A New Effective Asymptotic Formula for the Stieltjes Constants

Lazhar Fekih-Ahmed

A NEW EFFECTIVE ASYMPTOTIC FORMULA FOR THE STIELTJES CONSTANTS

LAZHAR FEKIH-AHMED

Abstract. We derive a new integral formula for the Stieltjes constants. The new formula permits easy computations using an effective asymptotic formula. Both the sign oscillations and the leading order of growth are provided. The formula can also be easily extended to some generalized Euler constants.

1. Introduction

The Stieltjes constants γ_n are defined as the coefficients of Laurent series expansion of the Riemann zeta function at $s = 1$ [2]:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n (s-1)^n, \tag{1.1}$$

where $\gamma_0 = 0.5772156649$ is known as Euler's constant.

Exact and asymptotic formulas as well as upper bounds for the Stieltjes constants have been a subject of research for many decades [2, 3, 4, 8, 10, 11, 12, 13, 15]. The approach to estimate the Stieltjes constants is always deterministic except the paper [1] where a probabilistic approach is undertaken. The main reason to estimate the Stieltjes constants is that these constants and their generalization , known as Generalized Euler constants, have many applications in number theory.

This paper is a continuation of this line of research. We will give a new effective asymptotic formula for the Stieltjes constants. With the formula we obtain the sign oscillations and the leading order of growth of the Stieltjes constants. We will show that our results match those of [10] which may be considered very accurate compared to other results.

2. A New Formula for The Stieltjes Constants

Let $\phi(t)$ be the real function defined by

$$\phi(t) = \frac{d}{dt}\frac{-te^{-t}}{1-e^{-t}} = \frac{te^t}{(e^t-1)^2} - \frac{1}{e^t-1}. \tag{2.1}$$

In a previous article we have obtained the following integral representation of the Riemann zeta function:

Date: June 30, 2014.

2010 *Mathematics Subject Classification.* Primary 41A60, 30E15, 11M06, 11Y60.

Key words and phrases. Stieltjes constants; Riemann Zeta function; Laurent expansion; asymptotic expansion.

Theorem 2.1 ([7]). *With $\phi(t)$ as above, and for all s such that $\mathrm{Re}(s) > -k$, we have*

$$(s-1)\zeta(s) = \frac{(-1)^k}{\Gamma(s+k)} \int_0^\infty \frac{d^k\phi(t)}{dt^k} t^{s+k-1}\, dt. \tag{2.2}$$

If we chose $k = 1$ and we call

$$\mu(t) = -\frac{d\phi}{dt} = \frac{d^2}{dt^2}\frac{te^{-t}}{1-e^{-t}} = -\frac{(2+t)e^t}{(e^t-1)^2} + \frac{2te^{2t}}{(e^t-1)^3}, \tag{2.3}$$

then Theorem 2.1 provides the following formula valid for all s such that $\mathrm{Re}(s) > -1$

$$s(s-1)\zeta(s)\Gamma(s) = \int_0^\infty \mu(t)t^s\, dt. \tag{2.4}$$

If we now replace s by $1-s$ in equation (2.4) with the assumption that $\mathrm{Re}(1-s) < 2$, we get

$$s(s-1)\zeta(1-s)\Gamma(1-s) = \int_0^\infty \mu(t)t^{1-s}\, dt. \tag{2.5}$$

The functional equation for the Riemann zeta function states that

$$\zeta(s) = 2(2\pi)^{s-1}\sin(\frac{\pi s}{2})\zeta(1-s)\Gamma(1-s). \tag{2.6}$$

Multiplying both sides of the last equation by $s(s-1)$ and using (2.5), we obtain

$$\begin{aligned} s(s-1)\zeta(s) &= 2(2\pi)^{s-1}\sin(\frac{\pi s}{2})s(s-1)\zeta(1-s)\Gamma(1-s) \\ &= 2(2\pi)^{s-1}\sin(\frac{\pi s}{2})\int_0^\infty \mu(t)t^{1-s}\, dt. \end{aligned} \tag{2.7}$$

By observing that $(2\pi)^{s-1} = e^{(s-1)\log(2\pi)}$, that $t^{1-s} = e^{-(s-1)\log(t)}$ and that $2\sin(\frac{\pi s}{2}) = 2\cos(\pi\frac{(s-1)}{2}) = e^{i\pi\frac{(s-1)}{2}} + e^{-i\pi\frac{(s-1)}{2}}$, we can rewrite (2.7) as

$$s(s-1)\zeta(s) = \int_0^\infty \mu(t)\big[e^{(s-1)(a-\log(t))} + e^{(s-1)(\bar{a}-\log(t))}\big]\, dt, \tag{2.8}$$

where a is the fixed complex number $a = \log(2\pi) + i\frac{\pi}{2}$.

Finally, since the left hand side of (2.8) is analytic at $s = 1$ it has a Taylor series expansion

$$s(s-1)\zeta(s) = \sum_{n=0}^{\infty} \mu_n(s-1)^n, \tag{2.9}$$

where the coefficients μ_n are given by

$$\begin{aligned}\mu_n &= \frac{1}{n!}\lim_{s\to 1}\frac{d^n}{ds^n}\big\{s(s-1)\zeta(s)\big\}\\ &= \frac{1}{n!}\int_0^\infty \mu(t)\lim_{s\to 1}\frac{d^n}{ds^n}\big\{e^{(s-1)(a-\log(t))}+e^{(s-1)(\bar{a}-\log(t))}\big\}\,dt\\ &= \frac{1}{n!}\int_0^\infty \mu(t)\big\{(a-\log(t))^n+(\bar{a}-\log(t))^n\big\}\,dt. \qquad (2.10)\end{aligned}$$

This gives our first main result[1]:

Theorem 2.2. *With $\mu(t)$ and the constant a defined as above, the coefficients μ_n are given by*

$$\mu_n = \frac{2}{n!}\int_0^\infty \mu(t)\mathrm{Re}\big\{(a-\log t)^n\big\}\,dt. \qquad (2.11)$$

Once we have the coefficients μ_n of the power series for $s(s-1)\zeta(s)$, the Stieltjes coefficients γ_n can be calculated using power series multiplication:

$$(s-1)\zeta(s) = \sum_{n=0}^\infty (-1)^n(s-1)^n \times \sum_{n=0}^\infty \mu_n(s-1)^n \qquad (2.12)$$

since

$$\frac{1}{s} = \sum_{n=0}^\infty (-1)^n(s-1)^n. \qquad (2.13)$$

This immediately yields

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^\infty \left\{\sum_{k=0}^n (-1)^{n-k}\mu_k\right\}(s-1)^{n-1}; \qquad (2.14)$$

therefore,

$$\gamma_n = -n!\sum_{k=0}^{n+1}(-1)^k\mu_k = -n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{\sum_{k=0}^{n+1}\frac{(\log t-a)^k}{k!}\Big\}\,dt. \qquad (2.15)$$

The last formula can be simplified even further. Indeed, the sum inside the integral is a truncated sum of the exponential series $e^{\log t - a} = te^{-a}$. This yields,

[1]Note that the coefficients μ_n and the integral formula of $s(s-1)\zeta(s)$ are as important as the Stieltjes constants and the function $(s-1)\zeta(s)$. In fact, like the Riemann $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$ function, $s(s-1)\zeta(s)$ possess some symmetry and can play an important role in the theory of the Riemann zeta function.

$$\begin{aligned}\gamma_n &= -n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{\sum_{k=0}^{n+1}\frac{(\log t-a)^k}{k!}\Big\}\,dt\\ &= -n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{te^{-a}-\sum_{k=n+2}^{\infty}\frac{(\log t-a)^k}{k!}\Big\}\,dt\\ &= n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{\sum_{k=n+2}^{\infty}\frac{(\log t-a)^k}{k!}\Big\}\,dt \qquad (2.16)\end{aligned}$$

since $\mathrm{Re}\{e^{-a}\}=\mathrm{Re}\{\frac{-i}{2\pi}\}=0$. Hence, with

$$I(n)=\int_0^\infty \mu(t)(\log t-a)^n\,dt, \qquad (2.17)$$

we can write

$$\gamma_n = n!\left[\frac{I(n+2)}{(n+2)!}+\frac{I(n+3)}{(n+3)!}+\ldots\right] \qquad (2.18)$$

$$= n!\left[(-1)^{n+2}\mu_{n+2}+(-1)^{n+3}\mu_{n+3}+\ldots\right], \qquad (2.19)$$

and we have our second main result:

Theorem 2.3. *With $\mu(t)$ and μ_n defined as above, the Stieltjes constants are given by*

$$\gamma_n = n!(-1)^n\left[\mu_{n+2}-\mu_{n+3}+\mu_{n+4}-\ldots\right]. \qquad (2.20)$$

We do not know yet that the leading term $n!(-1)^n\mu_{n+2}=\frac{n!}{(n+2)!}I(n+2)$ is the dominant term for approximating γ_n. All we know for now is that the μ_n's are the Taylor coefficients of an entire function and that $\mu_n\to 0$ as $n\to\infty$. In the next section, we will show that $|\mu_{n+2}|\gg|\mu_{n+3}|\gg\ldots$ for large n so that $\{|\mu_k|\}_{n+2}^{\infty}$ form an asymptotic sequence[2]. This implies that γ_n can be written as

$$\gamma_n=\frac{1}{(n+1)(n+2)}\int_0^\infty 2\mu(t)\mathrm{Re}\big\{(\log t-a)^{n+2}\big\}\,dt+\text{higher order terms}, \qquad (2.21)$$

and that the leading term provides an asymptotic approximation of γ_n:

$$\gamma_n\approx\frac{1}{(n+1)(n+2)}\int_0^\infty 2\mu(t)\mathrm{Re}\big\{(\log t-a)^{n+2}\big\}\,dt=n!(-1)^n\mu_{n+2}. \qquad (2.22)$$

Therefore, Theorem 2.3 permits an asymptotic expansion of the constants γ_n. It is the subject of the next section.

[2]We write $f(n)\gg g(n)$, or f is "much greater than" g, if $g=o(f)$ as $n\to\infty$.

3. Asymptotic Estimates of The Stieltjes Constants

This section is dedicated to approximating the complex-valued integral

$$I(n) = \int_0^\infty \mu(t)(\log t - a)^n \, dt. \tag{3.1}$$

There are mainly two methods used for the asymptotic evaluation of complex integrals of the form (3.1) when n is large: the steepest descent method or Debye's method and the saddle-point method [5]. By rewriting I_n in a suitable form, we find that the saddle-point method provides the solution to our asymptotic analysis.

Let

$$g(t) = \mu(t)e^t, \tag{3.2}$$

then by the change of variables $t = nz$, our integral becomes

$$\begin{aligned} I(n) &= n\int_0^\infty g(nz)e^{-nz}\left\{\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right\}^n dz \\ &= n\int_0^\infty g(nz)e^{n\left\{-z+\log\left[\log\left(\frac{nz}{2\pi}\right)-i\frac{\pi}{2}\right]\right\}} dz. \end{aligned} \tag{3.3}$$

If we define

$$f(z) = -z + \log\left[\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right], \tag{3.4}$$

then the saddle-point method consists in deforming the path of integration into a path which goes through a saddle-point at which the derivative $f'(z)$, vanishes. If z_0 is the saddle-point at which the real part of $f(z)$ takes the greatest value, the neighborhood of z_0 provides the dominant part of the integral as $n \to \infty$ [5, p. 91-93]. This dominant part provides an approximation of the integral and it is given by the formula

$$I(n) \approx ng(nz_0)e^{nf(z_0)}\left(\frac{-2\pi}{nf''(z_0)}\right)^{\frac{1}{2}}. \tag{3.5}$$

In our case, we have

$$f'(z) = -1 + \frac{1}{z\left[\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right]}, \quad \text{and} \tag{3.6}$$

$$f''(z) = \frac{-1}{z^2\left[\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right]} - \frac{1}{z^2\left[\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right]^2}. \tag{3.7}$$

The saddle-point z_0 should verify the equation

$$\begin{aligned} & z_0\left[\log\left(\frac{nz_0}{2\pi}\right) - i\frac{\pi}{2}\right] = 1 \\ \Leftrightarrow \quad & \frac{nz_0}{2\pi}\left[\log\left(\frac{nz_0}{2\pi}\right) - i\frac{\pi}{2}\right] = \frac{n}{2\pi} \\ \Leftrightarrow \quad & \frac{nz_0}{2\pi}\log\left(\frac{nz_0}{2\pi}e^{-i\frac{\pi}{2}}\right) = \frac{n}{2\pi} \\ \Leftrightarrow \quad & \frac{nz_0}{2\pi}e^{-i\frac{\pi}{2}}\log\left(\frac{nz_0}{2\pi}e^{-i\frac{\pi}{2}}\right) = \frac{n}{2\pi}e^{-i\frac{\pi}{2}}. \end{aligned} \tag{3.8}$$

The last equation is of the form $v \log v = b$ whose solution can be explicitly written using the principal branch[3] of the Lambert W-function [6]:

$$v = e^{W(b)}. \tag{3.9}$$

After some algebra, the saddle-point solution to our equation (3.8) is thus given by

$$z_0 = \frac{2\pi}{ni} e^{W\left(\frac{ni}{2\pi}\right)}, \tag{3.10}$$

and at the saddle-point, we have the values

$$f(z_0) = -z_0 - \log z_0 \tag{3.11}$$

$$f''(z_0) = -1 - \frac{1}{z_0}. \tag{3.12}$$

The saddle-point approximation of our integral (3.1) is given by the formula:

$$\begin{aligned} I(n) &= n\sqrt{\frac{2\pi}{n}} g(nz_0) e^{-nz_0 - n\log(z_0)} \frac{1}{\sqrt{1+\frac{1}{z_0}}} \\ &= n\sqrt{\frac{2\pi}{n}} \mu(nz_0) \frac{z_0^{\frac{1}{2}-n}}{\sqrt{1+z_0}}. \end{aligned} \tag{3.13}$$

It turns out that $g(t)$ can be very well approximated by

$$g(t) = \begin{cases} \frac{1}{6} e^{-\frac{1}{10}t^2} & \text{if } 0 \leq t \leq 1 \\ -2 + t & \text{if } t \gg 1. \end{cases} \tag{3.14}$$

Moreover, when n is large, $g(nz_0)$ can also be very well approximated[4] by

$$g(nz_0) \approx nz_0 - 1, \tag{3.15}$$

so that we obtain the final approximation

$$I(n) \approx n\sqrt{\frac{2\pi}{n}} (nz_0 - 1) \frac{z_0^{\frac{1}{2}-n}}{e^{nz_0}\sqrt{1+z_0}}. \tag{3.16}$$

To obtain an approximation of the coefficient $\mu_n = \frac{2}{n!}\mathrm{Re}\{I(n)\}$, we use Stirling approximation of $n!$ and we further simplify $I(n)$ by resorting to the following asymptotic development of the principal branch of $W(z)$ [6]:

$$W(z) = \log(z) - \log(\log z) + \cdots \tag{3.17}$$

For $n \gg 1$, we can rewrite (3.10) as

[3]The principal branch of the Lambert W-function is denoted by $W_0(z) = W(z)$. See [6] for a thorough explanation of the definition of all the branches.

[4]The approximations of $g(t)$ and $g(nz_0)$ are of course not necessary. We can keep the original functions $g(nz_0)$ or $\mu(nz_0)$ for the final asymptotic formula.

$$z_0 \sim \frac{1}{\log\left(\frac{n}{2\pi}\right)} e^{-i\arctan\left(\frac{\pi}{2\log n}\right)} \sim \frac{1}{\log\left(\frac{n}{2\pi}\right)} e^{-i\frac{\pi}{2\log n}} \tag{3.18}$$

so that

$$\frac{1}{z_0^{n-\frac{1}{2}}} \sim \log\left(\frac{n}{2\pi}\right)^{n-\frac{1}{2}} e^{-i(n-\frac{1}{2})\frac{\pi}{2\log n}}, \tag{3.19}$$

and

$$e^{-nz_0} \sim e^{-\frac{n}{\log\left(\frac{n}{2\pi}\right)} e^{-i\frac{\pi}{2\log n}}} \sim e^{-\frac{n}{\log\left(\frac{n}{2\pi}\right)}}. \tag{3.20}$$

Using Stirling formula

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}, \tag{3.21}$$

we obtain for large n

$$|\mu_n| \sim \frac{n\log n}{e^{n\log n}}. \tag{3.22}$$

The last equation proves that $\lim_{n\to\infty} \frac{|\mu_{n+3}|}{|\mu_{n+2}|} = 0$, or equivalently that $|\mu_{n+2}| \gg |\mu_{n+3}| \gg \ldots$ for large n. Hence, Theorem 2.3 also provides an asymptotic expansion of γ_n, and we deduce the following one-term asymptotic approximation of $\gamma_n \approx n!(-1)^n\mu_{n+2}$:

Theorem 3.1. *Let $z_0^* = \frac{2\pi}{(n+2)i} e^{W\left(\frac{(n+2)i}{2\pi}\right)}$, where W is the Lambert W-function. An approximate formula for the Stieltjes constants for large n is*

$$\gamma_n \approx \frac{2}{(n+1)}\sqrt{\frac{2\pi}{n+2}}\mathrm{Re}\left\{\left((n+2)z_0^* - 1\right)\frac{{z_0^*}^{\frac{1}{2}-n-2}}{e^{(n+2)z_0^*}\sqrt{1+z_0^*}}\right\}. \tag{3.23}$$

We can also find an asymptotic formula of γ_n as a function of n only by using approximations similar to equations (3.17-3.19). For $n \gg 1$ we can write

$$z_0^* \sim \frac{1}{\log\left(\frac{n+2}{2\pi}\right)} e^{-i\frac{\pi}{2\log(n+2)}}, \tag{3.24}$$

$$\frac{1}{{z_0^*}^{n+\frac{1}{2}}} \sim \log\left(\frac{n+2}{2\pi}\right)^{n+\frac{1}{2}} e^{-i(n+\frac{1}{2})\frac{\pi}{2\log(n+2)}}, \tag{3.25}$$

and

$$e^{-(n+2)z_0^*} \sim e^{-\frac{(n+2)}{\log\left(\frac{n+2}{2\pi}\right)}}, \tag{3.26}$$

and after some easy algebraic manipulations, we obtain the oscillations and the leading order of growth of the Stieltjes constants:

(3.27)

$$\gamma_n \sim 2\frac{\sqrt{2\pi}}{\sqrt{n+2}}e^{(n+\frac{1}{2})\log(\log(n+2)-\log(2\pi))-\frac{(n+2)}{\log\left(\frac{n+2}{2\pi}\right)}}\cos\left((n+\frac{1}{2})\frac{\pi}{2\log(n+2)}\right).$$

Both the oscillations and the leading order of growth match the results of [10].

We also note that several terms of (2.20) can also be added to the one-term approximation given by Theorem 3.1. This leads to the following multi-term approximation of γ_n:

Theorem 3.2. *Let $z_0^* = \frac{2\pi}{(n+2)i}e^{W\left(\frac{(n+2)i}{2\pi}\right)}$, where W is the Lambert W-function. An M-term approximate formula for the Stieltjes constants for large n is*

(3.28)

$$\gamma_n \approx \sum_{k=0}^{M-1}\frac{2n!}{(n+1+k)!}\sqrt{\frac{2\pi}{n+2+k}}\mathrm{Re}\left\{((n+2+k)z_0^*-1)\frac{{z_0^*}^{\frac{1}{2}-n-2-k}}{e^{(n+2+k)z_0^*}\sqrt{1+z_0^*}}\right\}.$$

When $M = 1$, Theorem 3.2 reduces to Theorem 3.1. We will see in the next section that a three-term ($M = 3$) approximation provide satisfactory results for large and small values of n.

4. Numerical Results

We implemented the formula of Theorem 3.2 in Maple™ . For a given value of n, the following procedure computes the value of the M-term approximation formula of the n^{th} Stieltjes constant γ_n:

```
gamman := proc (n, M)
#Input n:  the desired nth Stieltjes constant
#Input M: the number of terms in asymptotic formula
#An example call:  gamman(137,3)
local k, coef, w0, z0, f, fpp;
coef := 0;
for k from 0 to M-1 do
w0 := LambertW(((1/2)*I)*(n+2+k)/Pi):
z0 := -(2*I)*Pi*exp(w0)/(n+2+k):
f := -z0-ln(z0):
fpp := -1-1/z0:
coef := coef+Re(2*factorial(n)*((n+2+k).z0-1)
*sqrt(-2*Pi/((n+2+k)*fpp))*exp((n+2+k)*f)/factorial(n+1+k)):
end do:
evalf(coef);
end proc:
```

The approximations (3.23) and (3.28) with $M = 3$ were examined and compared to the exact values for n from 2 to 100000 given in [8, 9] and the values of the asymptotic formula of Knessl and Coffey [10].

Table 1 below displays the approximate value of γ_n using Theorem 3.1 and Theorem 3.2 and $M = 3$, the approximation using the formula of [10] and the exact known values for n from 2 to 20. Table 2 displays the approximate value of γ_n and the exact known values for some higher values of n.

™ Maple is a trademark of Waterloo Maple Inc.

n	Exact γ_n	Theorem 3.2 Eq. (3.28), $M=3$	Theorem 3.1 Eq. (3.23)	Knessl-Coffey Formula [10]
2	-0.009690363192	-0.008382380783	-0.008909030193	–
3	0.002053834420	0.001621242634	0.001073584137	0.00190188
4	0.002325370065	0.002185636219	0.002025456323	0.00231644
5	0.000793323817	0.0007895679944	0.000825888315	0.000812965
6	-0.000238769345	-0.0002241100338	-0.000149933239	-0.000242081
7	-0.000527289567	-0.0005200052551	-0.000475920788	-0.000541476
8	-0.000352123353	-0.0003506762586	-0.000346534072	-0.00036176
9	-0.000034394774	-0.0000349578308	-0.000055274760	-0.000035070
10	0.000205332814	0.0002044473764	0.000179950900	0.000210539
11	0.000270184439	0.0002693789419	0.000255402785	0.00027624
12	0.000167272912	0.0001666692377	0.000168701645	0.000170507
13	-0.000027463806	-0.0000277087054	-0.000012840713	-0.000028263
14	-0.000209209262	-0.0002089871741	-0.000190127572	-0.000213064
15	-0.000283468655	-0.0002828583838	-0.000270364310	-0.000288108
16	-0.000199696858	-0.0001989876591	-0.000200475577	-0.000202633
17	0.000026277037	0.0000266966357	0.00000969746	0.0000267683
18	0.000307368408	0.0003071961365	0.000280749078	0.000311543
19	0.000503605453	0.0005027990007	0.000479486029	0.000509981
20	0.000466343561	0.0004652039644	0.000460162247	0.000471981

TABLE 1. First 20 Stieltjes constants γ_n and their approximate values given by Theorem 3.2, Theorem 3.1, and by the formula of Knessl-Coffey.

We can see that both asymptotic formulas given by Theorem 3.2 with $M = 3$ or by Theorem 3.1 provide good approximations of the exact Stieltjes constants except at $n = 137$ where the approximation of Theorem 3.1 fails to give the correct sign of γ_{137}. Curiously, the asymptotic formula of Knessl-Coffey also fails to give the correct sign of γ_{137}. It seems that the two asymptotic formulas are unrelated to each other[5]. Thus, the point $n = 137$ is inherently a badly conditioned point for both asymptotic formulas. For instance, with a small perturbation of $n = 137$, formula (3.23) gives the value $0.001041695409.10^{29}$ for $n = 137.017$, and the value $-0.1059515438.10^{29}$ for $n = 137.018$. This shows that the point $n = 137$ is numerically ill-conditioned. This ill-conditioning can be explained by the fact that the saddle-point equation of [10, eq. (2.4)] and the saddle-point equation (3.2) both involve the evaluation of $W(\frac{ni}{2\pi})$.

We also observe that except for the the specific values of $n = 2$ and $n = 137$, the approximation error of γ_n using the formula of Knessl-Coffey is less than that of the single-term formula given by Theorem 3.1. However, at the expense of adding two extra terms to the approximation, the formula of Theorem 3.2 outperforms both

[5]The approximation formula of [10] is given by $\gamma_n \approx -\int_0^\infty \frac{\sin(\pi e^t)}{\pi} t^{n-1} e^{-t}(n-t)\,dt$, whereas our approximation formula is $\gamma_n \approx -\frac{1}{(n+1)(n+2)} \int_0^\infty 2\mu(t)\mathrm{Re}\big\{(\log t - a)^{n+2}\big\}\,dt$. The author unsuccessfully tried to derive a relationship between the two formulas.

n	Exact γ_n	Theorem 3.2 Eq. (3.28), $M=3$	Theorem 3.1 Eq. (3.23)	Knessl-Coffey Formula [10]
30	0.003557728	0.0035491	0.003790	0.00359535
35	-0.02037304	-0.0203320	-0.022336	-0.0205982
40	0.248721559	0.2484162	0.265889	0.251108
45	-5.07234458	-5.0686103	-5.211491	-5.10969
50	126.8236026	126.7545688	127.121	127.549
100	$-4.253401.10^{17}$	$-4.251316.10^{17}$	$-4.14170.10^{17}$	$-4.25941.10^{17}$
136	$4.226701.10^{30}$	$4.226998.10^{30}$	$4.22698.10^{30}$	$4.22698.10^{30}$
137	$\mathbf{-0.00079.10^{29}}$	$\mathbf{-0.03484.10^{29}}$	$\mathbf{1.79099.10^{29}}$	$\mathbf{3.89874.10^{29}}$
138	$-2.523130.10^{31}$	$-2.521344.10^{31}$	$-2.4420176.10^{31}$	$-2.52354.10^{31}$
150	$8.028853.10^{35}$	$8.031999.10^{35}$	$8.1242241.10^{35}$	$8.05143.10^{35}$
250	$3.059212.10^{79}$	$3.058889.10^{79}$	$3.038525.10^{79}$	$3.06165.10^{79}$
300	$-5.55672.10^{102}$	$-5.55436.10^{102}$	$-5.47283.10^{102}$	$-5.55679.10^{102}$
800	$4.91354.10^{369}$	$4.91329.10^{369}$	$4.899488.10^{369}$	$4.91452.10^{369}$
1400	$-4.09728.10^{728}$	$-4.09772.10^{728}$	$-4.10081.10^{728}$	$-4.09851.10^{728}$

TABLE 2. Stieltjes constants γ_n and their approximate values given by Theorem 3.2, Theorem 3.1 and by Knessl-Coffey formula for different values of n.

n	Relative error
2	-13.5 %
3	-21.06 %
4	-6.01 %
6	-6.14 %
137	-56.41 %
821	7.95 %
1090	8.01 %
7259	9.12 %
8815	5.35 %

TABLE 3. The values of n ($2 \leq n \leq 100000$) for which the relative error of the Stieltjes constants γ_n computed using Theorem 3.2 with $M = 3$ exceeds 5%.

formulas: the sign error of the $n = 137$ disappears and the approximation error is greatly reduced.

Table 3 displays the values of n ($2 \leq n \leq 100000$) for which the relative error of the Stieltjes constants γ_n computed using Theorem 3.2 with $M = 3$ exceeds 5%. The relative errors rarely exceed 5%. In fact, for $n \geq 8816$, the errors are all less than 1.6% with two exceptions at $n = 71158$ and $n = 84589$ where the errors are equal to -2.4% and 4.5% respectively. It appears that with this accuracy of the approximation, the three-term asymptotic formula will hopefully be robust and work for all values of n.

5. Conclusion and Extensions

It is possible[6] that the analysis of this paper can be generalized to find an effective asymptotic formulas for the generalized Euler constants $\gamma_n(a)$ defined as the coefficients of the Laurent series of the Hurwitz zeta function $\zeta(s,a)$ at the point $s=1$ or at any other point of the complex plane. Instead of formula (2.2) of Theorem 2.1, we use the formula from [7]:

$$(s-1)\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \psi(t) e^{-(a-1)t} t^{s-1}\, dt, \tag{5.1}$$

which is valid for all s such that $\mathrm{Re}(s) > 0$ and all $0 < a \leq 1$, and where the real function $\psi(t)$ is defined by

$$\psi(t) = \frac{te^t}{(e^t-1)^2} - \frac{1}{e^t-1} + \frac{(a-1)t}{e^t-1}. \tag{5.2}$$

It would be interesting to compare the formulas with the results and conjectures of Kreminski who has done extensive computations on the generalized Euler constants [11].

Acknowledgement

I am grateful to Prof. Mark W. Coffey for his useful comments and for providing the values of the last column in Tables 1-2 using the formula of [10].

References

1. J. A. Adell, Asymptotic estimates for Stieltjes constants. A probabilistic approach, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467, pp. 954963, (2011).
2. W. E. Briggs, Some constants associated with the Riemann zeta-function, Mich. Math. J. 3, pp. 117-121 (1955).
3. W. E. Briggs and S. Chowla, The power series coefficients of $\zeta(s)$, Amer. Math. Monthly, Vol. 62, pp. 323-325, (1955).
4. M. W. Coffey, New results on the Stieltjes constants: Asymptotic and exact evaluation, J. Math. Anal. Appl. Vol. 317, pp. 603612, (2006).
5. E. T. Copson, Asymptotic Expansions, Cambridge University Press, (1965).
6. R. M. Corles et al., On the Lambert W Function, Advances in Computational Mathematics, Vol. 5, No. 1, pp. 329-359, (1996).
7. L. Fekih-Ahmed, On the Hurwitz Zeta Function, `http://hal.archives-ouvertes.fr/hal-00602532`, (2011).
8. F. Johansson, Rigorous high-precision computation of the Hurwitz zeta function and its derivatives, `http://arxiv.org/abs/1309.2877`, (2013).
9. F. Johansson, The Stieltjes constants $\gamma_0 \ldots \gamma_{100000}$ rounded to 20 decimal digits, `http://fredrikj.net/math/stieltjes100k20d.txt`, Jan. (2014).
10. C. Knessl, M.W. Coffey, An effective asymptotic formula for the Stieltjes constants, Mathematics of Computation, Vol. 80, No. 273, pp. 379-386, (2011).
11. R. Kreminski, Newton-Cotes Integration for Approximating Stieltjes (Generalized Euler) Constants, Mathematics of Computation, Vol. 72, No. 243, pp. 1379-1397, (2003).
12. Y. Matsuoka, On the power series coefficients of the Riemann zeta function, Tokyo J. Math. 12, pp. 49-58 (1989).
13. D. Mitrović, The signs of some constants associated with the Riemann zeta function, Mich. Math. J. 9, pp. 395-397 (1962).

[6]The functional equation artifice used in this paper to find the asymptotic formula for $\gamma_n(a)$ when $a=1$ extends easily to $a=\frac{1}{2}$. For irrational a or other rationals the extension is not obvious.

14. S. Saad Eddin, Explicit upper bounds for the Stieltjes constants, Journal of Number Theory, Vol. 133, No. 3, pp. 1027-1044, (2013).
15. N.-Y. Zhang and K. S. Williams, Some results on the generalized Stieltjes constants, Analysis 14, pp. 147-162 (1994).

École Nationale d'Ingénieurs de Tunis, BP 37, Le Belvédère 1002 , Tunis, Tunisia
E-mail address: lazhar.fekihahmed@enit.rnu.tn