On the Power Series Expansion of the Reciprocal Gamma Function

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Abstract. Using the reflection formula of the Gamma function, we derive a new formula for the Taylor coefficients of the reciprocal Gamma function. The new formula provides effective asymptotic values for the coefficients even for very small values of the indices. Both the sign oscillations and the leading order of growth are given.

1. Introduction

The reciprocal Gamma function is an entire function with a Taylor series given by

$$\frac{1}{\Gamma(z)} = \sum_{n=1}^{\infty} a_n z^n. \tag{1.1}$$

It has been a challenge since the time of Weierstrass to compute or at least estimate the coefficients of the reciprocal Gamma function. The main reason is the ubiquitous presence of the reciprocal gamma function in analytic number theory and its various connections to other transcendental functions (for example the Riemann zeta function). Since Bourguet [3] who was the first to calculate the first 23 coefficients, there has been very few publications, to the author's knowledge, on accurate calculations of the coefficients beyond a_{50}.

Knowing that an effective asymptotic formula is always useful as an independent check for the sign and value of the coefficients for very large values of n, it is important to have such a formula in order to enhance the calculations. The only asymptotic formula that is known to date is that of Hayman [9].

With regard to the computation of the coefficients of the reciprocal Gamma function, there are basically three known methods [15, 3, 2]. The first method is due to Bourguet [3]. It consists in exploiting the recursive formula

$$na_n = \gamma a_{n-1} - \zeta(2)a_{n-2} + \zeta(3)a_{n-3} - \cdots + (-1)^{n+1}\zeta(k); n > 2, \tag{1.2}$$

with $a_1 = 1$, $a_2 = \gamma$, the Euler constant, and $\zeta(k)$ is the zeta function of Riemann.

It has been noticed in [2] that this method suffers from severe numerical instability; all digits are lost from $n \geq 27$.

The second method is based on Cauchy's formula for the coefficients of Taylor series using circular contours:

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$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{1}{z^{n+1}\Gamma(z)}\,dz$$
$$= \frac{1}{2\pi r^n}\int_0^{2\pi} \frac{e^{-in\theta}}{\Gamma(re^{i\theta})}\,d\theta, \tag{1.3}$$

where r can be between 0 and ∞ since the reciprocal Gamma function is entire.

The integral (1.3) can be evaluated with the many existing quadrature rules such as the trapezoidal rule or the Gauss-Legendre quadrature. Particular attention is given to the method discovered by Lyness [12] which uses the trapezoidal rule in conjunction with the discrete Fourier transform. It is very fast and provides good results [14] as long as the radius of the contour is properly selected.

Although the radius r of the contour can theoretically be arbitrarily chosen, the effects of the value of r on approximation and round-off errors are numerically very different. A comprehensive investigation for choosing a good radius r has been carried out in [2], where it has been shown that as n increases so does the good r.

In [13], different quadrature formulas, using also the method of contour integration, have been investigated for the calculation of a_n. The contour chosen is no longer circular but chosen as the Hankel contour. The reciprocal Gamma function is represented using Heine's formula [10]:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\mathcal{C}} e^t t^{-z}\,dt, \tag{1.4}$$

where $\mathcal{C}$ consists of the three parts $C = C_+ \cup C_\epsilon \cup C_-$: a path which extends from (∞, ϵ), around the origin counter clockwise on a circle of center the origin and of radius ϵ and back to (ϵ, ∞), where ϵ is an arbitrarily small positive number.

Lastly, the third method for calculating the coefficients of the reciprocal Gamma function for large values of n is not a numerical one. It consists in approximating the coefficients using an asymptotic formula. The first attempt was initiated by Bourguet [3] who found the following upper bound

$$a_n \le \frac{(-1)^n}{\pi\Gamma(n+1)}\frac{e\pi^{n+1}}{n+1} + \frac{4}{\pi^2\sqrt{\Gamma(n+1)}}$$
$$\lesssim \frac{4}{\pi^2\sqrt{\Gamma(n+1)}}. \tag{1.5}$$

But the first systematic study to obtain an asymptotic formula for the coefficients was carried out by Hayman [9] theoretically, and by Bornemann [2] numerically (see also [1] for the related phenomenon of oscillations of the derivatives).

In this paper, we will give a new effective asymptotic formula for the coefficients a_n. With the formula, we obtain the sign oscillations and the leading order of growth of the coefficients. We will show that our results can be considered very accurate even for very small values of n.

2. An Integral Formula For The Coefficients a_n

Let's replace z by $z-1$ into the series (1.1), we have

$$\frac{1}{\Gamma(z-1)} = a_1(z-1) + a_2(z-1)^2 + a_3(z-1)^3 \cdots, \tag{2.1}$$

and dividing both sides by $z-1$, we get

$$\frac{1}{\Gamma(z)} = \frac{1}{z-1}\left[a_1(z-1) + a_2(z-1)^2 + a_3(z-1)^3 \cdots\right] \tag{2.2}$$

To obtain an integral formula for the reciprocal Gamma function, we start from Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.3}$$

to get

$$\frac{1}{\Gamma(z)} = \frac{\sin(\pi z)}{\pi}\Gamma(1-z). \tag{2.4}$$

Now, for $\mathrm{Re}(z) < 2$, we can write

$$\begin{aligned}\frac{1}{\Gamma(z)} &= \frac{\sin(\pi z)}{\pi}\Gamma(1-z)\\ &= \frac{\sin(\pi(z-1))}{\pi(z-1)}\Gamma(2-z)\\ &= \frac{\sin(\pi(z-1))}{\pi(z-1)}\int_0^\infty e^{-t}t^{1-z}\,dt. \qquad (2.5)\end{aligned}$$

By observing that $\sin(\pi(z-1)) = \frac{e^{i\pi(z-1)} - e^{-i\pi(z-1)}}{2i}$, we can rewrite (2.5) as

$$\frac{1}{\Gamma(z)} = \frac{1}{z-1}\frac{1}{2\pi i}\int_0^\infty e^{-t}\left[e^{(z-1)(-\log(t)-i\pi)} - e^{(z-1)(-\log(t)+i\pi)}\right]dt. \tag{2.6}$$

And if we compare the two equations (2.6) and (2.2), we deduce that the coefficients a_n for $n \geq 1$ are given by

$$\begin{aligned}a_n &= \frac{1}{2\pi i n!}\int_0^\infty e^{-t}\lim_{z\to 1}\frac{d^n}{dz^n}\left\{e^{(z-1)(-\log(t)+i\pi)} - e^{(z-1)(-\log(t)-i\pi)}\right\}dt\\ &= \frac{1}{2\pi i n!}\int_0^\infty e^{-t}\left\{(-\log(t)+i\pi)^n - (-\log(t)-i\pi)^n\right\}dt\\ &= \frac{1}{\pi n!}\int_0^\infty e^{-t}\mathrm{Im}\left\{(-\log(t)+i\pi)^n\right\}dt, \qquad (2.7)\end{aligned}$$

where Im stands for the imaginary part. This is our expression of the coefficients a_n, described in the following

Theorem 2.1. *The coefficients a_n are given by*

$$a_n = \frac{(-1)^n}{\pi n!}\int_0^\infty e^{-t}\mathrm{Im}\big\{(\log t - i\pi)^n\big\}\,dt. \tag{2.8}$$

Theorem 2.1 permits an exact asymptotic evaluation of the constants a_n[1]. It is the subject of the next section.

3. Asymptotic Estimates of the Coefficients

This section is dedicated to approximating the complex-valued integral

$$I(n) = \int_0^\infty e^{-t} (\log t - i\pi)^n \, dt \tag{3.1}$$

using the saddle-point method [4, 6]. By the change of variables $t = nz$, our integral becomes

$$\begin{aligned} I(n) &= n \int_0^\infty e^{-nz} \left\{ \log(nz) - i\pi \right\}^n \, dz \\ &= n \int_0^\infty e^{n\{-z + \log[\log(nz) - i\pi]\}} \, dz. \end{aligned} \tag{3.2}$$

If we define

$$f(z) = -z + \log\left(\log(nz) - i\pi\right), \tag{3.3}$$

then the saddle-point method consists in deforming the path of integration into a path which goes through a saddle-point at which the derivative $f'(z)$, vanishes. If z_0 is the saddle-point at which the real part of $f(z)$ takes the greatest value, the neighborhood of z_0 provides the dominant part of the integral as $n \to \infty$ [4, p. 91-93]. This dominant part provides an approximation of the integral and it is given by the formula

$$I(n) \approx n e^{n f(z_0)} \left(\frac{-2\pi}{n f''(z_0)} \right)^{\frac{1}{2}}. \tag{3.4}$$

In our case, we have

$$f'(z) = -1 + \frac{1}{z\left(\log(nz) - i\pi\right)}, \quad \text{and} \tag{3.5}$$

$$f''(z) = \frac{-1}{z^2\left(\log(nz) - i\pi\right)} - \frac{1}{z^2\left(\log(nz) - i\pi\right)^2}. \tag{3.6}$$

The saddle-point z_0 should verify the equation

$$\begin{aligned} & z_0\left(\log(nz_0) - i\pi\right) = 1 \\ \Leftrightarrow \quad & nz_0 e^{-i\pi} \log\left(nz_0 e^{-i\pi}\right) = n e^{-i\pi}. \end{aligned} \tag{3.7}$$

[1]The theorem is almost evident and easy to derive. It is hard to believe that it has not been discovered before. To the author's knowledge, the integral formula (2.7) is new and seems to be inexistant in the literature.

The last equation is of the form $v \log v = b$ whose solution can be explicitly written using the branch $k = -1$ of the Lambert W-function[2] [5]:

$$v = e^{W_{-1}(b)}. \tag{3.8}$$

The saddle-point solution to our equation (3.7) is given by

$$z_0 = \frac{e^{-i\pi}}{n} e^{W_{-1}\left(ne^{-i\pi}\right)} = \frac{e^{W_{-1}}(-n)}{-n}, \tag{3.9}$$

and at the saddle-point, we have the values

$$f(z_0) = -z0 - \log z_0 \tag{3.10}$$

$$f''(z_0) = -1 - \frac{1}{z_0}. \tag{3.11}$$

Therefore, the saddle-point approximation of our integral (3.1) is given by

$$I(n) \approx \sqrt{2\pi n} e^{-nz_0} \frac{z_0^{\frac{1}{2}-n}}{\sqrt{1+z_0}}. \tag{3.12}$$

Now since $a_n = \frac{(-1)^n}{\pi n!} \mathrm{Im}\left\{I(n)\right\}$, we arrive at our main result:

Theorem 3.1. *Let* $z_0 = \frac{e^{W_{-1}(-n)}}{-n}$, *where* W_{-1} *is the branch* $k = -1$ *of the Lambert* W*-function. For* n *large enough, the Taylor coefficients of the reciprocal Gamma function can be approximated by*

$$a_n \approx (-1)^n \sqrt{\frac{2}{\pi}} \frac{\sqrt{n}}{n!} \mathrm{Im}\left\{ e^{-nz_0} \frac{z_0^{\frac{1}{2}-n}}{\sqrt{1+z_0}} \right\}. \tag{3.13}$$

Bornemann's derivation [2] of Hayman's asymptotic formula for the coefficients a_n is given by[3]

$$a_n \sim \frac{\sqrt{2}}{\pi n} \frac{1}{\left|\Gamma\left(r_n e^{i\theta_n}\right)\right| r_n^n} \cos\phi_n, \tag{3.14}$$

where

$$z_n = r_n e^{i\theta_n} = e^{W\left(\frac{1}{2}-n\right)} \tag{3.15}$$

$$\phi_n = \left(n - \frac{1}{2}\right)\left(\frac{\sin^2\theta_n}{\theta_n} - \theta_n\right) - \frac{1}{2}\left(\cot\theta_n - \theta_n \csc^2\theta_n\right). \tag{3.16}$$

[2]The principal branch of the Lambert W-function is denoted by $W_0(z) = W(z)$. The principal branch $W_0(z)$ and the branch $W_{-1}(z)$ are the only branches of W that take on real values. The other branches of W have the negative real axis as the only branch cut closed on the top for counter clockwise continuity. In our equation (3.7), the argument is $-\pi$ and not π and so the solution belongs to the branch of W_{-1}. See [5] for a excellent discussion and explanation of all the branches of W.

[3]The formula of Bornemann differs from that of Hayman in the phase approximation. The orignal approximation given by Hayman is $\phi_n = \left(n - \frac{1}{2}\right)\left(\frac{\sin^2\theta_n}{\theta_n} - \theta_n\right)$. For the calculations, both phase approximations give essentially the same results.

Note that both formulas use the lambert W-function. Our formula will be compared to Hayman's formula in the next section.

We can also find an asymptotic formula of our a_n as a function of n only by resorting to the following asymptotic development of the branch of $W_{-1}(z)$ [5]:

$$W_{-1}(z) = \log(z - 2\pi i) - \log(\log z - 2\pi i) + \cdots \tag{3.17}$$

For $n \gg 1$ we can write

$$z_0 \sim \frac{-n - 2\pi i}{-n\log(-n - 2\pi i)} \sim \frac{1}{\log n - \pi i} \sim \frac{e^{-i\arctan\left(\frac{\pi}{\log n}\right)}}{\sqrt{(\log n)^2 + \pi^2}} \sim \frac{e^{i\frac{\pi}{\log n}}}{\log n}, \tag{3.18}$$

$$\frac{z_0^{\frac{1}{2}-n}}{\sqrt{1+z_0}} \sim \frac{1}{z_0^n} \sim \frac{e^{i\frac{n\pi}{\log n}}}{(\log n)^n}, \tag{3.19}$$

and

$$e^{-nz_0} \sim e^{\frac{-n}{\log n}e^{-i\frac{\pi}{\log n}}} \sim e^{\frac{-n}{\log n}}, \tag{3.20}$$

and using Stirling formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, this yields the second approximation

$$a_n \sim \frac{(-1)^n}{\pi} e^{-n\log n - n\log\log n + n - \frac{n}{\log n}} \sin\left(\frac{n\pi}{\log n}\right). \tag{3.21}$$

Equation (3.21) is a rough approximation. It will not be used for calculations. It only provides the leading order of growth and the sign oscillations of the coefficients.

4. Numerical Results and Conclusion

We implemented the formula of Theorem 3.1 and Hayman's formula (3.14) in Maple$^{\text{TM}}$. The approximations (3.13) and (3.14) were examined and compared to the exact values for n from 1 to 20 given in [15]. They are displayed in Table 1.

Table 2, displays the approximate value of a_n and some known exact values for higher values of n. We do not have the exact values of a_n when $n \geq 41$. So only the asymptotic values of both formulas are compared.

For $n = 4$ Hayman's formula gives the wrong sign. It provides a value of the coefficient with an error of 18.5% for $n = 15$, and for $n = 13, 16$, the error is almost 96%. Moreover, we can see that for at least the values of $2 \leq n \leq 20$ Hayman's formula is not as good an approximation to the exact value as the formula of Theorem 3.1.

From the trend of the values in Table 1 and Table 2, we conclude that for small values of n our formula outperforms Hayman's formula and that for larger values of n both formulas give the same sign but differ slightly in magnitude. The asymptotic formula of this paper has the advantage that it does not depend on the radius r_n of the circular contour, a real advancement in estimating the coefficients of the Taylor series of the reciprocal Gamma function.

As a final remark, the asymptotic formulas of this paper can of course be used to find an asymptotic formula for the related constants b_n defined by the power series

$^{\text{TM}}$Maple is a trademark of Waterloo Maple Inc.

n	a_n	Formula of Theorem 3.1	Hayman's formula
2	0.577215664	0.471315586	0.318527853
3	-0.655878071	-0.634156618	-0.745580393
4	**-0.042002635**	**-0.024878383**	**0.035835755**
5	0.166538611	0.1586548367	0.170422513
6	-0.042197734	-0.0422409922	-0.055165293
7	-0.009621971	-0.0088055266	-0.006842089
8	0.007218943	0.0070070400	0.007791124
9	-0.001165167	-0.0011689459	-0.001538105
10	-0.000215241	-0.0002013214	-0.000162310
11	0.000128050	0.0001248855	0.000137477
12	-0.000020134	-0.0000200451	-0.000025104
13	**-0.00000125**	**-0.000001139**	**-0.000000054**
14	0.000001133	0.0000011053	0.000001178
15	**-2.0563384.10^{-7}**	**-2.034656492.10^{-7}**	**$-$2.410634519.10^{-7}**
16	**6.11609510.10^{-9}**	**6.506886194.10^{-9}**	**1.201994777.10^{-8}**
17	5.00200764.10^{-9}	4.864046460.10^{-9}	4.859838872.10^{-9}
18	-1.18127457.10^{-9}	-1.164373917.10^{-9}	$-$1.3136121.10^{-9}
19	1.043426711.10^{-10}	1.043634325.10^{-10}	1.3322234.10^{-10}
20	7.782263439.10^{-12}	7.415156531.10^{-12}	5.436583518.10^{-12}

Table 1. First 20 coefficients and their approximate values given by Theorem 3.1 and formula (3.14).

n	a_n	Formula of Theorem 3.1	Hayman's formula
30	1.7144063219.10^{-20}	1.708720889.10^{-20}	2.072558647.10^{-20}
40	$-$1.1245843492.10^{-30}	$-$1.110270738.10^{-30}	$-$1.143814145.10^{-30}
50	$-$1.0562331785.10^{-41}	$-$1.051407032.10^{-41}	$-$1.211991030.10^{-41}
100	6.6158100911.10^{-106}	6.599969140.10^{-106}	7.567580120.10^{-106}
150	1.1936904502.10^{-179}	1.193587226.10^{-179}	1.441258864.10^{-179}
250	$-$2.4488582032.10^{-343}	$-$2.446740476.10^{-343}	$-$2.802890936.10^{-343}
300	2.90203183445.10^{-431}	2.900143434.10^{-431}	3.330671268.10^{-431}
800	$-$2.46251758839.10^{-1431}	$-$2.460396773.10^{-1431}	$-$2.585278168.$^{-1431}$
1400	$-$6.07622638292.10^{-2792}	$-$6.074000773.10^{-2792}	$-$6.575937595.10^{-2792}

Table 2. The coefficients a_n using the asymptotic formula of Hayman [2, 9] and the asymptotic formula of Theorem 3.1 for different values of n.

$$\frac{1}{\Gamma(z)} = z(1+z)\left[b_0 + b_1 z + b_2 z^2 \cdots\right], \tag{4.1}$$

where the coefficients a_n and b_n are connected by the relation

$$a_n = b_{n-1} + b_{n-2}; n \geq 2. \tag{4.2}$$

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