A New Effective Asymptotic Formula for the Stieltjes Constants

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A NEW EFFECTIVE ASYMPTOTIC FORMULA FOR THE STIELTJES CONSTANTS

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Abstract. We derive a new integral formula for the Stieltjes constants. The new formula permits easy computations as well as an exact approximate asymptotic formula. Both the sign oscillations and the leading order of growth are provided. The formula can also be easily extended to generalized Euler constants.

1. Introduction

The Stieltjes constants γ_n are defined as the coefficients of Laurent series expansion of the Riemann zeta function at $s = 1$ [2]:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n(s-1)^n, \tag{1.1}$$

where $\gamma_0 = 0.5772156649$ is known as Euler's constant.

Exact and asymptotic formulas as well as upper bounds for the Stieltjes constants have been a subject of research for many decades [3, 2, 7, 9, 11, 10, 12, 14]. The approach to estimate the Stieltjes constants is always deterministic except the paper [1] where a probabilistic approach is undertaken. The main reason to estimate the Stieltjes constants is that these constants and their generalization , known as Generalized Euler constants, have many applications in number theory.

This paper is a continuation of this line of research. We will give a new effective asymptotic formula for the Stieltjes constants. With the formula we obtain the sign oscillations and the leading order of growth of the Stieltjes constants. We will show that our results match those of [9] which may be considered very accurate compared to other results.

2. An Integral Formula for $s(s-1)\zeta(s)$

Let $\phi(t)$ be the real function defined by

$$\phi(t) = \frac{d}{dt}\frac{-te^{-t}}{1-e^{-t}} = \frac{te^t}{(e^t-1)^2} - \frac{1}{e^t-1}. \tag{2.1}$$

In a previous article we have obtained the following integral representation of the Riemann zeta function.

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$$\zeta(s) = \frac{(-1)^k}{\Gamma(s+k)} \int_0^\infty \frac{d^k}{dt^k}(\phi(t)t^{s+k-1}\, dt. \tag{2.2}$$

Theorem 2.1 ([6]). *With $\phi(t)$ as above, and for all s such that $\mathrm{Re}(s) > -k$, we have*

$$(s-1)\zeta(s) = \frac{(-1)^k}{\Gamma(s+k)} \int_0^\infty \frac{d^k}{dt^k}(\phi(t)t^{s+k-1}\, dt. \tag{2.3}$$

If we chose $k = 1$ and we call

$$\mu(t) = -\frac{d\phi}{dt} = \frac{d^2}{dt^2}\frac{te^{-t}}{1-e^{-t}} = \frac{(2+t)e^t}{(e^t-1)^2} - \frac{2te^{2t}}{(e^t-1)^3}, \tag{2.4}$$

then Theorem 2.1 provides the following formula valid for all s such that $\mathrm{Re}(s) > -1$

$$s(s-1)\zeta(s)\Gamma(s) = \int_0^\infty \mu(t)t^s\, dt. \tag{2.5}$$

If we now replace s by $1-s$ in equation (2.5) with the assumption that $\mathrm{Re}(1-s) < 2$, we get

$$s(s-1)\zeta(1-s)\Gamma(1-s) = \int_0^\infty \mu(t)t^{1-s}\, dt. \tag{2.6}$$

The functional equation for the Riemann zeta function states that

$$\zeta(s) = 2(2\pi)^{s-1}\sin(\frac{\pi s}{2})\zeta(1-s)\Gamma(1-s). \tag{2.7}$$

Multiplying both sides of the last equation by $s(s-1)$ and using (2.6), we obtain

$$\begin{aligned} s(s-1)\zeta(s) &= 2(2\pi)^{s-1}\sin(\frac{\pi s}{2})s(s-1)\zeta(1-s)\Gamma(1-s) \\ &= 2(2\pi)^{s-1}\sin(\frac{\pi s}{2})\int_0^\infty \mu(t)t^{1-s}\, dt. \end{aligned} \tag{2.8}$$

By observing that $(2\pi)^{s-1} = e^{(s-1)\log(2\pi)}$, that $t^{1-s} = e^{-(s-1)\log(t)}$ and that $2\sin(\frac{\pi s}{2}) = 2\cos(\pi\frac{(s-1)}{2}) = e^{i\pi\frac{(s-1)}{2}} + e^{-i\pi\frac{(s-1)}{2}}$, we can rewrite (2.8) as

$$s(s-1)\zeta(s) = \int_0^\infty \mu(t)\big[e^{(s-1)(a-\log(t))} + e^{(s-1)(\bar{a}-\log(t))}\big]\, dt, \tag{2.9}$$

where a is the fixed complex number $a = \log(2\pi) + i\frac{\pi}{2}$.

Finally, since the left hand side of (2.9) is analytic at $s = 1$ it has a Taylor series expansion

$$s(s-1)\zeta(s) = \sum_{n=0}^{\infty} \mu_n(s-1)^n, \tag{2.10}$$

where the coefficients μ_n are given by

$$\begin{aligned}\mu_n &= \lim_{s\to 1}\frac{d^n}{ds^n}\big\{s(s-1)\zeta(s)\big\}\\ &= \int_0^\infty \mu(t)\lim_{s\to 1}\frac{d^n}{ds^n}\big\{e^{(s-1)(a-\log(t))}+e^{(s-1)(\bar{a}-\log(t))}\big\}\,dt\\ &= \int_0^\infty \mu(t)\big\{(a-\log(t))^n+(\bar{a}-\log(t))^n\big\}\,dt. \qquad (2.11)\end{aligned}$$

This gives our first main result[1]:

Theorem 2.2. *With $\mu(t)$ and the constant a defined as above, the coefficients μ_n are given by*

$$\mu_n = \frac{2}{n!}\int_0^\infty \mu(t)\mathrm{Re}\big\{(a-\log t)^n\big\}\,dt. \qquad (2.12)$$

Once we have the coefficients μ_n of the power series for $s(s-1)\zeta(s)$, the Stieltjes coefficients γ_n can be calculated using power series multiplication:

$$(s-1)\zeta(s) = \sum_{n=0}^\infty (-1)^n(s-1)^n \times \sum_{n=0}^\infty \mu_n(s-1)^n \qquad (2.13)$$

since

$$\frac{1}{s} = \sum_{n=0}^\infty (-1)^n(s-1)^n. \qquad (2.14)$$

This immediately yields

$$\zeta(s) = \frac{1}{s-1}+\sum_{n=1}^\infty \left\{\sum_{k=0}^n (-1)^{n-k}\mu_k\right\}(-1)^n(s-1)^{n-1}; \qquad (2.15)$$

therefore,

$$\gamma_n = n!\sum_{k=0}^{n+1}(-1)^k\mu_k = n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{\sum_{k=0}^{n+1}\frac{(\log t-a)^k}{k!}\Big\}\,dt. \qquad (2.16)$$

The last formula can be simplified even further. Indeed, the sum inside the integral is a truncated sum of the exponential series $e^{\log t-a} = te^{-a}$. This yields,

[1]Note that the coefficients μ_n and the integral formula of $s(s-1)\zeta(s)$ are as important as the Stieltjes constants and the function $(s-1)\zeta(s)$. In fact, like the Riemann $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$ function, $s(s-1)\zeta(s)$ possess some symmetry and can play an important role in the theory of the Riemann zeta function.

$$\begin{aligned}\gamma_n &= n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{\sum_{k=0}^{n}\frac{(\log t-a)^k}{k!}\Big\}\,dt\\ &= n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{te^{-a}+\sum_{k=n+2}^{\infty}\frac{(\log t-a)^k}{k!}\Big\}\,dt\\ &= n!\int_0^\infty 2\mu(t)\mathrm{Re}\Big\{\sum_{k=n+2}^{\infty}\frac{(\log t-a)^k}{k!}\Big\}\,dt \qquad (2.17)\end{aligned}$$

since $\mathrm{Re}\{e^{-a}\}=\mathrm{Re}\{\frac{-i}{2\pi}\}=0$. Hence, the leading term of an asymptotic expansion of γ_n is equal to

$$\gamma_n=\frac{1}{(n+1)(n+2)}\int_0^\infty 2\mu(t)\mathrm{Re}\big\{(\log t-a)^{n+2}\big\}\,dt+\text{higher order terms}, \qquad (2.18)$$

and we have our second main result:

Theorem 2.3. *With $\mu(t)$ defined as above, the Stieltjes constants γ_n can be approximated by*

$$\frac{1}{(n+1)(n+2)}\int_0^\infty 2\mu(t)\mathrm{Re}\big\{(\log t-a)^{n+2}\big\}\,dt=n!\mu_{n+2}. \qquad (2.19)$$

Theorem 2.3 permits an exact asymptotic evaluation of the constants μ_n and γ_n. It is the subject of the next section.

3. Asymptotic Estimates of The Stieltjes Constants

This section is dedicated to approximating the complex-valued integral

$$I(n)=\int_0^\infty \mu(t)(\log t-a)^n\,dt. \qquad (3.1)$$

There are mainly two methods used for the asymptotic evaluation of complex integrals of the form (3.1) when n is large: the steepest descent method or Debye's method and the saddle-point method [4]. By rewriting I_n in a suitable form, we find that the saddle-point method provides the solution to our asymptotic analysis.

Let

$$g(t)=\mu(t)e^t, \qquad (3.2)$$

then by the change of variables $t=nz$, our integral becomes

$$\begin{aligned}I(n)&=n\int_0^\infty g(nz)e^{-nz}\left\{\log\left(\frac{nz}{2\pi}\right)-i\frac{\pi}{2}\right\}^n\,dz\\ &=n\int_0^\infty g(nz)e^{n\left\{-z+\log\left[\log\left(\frac{nz}{2\pi}\right)-i\frac{\pi}{2}\right]\right\}}\,dz. \qquad (3.3)\end{aligned}$$

If we define

$$f(z)=-z+\log\left[\log\left(\frac{nz}{2\pi}\right)-i\frac{\pi}{2}\right], \qquad (3.4)$$

then the saddle-point method consists in deforming the path of integration into a path which goes through a saddle-point at which the derivative $f'(z)$, vanishes.

If z_0 is the saddle-point at which the real part of $f(z)$ takes the greatest value, the neighborhood of z_0 provides the dominant part of the integral as $n \to \infty$ [4, p. 91-93]. This dominant part provides an approximation of the integral and it is given by the formula

$$I(n) \approx ng(nz_0)e^{nf(z_0)} \left(\frac{-2\pi}{nf''(z_0)} \right)^{\frac{1}{2}}. \tag{3.5}$$

In our case, we have

$$f'(z) = -1 + \frac{1}{z\left[\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right]}, \quad \text{and} \tag{3.6}$$

$$f''(z) = \frac{-1}{z^2\left[\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right]} - \frac{1}{z^2\left[\log\left(\frac{nz}{2\pi}\right) - i\frac{\pi}{2}\right]^2}. \tag{3.7}$$

The saddle-point z_0 should verify the equation

$$\begin{aligned} & z_0\left[\log\left(\frac{nz_0}{2\pi}\right) - i\frac{\pi}{2}\right] = 1 \\ \Leftrightarrow \quad & \frac{nz_0}{2\pi}\left[\log\left(\frac{nz_0}{2\pi}\right) - i\frac{\pi}{2}\right] = \frac{n}{2\pi} \\ \Leftrightarrow \quad & \frac{nz_0}{2\pi}\log\left(\frac{nz_0}{2\pi}e^{-i\frac{\pi}{2}}\right) = \frac{n}{2\pi} \\ \Leftrightarrow \quad & \frac{nz_0}{2\pi}e^{-i\frac{\pi}{2}}\log\left(\frac{nz_0}{2\pi}e^{-i\frac{\pi}{2}}\right) = \frac{n}{2\pi}e^{-i\frac{\pi}{2}}. \end{aligned} \tag{3.8}$$

The last equation is of the form $v \log v = b$ whose solution can be explicitly written using the principal branch[2] of the Lambert W-function [5]:

$$v = e^{W(b)}. \tag{3.9}$$

After some algebra, the saddle-point solution to our equation (3.8) is thus given by

$$z_0 = \frac{2\pi}{ni}e^{W\left(\frac{ni}{2\pi}\right)}, \tag{3.10}$$

and at the saddle-point, we have the values

$$f(z_0) = -z0 - \log z_0 \tag{3.11}$$

$$f''(z_0) = -1 - \frac{1}{z_0}. \tag{3.12}$$

The saddle-point approximation of our integral (3.1) is given by the formula:

[2]The principal branch of the Lambert W-function is denoted by $W_0(z) = W(z)$. See [5] for a thorough explanation of the definition of all the branches.

$$I(n) = n\sqrt{\frac{2\pi}{n}} g(nz_0) e^{-nz_0 - n\log(z_0)} \frac{1}{\sqrt{1+\frac{1}{z_0}}}$$

$$= n\sqrt{\frac{2\pi}{n}} \mu(nz_0) \frac{z_0^{\frac{1}{2}-n}}{\sqrt{1+z_0}}. \tag{3.13}$$

It turns out that $g(t)$ can be very well approximated by

$$g(t) = \begin{cases} \frac{1}{6} e^{-\frac{1}{10}t^2} & \text{if } 0 \leq t \leq 1 \\ -2 + t & \text{if } t \gg 1. \end{cases} \tag{3.14}$$

Moreover, when n is large, $g(nz_0)$ can also be very well approximated[3] by

$$g(nz_0) \approx nz_0 - 1, \tag{3.15}$$

so that we obtain the final approximation

$$I(n) \approx n\sqrt{\frac{2\pi}{n}} (nz_0 - 1) \frac{z_0^{\frac{1}{2}-n}}{e^{nz_0}\sqrt{1+z_0}}. \tag{3.16}$$

If we recall the formulas

$$\mu_n = \frac{2}{n!} \mathrm{Re}\left\{ I(n) \right\}, \text{and} \tag{3.17}$$

$$\gamma_n = n! \mu_{n+2}, \tag{3.18}$$

then by Theorem 2.3, we deduce an approximation of the Stieltjes constants γ_n:

Theorem 3.1. *Let $z_0 = \frac{2\pi}{(n+2)i} e^{W\left(\frac{(n+2)i}{2\pi}\right)}$, where W is the Lambert W-function. An approximate formula for the Stieltjes constants for large n is*

$$\gamma_n \approx \frac{2}{(n+1)} \sqrt{\frac{2\pi}{n+2}} \mathrm{Re}\left\{ \left((n+2)z_0 - 1\right) \frac{z_0^{\frac{1}{2}-n-2}}{e^{(n+2)z_0}\sqrt{1+z_0}} \right\}. \tag{3.19}$$

We can also find an asymptotic formula of γ_n as a function of n only by resorting to the following asymptotic development of the principal branch of $W(z)$ [5]:

$$W(z) = \log(z) - \log\left(\log z\right) + \cdots \tag{3.20}$$

For $n \gg 1$ we can write

$$z_0 \sim \frac{1}{\log\left(\frac{n+2}{2\pi}\right)} e^{-i \arctan\left(\frac{\pi}{2\log(n+2)}\right)} \sim \frac{1}{\log\left(\frac{n+2}{2\pi}\right)} e^{-i\frac{\pi}{2\log(n+2)}}, \tag{3.21}$$

$$\frac{1}{z_0^{n+\frac{1}{2}}} \sim \log\left(\frac{n+2}{2\pi}\right)^{n+\frac{1}{2}} e^{-i(n+\frac{1}{2})\frac{\pi}{2\log(n+2)}}, \tag{3.22}$$

[3]The approximations of $g(t)$ and $g(nz_0)$ are of course not necessary. We can keep the original functions $g(nz_0)$ or $\mu(nz_0)$ for the final asymptotic formula.

and

$$e^{-(n+2)z_0} \sim e^{-\frac{(n+2)}{\log\left(\frac{n+2}{2\pi}\right)} e^{-i\frac{\pi}{2\log(n+2)}}} \sim e^{-\frac{(n+2)}{\log\left(\frac{n+2}{2\pi}\right)}}, \tag{3.23}$$

and after some easy algebraic manipulations, we obtain the oscillations and the leading order of growth of the Stieltjes constants:

$$\gamma_n \sim 2\frac{\sqrt{2\pi}}{\sqrt{n+2}} e^{(n+\frac{1}{2})\log(\log(n+2)-\log(2\pi))-\frac{(n+2)}{\log\left(\frac{n+2}{2\pi}\right)}} \cos\left((n+\frac{1}{2})\frac{\pi}{2\log(n+2)}\right). \tag{3.24}$$

Both the oscillations and the leading order of growth match the results of [9].

4. Numerical Results

We implemented the formula of Theorem 3.1 in MAPLE. For a given value of n, the following MAPLE code computes the value of the n^{th} Stieltjes constant γ_n:

```
w0 := LambertW(I*(n+2)/(2*Pi)):
z0:=2*Pi*exp(w0)/(I*(n+2)):
f:=-z0-ln(z0): fpp := -1-1/z0:
Re(2*((n+2)*z0-1)*sqrt((-2*Pi)*(1/((n+2)*fpp)))*exp((n+2)*f)/(n+1));
```

The approximation (3.19) was examined and compared to the exact values for n from 1 to 1000 given in [7, 8]. For small values of n ($n \leq 150$), we have also used the following MAPLE code to verify the n^{th} Stieltjes coefficient:

```
coef:=n!.(-1)^n.evalf(coeftayl((s-1)*Zeta(s), s = 1,n+1));
```

Table 1 below displays the approximate value of γ_n and the exact known values for n from 2 to 20. Table 2 displays the approximate value of γ_n and the exact known values for higher values of n.

We can see that the asymptotic formula is a good approximation of the exact Stieltjes constants except at $n = 137$ where the approximation fails to give the correct sign of γ_{137}. Curiously, the asymptotic formula of Knessl et al [9] fails also to give the correct sign of γ_{137}. It seems that the two asymptotic formulas are unrelated to each other[4]. Thus, the point $n = 137$ is inherently a badly conditioned point for both asymptotic formulas. For instance, with a small perturbation of $n = 137$, the formula above gives the value $0.001041695409.10^{29}$ for $n = 137.017$, and the value $-0.1059515438.10^{29}$ for $n = 137.018$. This shows that the point $n = 137$ is numerically ill-conditioned. This ill-conditioning can be explained by the fact that the saddle-point equation of [9, eq. (2.4)] and the saddle-point equation (3.2) both involve the evaluation of $W(\frac{ni}{2\pi})$.

[4]The approximation formula of [9] is given by $\gamma_n \approx -\int_0^\infty \frac{\sin(\pi e^t)}{\pi} t^{n-1} e^{-t}(n-t)\,dt$, whereas our approximation formula is $\gamma_n \approx -\frac{1}{(n+1)(n+2)} \int_0^\infty 2\mu(t)\mathrm{Re}\{(\log t - a)^{n+2}\}\,dt$. The author unsuccessfully tried to derive a relationship between the two formulas.

n	γ_n	Approximate γ_n
2	-0.009690363192	-0.008909030193
3	0.002053834420	0.001073584137
4	0.002325370065	0.002025456323
5	0.000793323817	0.000825888315
6	-0.000238769345	-0.000149933239
7	-0.000527289567	-0.000475920788
8	-0.000352123353	-0.000346534072
9	-0.000034394774	-0.000055274760
10	0.000205332814	0.000179950900
11	0.000270184439	0.000255402785
12	0.000167272912	0.000168701645
13	-0.000027463806	-0.000012840713
14	-0.000209209262	-0.000190127572
15	-0.000283468655	-0.000270364310
16	-0.000199696858	-0.000200475577
17	0.000026277037	0.000009697462
18	0.000307368408	0.000280749078
19	0.000503605453	0.000479486029
20	0.000466343561	0.000460162247

TABLE 1. First 20 Stieltjes constants γ_n and their approximate values given by Theorem 3.1.

n	γ_n	Approximate γ_n
30	0.0035577288	0.0037901372
35	-0.020373043	-0.022336513
40	0.2487215593	0.2658897332
45	-5.072344589	-5.211491845
50	126.82360265	127.1212577
100	$-4.253401571.10^{17}$	$-4.141706755.10^{17}$
136	$4.2267012858.10^{30}$	$4.226982515.10^{30}$
137	$\mathbf{-0.0007995.10^{29}}$	$\mathbf{1.790999409.10^{29}}$
138	$-2.52313010.10^{31}$	$-2.442017698.10^{31}$
150	$8.028853731.10^{35}$	$8.124224157.10^{35}$
250	$3.059212855.10^{79}$	$3.038525930.10^{79}$
300	$-5.55672822.10^{102}$	$-5.47283964.10^{102}$
800	$4.9135405617.10^{369}$	$4.899488755.10^{369}$
1400	$-4.097287334.10^{728}$	$-4.10081098.10^{728}$

TABLE 2. Stieltjes constants γ_n and their approximate values given by Theorem 3.1 for different values of n.

5. Conclusion and Extensions

The analysis of this paper can be generalized to find effective asymptotic formulas for the generalized Euler constants $\gamma_n(a)$ defined as the coefficients of the Laurent

series of the Hurwitz zeta function $\zeta(s,a)$ at the point $s=1$ or at any other point of the complex plane. Instead of formula (2.3) of Theorem 2.1, we use the formula from [6]:

$$(s-1)\zeta(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \psi(t) e^{-(a-1)t} t^{s-1}\, dt, \tag{5.1}$$

which is valid for all s such that $\mathrm{Re}(s) > 0$ and all $0 < a \leq 1$, and where the real function $\psi(t)$ is defined by

$$\psi(t) = \frac{te^t}{(e^t-1)^2} - \frac{1}{e^t-1} + \frac{(a-1)t}{e^t-1}. \tag{5.2}$$

It would be interesting to compare the formulas with the results and conjectures of Kreminski who has done extensive computations on the generalized Euler constants [10].

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