the construction of the so called *structural state* (see [23] and [5]) in our stochastic case: at each time  $t$ , we expect to be able to describe the essential state of the system through  $\Gamma(X(t), u_t) \in L^2(\Omega, \mathcal{F}_{t+d}, \mathbb{P}; D(A)')$ . Note that, since our system does not depend on the past of the state, as in the deterministic case (see [5], Chapter II.4) the structural state depends only on the present of the state and of the whole past of the control.

Now denote by  $y_I$  the random element of  $D(A)'$  defined as

$$y_I := \Gamma(x_I, u_I). \quad (20)$$

Then  $y_I$  corresponds to the structural state for the initial datum and it will be the initial datum of our evolution equation. Notice that it is a stochastic datum even though  $(x_I, u_I)$  is deterministic.

In the space  $D(A)'$  we consider the following SDE

$$\begin{cases} dY(t) = (A^*Y(t) + \lambda \sigma u(t) B^*) dt + \sigma u(t) B^* dW(t + d), \\ Y(0) = y_I. \end{cases} \quad (21)$$

We call *mild solution* to (21) the  $D(A)'$ -valued process  $Y(t)$ ,  $t \geq 0$ , by

$$Y(t) = e^{tA^*} y_I + \int_0^t \lambda \sigma u(s) e^{(t-s)A^*} B^* ds + \int_0^t \sigma u(s) e^{(t-s)A^*} B^* dW(s + d). \quad (22)$$

We notice that

$$Y(t) \in L^2(\Omega, \mathcal{F}_{t+d}, \mathbb{P}; D(A)'), \quad \forall t \geq 0,$$

so  $Y$  is adapted with respect to  $(\mathcal{G}_t)_{t \geq 0} := (\mathcal{F}_{t+d})_{t \geq 0}$ . Although a priori the process  $Y$  takes values in  $D(A)'$ , actually we have the following.

**Proposition 3.2** *Assume that  $u_I \in L^\infty([-d, 0]; \mathbb{R})$  and that  $u \in L_{loc}^\infty([0, +\infty); L^\infty(\Omega; \mathbb{R}))$ . Let  $Y$  be the mild solution to (21). Then  $Y(t) \in L^2(\Omega, \mathcal{F}_{t+d}, \mathbb{P}; H)$  for all  $t \in [0, +\infty)$ . In particular  $Y(t) \in H$  a.s. for all  $t \in [0, +\infty)$ .*

*Proof.* First we shall show that  $y_I \in L^2(\Omega, \mathcal{F}_d; H)$ . By definition

$$y_I \in L^2(\Omega, \mathcal{F}_d, \mathbb{P}; D(A)') = (L^2(\Omega, \mathcal{F}_d, \mathbb{P}; D(A)))'.$$

Due to Lemma 2.2,  $e^{tA^*}$  is a  $C_0$ -semigroup on  $D(A)'$  and then there exists  $C, \alpha > 0$

$$\|e^{tA^*}\|_{\mathcal{L}(D(A)')} \leq C e^{\alpha t}. \quad (23)$$

Taking into account (16) and (23), we can find  $c, \tilde{c} > 0$  such that, for any  $\psi = (\psi_0, \psi_1) \in$



$L^2(\Omega, \mathcal{F}_d, \mathbb{P}; D(A))$ , we have

$$\begin{aligned}
\langle y_I, \psi \rangle_{L^2(\Omega; D(A))' \times L^2(\Omega; D(A))} &= \mathbb{E} \langle y_I, \psi \rangle_{D(A)' \times D(A)} \\
&\leq c \left( |x_I| \sqrt{\mathbb{E}|\psi_0|^2} + \mathbb{E} \left| \int_{-d}^0 B e^{-\tau A} \psi u_I(\tau) d\tau \right| + \mathbb{E} \left| \int_{-d}^0 B e^{-\tau A} \psi u_I(\tau) dW(\tau + d) \right| \right) \\
&\leq c \left( |x_I| \|\psi_0\|_{L^2(\Omega; \mathbb{R})} + \mathbb{E} \left| \int_{-d}^0 \psi_1(-d - \tau) u_I(\tau) d\tau \right| \right. \\
&\quad \left. + \left( \mathbb{E} \left| \int_{-d}^0 \psi_1(-d - \tau) u_I(\tau) dW(\tau + d) \right|^2 \right)^{1/2} \right) \\
&\leq \tilde{c} \left( |x_I| \|\psi_0\|_{L^2(\Omega; \mathbb{R})} + \|\psi\|_{L^2(\Omega; H)} \|u_I\|_{L^\infty} + \|\psi\|_{L^2(\Omega; H)} \|u_I\|_{L^\infty}^{1/2} \right) \\
&= \tilde{c} \left( |x_I| + \|u_I\|_{L^\infty} + \|u_I\|_{L^\infty}^{1/2} \right) \|\psi\|_{L^2(\Omega; H)}, \quad (24)
\end{aligned}$$

where we have suppressed the  $\sigma$ -algebra  $\mathcal{F}_d$  and the probability  $\mathbb{P}$  in the notations, we have set  $L^\infty := L^\infty([-d, 0]; \mathbb{R})$ , and we have used the Holder inequality to estimate the term in  $d\tau$  and the Jensen inequality and the Ito isometry to estimate the term in  $dW(\tau + d)$ . Since  $L^2(\Omega, \mathcal{F}_d, \mathbb{P}; D(A))$  is a dense subset of  $L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H)$ , by (24) we can extend  $y_I$  to a bounded functional on  $L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H)$ . Since we are identifying  $H$  with its dual, we conclude that

$$y_I \in (L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H))' = L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H') = L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H).$$

Consider now  $Y(t)$  whose expression is given in (22) with  $t \in [0, d]$ . Since we have proved that  $y_I \in L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H)$ , also  $e^{tA^*} y_I \in L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H)$ . So, in order to prove the claim for  $t \in [0, d]$ , we need to prove that

$$\int_0^t \lambda \sigma u(s) e^{(t-s)A^*} B^* ds + \int_0^t \sigma u(s) e^{(t-s)A^*} B^* dW(s + d) \in L^2(\Omega, \mathcal{F}_{t+d}; H), \quad \forall t \in [0, d].$$

We can use the same arguments that we have used above for estimating the term  $\int_{-d}^0 \lambda \sigma u_I(\tau) e^{-\tau A^*} B^* d\tau + \int_{-d}^0 \sigma u_I(\tau) e^{-\tau A^*} B^* dW(\tau + d)$  appearing in the expression of  $y_I$ , getting the result for  $t \in [0, d]$ .

In particular one gets also  $Y(d) \in L^2(\Omega, \mathcal{F}_d, \mathbb{P}; H)$ . So, if now  $t \in [d, 2d]$ , we observe first that  $Y(t)$  solves

$$\begin{aligned}
Y(t) &= e^{(t-d)A^*} Y(d) + \int_0^{t-d} u(d+s) e^{(t-d-s)A^*} B^* ds \\
&\quad + \int_0^{t-d} u(d+s) e^{(t-d-s)A^*} B^* dW(2d+s).
\end{aligned}$$

and then we use once again the same kind of estimates to prove the claim for  $t \in [d, 2d]$ . Iterating the argument we conclude the proof.  $\square$

A fundamental consequence of Proposition 3.2 is that for the mild solution  $Y = (Y_0, Y_1)$  of (21) we have

$$Y_0(t) = \langle Y(t), (1, 0) \rangle, \quad \text{a.s. } \forall t \geq 0. \quad (25)$$

We prove now that the solution of the initial stochastic DDE (7) is indeed linked with the solution of the stochastic evolution equation (21). We need first a simple lemma that we will use to approximate the solution. Consider the functions

$$(1, \varphi^\varepsilon) \in D(A), \quad \varphi^\varepsilon(s) := (1 + s/\varepsilon) \mathbf{1}_{[-\varepsilon, 0]}(s). \quad (26)$$

and notice that

$$(1, \varphi^\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} (1, 0), \quad \text{in } H. \quad (27)$$

**Lemma 3.3**

(i) For  $t \geq 0$  we have

$$e^{tA}(1, 0) = \left( e^{rt}, \mathbf{1}_{[(-t) \vee (-d), 0]} e^{r(t+\cdot)} \right). \quad (28)$$

(ii) Let  $\varphi^\varepsilon$  be the function defined in (26). We have

$$Be^{tA}(1, \varphi^\varepsilon) = \begin{cases} 0, & t \in [0, d - \varepsilon], \\ 1 + \frac{t-d}{\varepsilon}, & t \in (d - \varepsilon, d], \\ e^{r(t-d)}, & t > d. \end{cases} \quad (29)$$

*Proof.* Claim (i) follows from the explicit expression of  $e^{tA}$  given in (11). Claim (ii) follows from (16).  $\square$

**Theorem 3.4** *Let the assumptions of Proposition 3.2 be verified. Let  $X(t)$  be the solution of (7) and let  $Y(t) = (Y_0(t), Y_1(t))$  be the mild solution of (21). Then  $Y_0(t) = X(t)$  a.s. for all  $t \geq 0$ .*

*Proof.* Choose  $t > 0$ . Let  $\varphi^\varepsilon$  be the function defined in Lemma 3.3. Using (22), (27) and (25), we can write (the limits below are intended holding a.s.)

$$\begin{aligned} Y_0(t) &= \langle Y(t), (1, 0) \rangle = \left\langle Y(t), \lim_{\varepsilon \rightarrow 0} (1, \varphi^\varepsilon) \right\rangle = \lim_{\varepsilon \rightarrow 0} \langle Y(t), (1, \varphi^\varepsilon) \rangle \\ &= \lim_{\varepsilon \rightarrow 0} \left[ \left\langle e^{tA} \begin{pmatrix} x_I \\ 0 \end{pmatrix}, (1, \varphi^\varepsilon) \right\rangle + \left\langle \int_{-d}^0 \lambda \sigma u_I(\tau) e^{(t-\tau)A^*} B^* d\tau, (1, \varphi^\varepsilon) \right\rangle \right] \\ &+ \left\langle \int_{-d}^0 \sigma u_I(\tau) e^{(t-\tau)A^*} B^* dW(\tau + d), (1, \varphi^\varepsilon) \right\rangle + \left\langle \int_0^t \lambda \sigma u(\tau) e^{(t-\tau)A^*} B^* d\tau, (1, \varphi^\varepsilon) \right\rangle \\ &+ \left\langle \int_0^t \sigma u_I(\tau) e^{(t-\tau)A^*} B^* dW(\tau + d), (1, \varphi^\varepsilon) \right\rangle. \quad (30) \end{aligned}$$

Passing to the adjoint,

$$\begin{aligned} Y_0(t) &= \lim_{\varepsilon \rightarrow 0} \left[ \left\langle \begin{pmatrix} x_I \\ 0 \end{pmatrix}, e^{tA}(1, \varphi^\varepsilon) \right\rangle + \int_{-d}^0 \lambda \sigma u_I(\tau) B e^{(t-\tau)A}(1, \varphi^\varepsilon) d\tau \right. \\ &+ \int_{-d}^0 \sigma u_I(\tau) B e^{(t-\tau)A}(1, \varphi^\varepsilon) dW(\tau + d) + \int_0^t \lambda \sigma u(\tau) B e^{(t-\tau)A}(1, \varphi^\varepsilon) d\tau \\ &\left. + \int_0^t \sigma u(\tau) B e^{(t-\tau)A}(1, \varphi^\varepsilon) dW(\tau + d) \right]. \quad (31) \end{aligned}$$

Now, setting  $u(\tau) = u_I(\tau)$  for  $\tau \in [-d, 0)$ , using (28) and (29) and taking the limit we obtain

$$\begin{aligned} Y_0(t) &= x_I e^{rt} + \int_{-d}^{t-d} \lambda \sigma u(\tau) e^{r(t-\tau-d)} d\tau + \int_{-d}^{t-d} \sigma u(\tau) e^{r(t-\tau-d)} dW(\tau + d) \\ &= \int_0^t \lambda \sigma u(\tau - d) e^{r(t-\tau)} d\tau + \int_0^t \sigma u(\tau - d) e^{r(t-\tau)} dW(\tau) = X(t), \quad (32) \end{aligned}$$

where the last equality follows from the fact that  $X$  solves the original state equation (7). Thus we have the claim.  $\square$

## 4 Some comments on the dynamic programming approach to the infinite dimensional formulation

By Theorem 3.4, we can try to rewrite the optimal control problems we described in the introduction in a equivalent way as optimal control problems in  $H$  and then try to apply the tools of the dynamic programming to them.

For example, let us focus on the optimization problem (7)-(8) and define the value function at time  $t = 0$

$$V(0, p, x_I, u_I) := \inf_{u(\cdot)} \mathbb{E} \left[ (g(P(T)) - X(T))^2 \right].$$

At least three nontrivial questions arise.

1. Let us consider the optimal control problem in  $H$  having as state equations (21) with deterministic initial datum  $y$  and (5). Then define the value function at time  $t = 0$

$$V^H(0, p, y) := \inf_{u(\cdot)} \mathbb{E} \left[ (g(P(T)) - Y_0(T))^2 \right]. \quad (33)$$

When is it true the desirable equality

$$V(0, p, x_I, u_I) = \mathbb{E} [V^H(0, p, y_I)]? \quad (34)$$

2. Consider as initial time  $t \in [0, T]$  and denote by  $P^{t,p}$  the process with dynamics (5) starting from  $p$  at time  $t$  and by  $Y^{t,y,u(\cdot)}$  the process with dynamics (21) starting from the deterministic datum  $y$  at time  $t$  and under the control  $u(\cdot)$ . The dynamic programming principle formally writes as (with clear meaning of  $V^H(t, \cdot)$ )

$$V^H(t, p, y) = \inf_{u(\cdot)} \mathbb{E} \left[ V^H(\tau, P^{t,p}(\tau), Y^{t,y,u(\cdot)}(\tau)) \right].$$

where  $\tau \in [t, T]$  is a stopping time. Is it always true?

3. What is the HJB equation associated to  $V^H$ ? Clearly it should be derived formally by the dynamic programming principle. But this passes through the use of Itô's formula. However the variation  $dY(s)$  depends on  $dW(s+d)$ , while the variation  $dP(s)$  depends on  $dW(s)$ . So how does Itô's calculus apply to this situation? Clearly there are some specific cases where this difficulty disappears, for instance when  $g$ , hence  $V^H$ , does not depend on  $p$ .

All these questions (and possibly more) are out of the scope of the present letter and left for future research. Probably positive answers can be obtained in some specific problems.

## References

- [1] Asea P. K., Zak P. J., *Time-to-build and cycles*, Journal of Economic Dynamics and Control, Vol. 23, No.8, pp. 1155–1175, 1999.
- [2] Bambi M., *Endogenous growth and time to build: the AK case*, Journal of Economic Dynamics and Control, Vol. 32, No. 4, pp. 1015–1040, 2008.
- [3] Bruder B., Pham H. *Impulse Control Problem on Finite Horizon with Execution Delay*. Stochastic Processes and their Applications, 119, pp. 1436–1469, 2009.
- [4] Bambi M., Fabbri G., Gozzi F., *Optimal policy and consumption smoothing effects in the time-to-build AK model*, Economic Theory, vol. 50, No. 3, pp. 635–669, 2012.
- [5] Bensoussan A., Da Prato G., Delfour M. C., Mitter S. K., “Representation and Control of Infinite Dimensional System”, Birkhäuser Boston, Second Edition 2007.
- [6] Boucekine R., Licandro O., Puch Luis A., Fernando Del Rio F., *Vintage capital and the dynamics of the AK model*, Journal of Economic Theory, Vol. 120, pp. 39–72, 2005.
- [7] d’Albis, H., Augeraud-Veron, E., Venditti, A., *Business cycle fluctuations and learning-by-doing externalities in a one-sector model*, Journal of Mathematical Economics, Vol. 48, No. 5, pp. 295–308, 2012.

- [8] Delfour M. C. *State theory of linear hereditary differential systems*, Journal of Mathematical Analysis and Applications, Vol. 60, No. 1, pp. 8–35, 1977.
- [9] Delfour M. C., Mitter, S. K. *Hereditary differential systems with constant delays. II. A class of affine systems and the adjoint problem*, Journal of Differential Equations Vol. 18, No. , pp. 18–28, 1975.
- [10] Di Nunno, G., Øksendal, B., *Optimal portfolio, partial information and Malliavin calculus*, Stochastics, Vol. 81, No. 3-4 pp. 303–322, 2009.
- [11] Federico S., Goldys B., Gozzi F., *HJB Equations for the Optimal Control of Differential Equations with Delays and State Constraints, I: Regularity of Viscosity*, SIAM Journal on Control and Optimization, Vol. 48, No. 8, pp. 4910–4937, 2010.
- [12] Federico S., Goldys B., Gozzi F., *HJB Equations for the Optimal Control of Differential Equations with Delays and State Constraints, II: Verification and Optimal*, SIAM Journal on Control and Optimization, Vol. 49, No. 6, pp. 2378–2414, 2011.
- [13] Federico S. Tankov P., *Finite-dimensional representation for controlled diffusions with delay*, Applied Mathematics and Optimization, to appear.
- [14] Federico S. Tacconi E., *Dynamic programming for optimal control problems with delays in the control variable*, SIAM - Journal on Control and Optimization, Vol. 52, No. 2, pp. 1203-1236 (2014).
- [15] Feichtinger G., Hartl R., Sethi S., *Dynamic Optimal Control Models in Advertising: Recent Developments*, Management Science, Vol. 40, No. 2, pp.195–226, 1994.
- [16] Fabbri G., Gozzi F., *Solving optimal growth models with vintage capital: The dynamic programming approach*, Journal of Economic Theory 143, pp. 331–373, 2008.
- [17] Gozzi F., Marinelli C., Savin S., *On controlled linear diffusions with delay in a model of optimal advertising under uncertainty with memory effects*, Journal of Optimization, Theory and Applications, Vol. 142, No. 2, pp. 291–321, 2009.
- [18] Ichikawa A., *Quadratic control of evolution equations with delays in the control*, SIAM Journal on control and optimization, Vol. 20, No.5, pp. 645–668, 1982.
- [19] Kydland, F. E., Prescott, E. C., *Time-to-build and aggregate fluctuations*, Econometrica, Vol. 50, No. 6, pp. 1345–1370, 1982.
- [20] Pauwels W., *Optimal Dynamic Advertising Policies in the Presence of Continuously Distributed Time Lags*, Journal of Optimization Theory and Applications, Vol. 22, No. 1, pp. 79–89, 1977.
- [21] Sethi S. P., *Sufficient Conditions for the Optimal Control of a Class of Systems with Continuous Lags*, Journal of Optimization Theory and Applications, Vol. 13, No. 5, pp. 545–552, 1974.
- [22] Tsoukalas J., *Time to Build Capital: Revisiting Investment-Cash Flow Sensitivities*, Journal of Economic Dynamics and Control Vol. 35, No. 7, pp. 10001016, 2011.
- [23] Vinter R.B., Kwong R.H., *The infinite dimensional quadratic control problem for linear systems with state and control delay: an evolution equation approach*, SIAM Journal on Control and Optimization, Vol. 19, No. 1, pp. 139–153, 1981.
- [24] Yong J., and X.Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer Verlag, New York, 1999.