Kneading with weights

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Abstract

We generalise Milnor-Thurston's kneading theory to the setting of piecewise continuous and monotone interval maps with weights. We define a weighted kneading determinant $\mathcal{D}(t)$ and establish combinatorially two kneading identities, one with the cutting invariant and one with the dynamical zeta function. For the pressure $\log \rho_1$ of the weighted system, playing the role of entropy, we prove that $\mathcal{D}(t)$ is non-zero when $|t| < 1/\rho_1$ and has a zero at $1/\rho_1$. Furthermore, our map is semi-conjugate to an analytic family $h_t, 0 < t < 1/\rho_1$ of Cantor PL maps converging to an interval PL map h_{1/ρ_1} with equal pressure[1].

1 Introduction

Let $I = [a, b]$. Let $a = c_0 < c_1 < \cdots < c_{\ell+1} = b$. Set $S = \{0, 1, \cdots, \ell\}$. For each $i \in S$, set $I_i =]c_i, c_{i+1}[$ and let $f_i : I_i \to I$ be a strictly monotone continuous map extending continuously to the closure, and finally assign a constant weight $g_i \in \mathbb{C}$.

We say that $(I_i, f_i, g_i)_{i\in S}$ is a **weighted system**. In the particular case that each g_i equals 1, we say also that the system is **unweighted**.

Milnor-Thurston [MT] developed a widely used kneading theory on unweighted systems so that the maps f_i glue together to a single continuous map f. Let us recall a list of their results (see also [Ha] for an enlightening introduction to the subject).

Milnor-Thurston introduce a power series matrix $\mathcal{N}(t)$, called the *kneading matrix*, which records combinatorially the forward orbits of the cutting points. They establish two identities:

1. The **Main Kneading Identity**, relating $\mathcal{N}(t)$ to the growth of the cutting points of f^n on any subinterval J, and taking the form

$$\gamma_J(t) \cdot \mathcal{N}(t) = \text{terms involving boundaries of } J; \tag{1}$$

[1] 2010 MSC 37E05, 37E15, 37C30

Key words: Milnor-Thurston kneading theory, maps of the interval, dynamical zeta functions, kneading determinant, pressure, entropy, semi-conjugacies.

2. The **zeta-function identity**, relating $\mathcal{N}(t)$ to a dynamical Artin-Mazur zeta function that counts the global growth of the f^n-fixed points, taking the form

$$\zeta(t) \cdot \det \mathcal{N}(t) = 1. \tag{2}$$

Using these identities, Milnor-Thurston derive the following important consequences:

3. For $\log s$ the topological entropy of the map, the matrix $\mathcal{N}(t)$ is invertible when $|t| < 1/s$. If $s > 1$ the matrix $\mathcal{N}(t)$ is singular at $t = 1/s$ and the growth rate of the periodic points is precisely s.

4. If $s > 1$, the map is semi-conjugate to a simple model dynamical system which is a continuous PL (i.e. piecewise-linear) map of slope s.

Most of this theory has been extended by Preston [Pr] to the general unweighted setting without the assumption of global continuity. An advantage to allow discontinuity at the cutting points is that one can treat tree and graph maps as interval unweighted systems after edge concatenation. See for example Tiozzo [Ti]. There exist also works that treat tree maps as they are. See for example Alves-SousaRamos, Baillif and Baillif-deCarvalho [AS, Ba, BC].

An essential difference in Preston's approach as compared to Milnor-Thurston's lies in the proof of the zeta-function identity. Preston's method is purely combinatorial whereas the original proof tests on a concrete example and then studies behaviours under perturbations.

In this work we will generalise all four results above to weighted systems, where the pressure $\log \rho_1$ will play the role of entropy. Points 1-4 will become Theorems 2.1, 2.2, 2.3, 2.5 below.

Our setting is identical to that of Baladi-Ruelle [BR]. In their work they define a weighted kneading matrix $\mathcal{B}$ and a weighted zeta function, and establish a version the zeta-function identity using a perturbative method similar to that of Milnor-Thurston. For our purpose we will define a somewhat different kneading matrix $\mathcal{R}$.

We will not rely on previous established results but instead provide self-contained proofs. In a way our results recover partially results in [BR, MT, Pr].

Our proofs will be fairly elementary, with, as the only background, some basic knowledge of complex analysis. The rest is to play carefully with the combinatorics of iterations, following mostly Milnor-Thurston.

There is however a notable exception, which is about the proof of the zeta-function identity. For this we choose to follow the combinatorial method of Preston, along with several significant differences. Preston cuts off the graph above the diagonal in order to count the intersections, instead we keep the graph intact but change signs across the diagonal. Preston's kneading matrix is similar to that of Milnor-Thurston, by recording the sequence of visited intervals of a critical orbit. Instead we take the kneading matrix $\mathcal{B}$ of Baladi-Ruelle, which records the orbit's position relative to every given critical point. We then add one more dimension to $\mathcal{B}$ to obtain our kneading matrix $\mathcal{R}$, by incorporating

the influence of the boundary cutting points (with a somewhat different choice of sign). These modifications are designed to simplify, even in the unweighted case, Preston's proof of the zeta-function identity. Preston's idea is to express $-\Big(\log\zeta(t)\Big)'_t$ as the trace of a certain matrix $\mathcal{F}$, and then use repeatedly the Main Kneading Identity to connect $\mathcal{F}$ with the derivative of the kneading matrix. Here, many choices are possible but most give rise to additional correcting terms. Having tested various possibilities we came up with the current choice of the kneading matrix $\mathcal{R}$ and a matrix $\mathcal{F}$ for which we have the simplest relation possible, i.e. $\mathcal{F}\mathcal{R} = \mathcal{R}'$ (see Theorem 4.1). Once this relation established, the zeta-function identity is a one-line computation:

$$-\frac{d}{dt}\log\zeta(t) = Tr\mathcal{F} = Tr\mathcal{R}'\mathcal{R}^{-1} = \frac{d}{dt}\log\det\mathcal{R}.$$

The kneading matrix and its smallest positive zero cost relatively little to evaluate. This enables a fast and accurate computation of the pressure/entropy as well as the semi-conjugacy and the PL model map.

While experimenting these ideas we noticed that the system is also semi-conjugate to a PL map for every $0 < t < \rho_1$, although the conjugated system acts on a Cantor set instead of an interval. This numerical observation can easily be proved and has now become our Theorem 2.4. To the best of our knowledge this statement is new, also in the unweighted setting, even though its proof does not require any new ideas.

A further justification of our choice of the kneading determinant $\det\mathcal{R}$ as compared to $\det\mathcal{B}$, is that the latter may have a spurious small zero unrelated to the pressure (in Appendix C we give an example).

Another originality of this work is the systematic treatment of point-germs relative to points. Each point x in the interior of the interval generates two point-germs: x^+ and x^-. They have often distinct dynamical behaviour and it is convenient to treat the two germs independently. The idea is certainly present to all the papers in the theory. But highlighting the notion transforms our computations in more concise forms.

Why adding weights to piecewise continuous and monotone maps? One motivation is that one can prescribe slope ratios for the PL model maps, the other is that one can choose to ignore some parts of a dynamical system by assigning zero weights, so to reveal deeper entropies hidden for example in renormalisation pieces.

A further application, not pursued in the current work, is to construct various invariant measures by playing with weights and following Preston's construction of measures maximizing the entropy.

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2 Notation and results

Let $I = [a, b]$. Let $a = c_0 < c_1 < \cdots < c_{\ell+1} = b$. Set $I_i =]c_i, c_{i+1}[$ and let $f_i : I_i \to I$ be strictly monotone continuous maps for $i = 0, \cdots, \ell$. We write $f = (f_0|_{I_0}, \cdots, f_\ell|_{I_\ell})$ and let $s_i = \mathrm{sign}(f_i(c_{i+1}^-) - f_i(c_u^+)) = \pm 1$ denote the sign of monotonicity. We consider f as undefined at the cutting points. On the other hand, each f_i extends to a continuous map from the closed interval $[c_i, c_{i+1}]$ to $[a, b]$.

We call $\mathcal{C}(f) = \{c_i : 1 \le i \le \ell\}$ the interior cutting points of the interval. The set of cutting points, $\mathcal{C}_*(f)$, includes c_0 and $c_{\ell+1}$.

In order to treat monotonicity and discontinuities in a consistent manner it is convenient to extend our base interval I to its unit-tangent bundle, also denoted the space of point-germs $\widehat{I}$: each point $x \in I \smallsetminus \{a, b\}$ generates two point-germs denoted $x^+ = (x, +1)$ and $x^- = (x, -1)$ while the boundary points a, b each has only one point-germ a^+ and b^-. We write

$$\varepsilon(x^+) := 1 \quad \text{and} \quad \varepsilon(x^-) := -1$$

for the direction of the germ. In order to make some formulae in Section 4 more concise, we set (artificially) $c_0^- = b^-$ so that $\{c_0^+, c_0^-\} = \{a^+, b^-\}$. For $x \in I$ we denote by $\widehat{x} = (x, \sigma)$ the point-germ based at x and in the direction $\sigma \in \{\pm 1\}$.

It is notationally convenient to define an order $<$ on the collection of point-germs together with base points, $I \cup \widehat{I}$, by declaring that for two base points $x < y$ we have $x < x^+ < y^- < y < y^+$. Given two point-germs $\widehat{u}, \widehat{v} \in \widehat{I}$ with $\widehat{u} < \widehat{v}$, we define

$$\langle \widehat{u}, \widehat{v} \rangle := \left\{ x \in I \mid \widehat{u} < x < \widehat{v} \right\}$$

as a sub interval of I. It is then consistent to write e.g. $[u, v[= \langle u^-, v^- \rangle$ and $]u, v[= \langle u^+, v^- \rangle$. Note that the boundary points a, b never belong to an interval of the form $\langle \widehat{u}, \widehat{v} \rangle$. When $J =]u, v[$ is an open interval we set $\widehat{J} = \{\widehat{x} : u < x < v\} \cup \{u^+\} \cup \{v^-\}$. In particular, $\widehat{I_i} = \{\widehat{x} : c_i < x < c_{i+1}\} \cup \{c_i^+\} \cup \{c_{i+1}^-\}$, $0 \le i \le \ell$. We observe that the $\widehat{I_i}$'s are disjoint and their union is $\widehat{I}$.

Our original map f induces a well-defined map $\widehat{f} : \widehat{I} \to \widehat{I}$. When $\widehat{x} = (x, \sigma) \in \widehat{I_i}$ then $f(\widehat{x}) = (y, \sigma')$ is simply the germ based at $y = \lim_{t \to 0^+} f_i(x + \sigma t)$ whose direction is $\sigma' = s_i \sigma$. Note that on each $\widehat{I_i}$, $\widehat{f}$ is monotone because f is *strictly* monotone. We will usually write f also for the extended map $\widehat{f}$.

For each $0 \le i \le \ell$ we let $g_i \in \mathbb{C}$ be a weight associated with the interval I_i. Both g_i and s_i gives rise to functions on $\widehat{I_i}$ by declaring $s(\widehat{x}) = s_i$ and $g(\widehat{x}) = g_i$ whenever $\widehat{x} \in \widehat{I_i}$. We may define products along orbits, $s^n, g^n, [sg]^n$ by setting $s^0 = g^0 = 1$, and

$$\forall\, n \ge 1, \quad s^n(\widehat{x}) := \prod_{k=0}^{n-1} s(f^k(\widehat{x})), \quad g^n(\widehat{x}) := \prod_{k=0}^{n-1} g(f^k(\widehat{x})), \quad [sg]^n := s^n g^n.$$

Note that $s^n(\widehat{x})$ is the sense of monotonicity of f^n at $\widehat{x}$.

We define a half-sign function:

$$\forall\ \widehat{x} \in \widehat{I},\ y \in I, \quad \sigma(\widehat{x}, y) := \frac{1}{2}\operatorname{sgn}(\widehat{x} - y) = \begin{cases} +1/2 & \text{if } \widehat{x} > y \\ -1/2 & \text{if } \widehat{x} < y \end{cases} .$$

Concerning forward orbits of point-germs we set for $j, k = 0, \cdots, \ell$:

$$\theta(\widehat{x}, t; c_k) = \sum_{m\geq 0} t^m [sg]^m(\widehat{x}) \cdot \sigma(f^m \widehat{x}, c_k), \text{ so in particular} \tag{3}$$

$$\theta(\widehat{x}, t; c_0) = \sum_{m\geq 0} t^m [sg]^m(\widehat{x}) \cdot \sigma(f^m \widehat{x}, c_0) = \frac{1}{2}\sum_{m\geq 0} t^m [sg]^m(\widehat{x}) \tag{4}$$

$$\varepsilon^*(\widehat{c}) = \varepsilon(\widehat{c}) \quad \text{if } \widehat{c} \neq c_0^{\pm} \text{ and } \varepsilon^*(\widehat{c}) = +1 \quad \text{if } \widehat{c} = c_0^{\pm} \tag{5}$$

$$R_{jk}(t) = \sum_{\widehat{c}_j = c_j^+, c_j^-} \varepsilon^*(\widehat{c}_j) \cdot \theta(\widehat{c}_j, t; c_k), \tag{6}$$

$$\mathcal{R}(t) = (R_{jk}(t))_{0\leq j,k\leq \ell} \text{ (the } \textbf{kneading matrix}\text{)} \tag{7}$$

$$\mathcal{B}(t) = (R_{jk}(t))_{1\leq j,k\leq \ell} \text{ (the } \textbf{reduced kneading matrix}\text{)} \tag{8}$$

In particular, one has (note the signs):

$$R_{jk}(t) = \theta(c_j^+, t; c_k) - \theta(c_j^-, t; c_k) =: \Delta_{c_j}\theta(\cdot, t; c_k) \ (j > 0) \text{ while} \tag{9}$$

$$R_{0k}(t) = \theta(a^+, t; c_k) + \theta(b^-, t; c_k) \tag{10}$$

(this choice of signs is designed to absorb boundary correcting terms in later calculations).

Regarding 'backward'-orbits we define Z_1 as the set of level-1 cylinders $(j) := I_j =]c_j, c_{j+1}[,\ j = 0, 1, \cdots, \ell$. Define then recursively Z_n as the set of non-empty level-n cylinders of the form : $(i_0 i_1 \cdots i_{n-1}) := I_{i_0} \cap f_{i_0}^{-1}(i_1 \cdots i_{n-1})$. Each $\alpha = (i_0 i_1 \cdots i_{n-1})$ is an open interval $]u, v[= \langle u^+, v^-\rangle$. We set $\widehat{\partial}\alpha = \{u^+, v^-\}$. For $0 \leq j < n$, $f^j(\alpha) \subset I_{i_j}$. So f^n maps α homeomorphically onto its image, in particular each of the functions s^j and g^j, $0 \leq j < n$, is constant on α.

Definition 2.1. *We call* $(I_i, f_i)_{0\leq i\leq \ell}$ *expansive if* $\lim\limits_{n\to\infty} \sup\limits_{\alpha\in Z_n} \operatorname{diam}(\alpha) = 0$.

For any $y \in I$, set $\Gamma_{0,y} = \{y\}$, and for $p > 0$,

$$\Gamma_{p,y} = \Big\{ x \in \bigcup_{\alpha\in Z_p} \alpha \ \Big|\ f^p(x) = y \Big\}.$$

Note that $x \in \Gamma_{p,y}$ implies that $g^p(x^-) = g^p(x^+)$, for which we simply write $g^p(x)$. This is because $g^0(x) \equiv 1$ and every j-iterate $(0 \leq j < p)$ of a p-cylinder $\alpha \in Z_p$ belongs to some level-1 cylinder. Define

$$\gamma_y(t) = \sum_{p\geq 0} \sum_{x\in\Gamma_{p,y}} t^p g^p(x) \quad \text{and, for } J \subset]a, b[, \quad \gamma_{y,J}(t) = \sum_{p\geq 0} \sum_{x\in\Gamma_{p,y}} t^p g^p(x)\chi_J(x).$$

This function counts the (weighted) number[2] of preimages of y.

Clearly when J and J' are disjoint subsets we have

$$\gamma_{y,J}(t) + \gamma_{y,J'}(t) = \gamma_{y,J\cup J'}(t).$$

Theorem 2.1. *(Main Kneading Identity, or MKI in short) For any interval $J = \langle \widehat{u}, \widehat{v}\rangle$ in I,*

$$\forall\ k \in \{0, \cdots, \ell\}, \quad \sum_{j=1}^{\ell} \gamma_{c_j,J}(t) R_{jk}(t) = \theta(\widehat{v}, t; c_k) - \theta(\widehat{u}, t; c_k) =: \Delta_J \theta(\cdot, t; c_k) \tag{11}$$

(the term $j = 0$ is not included in the sum, but we do allow $k = 0$).

We also need a particular way to count the fixed points of f^n. Fix $n \geq 1$ and an n-cylinder α. The value of $g^n(x)$ is a constant on α, denoted by $g^n_{|\alpha}$. We define the (fixed point counting) weight $\omega(\alpha)$ by

$$\omega(\alpha) = -g^n_{|\alpha} \sum_{\widehat{x}\in\widehat{\partial}\alpha} \sigma(f^n\widehat{x}, x) \cdot \varepsilon(f^n\widehat{x}).$$

We refer to Appendix A for an account of the geometric meaning of this weight. We then introduce the following weighted counting of fixed points of f^n:

$$N_n := \sum_{\alpha\in Z_n} \omega(\alpha).$$

Set $Z(t) := \exp\Big(\sum_{n\geq 1} \frac{1}{n} N_n t^n\Big)$, $\quad N_f(t) := \sum_{n\geq 1} N_n t^{n-1} = (\log Z)'$.

Theorem 2.2. *(zeta-function identity) Let $\mathcal{D}(t) = \det \mathcal{R}(t)$. We then have $Z(t)\cdot\mathcal{D}(t) = 1$, or equivalently*

$$N_f(t) + \frac{\mathcal{D}'(t)}{\mathcal{D}(t)} = 0.$$

Definition 2.2. *For every $n \geq 0$ we write $\big\|g^n\big\|_\infty = \sup_{\alpha\in Z_n} \big|g^n_{|\alpha}\big|$ and $\big\|g^n\big\|_1 = \sum_{\alpha\in Z_n} \big|g^n_{|\alpha}\big|$. We then set*

$$\rho_\infty := \limsup_{n\to\infty} \big\|g^n\big\|_\infty^{1/n} \quad \leq \quad \rho_1 := \limsup_{n\to\infty} \big\|g^n\big\|_1^{1/n}. \tag{12}$$

We also call $\log\rho_1$ the **pressure**[3] *of the weighted system $(I_i, f_i, g_i)_{i\in S}$. This is consistent with usual "thermodynamic formalism" for dynamical systems.*

[2] In the case $g_i \equiv 1$, we have $\gamma_{y,J}(t) = \sum_{p\geq 0} \#(\Gamma_{p,y} \cap J)\, t^p$.

[3] In the case $g_i \equiv 1$, we have $\rho_\infty = 1$ and ρ_1 is the growth rate of the n-cylinders. By Misiurewicz-Szlenk ([MS]) $\log\rho_1$ is equal to the topological entropy of the unweighted system.

Theorem 2.3. *We have*

1. *The power series for $\theta(\widehat{x}, t; c_k)$, $R_{jk}(t)$ define analytic functions of t on the disc $\{|t| < 1/\rho_\infty\}$.*
2. *The kneading matrix $\mathcal{R}(t)$ is invertible when $|t| < 1/\rho_1$.*
3. *Suppose that $\rho_1 > \rho_\infty$ and all $g_i \geq 0$. Then $\mathcal{R}(t)$ is non-invertible at $t = 1/\rho_1$ and $1/\rho_1$ coincides with the radius of convergence of $Z(t)$.*

Theorem 2.4. *Assume $\rho_1 > \rho_\infty$ and all $g_i > 0$. For each $0 < t < 1/\rho_1$ there is a monotone (non-continuous) map $\phi_t : \widehat{I} \to [0, 1]$ with the following properties*

A. *For $0 \leq i \leq \ell$, let $\widetilde{I}_{t,i} = [\phi_t(c_i^+), \phi_t(c_{i+1}^-)]$ (this is an interval or a point). The collection $\widetilde{I}_{t,i}$, $0 \leq i \leq \ell$ is pairwise disjoint.*

B. *For each i there is an affine map $\widetilde{f}_{t,i} : \widetilde{I}_{t,i} \to [0, 1]$ of slope $s_i/(tg_i)$ such that*

$$\phi_t \circ f_{\mid \widehat{I}_i} = \widetilde{f}_{t,i} \circ \phi_{t \mid \widehat{I}_i} \tag{13}$$

C. *The partially defined dynamical system $(\widetilde{I}_{t,i}, \widetilde{f}_{t,i})_{0 \leq i \leq \ell}$ is uniformly expanding and its maximal invariant domain is precisely $\Omega_t = \phi_t(\widehat{I})$.*

Remark. Thus, ϕ_t is semi-conjugating the dynamical system $(\widehat{I}, f)$ to a uniformly expanding dynamical system $(\Omega_{t,i}, \widetilde{f}_{t,i})_{0 \leq i \leq \ell}$. In general, Ω_t is a Cantor set. The subset $\Omega_{t,i} = \phi_t(\widehat{I}_i)$ is trivial, i.e. reduced to a point, precisely when the forward orbit of I_i never encounters a cutting point. This can not happen if the original system is expansive.

The proof will show that the semi-conjugacy ϕ_t can be explicitly expressed as

$$\frac{h(\widehat{x}) - h(a^+)}{h(b^-) - h(a^+)} \quad \text{with} \quad h(\widehat{x}) = \Big(\theta(\widehat{x}, t; c_k)\Big)_{k=0,\cdots \ell} \cdot \mathcal{R}^{-1} \cdot \begin{pmatrix} 0 \\ G(c_1, t) \\ \vdots \\ G(c_\ell, t) \end{pmatrix},$$

where $G(x, t)$ is the average of the generating functions for $g^n(x^-)$ and $g^n(x^+)$ (see (15), (21) and (25)). If $\ell = 1$, one can replace $h(\widehat{x})$ by $\theta(\widehat{x}, t; c_1)$, which is particularly simple to implement numerically.

When taking the limit as $t \nearrow 1/\rho_1$ we obtain a different type of semi-conjugacy:

Theorem 2.5. *Assume $\rho_1 > \rho_\infty$ and all $g_i \geq 0$. There is a monotone continuous surjective map $\phi : I \to [0, 1]$ with the following properties: Denote by $\widetilde{S} \subset S := \{0, \cdots, \ell\}$ the subset of i's for which $\widetilde{I}_i = \text{Int } \phi(I_i)$ is non-empty. Then*

A. *For every $i \in \widetilde{S}$, there is an affine map $\widetilde{f}_i$ of slope $s_i\rho_1/g_i$ such that*

$$\widetilde{f}_i(\phi(x)) = \phi(f_i(x)), x \in I_i.$$

B. *The two weighted systems $(I_i, f_i, g_i)_{i\in S}$ and $(\widetilde{I}_i, \widetilde{f}_i, g_i)_{i\in\widetilde{S}}$ have equal pressures.*

C. *If the sytem $f = (I_i, f_i)_{i\in S}$ extends continuously to a map on $[a, b]$ then so does $\widetilde{f} = (\widetilde{I}_i, \widetilde{f}_i)_{i\in\widetilde{S}}$ on $[0, 1]$ and ϕ gives a genuine topological semi-conjugacy. We have in this case for every $x \in [a, b]$:*

$$\widetilde{f}(\phi(x)) = \phi(f(x)).$$

For the last theorem, some intervals may disappear under the semi-conjugacy, i.e. the set $\widetilde{S}$ becomes a strict subset of S. This happens in particular, when the original system is not transitive and contains sub-systems of a smaller pressure. The set $\widetilde{S}$ may even depend on the choice of the weights g_i. In particular, intervals for which $g_i = 0$ disappear under the conjugacy.

3 The Main Kneading Identity

Lemma 3.1. *We have $\mathcal{R}(0) = id$.*

Proof. Note that

$$R_{jk}(t) = \sum_{\widehat{c}_j=c_j^+,c_j^-} \varepsilon^*(\widehat{c}_j)\cdot\theta(\widehat{c}_j, t; c_k) = \sum_{n\geq 0} t^n \sum_{\widehat{c}_j=c_j^+,c_j^-} \varepsilon^*(\widehat{c}_j)[sg]^n(\widehat{c}_j)\cdot\sigma(f^n\widehat{c}_j, c_k)$$

By convention $f^0 = id$. Recall that $\varepsilon^*(\widehat{c}_j) = \varepsilon(\widehat{c}_j)$ if $j \neq 0$ and $\varepsilon^*(\widehat{c}_0) = 1$.

Assume first $j > 0$. Then, for all $k = 0, \cdots, \ell$,

$R_{jk}(0) = \sum_{\widehat{c}_j=c_j^+,c_j^-} \varepsilon^*(\widehat{c}_j)\cdot[sg]^0(\widehat{c}_j)\cdot\sigma(f^0\widehat{c}_j, c_k) = \sum_{\widehat{c}_j=c_j^+,c_j^-} \varepsilon(\widehat{c}_j)\cdot\sigma(\widehat{c}_j, c_k) = \delta_{jk}.$

Also, $R_{0k}(0) = \theta(a^+, 0; c_k) + \theta(b^-, 0; c_k) = \sigma(a^+, c_k) + \sigma(b^-, c_k) = \delta_{0k}$. □

3.1 Proof of Theorem 2.1.

Consider first an open interval $J =]u, v[\subset]a, b[$, and a c_k for some $k \in \{0, \cdots, \ell\}$. For each $n \geq 0$ and each $(n+1)$-cylinder $\alpha \in Z_{n+1}$, the functions $[sg]^n(\widehat{x}) = \prod_{j=0}^{n-1} s(f^j\widehat{x})g(f^j\widehat{x})$

and $\sigma(f^n\widehat{x}, c_k)$, $\widehat{x} \in \widehat{\alpha}$ are constants. When $\alpha \in Z_{n+1}$ and $\alpha \cap J \neq \emptyset$, then obviously

$$\sum_{\widehat{x}\in\widehat{\partial}(J\cap\alpha)} \varepsilon(\widehat{x}) = 1 + (-1) = 0.$$

So the following power series vanishes identically:

$$\sum_{n\geq 0} t^n \sum_{\alpha\in Z_{n+1}, \widehat{x}\in\widehat{\partial}(J\cap\alpha)} \varepsilon(\widehat{x}) \cdot [sg]^n(\widehat{x}) \cdot \sigma(f^n\widehat{x}, c_k) = 0.$$

In this sum, $\widehat{x} = u^+, v^-$ appears for every $n \geq 0$. Extracting their contributions we write:

$$\sum_{\widehat{x}\in\widehat{\partial}J} \theta(\widehat{x}, t; c_k) \cdot \varepsilon(\widehat{x}) + \sum_{n\geq 0} t^n \sum_{\alpha\in Z_{n+1}, \widehat{x}\in\widehat{\partial}\alpha} \chi_J(x) \cdot \varepsilon(\widehat{x}) \cdot [sg]^n(\widehat{x}) \cdot \sigma(f^n\widehat{x}, c_k) = 0. \qquad (14)$$

Now when $\alpha \in Z_{n+1}, \widehat{x} \in \widehat{\partial}(J \cap \alpha)$, there is a unique minimal integer $0 \leq p \leq n$ for which $f^p(\widehat{x}) = \widehat{c}$ for some $c \in \{c_1, \cdots, c_\ell\} =: \mathcal{C}(f)$ and $\widehat{c} = c^+$ or c^- (note that the boundary points a, b are excluded here, since for an interior point to be mapped to them, it has to pass an interior cutting point just before). Recall that $\Gamma_{p,c} = \{x \in \bigcup_{\alpha\in Z_p} \alpha \mid f^p x = c\}$ and $\Gamma_{0,c} = \{c\}$. When $x \in \Gamma_{p,c}$ and $f^p\widehat{x} = \widehat{c}$, then $g^p(\widehat{x}) = g^p(x)$, $\sigma(f^p\widehat{x}, c_k) = \sigma(f^{n-p}\widehat{c}, c_k)$ and also (the essential point here is that the sign $s^p(\widehat{x})$ is absorbed in $\varepsilon(\widehat{c})$)

$$\varepsilon(\widehat{x}) \cdot [sg]^n(\widehat{x}) = g^p(x)\Big(\varepsilon(\widehat{x})s^p(\widehat{x})\Big)[sg]^{n-p}(\widehat{c}) = g^p(x) \cdot \varepsilon(\widehat{c}) \cdot [sg]^{n-p}(\widehat{c}).$$

So we obtain, for the second term in (14) (writing $t^n = t^p t^q$),

$$\sum_{c\in\mathcal{C}(f)} \Big[\Big(\sum_{p\geq 0} t^p \sum_{x\in\Gamma_{p,c}} g^p(x)\chi_J(x)\Big) \cdot \sum_{\widehat{c}=c^\pm, q\geq 0} t^q \cdot \varepsilon(\widehat{c}) \cdot [sg]^q(\widehat{c})\sigma(f^q\widehat{c}, c_k)\Big]$$

$$= \sum_{c\in\mathcal{C}(f)} \gamma_{J,c}(t) \cdot \Delta_c\theta(\cdot, t; c_k).$$

Combining with (14) we get the Main kneading Identity when J is an open interval.

It remains to prove the case that J is half closed or closed. Consider for example $J = \langle u^-, v^-\rangle$ with $a < u < v \leq b$. We have $\langle a^+, v^-\rangle = \langle a^+, u^-\rangle \sqcup J$ and the additivity $\gamma_{c,\langle a^+,v^-\rangle} = \gamma_{c,\langle a^+,u^-\rangle} + \gamma_{c,J}$. The result then follows by applying the identity to the two intervals $\langle a^+, u^-\rangle$ and $\langle a^+, v^-\rangle$ and subtracting. □

4 Zeta functions and kneading determinants

In this section we prove Theorems 2.2 and 2.3.

4.1 Relating $N_f(t)$ to $\mathcal{R}'(t)$

Set $\widehat{\mathcal{C}}(f) := \{a^+ = c_0^+, c_1^-, c_1^+, \cdots, c_\ell^-, c_\ell^+, b^- = c_0^-\}$. For all $\widehat{c} \in \widehat{\mathcal{C}}(f)$, set $\Gamma_{0,\widehat{c}} = \{\widehat{c}\}$ and, for $p \geq 1$,

$$\Gamma_{p,\widehat{c}} = \{\widehat{x} \in \widehat{I} \mid f^p\widehat{x} = \widehat{c},\ , f^j\widehat{x} \notin \widehat{\mathcal{C}}(f) \text{ for } 0 \leq j < p\}.$$

If $\widehat{c} \neq c_0^\pm$, then for any $\widehat{x} \in \Gamma_{p,\widehat{c}}$ we have $x \in \Gamma_{p,c}$. Conversely for any $x \in \Gamma_{p,c}$ exactly one of $x^\pm$ belongs to $\Gamma_{p,\widehat{c}}$.

Notice that if $\widehat{c} = c_0^\pm$ then $\Gamma_{p,\widehat{c}} = \emptyset$ when $p \geq 1$: due to the forward invariance of I we have $f^{-1}(\{a,b\}) \subset \{a, b, c_1, \cdots, c_\ell\}$, so every orbit passing though $\{a,b\}$ must pass through $\{c_1, \cdots, c_\ell\}$ just before.

Fix $n \geq 1$ and an n-cylinder α. Note that for each $\widehat{x} \in \widehat{\partial}\alpha$, we have $g^n_{|\alpha} \cdot \varepsilon(f^n\widehat{x}) = [sg]^n(\widehat{x}) \cdot \varepsilon(\widehat{x})$, so

$$-\omega(\alpha) := g^n_{|\alpha} \sum_{\widehat{x} \in \widehat{\partial}\alpha} \sigma(f^n\widehat{x}, x) \cdot \varepsilon(f^n\widehat{x}) = \sum_{\widehat{x} \in \widehat{\partial}\alpha} \sigma(f^n\widehat{x}, x)[sg]^n(\widehat{x}) \cdot \varepsilon(\widehat{x}).$$

To each $\widehat{x} \in \widehat{\partial}\alpha$, there is a unique $\widehat{c} \in \widehat{\mathcal{C}}(f)$ and $0 \leq p < n$ such that $x \in \Gamma_{p,\widehat{c}}$. Schematically,

$$\widehat{x} \xrightarrow[p \text{ minimal}]{f^p} \widehat{c} \xrightarrow{f^{q+1}} f^n\widehat{x} = f^{q+1}\widehat{c}$$

Setting q such that $p + q = n - 1$, we have the "co-cycle" properties:

$$s^n(\widehat{x})\varepsilon(\widehat{x}) = s^{q+1}(\widehat{c})\varepsilon(\widehat{c}),\ g^0(\widehat{x}) = 1,\ g^n(\widehat{x}) = g^{q+1}(\widehat{c})g^p(\widehat{x}).$$

Now

$$\begin{aligned}
-\sum_{n\geq 1} t^{n-1} N_n &= -\sum_{n\geq 1} t^{n-1} \sum_{\alpha \in Z_n} \omega(\alpha) \\
&= \sum_{n\geq 1} t^{n-1} \sum_{\alpha \in Z_n, \widehat{x} \in \widehat{\partial}\alpha} \sigma(f^n\widehat{x}, x)[sg]^n(\widehat{x}) \cdot \varepsilon(\widehat{x}) \\
&= \sum_{\widehat{c} \in \widehat{\mathcal{C}}(f)} \sum_{q \geq 0} t^q [sg]^{q+1}(\widehat{c}) \cdot \varepsilon(\widehat{c}) \sum_{p \geq 0, \widehat{x} \in \Gamma_{p,\widehat{c}}} t^p g^p(\widehat{x}) \sigma(f^{q+1}\widehat{c}, x) \\
&= \sum_{\widehat{c} \in \widehat{\mathcal{C}}(f) \smallsetminus \{c_0^\pm\}} \sum_{q \geq 0} t^q [sg]^{q+1}(\widehat{c}) \cdot \varepsilon(\widehat{c}) \Big(\sum_{p \geq 0, \widehat{x} \in \Gamma_{p,\widehat{c}}} t^p g^p(\widehat{x}) \sigma(f^{q+1}\widehat{c}, x) \Big) \\
&\quad + \sum_{\widehat{c} = c_0^\pm} \sum_{q \geq 0} t^q [sg]^{q+1}(\widehat{c}) \cdot \Big(\varepsilon(\widehat{c}) \sum_{p \geq 0, \widehat{x} \in \Gamma_{p,\widehat{c}}} t^p g^p(\widehat{x}) \sigma(f^{q+1}\widehat{c}, x) \Big)
\end{aligned}$$

Note that the $\varepsilon(\widehat{c})$ factor in the last expression is treated differently for $\widehat{c} = c_1^\pm, \cdots, c_\ell^\pm$ and $\widehat{c} = c_0^\pm$. The reason for this is that we want the two expressions in the parenthesis to be independent of the direction of $\widehat{c}$. Indeed, for any $\widehat{u} \in \widehat{I}$,

$$\text{for} \quad \widehat{c} = c_1^\pm, \cdots, c_\ell^\pm, \quad m_{\widehat{c}}(\widehat{u}, t) := \sum_{p \geq 0, \widehat{x} \in \Gamma_{p,\widehat{c}}} t^p g^p(\widehat{x}) \sigma(\widehat{u}, x) = \sum_{p \geq 0, x \in \Gamma_{p,c}} t^p g^p(x) \sigma(\widehat{u}, x)$$

$$\text{and for} \quad \widehat{c} = c_0^{\pm}, \quad m_{\widehat{c}}(\widehat{u}, t) := \varepsilon(\widehat{c}) \sum_{p \geq 0,\, \widehat{x} \in \Gamma_{p,\widehat{c}}} t^p g^p(\widehat{x}) \sigma(\widehat{u}, x) = \varepsilon(\widehat{c}) \sigma(\widehat{u}, c) \equiv \frac{1}{2}$$

where we have used the facts that $\Gamma_{p,c_0^{\pm}} = \emptyset$ for $p > 0$, $g^0(\widehat{x}) \equiv 1$ and $g^p(x^+) = g^p(x^-) =: g^p(x)$ for $x \in \Gamma_{p,c_j}$, $j > 0$.

In both cases $m_{\widehat{c}}(\widehat{u}, t)$ is independent of $\varepsilon(\widehat{c}) = +$ or $-$, so we may safely write $m_c(\widehat{u}, t)$ for this quantity. To compactify the two cases we set $\mathcal{C}^*(f) = \{c_0, c_1, \cdots, c_\ell\}$. Recall that $c_0^+ = a^+$, $c_0^- = b^-$ and $\varepsilon^*(\widehat{c}) := \varepsilon(\widehat{c})$ if $\widehat{c} \neq c_0^{\pm}$ and $\varepsilon^*(\widehat{c}) = 1$ otherwise. Then

$$-\sum_{n \geq 1} t^{n-1} N_n \;\; = \;\; \sum_{c \in \mathcal{C}^*(f)} \sum_{\widehat{c} = c^{\pm},\, q \geq 0} t^q [sg]^{q+1}(\widehat{c}) \cdot \varepsilon^*(\widehat{c}) \cdot m_c(f^{q+1}\widehat{c}, t)$$

A central idea (due to Preston) is to consider the right hand side as the trace of an $(\ell+1) \times (\ell+1)$ matrix $\mathcal{F}$, and to define $\mathcal{F}$ in a way so that $\mathcal{F}\mathcal{R}$ becomes related to $\mathcal{R}'$. There are many choices suitable for this purpose with most choices giving rise to additional correcting terms. There is, however, a choice for which the relationship becomes particularly simple (note the $*$ in the epsilon factor):

For $i, j \in \{0, 1, \cdots, \ell\}$, define

$$F_{ij}(t) = \sum_{q \geq 0,\, \widehat{c}_i = c_i^{\pm}} t^q [sg]^{q+1}(\widehat{c}_i) \cdot \varepsilon^*(\widehat{c}_i) \cdot m_{c_j}(f^{q+1}\widehat{c}_i, t).$$

We then have:

Theorem 4.1. $\mathcal{F}\mathcal{R} = \mathcal{R}'$.

Proof. We establish at first a consequence of the Main Kneading Identity:

Claim. For every $\widehat{w} \in \widehat{I}$, $k = 0, 1, \cdots, \ell$,

$$\sum_{j=0}^{\ell} m_{c_j}(\widehat{w}, t) R_{jk}(t) = \theta(\widehat{w}, t; c_k).$$

Proof. By the Main Kneading Identity, we sum first over interior cutting points:

$$\begin{aligned}
\sum_{j=1}^{\ell} m_{c_j}(\widehat{w}, t) R_{jk}(t) &= \sum_{j=1}^{\ell} \sum_{p \geq 0,\, x \in \Gamma_{p,c_j}} t^p g^p(x) \sigma(\widehat{w}, x) \cdot R_{jk}(t) \\
&= \sum_{j=1}^{\ell} \sum_{p \geq 0,\, x \in \Gamma_{p,c_j}} t^p g^p(x) \frac{1}{2} \Big(\chi_{(a^+, \widehat{w})}(x) - \chi_{(\widehat{w}, b^-)}(x) \Big) R_{jk}(t) \\
&= \frac{1}{2} \Big(2\theta(\widehat{w}, t; c_k) - \theta(a^+, t; c_k) - \theta(b^-, t; c_k) \Big)
\end{aligned}$$

Adding the boundary term $m_{c_0}(\widehat{w}, t) R_{0k}(t) = \frac{1}{2} \Big(\theta(a^+, t; c_k) + \theta(b^-, t; c_k) \Big)$ we get the desired result and end the proof of the claim.

Now, for $i,k \in \{0,\cdots,\ell\}$,

$$\begin{aligned}
\sum_{j=0}^{\ell} F_{ij}R_{jk} &= \sum_{q\geq 0,\,\widehat{c}_i=c_i^{\pm}} t^q[sg]^{q+1}(\widehat{c}_i)\cdot\varepsilon^*(\widehat{c}_i)\cdot\sum_{j=0}^{\ell}\Big(m_{c_j}(f^{q+1}\widehat{c}_i,t)R_{jk}\Big)\\
&= \sum_{q\geq 0,\,\widehat{c}_i=c_i^{\pm}} t^q[sg]^{q+1}(\widehat{c}_i)\cdot\varepsilon^*(\widehat{c}_i)\cdot\theta(f^{q+1}(\widehat{c}_i),t;c_k)\\
&= \sum_{q\geq 0,\,\widehat{c}_i=c_i^{\pm}} t^q[sg]^{q+1}(\widehat{c}_i)\cdot\varepsilon^*(\widehat{c}_i)\sum_{p\geq 0}t^p[sg]^p(f^{q+1}\widehat{c}_i)\sigma(f^p(f^{q+1}\widehat{c}_i),c_k)\\
&= \sum_{\widehat{c}_i=c_i^{\pm}}\sum_{p,q\geq 0} t^{p+q}[sg]^{p+q+1}(\widehat{c}_i)\cdot\sigma(f^{p+q+1}\widehat{c}_i,c_k)\cdot\varepsilon^*(\widehat{c}_i)\\
&= \sum_{\widehat{c}_i=c_i^{\pm}}\Big(\sum_{n\geq 1} n\cdot t^{n-1}[sg]^n(\widehat{c}_i)\cdot\sigma(f^n\widehat{c}_i,c_k)\Big)\varepsilon^*(\widehat{c}_i)\\
&= \sum_{\widehat{c}_i=c_i^{\pm}}\Big(\frac{d}{dt}\theta(\widehat{c}_j,t;c_k)\Big)\varepsilon^*(\widehat{c}_i) \;=\; \frac{d}{dt}R_{ik}(t)
\end{aligned}$$

in which we recall that $R_{jk}(t)=\sum_{\widehat{c}_j=c_j^{\pm}}\theta(\widehat{c}_j,t;c_k)\cdot\varepsilon^*(\widehat{c}_j)$. □

Proof of Theorem 2.2. We have

$$0 = N_f(t) + Tr\mathcal{F} = N_f(t) + Tr\mathcal{R}'\mathcal{R}^{-1} = N_f(t) + \frac{\mathcal{D}'}{\mathcal{D}}.$$

□

4.2 Weighted lap function and proof of Theorem 2.3

Let us consider the generating function of $g^n(\widehat{x})$:

$$\begin{aligned}
G(\widehat{x},t) &= \sum_{n\geq 0} t^n g^n(\widehat{x}) \;\text{ for } \widehat{x}\in\widehat{I} \text{ and then}\\
G(x,t) &= \frac{1}{2}\Big(G(x^-,t)+G(x^+,t)\Big) \;\text{ when } a<x<b.
\end{aligned} \tag{15}$$

Let $J=\langle\widehat{u},\widehat{v}\rangle\subset]a,b[$ be an (open, closed or half-closed) interval or a point. We define the **weighted lap function**[4]

$$L(J,t) \;:=\; \frac{1}{2}\sum_{n\geq 0} t^n \sum_{\alpha\in Z_{n+1}}\sum_{\widehat{x}\in\widehat{\partial}\alpha} g^n(\widehat{x})\chi_J(x). \tag{16}$$

[4]If $g_i\equiv 1$ the G-functions are $\dfrac{1}{1-t}$ and the function $L(J,t)$ is the generating function for the numbers of $(n+1)$-cylinders in J, and $L(]a,b[,t)$ has radius of convergence equal to $1/\rho_1$.

Repeating the calculation in our proof of the Main Kneading Identity without the sign factors s, ε and σ, it follows easily that

$$L(J,t) = \sum_{j=1}^{\ell} \Big(\sum_{p\geq 0,\, x\in\Gamma_{p,c_j}} t^p g^p(x)\chi_J(x) \Big) \Big(\sum_{\widehat{c}=c_j^{\pm}} \frac{1}{2} \sum_{q\geq 0} t^q \cdot g^q(\widehat{c}) \Big) \tag{17}$$

$$= \sum_{j=1}^{\ell} \gamma_{c_j,J}(t) \cdot G(c_j,t) \tag{18}$$

In particular, for a one-point set $J = \{x\}$ we have simply

$$L(\{x\},t) = \begin{cases} t^p g^p(x) \cdot G(c_i,t) & \text{for } x \in \Gamma_{p,c_i},\ p \geq 0,\ 1 \leq i \leq \ell \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Lemma 4.2. *Fix any subinterval $J = \langle \widehat{u}, \widehat{v} \rangle$. The functions G, θ, $\Delta_J\theta$, $\mathcal{R}_{jk}$ are all analytic functions of t on the disc $\{|t| < 1/\rho_\infty\}$. The kneading matrix is invertible when $|t| < 1/\rho_1$. The function $L(J,t)$ is meromorphic on $\{|t| < 1/\rho_\infty\}$ and analytic on $\{|t| < 1/\rho_1\}$.*

Proof. The first claim follows from the definition of ρ_∞ and the following estimates:

$$\forall\, \widehat{x} \in \widehat{I}, \quad |G(\widehat{x},t)| \leq \sum_{n\geq 0} |t|^n \|g^n\|_\infty < \infty \quad \text{for} \quad |t| < 1/\rho_\infty$$

Similarly $\forall\, k$,

$$|\Delta_J\theta(\cdot,t;c_k)| \leq \sum_{n\geq 0} |t|^n \|g^n\|_\infty < \infty \quad \text{for} \quad |t| < 1/\rho_\infty.$$

To see that the kneading matrix is invertible when $|t| < 1/\rho_1$ we use the relationship to the zeta function. By Theorem 2.2 we have $Z(t) \cdot \det \mathcal{R}(t) = 1$, where

$$Z(t) = \exp\Big(\sum_{n\geq 1} \frac{N_n}{n} t^n \Big)$$

and each $|N_n| \leq \big\|g^n\big\|_1$. So $Z(t)$ is analytic and non-zero for $|t| < 1/\rho_1$ whence $\mathcal{R}(t)$ is invertible for $|t| < 1/\rho_1$.

We have

$$|L(J,t)| \leq \sum_{n\geq 0} |t|^n \sum_{\alpha\in Z_{n+1}} \big|g^n_{|\alpha}\big| \leq \sum_{n\geq 0} |t|^n \sum_{\alpha\in Z_n} \big|g^n_{|\alpha}\big|(\ell+1) = (\ell+1) \sum_{n\geq 0} |t|^n \|g^n\|_1 \tag{20}$$

which shows that $L(J,t)$ has radius of convergence at least $1/\rho_1$.

Using the MKI itself for the γ factor in (18) we get:

$$L(J,t) = \sum_{k=0}^{\ell} \Delta_J\theta(\cdot,t;c_k)\Big(\sum_{j=1}^{\ell} \mathcal{R}^{-1}(t)_{kj}\cdot G(c_j,t)\Big) \tag{21}$$

The above identities are valid as formal power series but also when the functions involved are analytic and $\mathcal{R}(t)$ is invertible. As $1/\rho_1 \le 1/\rho_\infty$, so when $|t| < 1/\rho_1$, the identity (21) is valid. □

Proof of Theorem 2.3

The first two claims have already been proved in Lemma 4.2.

We proceed to prove the last claim. When all g_i's are positive and $t \ge 0$ we have

$$L(]a,b[,t) + G(a^+,t) + G(b^-,t) = \sum_{n\ge 0} t^n \sum_{\alpha\in Z_{n+1}} g^n_{|\alpha} \ge \sum_{n\ge 0} t^n \|g^n\|_1.$$

By definition the RHS has radius of convergence equal to $1/\rho_1$. Being a power-series with positive coefficients it follows that the RHS diverges as $t \nearrow 1/\rho_1$.

Under the further assumption $1/\rho_1 < 1/\rho_\infty$, the functions $t \mapsto G(\widehat{x},t)$, in particular $G(a^+,t)$ and $G(b^-,t)$, remain bounded at $t = 1/\rho_1$. So $L(]a,b[,t)$ must diverge as $t \nearrow 1/\rho_1$. Combining with (20) we know that the radius of convergence of $L(]a,b[,t)$ is equal to $1/\rho_1$. Now, the functions $\Delta_J\theta$ and G involved in (21) remain bounded on $|t| \le 1/\rho_1$. Letting $t \nearrow 1/\rho_1$ in (21) we conclude that $\mathcal{R}(t)$ must be non-invertible at $t = 1/\rho_1$. □

5 Semi-conjugacies to piecewise linear models

In this section we prove Theorems 2.4 and 2.5.

Lemma 5.1. *Fix $J = \langle\widehat{u},\widehat{v}\rangle \subset I_j =]c_j^+, c_{j+1}^-[$. We have for $k = 0,\cdots,\ell$ and $|t| < 1/\rho_\infty$:*

$$\theta(\widehat{v},t;c_k) - \theta(\widehat{u},t;c_k) = t\cdot s_j g_j\Big(\theta(f\widehat{v},t;c_k) - \theta(f\widehat{u},t;c_k)\Big) \tag{22}$$

When also $|t| < 1/\rho_1$ we have for the weighted lap function :

$$L(J,t) = t\, g_j \cdot L(f_j J,t) \tag{23}$$

Proof. Let us fix $k \in \{0,\cdots,\ell\}$. By definition, we have the following relation for $\theta(\cdot,t;c_k)$ when applied to $\widehat{x}$ and $f\widehat{x}$:

$$\forall\,\widehat{x}\in\widehat{I},\quad \theta(\widehat{x},t;c_k) = \sum_{m\ge 0} t^m [sg]^m(\widehat{x})\cdot\sigma(f^m\widehat{x},c_k) = \sigma(\widehat{x},c_k) + t\cdot[sg](\widehat{x})\cdot\theta(f\widehat{x},t;c_k).$$

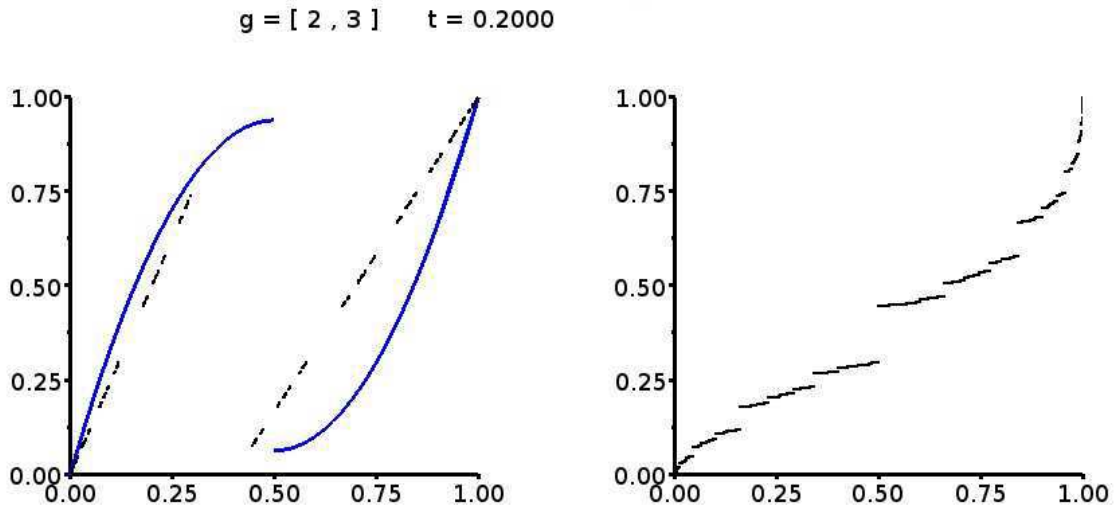

Figure 1: Left: Example of a discontinuous map f and the graph restricted to Ω_t of its conjugated map $\widetilde{f}_t$. Right: The graph of ϕ_t. Here, $t = 0.2 < 1/\rho_1 = 0.2684$. The ratio of slopes of the two branches is $3 : 2$ (coming from the choice of weights).

This implies (22) when restricting to $\widehat{I}_j$. Now, $\Delta_J\theta(\cdot, t; c_k) = \theta(\widehat{v}, t; c_k) - \theta(\widehat{u}, t; c_k)$ and (as f may reverse the orientation) $\Delta_{f_jJ}\theta(\cdot, t; c_k), = s_j\Big(\theta(f\widehat{v}, t; c_k) - \theta(f\widehat{u}, t; c_k)\Big)$, so

$$\Delta_J\theta(\cdot, t; c_k) = tg_j\Delta_{f_jJ}\theta(\cdot, t; c_k). \tag{24}$$

The result for $L(J, t)$ now follows by linearity in equation (21) which is valid when $|t| < 1/\rho_1$. □

Proof of Theorem 2.4.

We assume here that all $g_i > 0$ and that $\rho_1 > \rho_\infty$. Fix $0 < t < 1/\rho_1 < 1/\rho_\infty$. Noting that $0 < L(]a, b[, t) < +\infty$ we define our conjugating map $\phi_t : \widehat{I} \to \mathbb{R}$ by setting

$$\phi_t(\widehat{x}) = \frac{L(\langle a^+, \widehat{x}\rangle, t)}{L(]a, b[, t)}, \quad \widehat{x} \in \widehat{I}.\text{`} \tag{25}$$

Notice that ϕ_t maps point-germs to genuine real numbers.

Part A: Using (23) we get for any $\widehat{x}_1, \widehat{x}_2 \in \widehat{I}_j$ (the sign enters again) :

$$\phi_t(\widehat{x}_2) - \phi_t(\widehat{x}_1) = ts_jg_j\Big(\phi_t(f\widehat{x}_2) - \phi_t(f\widehat{x}_1)\Big). \tag{26}$$

Similarly, we get by iterating this argument for $\widehat{x}_1, \widehat{x}_2 \in \widehat{\alpha}$ with $\alpha \in Z_n$:

$$\phi_t(\widehat{x}_2) - \phi_t(\widehat{x}_1) = t^n\, s^n_{|\alpha}\, g^n_{|\alpha}\, \Big(\phi_t(f^n\widehat{x}_2) - \phi_t(f^n\widehat{x}_1)\Big). \tag{27}$$

When the g_i's are non-negative, we clearly have $L(J,t) \geq 0$ for any interval J so by set-additivity with respect to J it follows that ϕ_t is monotone increasing and takes values in $[0,1]$. Let $\Omega_t = \phi_t(\widehat{I}) \subset [0,1]$ and set $\Omega_{t,i} = \phi_t(\widehat{I}_i)$. By monotonicity of ϕ_t the convex hull of $\Omega_{t,i}$ is precisely $\widetilde{I}_{t,i} = [\phi_t(c_i^+), \phi_t(c_{i+1}^-)]$

Let now $a < x < b$. As all $g_i > 0$, by (19)

$$\phi_t(x^+) - \phi_t(x^-) = \frac{L(\{x\},t)}{L(]a,b[,t)} > 0 \tag{28}$$

precisely when x is an interior cutting point or a pre-image of such. We have in particular $L(\{c_i\},t) \geq 1$ so that $\sup \Omega_{t,i} < \inf \Omega_{t,i+1}$ and also $\sup \widetilde{I}_{t,i} < \inf \widetilde{I}_{t,i+1}$, proving claim **A.**

Part B: Given $y \in \Omega_t$ suppose that $y = \phi_t(\widehat{x}_1) = \phi_t(\widehat{x}_2)$ with $\widehat{x}_1 < \widehat{x}_2$. By the previous paragraph $\widehat{x}_1$ and $\widehat{x}_2$ must belong to the same $\widehat{I}_j$. (In fact they even belong to the same n-cylinder for all n). So by the identity (26) we must have $\phi_t(f\widehat{x}_2) - \phi_t(f\widehat{x}_1) = 0$. This implies that there is a well-defined map $\widetilde{f}_t : \Omega_t \to \Omega_t$ given by:

$$\widetilde{f}_t(y) := \phi_t(f\widehat{x}), \quad y = \phi_t(\widehat{x}) \in \Omega_t \tag{29}$$

(since the value is independent of the choice of $\widehat{x}$ in the pre-image of y).

Equation (26) shows that the conjugated map has (finite) slope $(ts_jg_j)^{-1} = s_j/tg_j$ on each $\Omega_{t,j} = \phi(I_j)$ (if it is not reduced to a point). The map $\widetilde{f}_{t,i}$ is defined to be this affine map extended to $\widetilde{I}_{t,i} = [\phi_t(c_i^+), \phi_t(c_{i+1}^-)]$. □

Part C: The last part of the theorem is tricky due to the fact that ϕ_t is neither continuous nor injective. For an open interval $J =]u,v[\subset]a,b[$ we will in the following use the shorthand notation:

$$\Xi_t(J) := [\phi_t(u^+), \phi_t(v^-)] \tag{30}$$

For $0 \leq i \leq \ell$ we define $\widetilde{I}_t(i) = \widetilde{I}_{t,i} = \Xi_t(I_i)$ and then recursively $\widetilde{I}_t(i_0, \ldots, i_{n-1}) = \widetilde{I}_{t,i_0} \cap \widetilde{f}_t^{-1}\widetilde{I}_t(i_1, \ldots, i_{n-1})$ which is either empty, a point or a closed interval. We write $\widetilde{Z}_{t,n}$ for the collection of non-empty sets of this form. They form a partition for the domain of definition of $(\widetilde{f}_t)^n$. The maximal invariant domain for $\widetilde{f}_t$ is the compact set $\Omega_t = \bigcap_{n \geq 1} \left(\bigcup \widetilde{Z}_{t,n} \right) \subset [0,1]$. Our first goal is to exhibit a simple relationship between cylinders and the above sets.

Lemma 5.2. *There is a bijection between $\alpha \in Z_n$ and $\widetilde{\alpha} \in \widetilde{Z}_{t,n}$ given by $\widetilde{\alpha} = \Xi_t(\alpha)$.*

Proof: For $n = 1$ this is the very definition: Z_1 consists of the intervals $\{I_i : 0 \leq i \leq \ell\}$ and $\widetilde{I}_t(i) = \Xi_t(I_i) = [\phi_t(c_i^+), \phi_t(c_{i+1}^-)]$.

When $J =]u,v[\subset I_i$ the definition of $\widetilde{f}_t$ shows that $\widetilde{f}_t\ \Xi_t(J) = \widetilde{f}_{t,i}[\phi_t(u^+), \phi_t(v^-)] = [\phi_t(f_iu^+); \phi_t(f_iv^-)] = \Xi_t(fJ)$. For $\alpha \in Z_n$ this implies $(\widetilde{f}_t)^k\ \Xi_t(\alpha) = \Xi_t(f^k\alpha) \subset \Xi_t(I_{i_k})$. It follows by recursion that $\Xi_t(\alpha) \subset \widetilde{\alpha} = \widetilde{I}_t(i_0, \ldots, i_n)$.

In order to show equality we proceed by induction in n. Let $\beta = (i_1 \dots i_n) =]u_1, u_2[\in Z_n$ (with $a \le u_1 < u_2 \le b$). Our induction hypothesis is that $\widetilde{I}_t(i_1, \dots, i_n) = \Xi_t(\beta) = [\phi_t(u_1^+), \phi_t(u_2^-)]$. Here, u_1 and u_2 are necessarily (eventual) cutting points so by (28) we have when $a < u_1$ and $u_2 < b$, respectively :

$$\phi_t(u_1^-) < \phi_t(u_1^+) \quad \text{and} \quad \phi_t(u_2^-) < \phi_t(u_2^+). \tag{31}$$

Suppose that $\widetilde{f}_t\, \widetilde{I}_{t,i_0} = \Xi_t(fI_{i_0})$ intersects $\Xi_t(\beta)$ non-trivially and write $\widetilde{\alpha} = \widetilde{I}_t(i_0, \dots, i_n) = [\xi_1, \xi_2]$. We claim that also $]v_1, v_2[\equiv fI_{i_0}$ intersects β. If this were not the case, then e.g. $v_1 < v_2 \le u_1 < u_2$ in which case the first strict inequality in (31) shows that $\phi_t(v_2^-) \le \phi_t(u_1^-) < \phi_t(u_1^+)$ so that $\widetilde{\alpha}$ was empty in the first place.

Assume $s_{i_0} = +1$. We have $\widetilde{\alpha} = [\xi_1, \xi_2] = \widetilde{I}_{t,i_0} \cap \widetilde{f}_t^{-1}\Xi_t(\beta)$ (using the induction hypothesis) and we write $\alpha = I_{i_0} \cap f^{-1}\beta =]w_1, w_2[$ (which is non-empty as just shown). We will calculate the left end points of α and $\widetilde{\alpha}$. We consider the two possibilities: Either $u_1 \le v_1 (< u_2, v_2)$ or $v_1 < u_1 (< u_2, v_2)$.

In the first case $w_1^+ = f_{i_0}^{-1} \max\{u_1^+, fc_{i_0}^+\} = c_{i_0}^+$ and since $\phi_t(u_1^+) \le \phi_t(v_1^+) = \widetilde{f}_t\phi_t(c_{i_0}^+)$ we get $\xi_1 = (\widetilde{f}_{t,i_0})^{-1} \max\{\phi_t(u_1^+), \widetilde{f}_t\phi_t(c_{i_0}^+)\} = \phi_t(c_{i_0}^+) = \phi_t(w_1^+)$. In the second case, continuity and strict monotonicity of f_i yields a unique value $w_1 \in]c_{i_0}, c_{i_0+1}[$ for which $fw_1 = u_1$. We have $\widetilde{f}_t\phi_t(w_1^+) = \phi_t(u_1^+) \le \phi_t(fc_{i_0}^+) \le \widetilde{f}_t\phi_t(c_{i_0}^+)$ so again $\xi_1 = \phi_t(w_1^+)$.

Thus in either case $\xi_1 = \phi_t(w_1^+)$. Similarly, $\xi_2 = \phi_t(w_2^-)$ thus implying $\widetilde{\alpha} = \Xi_t(\alpha)$ as we wanted to show. If $s_i = -1$ some intervals change direction but the conclusion remains the same. □

Returning now to the proof of Part C: Clearly $\phi_t(\widehat{I}) \subset \Omega_t$. In order to show surjectivity consider $\xi \in \Omega_t$. Assume that $\widetilde{f}_t^k \in \widetilde{I}_{t,i_k}$, $k \ge 0$. Then $\xi \in \widetilde{\alpha}_k = \widetilde{I}_t(i_0, \dots, i_{k-1}) = \Xi_t(\alpha_k)$ for all k (a nested sequence of intervals). First, if ξ is a boundary point of such an interval for some k then it is in the image of ϕ_t by the previous lemma. So assume that ξ is in the interior of $\widetilde{\alpha}_k = \Xi_t(\alpha_k)$ for all k. Let $\alpha_k =]u_k, v_k[$. Then $u_k \nearrow u_*$ and $v_k \searrow v_*$ with $u_* \le v_*$. None of the sequences are eventually constant. Now, $\phi_t(u_k^+) \le \xi \le \phi_t(v_k^-)$ and $0 \le \phi_t(v_k^-) - \phi_t(u_k^+) \le t^k g^k_{|\alpha_k}/L(]a, b[, t) \to 0$ as $k \to \infty$. For any $x \in [u^*, v^*]$ we conclude by monotonicity of ϕ_t that $\phi_t(x^+) = \phi_t(x^-) = \xi$. So $\phi_t : \widetilde{I} \to \Omega_t$ is surjective. □

In order to prove Theorem 2.5 we consider the limit $t \nearrow 1/\rho_1$. As the function $L(]a, b[, t)$ diverges the situation is a bit different. By Lemma 4.2 the lap-function $L(]a, b[, t)$ is meromorphic in the disc $\{|t| < 1/\rho_\infty\}$ and has a pole of some order $m \ge 1$ at $t = 1/\rho_1$. By positivity of $L(]a, b[, t)$ for $t > 0$ there is $c > 0$ so that

$$L(]a, b[, t) = \frac{c}{(1 - \rho_1 t)^m} + l.o.t.$$

For any interval $J \subset]a, b[$ we have $0 \le L(J, t) \le L(]a, b[, t)$. An eventual pole of $L(J, t)$ at $1/\rho_1$ is therefore of order at most m so $\dfrac{L(J, t)}{L(]a, b[, t)}$ extends analytically to $t = 1/\rho_1$ (the

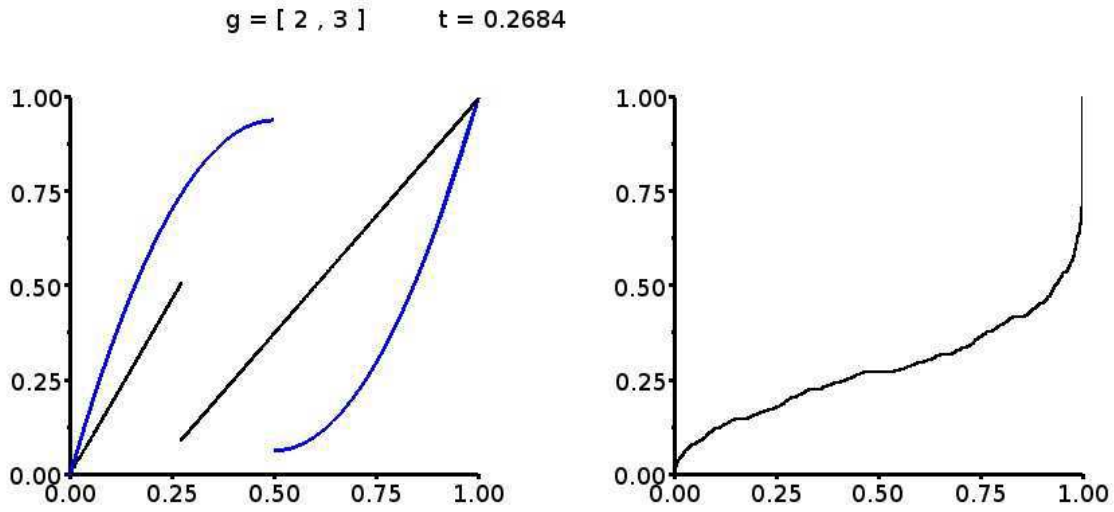

Figure 2: The same example as before but at the critical value: $t = 1/\rho_1 = 0.2684$. Note that Ω_t is no longer a Cantor set and that ϕ_t is continuous.

singularity is removable here). We denote the limit

$$\Lambda(J) := \lim_{t \nearrow 1/\rho_1} \frac{L(J,t)}{L(]a,b[,t)} \in [0,1]. \tag{32}$$

Lemma 5.3. *We have the following properties for* Λ*:*

1. *For any* $x \in I$ *:* $\Lambda(\{x\}) = 0$.

2. *For all* $\alpha \in Z_n$ *:* $\Lambda(\alpha) = \dfrac{1}{\rho_1^n}\, g^n_{|\alpha}\, \Lambda(f^n\alpha)$.

3. $\delta_n = \sup_{\alpha \in Z_n} \Lambda(\alpha) \xrightarrow[n\to\infty]{} 0$.

Proof. The expression (19) shows that the function $L(\{x\},t)$ is analytic on $\{|t| < 1/\rho_\infty\}$ in particular remains bounded on $\{|t| \le 1/\rho_1\}$. As $t \nearrow 1/\rho_1$, the denominator $L(]a,b[,t)$ diverges, the first claim follows.

For $J \subset I_j$ for some j we divide (23) by $L(]a,b[,t)$ and take the limit $t \nearrow 1/\rho_1$ to obtain:

$$\Lambda(J) = \frac{1}{\rho_1} g_j \Lambda(f_j J). \tag{33}$$

In particular, for $\alpha = (i_0 i_1 \cdots i_{n-1}) \in Z_n$ we have $\alpha \in I_{i_0}$ so that $\Lambda(\alpha) = \dfrac{1}{\rho_1} g_{i_0} \cdot \Lambda(f\alpha)$. Iterating this we get the formula.

The last claim follows from

$$\Lambda(\alpha) = \frac{1}{\rho_1^n}\, g_{|\alpha}^n\, \Lambda(f^n\alpha) \leq \frac{\rho_\infty^n}{\rho_1^n} \xrightarrow[n\to\infty]{\rho_1>\rho_\infty} 0.$$

□

Lemma 5.4. *The map $\phi : [a,b] \to [0,1]$ defined by*

$$\phi(x) = \Lambda(]a,x[),\ x \in [a,b]$$

is non-decreasing, continuous and surjective. One has for $x \in]a,b[$:

$$\phi(x) = \lim_{t\nearrow 1/\rho_1} \phi_t(x^-) = \lim_{t\nearrow 1/\rho_1} \phi_t(x^+) \tag{34}$$

Proof. Monotonicity follows from positivity and additivity of $\Lambda(J)$, $J \subset]a,b[$. Let $x \in [a,b]$ and $\varepsilon > 0$. Choose n so that $\delta_n < \varepsilon/2$ (δ_n from the previous lemma). Either x is inside some n-cylinder or on the boundary of two such cylinders. In any case, we may find at most two n-cylinders α_1, α_2 with $\overline{\alpha_1} \cap \overline{\alpha_2} = \{x'\}$ so that $J = \alpha_1 \cup \{x'\} \cup \alpha_2$ is an open neighborhood of x and $\Lambda(J) < \varepsilon$. For $\delta > 0$ small enough $\phi(x+\delta) - \phi(x-\delta) \leq \Lambda(J) < \varepsilon$. As $\phi(a) = \Lambda(\emptyset) = 0$ and $\phi(b) = 1$ the map is surjective. The first equality in (34) is essentially the definition of ϕ and the second follows from the continuity just shown. □

We write $\widetilde{c}_i = \phi(c_i)$, $i = 0, \cdots, \ell+1$ and let $\widetilde{S} \subset S := \{0, \cdots, \ell\}$ denote the (possibly strict) subset of indices i for which $0 < \widetilde{c}_{i+1} - \widetilde{c}_i = \Lambda(]c_i, c_{i+1}[)$. For $i \in \widetilde{S}$ we set $\widetilde{I}_i =]\widetilde{c}_i, \widetilde{c}_{i+1}[$.

Proof of Theorem 2.5.

Part A: For $\widehat{x}_1, \widehat{x}_2 \in I_j$, taking the limit $t \nearrow 1/\rho_1$ in the identity (26) yields

$$\phi(\widehat{x}_2) - \phi(\widehat{x}_1) = t s_j g_j \Big(\phi(f_j\widehat{x}_2) - \phi(f_j\widehat{x}_1) \Big). \tag{35}$$

Continuity of ϕ and f_j shows that this identity is independent of the direction of the point-germs. The affine map $\widetilde{f}_j(y) = \widehat{c}_j + \dfrac{s_j}{t g_j}(y - \widehat{c}_j)$ then satisfies the required identity.

Part B: Recall that Z_n consists of the non-empty n-cylinder for $(I_i, f_i)_{i\in S}$. Let $\widetilde{Z}_n$ be the collection of non-empty open intervals of the form $\widetilde{\alpha} = \operatorname{Int} \phi(\alpha)$ where $\alpha = (i_0 \cdots i_{n-1}) \in Z_n$. Here each $i_k \in \widetilde{S}$, $0 \leq k < n$ (or else $\widetilde{\alpha}$ à fortiori empty) and $\widetilde{f}^k\widetilde{\alpha} \subset \widetilde{I}_{i_k}$. Therefore $\widetilde{\alpha}$ is contained in an n-cylinder for the dynamical system $(\widetilde{I}_i, \widetilde{f}_i)_{i\in\widetilde{S}}$. We claim that $\widetilde{\alpha}$ is actually equal to an n-cylinder for that system and $\widetilde{Z}_n$ is precisely the set of non-empty n-cylinders for the same system. To see this note that

$$1 = \sum_{\alpha\in Z_n} \Lambda(\alpha) = \sum_{\widetilde{\alpha}\in\widetilde{Z}_n} |\widetilde{\alpha}|, \tag{36}$$

where $|\cdot|$ denotes the length of intervals. There is no room for any other or any larger open cylinder.

Now, by Lemma 5.3 we have $|\widetilde{\alpha}| = \Lambda(\alpha) = \dfrac{g^n_{|\alpha}}{\rho_1^n}\Lambda(f^n\alpha) \le \dfrac{g^n_{|\alpha}}{\rho_1^n}$. So using (36) we get

$$\rho_1^n = \sum_{\widetilde{\alpha}\in\widetilde{Z}_n} |\widetilde{\alpha}|\rho_1^n = \sum_{\widetilde{\alpha}\in\widetilde{Z}_n} \Lambda(\alpha)\rho_1^n = \sum_{\widetilde{\alpha}\in\widetilde{Z}_n} g^n_{|\alpha}\Lambda(f^n\alpha) \le \sum_{\widetilde{\alpha}\in\widetilde{Z}_n} g^n_{|\alpha} \le \sum_{\alpha\in Z_n} g^n_{|\alpha} = \|g^n\|_1.$$

So $\rho_1 = \limsup_{n\to\infty} \Big(\sum_{\widetilde{\alpha}\in\widetilde{Z}_n} g^n_{|\alpha}\Big)^{1/n}$. The pressures of $(I_i, f_i, g_i)_{i\in S}$ and $(\widetilde{I}_i, \widetilde{f}_i, g_i)_{i\in\widetilde{S}}$ are therefore the same.

Part C: We assume here that f extends to a continuous map of $[a,b]$. When $J \subset [a,b]$ is an interval then $fJ \setminus \bigcup_i f(J\cap I_i)$ consists of a finite number of points. By Lemma 5.3 this set difference has zero mass. By the same lemma we get: $\Lambda(J) = \sum_{i=0}^{\ell} \dfrac{1}{\rho_1} g_i \Lambda(f_i(J\cap I_i))$. Thus,

$$\Big(\min_i g_i \frac{1}{\rho_1}\Big)\Lambda(fJ) \le \Lambda(J) \le \Big(\sum_i g_i \frac{1}{\rho_1}\Big)\Lambda(fJ).$$

In particular $\Lambda(J) = 0 \iff \Lambda(fJ) = 0$. (Note, however, that $\Lambda(J) > 0$ does not imply $\Lambda(f^{-1}J) > 0$ as the latter set might be empty).

Let us write $x \sim x'$ if $\phi(x) = \phi(x')$,

When $x, x' \in I$ and $x \sim x'$ then $\Lambda([x,x']) = 0$ so also $\Lambda(f[x,x']) = 0$. As we have assumed f continuous, $f([x,x'])$ is connected and contains $f(x)$, $f(x')$. Therefore, $\phi(f(x)) = \phi(f(x'))$, i.e. $f(x) \sim f(x')$. For $y \in [0,1]$, we may thus define $\widetilde{f}(y) = \phi(f(x))$ with $x \in \phi^{-1}(y)$ (independent of the choice of x). Then for every $x \in [a,b]$ we have:

$$\widetilde{f}(\phi(x)) = \phi(f(x))$$

The same argument also shows that for any two $x, x' \in I$ we have

$$\Big|\widetilde{f}(\phi(x)) - \widetilde{f}(\phi(x'))\Big| \le \max_{i\in\widetilde{S}} \frac{\rho_1}{g_i} |\phi(x) - \phi(x')|$$

so $\widetilde{f}$ is a continuous endomorphism of $[0,1]$. □

Remark: The set $\widetilde{S}$ may depend upon the weights g_i. If, however, f is transitive then $\widetilde{S} = S$ for any choice of non-zero weights and $\widetilde{Z}_n = Z_n$ for all n. We leave the exercise to the reader.

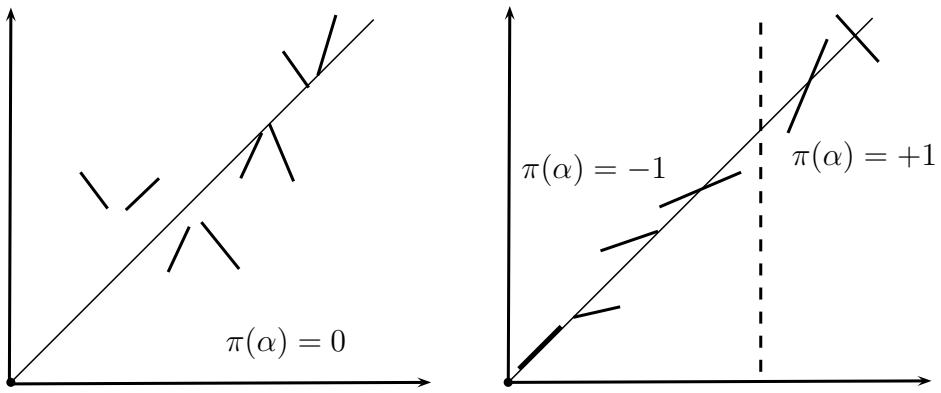

Figure 3: fixed points counting

A Geometry of the weight function $\omega(\alpha)$

Fix $n \geq 1$ and an n-cylinder $\alpha \in Z_n$. Recall that we have associated a weight

$$\omega(\alpha) = -\, g^n_{|\alpha} \sum_{\widehat{x} \in \widehat{\partial}\alpha} \sigma(f^n \widehat{x}, x) \cdot \varepsilon(f^n \widehat{x}).$$

Set

$$\pi(\alpha) := -\sum_{\widehat{x} \in \widehat{\partial}\alpha} \sigma(f^n \widehat{x}, x) \cdot \varepsilon(f^n \widehat{x}).$$

This quantity depends only on the boundary values and their positions relative to the diagonal. Let h be an affine map on α coinciding with f^n on the boundary.

Lemma A.1. $\pi(\alpha) = -\sum_{\widehat{x} \in \widehat{\partial}\alpha} \sigma(h(\widehat{x}), x) \cdot \varepsilon(h(\widehat{x}))$. *And,*

- $\pi(\alpha) = -1$ *if* $0 < \text{slope}(h) \leq 1$ *and* $\overline{h(\alpha)}$ *touches the diagonal*
- $\pi(\alpha) = 1$ *if* $h(\alpha)$ *transverses the diagonal with slope either* > 1 *or* < 0.
- $\pi(\alpha) = 0$ *in all other cases, namely*
  either $\overline{h(\alpha)}$ *does not touch the diagonal*
  or $\overline{h(\alpha)}$ *touches the diagonal at one end only, with slope* > 1 *or* < 0.

Proof. Since $f^n|_\alpha$ is a continuous strictly monotone map, we have $\sigma(f^n \widehat{x}, x) = \sigma(h(\widehat{x}), x)$ and $\varepsilon(f^n \widehat{x}) = \varepsilon(h(\widehat{x}))$ at the two ends of α. So we can replace f^n by h in $\pi(\alpha)$.

Extend h continuously to the boundary points. For $\widehat{x}$ a boundary germ. We check case by case the value of $-\sigma(h(\widehat{x}), x) \cdot \varepsilon(h(\widehat{x}))$

is $\dfrac{1}{2}$ if $h(x) < x$ and $h(\widehat{x}) > h(x)$, or if $h(x) > x$ and $h(\widehat{x}) < h(x)$

is $-\dfrac{1}{2}$ if $h(x) = x$, or $h(x) > x$ and $h(\widehat{x}) > h(x)$, or if $h(x) < x$ and $h(\widehat{x}) < h(x)$.

Adding the values at the two ends, we get the lemma. □

Notice that if f is expanding then $\pi(\alpha) \geq 0$ for all n and all $\alpha \in Z_n$. So in that case $N_n \geq 0$ for all n.

B Relation between $\det \mathcal{R}$, $\det \mathcal{B}$ and Milnor-Thurston's kneading determinant

We relate here our definition of the kneading determinant to that of Milnor-Thurston, modified by adding weights. Set $\theta(\widehat{x}, t; c_{\ell+1}) := -\theta(\widehat{x}, t; c_0)$

Lemma B.1. *(Minor-Thurston) We have* $\sum_{k=0}^{\ell}(1 - ts_k g_k)(\theta(\widehat{x}, t; c_k) - \theta(\widehat{x}, t; c_{k+1})) \equiv 1.$

For $k = 0, \cdots, \ell$, set $I_k =]c_k, c_{k+1}[$. Note that $\sigma(\widehat{x}, c_k) - \sigma(\widehat{x}, c_{k+1}) = \chi_{I_k}(\widehat{x})$. Set

$$\eta(\widehat{x}, t; I_k) := \theta(\widehat{x}, t; c_k) - \theta(\widehat{x}, t; c_{k+1}) = \sum_{m \geq 0} t^m [sg]^m(\widehat{x}) \chi_{I_k}(\widehat{x}).$$

$$\Delta_{c_i} \eta(\cdot, t; I_k) := \eta(c_i^+, t; , I_k) - \eta(c_i^-, t; I_k), \ i = 1, \cdots, \ell$$

And define the **Milnor-Thurston kneading matrix** $\ell \times (\ell + 1)$ matrix

$$\mathcal{N}(t) = \Big(\Delta_{c_i} \eta(\cdot, t; I_k)\Big)_{i=1,\cdots,\ell, \ k=0,1,\cdots,\ell}.$$

Denote by D_j the determinant of $\mathcal{N}(t)$ after deleting the j-th column.

Lemma B.2. *(Milnor-Thurston) The quantity* $\dfrac{(-1)^j D_j}{1 - s_j g_j t} =: D_{MT}(t)$ *is independent of* j *and is called the* **Milnor-Thurston kneading determinant**.

Proof. Let $\mathbf{v} = \begin{pmatrix} 1 - s_0 g_0 t \\ \vdots \\ 1 - s_\ell g_\ell t \end{pmatrix}$. By Lemma B.1, $\Big(\eta(\widehat{x}, t; I_0), \cdots, \eta(\widehat{x}, t; I_\ell)\Big)\mathbf{v} = 1$. So $\mathbf{v}$ is a kernel vector of $\mathcal{N}(t)$. Define an augmented kneading matrix $\mathcal{A}(t)$ by adding a line vector $(\dfrac{1}{1 - s_0 g_0 t}, \cdots, \dfrac{1}{1 - s_\ell g_\ell t})$ on top of $\mathcal{N}(t)$.

Then $\mathcal{A}\mathbf{v} = \begin{pmatrix} \ell + 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. By Cramer's solution form $1 - s_j g_j t = (\ell + 1)\dfrac{(-1)^i j D_j}{\det \mathcal{A}}$ and therefore $\dfrac{\det \mathcal{A}}{\ell + 1} = \dfrac{(-1)^j D_j}{1 - s_j g_j t}$ for all $j = 0, \cdots, \ell$. □

Lemma B.3. *Setting* $H(t) := 1 - t\dfrac{s_0 g_0 + s_\ell g_\ell}{2}$ *we have*

$$D_{MT}(t) = \det(\mathcal{R}(t)) \quad \textit{and} \quad H(t) \cdot \det \mathcal{R}(t) = \det \mathcal{B}(t).$$

Proof. Set $\kappa_i(t) = \dfrac{t}{2}(s_{i-1}g_{i-1} - s_i g_i)$, $i = 1, \cdots, \ell$. Grouping the terms about c_0 and $c_{\ell+1}$ in Lemma B.1 we get

$$2H(t) \cdot \theta(\widehat{x}, t; c_0) + \sum_{i=1}^{\ell} 2\kappa_i \cdot \theta(\widehat{x}, t; c_i) \equiv 1$$

It follows that

$$\begin{pmatrix} H(t) & \kappa_1(t) & \kappa_2(t) & \cdots & \kappa_\ell(t) \\ 0 & & & & \\ 0 & & & & \\ \vdots & & Id & & \\ 0 & & & & \end{pmatrix} \cdot \mathcal{R}(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & & & & \\ 1 & & & & \\ \vdots & & \mathcal{B} & & \\ 1 & & & & \end{pmatrix}$$

Therefore $H(t) \cdot \det \mathcal{R}(t) = \det \mathcal{B}(t)$. For the matrix $\mathcal{A}$ defined above,

$$\mathcal{A}(t) \cdot \frac{1}{2} \begin{pmatrix} 1 - s_0 g_0 t & -1 & -1 & \cdots & -1 \\ 1 - s_1 g_1 t & 1 & -1 & \cdots & -1 \\ 1 - s_2 g_2 t & 1 & 1 & \cdots & -1 \\ \vdots & & & & \\ 1 - s_\ell g_\ell t & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \frac{\ell+1}{2} & * & * & \cdots & * \\ 0 & & & & \\ 0 & & & & \\ \vdots & & \mathcal{B} & & \\ 0 & & & & \end{pmatrix}$$

In the second matrix on the left hand side add the last line to every other line one gets $\dfrac{\ell+1}{2} D_{MT}(t) \cdot H(t) = \dfrac{\det \mathcal{A}}{2} \cdot H(t) = \dfrac{\ell+1}{2} \det \mathcal{B}$.

Combining with Lemma B.3 we get $D_{MT}(t) \cdot H(t) = \det \mathcal{B} = \det \mathcal{R} \cdot H(t)$. Therefore $\det \mathcal{R}(t) = D_{MT}(t)$. □

Corollary B.4. *If all the weights* g_i *are equal to* 1*, all three determinants* D_{MT}*,* $\det \mathcal{R}$*,* $\det \mathcal{B}$ *have the same zeros in* $\mathbb{D}$*.*

Proof. In this case $H(t) = 1 - \dfrac{t}{2}(s_0 + s_\ell) = 1$ or $1 - t$ so $H(t)$ has no zeros in $\{|t| < 1/\rho_\infty\} = \mathbb{D}$. □

C The first zero of $\det \mathcal{B}$ may not correspond to the pressure

We have shown in Theorem 2.3, Point 3, that the first zero of $\det \mathcal{R}$ corresponds to the pressure. And in case all the weights g_i are 1, one can also use the first zero of $\det \mathcal{B}$

(Corollary B.4). This need not, however, be true with more general weights. Here is a counter example.

Let $I = [a, b] = [0, 3]$, $I_0 =]0, 1[$, $I_1 =]1, 2[$, $I_2 =]2, 3[$.

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 2 - 2(x-1) & 1 \leq x \leq 2 \\ 2(x-2) & 2 \leq x \leq 3 \end{cases}$$

Let us assign weights $g_0 = g_1 = 1$ and $g_2 = M$.

Note that $f(I_2) = [0, 2]$ and $f : [0, 2] \to [0, 2]$ is the full tent map. There is no periodic points in I_2. Using Lemma A.1 and the definition one obtains

$$Z(t) = \exp\Big(\sum_{n\geq 1} \frac{t^n}{n} 2^n\Big) = (1-2t)^{-1}.$$

So by Lemma B.3 and Theorem 2.2 we have $D_{MT}(t) = \det \mathcal{R}(t) = \dfrac{1}{Z(t)} = 1-2t$. The first zero being $1/2$ one obtains that the pressure is $\log 2$ (this pressure can also be computed directly). It is easily seen that the topological entropy is also $\log 2$.

On the other hand, $H(t) = 1 - \dfrac{t}{2}(s_0 g_0 + s_2 g_2) = 1 - \dfrac{t}{2}(1 + M)$. So by Lemma B.3 again

$$\det \mathcal{B}(t) = H(t) \det \mathcal{R}(t) = \Big(1 - \frac{t}{2}(1+M)\Big)(1-2t).$$

If $M > 3$, then $\det \mathcal{B}(t)$ has a 'spurious' zero at $\dfrac{2}{1+M}$ smaller than $\dfrac{1}{2}$.

So the first positive zero of $\mathcal{B}(t)$ does not correspond to the pressure in this case. By increasing M, one can make this first zero arbitrarily small without changing the pressure.

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