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OCTONION ALGEBRAS OVER RINGS ARE NOT DETERMINED BY THEIR NORMS

PHILIPPE GILLE

Résumé: Répondant à une question de H. Petersson, nous contruisons une classe d'exemples de paires d'algèbres d'octonions définies sur un anneau ayant des normes isométriques.

Abstract: Answering a question of H. Petersson, we provide a class of examples of pair of octonion algebras over a ring having isometric norms.

Keywords: Octonion algebras, torsors, descent.

MSC: 14L24, 20G41.

1. Introduction

If Q is a quaternion algebra over a field k, we know from Witt that Q is determined by its norm [15, §1.7]. This result has been extended over rings by Knus-Ojanguren-Sridharan ([10, prop. 4.4] , [9, V.4.3.2]) and holds actually over an arbitrary base (§2).

If C is a octonion algebra over k, we know from van der Blij-Springer that it is determined by its norm form [1, claim 2.3] (see also [15, §1.7]); more generally it is true over local rings (Bix, [2, lemma 1.1]). In his Lens lecture (May 21-25, 2012), H. Petersson raised the question whether it remains true over arbitrary commutative rings.

The goal of this note is to produce a counterexample to this question, namely an example of two non-isomorphic octonions algebras over some commutative ring R having isometric norms. Our argument is based on the study of fibrations of group schemes and uses topological fibrations which makes clear why it holds for quaternion algebras and not for octonions.

For the theory of reductive group schemes and related objects (e.g. Lie algebra sheaves, homogeneous spaces, quadratic spaces, ...) we refer to SGA3 [14] and to the book by Demazure-Gabriel [5]. The sheaves in sets or groups are denoted as $\underline{F}$ and are for the fppf (also called flat) topology over a base scheme S.

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2. Quaternion algebras and norms

Let S be a scheme. By a quaternion[1] algebra over S, we mean a rank 4 Azumaya $\mathcal{O}_S$–algebra $\mathcal{Q}$. Equivalently, it is an étale S–form of the matrix algebra $M_2(\mathcal{O}_S)$, namely the twist of $M_2(\mathcal{O}_S)$ by the PGL_2–torsor $E = \underline{\mathrm{Isom}}_{alg}(M_2(\mathcal{O}_S), \mathcal{Q})$.

By descent, it follows that isomorphism classes of quaternion S-algebras correspond to the étale cohomology set $H^1(S, \mathrm{PGL}_2)$.

The reduced norm (resp. trace) $\mathrm{Nrd} : \mathcal{Q} \to \mathcal{O}_S$ (resp: Trd) is the twist by E of the determinant map $M_2(\mathcal{O}_S) \to \mathcal{O}_S$ (resp. the trace), it is a quadratic (resp. linear) form over S.

Furthermore the canonical involution $X \mapsto \mathrm{tr}(X) - X$ on $M_2(\mathcal{O}_S)$ induces by descent the canonical involution of $\mathcal{Q}$.

The S–group scheme $\mathrm{SL}_1(\mathcal{Q})$ (resp. $\mathrm{PGL}_2(\mathcal{Q})$, $\mathrm{SO}(\mathcal{Q}, N_{\mathcal{Q}})$) is the twist by E of SL_2/S (resp. PGL_2/S, $\mathrm{SO}(M_2, \det)/S$).

The point is that the semisimple group scheme $\mathrm{SO}(\mathcal{Q}, N_{\mathcal{Q}})$ is of type $A_1 \times A_1$ and its universal cover is $\mathrm{SL}_1(\mathcal{Q}) \times \mathrm{SL}_1(\mathcal{Q})$.

2.1. **Lemma.** *We have an exact sequence of group schemes*

$$1 \to \mu_2 \to \mathrm{SL}_1(\mathcal{Q}) \times \mathrm{SL}_1(\mathcal{Q}) \xrightarrow{f} \mathrm{SO}(\mathcal{Q}, N_{\mathcal{Q}}) \to 1$$

where $f(x, y).q = xq\,y^{-1}$ *for every* $q \in \mathcal{Q}$.

Proof. We do first the case of $S = \mathrm{Spec}(\mathbb{Z})$ and $\mathcal{Q} = M_2(\mathbb{Z})$. We have $\mu_2 \subset \ker(f)$ and let us show the converse inclusion. Let R be a ring and pick $(x, y) \in \ker(f)(R)$. A such element satisfies $xAy^{-1} = A$ for each $A \in M_2(R)$. By taking $A = y$, we see that $x = y$ so that $xAx^{-1} = A$ for each $A \in M_2(R)$. By taking the canonical R-basis of $M_2(R)$, it follows that $x \in \mathbb{G}_m(R)$. Since $x \in \mathrm{SL}_2(R)$, we conclude that $(x, y) \in \mu_2(R)$. Thus $\mu_2 = \ker(f)$.

Since μ_2 is a central subgroup of $\mathrm{SL}_2 \times_R \mathrm{SL}_2$, we can mod out by μ_2 [14, XXII.4.3] and get a monomorphism $\widetilde{f} : (\mathrm{SL}_2 \times_R \mathrm{SL}_2)/\mu_2 \to \mathrm{SO}(M_2, det)$ of semisimple group schemes. According to [14, XVI.1.5.a], it is a closed immersion. On both sides, each $\mathbb{Q}$-fiber is smooth connected of dimension 6. It follows that $\widetilde{f}_{\mathbb{Q}}$ is an isomorphism. Since $\mathrm{SO}(M_2, det)$ is flat over $\mathbb{Z}$, we conclude that $\widetilde{f}$ is an isomorphism.

The general case follows again by twisting everything by the PGL_2–torsor E.

□

The adjoint map $Ad : \mathrm{PGL}_2 \to \mathrm{GL}(M_2)$ gives rise to the closed S–immersion $\mathrm{PGL}_2 \to \mathrm{O}(M_2, \det)$ where $\mathrm{O}(M_2, \det)$ stands for the orthogonal group scheme of the nonsingular quadratic form det [5, III.5.2]. It is equipped with the Dickson map $\mathrm{D} : \mathrm{O}(M_2, \det) \to \mathbb{Z}/2\mathbb{Z}$ whose kernel is by definition the special linear group $\mathrm{SO}(M_2, \det)$.

[1] Knus' definition requests less conditions [9, 1.3.7]; we deal here then with "separable quaternions algebras".

By twisting by the torsor E, it provides a closed S-immersion

$$Ad : \mathrm{PGL}_1(\mathcal{Q}) \to \mathrm{O}(\mathcal{Q}, \mathrm{Nrd}), q \mapsto Ad(q)$$

where $\mathrm{PGL}_1(\mathcal{Q})$ stands for the group scheme $\mathrm{GL}_1(\mathcal{Q})/\mathbb{G}_m$ of projective units.

On the other hand, the orthogonal S–group $\mathrm{O}(\mathcal{Q}, \mathrm{Nrd})$ acts on $\mathrm{SL}_1(\mathcal{Q}) = \mathrm{Ker}(\mathrm{GL}_1(\mathcal{Q})) \to \mathbb{G}_m)$ by the action induced from the standard action of $\mathrm{GL}_1(\mathcal{Q})$ on $\mathcal{Q}$.

2.2. **Proposition.** *(1) The S–scheme $\mathrm{SL}_1(\mathcal{Q})$ is a left homogeneous space (with respect to the flat topology) under the action of $\mathrm{SO}(\mathcal{Q}, \mathrm{Nrd})$ and a fortiori under the action of $\mathrm{O}(\mathcal{Q}, \mathrm{Nrd})$.*

(2) The orbit map

$$u : \mathrm{SO}(\mathcal{Q}, \mathrm{Nrd}) \to \mathrm{SL}_1(\mathcal{Q}), \quad g \mapsto g.1$$

is a split $\mathrm{PGL}_1(\mathcal{Q})$–torsor.

Proof. We put $G/S = \mathrm{SO}(\mathcal{Q}, \mathrm{Nrd})$, $H/S = \mathrm{PGL}_1(\mathcal{Q})$ and $X/S = \mathrm{SL}_1(\mathcal{Q})$.

(1) We have to check the definition [14, IV.6.7], namely to establish the properties

(a) the map $G \times_S X \to X \times_S X$, $(g, x) \mapsto (x, g.x)$ is an epimorphism of flat sheaves;

(b) $f : X \to S$ has sections locally with respect to the flat topology.

The condition (b) is obvious in our case since f has a global section given by the unit of $X = \mathrm{SL}_1(\mathcal{Q})$. Condition (a) will follow of the following stronger condition:

(c) $X(T)$ is homogeneous over $G(T)$ for each S-scheme T.

We are given T/S and a couple of quaternions $q_1, q_2 \in X(T)$ of reduced norm one. We put $q = q_2\, q_1^{-1} \in X(T)$. The left translation L_q is an element of $G(T)$ which satisfies $L_q.q_1 = q_2$. This shows (c).

(2) The map $u \circ f : \mathrm{SL}_1(\mathcal{Q}) \times \mathrm{SL}_1(\mathcal{Q}) \to \mathrm{SL}_1(\mathcal{Q})$ reads as follows: $(u \circ f)(x, y) = xy^{-1}$. Therefore $\mathrm{SL}_1(\mathcal{Q}) \times_S \mathrm{SL}_1(\mathcal{Q})/\mathrm{SL}_1(\mathcal{Q}) \xrightarrow{\sim} \mathrm{SL}_1(\mathcal{Q})$ where $\mathrm{SL}_1(\mathcal{Q})$ acts on $\mathrm{SL}_1(\mathcal{Q}) \times_S \mathrm{SL}_1(\mathcal{Q})$ by $z.(x, y) = (x\, z, z^{-1}\, x)$. By modding out by the diagonal μ_2 of $\mathrm{SL}_1(\mathcal{Q}) \times_S \mathrm{SL}_1(\mathcal{Q})$, we get an isomorphism of flat sheaves

$$\mathrm{SO}(\mathcal{Q}, \mathrm{Nrd})/\mathrm{PGL}_1(\mathcal{Q}) \xrightarrow{\sim} \mathrm{SL}_1(\mathcal{Q})$$

where $\mathrm{PGL}_1(\mathcal{Q})$ embeds by h in $\mathrm{SO}(\mathcal{Q}, \mathrm{Nrd})$. □

2.3. **Lemma.** $\mathrm{O}(\mathcal{Q}, \mathrm{Nrd}) = \mathrm{SO}(\mathcal{Q}, \mathrm{Nrd}) \times_S \mathbb{Z}/2\mathbb{Z}$ *where $\mathbb{Z}/2\mathbb{Z}$ is the S-subgroup $\mathrm{O}(\mathrm{Nrd})$ defined by the canonical involution.*

Proof. We have to show that the Dickson map $\mathrm{D} : \mathrm{O}(\mathcal{Q}, \mathrm{Nrd}) \to \mathbb{Z}/2\mathbb{Z}$ is split by applying 1 to the canonical involution. To check that the Dickson invariant of the canonical involution is 1, we can reason étale locally, that is to check it for each strict henselization $O^{sh}_{S,s}$ where s is a point of S. In particular, it enables us to assume that $\mathcal{Q}$ is the split quaternion algebra which is defined over $\mathbb{Z}$.

We can then deal with $S = \mathrm{Spec}(\mathbb{Z})$ and $\mathcal{Q} = M_2(\mathbb{Z})$ and it remains to show that $\mathrm{D}(\sigma) = 1$ where σ is the canonical involution of $M_2(\mathbb{Z})$. It is enough to check it over $\mathbb{Q}$ and then the Dickson invariant is nothing but the determinant by means of the identification $(\mathbb{Z}/2\mathbb{Z})_{\mathbb{Q}} \cong \mu_{2,\mathbb{Q}}$ [5, III.5.2.6]. The basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

of $M_2(\mathbb{Q})$ is a diagonalization basis for σ whose eigenvalues are $1, -1, -1, -1$. The determinant of σ is then -1, as desired.

□

If follows that we have an isomorphism of homogeneous $\mathrm{O}(\mathcal{Q}, \mathrm{Nrd})$-spaces $\mathrm{O}(\mathcal{Q}, \mathrm{Nrd})/(\mathrm{PGL}_1(\mathcal{Q}) \times_S \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathrm{SL}_1(\mathcal{Q})$.

2.4. **Theorem.** *Let $\mathcal{Q}'$ be a $\mathcal{O}_S$-quaternion algebra. Then $\mathcal{Q}'$ is isomorphic to $\mathcal{Q}$ if and only if the quadratic S–form* Nrd *and* Nrd$'$ *are isometric.*

Proof. Since $H^1(S, \mathrm{PGL}_1(\mathcal{Q}))$ classifies S–quaternion algebras and $H^1(S, \mathrm{O}(\mathcal{Q}, \mathrm{Nrd}))$ classifies the isometry classes of nonsingular quadratic forms of dimension 4, it follows that the kernel of the map

$$Ad_* : H^1(S, \mathrm{PGL}_1(\mathcal{Q})) \to H^1(S, \mathrm{O}(\mathcal{Q}, \mathrm{Nrd}))$$

classifies the isomorphism classes of quaternions S–algebras such that the quadratic S–form Nrd and Nrd$'$ are isometric. By applying [7, III.3.2.2] to the isomorphism $\mathrm{O}(\mathcal{Q}, \mathrm{Nrd})/(\mathrm{PGL}_1(\mathcal{Q}) \times_S \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} \mathrm{SL}_1(\mathcal{Q})$, we get an exact sequence of pointed sets

$$\mathrm{O}(\mathcal{Q}, \mathrm{Nrd})(S) \xrightarrow{f} \mathrm{SL}_1(\mathcal{Q})(S) \to H^1(S, \mathrm{PGL}_1(\mathcal{Q}) \times_S \mathbb{Z}/2\mathbb{Z}) \to H^1(S, \mathrm{O}(\mathcal{Q}, \mathrm{Nrd})).$$

By proposition 2.2, the map f admits a retraction so that the kernel of $H^1(S, \mathrm{PGL}_1(\mathcal{Q}) \times_S \mathbb{Z}/2\mathbb{Z}) \to H^1(S, \mathrm{O}(\mathcal{Q}, \mathrm{Nrd}))$ is trivial. A fortiori, the kernel of $H^1(S, \mathrm{PGL}_1(\mathcal{Q})) \to H^1(S, \mathrm{O}(\mathcal{Q}, \mathrm{Nrd}))$ is trivial, as desired. □

2.5. **Remark.** Knus-Ojanguren-Sridharan's proof uses the even Clifford algebra of the norm forms to encode the algebra. Somehow we use also the Clifford algebra by means of the Dickson invariant which is in the case related to the fact that the simply connected cover of $\mathrm{SO}(\mathcal{Q}, N_{\mathcal{Q}})$ is $\mathrm{SL}_1(\mathcal{Q}) \times_S \mathrm{SL}_1(\mathcal{Q})$.

3. Octonion algebras and norms

Let R be a commutative ring (with unit). From §4 of [11], a non-associative algebra C over R is called an octonion R-algebra[2] if it is a finitely generated projective R–module of rank 8, contains an identity element 1_C and admits a norm, i.e. a map $n_C : C \to R$ satisfying the two following conditions:

(1) n_C is a nonsingular quadratic form;

(2) $n_C(xy) = n_C(x)\, n_C(y)$ for all $x, y \in C$.

This notion is stable under base extension and descends under faithfully flat base change of rings.

The basic example of an octonion algebra is the split octonion algebra (*ibid*, 4.2) denoted C_0 and called the algebra of Zorn vector matrices, which is defined over $\mathbb{Z}$. There is another description of this algebra in §1.8 of [15] over fields by the "doubling process". It actually works over $\mathbb{Z}$, we take

$$C_0' = M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z})$$

with multiplication law $(x, y).(u.v) = (x\,u + v\,\sigma(y), \sigma(x)v + u\,y)$ (σ is the canonical involution of $M_2(\mathbb{Z})$) and norm $n_{C_0'}(x, y) = \det(x) - \det(y)$. We know that the fppf $\mathbb{Z}$–group sheaf $\underline{\mathrm{Aut}}(C_0) \cong \underline{\mathrm{Aut}}(C_0')$ is representable by an affine smooth group $\mathbb{Z}$–scheme $\mathrm{Aut}(C_0)$ [11, 4.10].

3.1. **Proposition.** *The $\mathbb{Z}$–group scheme* $\mathrm{Aut}(C_0)$ *is the Chevalley group of type G_2.*

Proof. Let us first show that $\mathrm{Aut}(C_0)$ is a semisimple group scheme of type G_2, that is by definition a smooth affine group scheme whose geometrical fibers are semisimple groups of type G_2 [14, XIX].

The fibers of the affine smooth group $\mathbb{Z}$–scheme $\mathrm{Aut}(C_0)$ are indeed semisimple groups of type G_2 according to theorem 2.3.5 of [15]. Hence $\mathrm{Aut}(C_0)$ is a semisimple group scheme of type G_2. By Demazure's unicity theorem [14, cor. 5.5] the Chevalley group of type G_2 is the unique split semisimple group scheme of type G_2, that is the unique semisimple group scheme of type G_2 admitting a split torus of rank two. Since $\mathrm{PGL}_2 \times \mathrm{PGL}_2$ embeds in $\mathrm{Aut}(C_0')$, $\mathrm{Aut}(C_0')$ contains a two dimensional split torus. Thus $\mathrm{Aut}(C_0) \cong \mathrm{Aut}(C_0')$ is the Chevalley group of type G_2. □

We come now to the question whether an octonion algebra is determined by its norm. Let C be an octonion algebra over R. We have natural closed group embeddings of group schemes

$$\mathrm{Aut}(C) \xrightarrow{j} \mathrm{O}(n_C) \subset \mathrm{GL}(C).$$

We get a map in cohomology

$$j_* : H^1(R, \mathrm{Aut}(C)) \to H^1(R, \mathrm{O}(n_C)).$$

[2]One can of course globalize this definition, see [13].

The left handside classifies octonion algebras over R while the right handside classifies 8-dimensional nonsingular quadratic R–forms. By descent, we have $j_*([C']) = [n_{C'}]$ for each octonion R–algebra C'. It follows that the kernel of j_* classifies the octonion algebras over R whose norm form is isometric to n_C.

3.2. **Lemma.** *The fppf quotient* $\mathrm{O}(n_C)/\operatorname{Aut}(C)$ *is representable by an affine scheme of finite presentation over* R.

Proof. According to [4, 6.12], the fppf quotient $\mathrm{GL}(C)/\operatorname{Aut}(C)$ is representable by an affine scheme of finite type over R. It is of finite presentation over R by the standard limit argument [14, VI_B.10.2]. In the other hand, the fppf sheaf $\mathrm{GL}(C)/\mathrm{O}(n_C)$ is representable by an affine scheme of finite presentation over R [16, lemme 2.26]. Therefore the "kernel" $\mathrm{O}(n_C)/\operatorname{Aut}(C)$ of the natural map $\mathrm{GL}(C)/\operatorname{Aut}(C) \to \mathrm{GL}(C)/\mathrm{O}(n_C)$ is representable by an affine scheme of finite type. □

We denote by $A(C)$ the coordinate ring of the affine scheme $\mathrm{O}(n_C)/\operatorname{Aut}(C)$.

3.3. **Theorem.** *Assume that* R *is a non trivial* $\mathbb{Q}$*-ring. Then the* $\operatorname{Aut}(C)$*-torsor* $\mathrm{O}(n_C) \to \operatorname{Spec}(A(C))$ *is not trivial, so that* $\ker(j_{*,A(C)})$ *is not trivial.*

3.4. **Remark.** By inspection of the proof, the result holds also for $\mathrm{SO}(n_C) \to \mathrm{SO}(n_C)/\operatorname{Aut}(C)$. If $R = \mathbb{C}$, it provides then a counterexample over a connected smooth complex affine variety.

Let us do first a special case.

3.5. **Proposition.** *Let* $C/\mathbb{R}$ *be the "compact" Cayley octonion algebra. Then theorem 3.3 holds is this case.*

Proof. In this case $G = \operatorname{Aut}(C)/\mathbb{R}$ is the anisotropic real form of G_2 and we consider its embedding in the "compact" O_8. We reason by contradiction assuming that the G–torsor $\mathrm{O}_8 \to \mathrm{O}_8/G$ is split. It follows that there is a G-equivariant isomorphism $\mathrm{O}_8 \cong \mathrm{O}_8/G \times_{\mathbb{R}} G$ over O_8/G. Hence the map $G \to \mathrm{O}_8$ admits a section. Taking the real points, it follows that the map $G(\mathbb{R}) \to \mathrm{O}_8(\mathbb{R})$ admits a continuous section, hence the homotopy group $\pi_n(G(\mathbb{R}), \bullet)$ is a direct summand of $\pi_n(\mathrm{O}_8(\mathbb{R}), \bullet)$ for all $n \geq 1$. From the tables [12, p. 970], we have $\pi_6(G(\mathbb{R}), \bullet) \cong \mathbb{Z}/3\mathbb{Z}$ and $\pi_n(\mathrm{O}_8(\mathbb{R}), \bullet) = \pi_n(\mathrm{SO}_8(\mathbb{R}), \bullet) = 0$, hence a contradiction.

□

We can proceed to the proof of theorem 3.3.

Proof. We claim that the above counterexample survives when extending the scalars to $\mathbb{C}$. According to the Cartan decomposition, there are homomeorphisms $G(\mathbb{C}) \cong G(\mathbb{R}) \times \mathbb{R}^m$ and $\mathrm{O}_8(\mathbb{C}) \cong \mathrm{O}_8(\mathbb{R}) \times \mathbb{R}^n$ Hence $\pi_6(G(\mathbb{C}), \bullet) = \mathbb{Z}/3\mathbb{Z}$ and does not inject in $\pi_6(\mathrm{O}_8(\mathbb{C}), \bullet) = 0$.

In other words, theorem 3.3 holds for the case $R = \operatorname{Spec}(\mathbb{C})$ and $C = C_0$. It holds over $\overline{\mathbb{Q}}$ and over an arbitrary algebraically closed field of characteristic zero.

For the general case, we consider a morphism $R \to F$ where F is an algebraically closed field. Since the $\mathrm{Aut}(C)_F$-torsor $\mathrm{O}(n_C)_F \to \mathrm{O}(n_C)_F/\mathrm{Aut}(C)_F$ is not split, it follows that the $\mathrm{Aut}(C)_F$-torsor $\mathrm{O}(n_C) \to \mathrm{O}(n_C)/\mathrm{Aut}(C)$ is not split. □

Concluding remarks. (1) The rings occuring in the examples are of dimension 14. The next question is to know the minimal dimension for the counterexamples. M. Brion communicates us a smaller example, say over the complex numbers. Since the action of map G_2 on the complex octonions C preserves 1_C and the octonions of trace 0, the map $G_2 \to \mathrm{SO}_8$ takes value in $\mathrm{SO}_7 \subset \mathrm{SO}_8$. A fortiori the G_2–torsor $\mathrm{SO}_7 \to \mathrm{SO}_7/G_2 = \mathrm{Spec}(B)$ provides an example of a non trivial octonion algebra over B having trivial norm. The dimension of B is then 7. Also the homogeneous space SO_7/G_2 occurs as the complement of a smooth quadric in $\mathbb{P}^7$. Let us explain this geometric fact. Firstly the map $G_2 \to \mathrm{SO}_7$ lifts in $G_2 \to \mathrm{Spin}_7$. The spinorial action of Spin_7 on $\mathbb{C}^7$ has been investigated by Igusa [8, prop. 4]. The Spin_7–orbits in $\mathbb{C}^7$ are 0, the orbit of a vector of highest weight and a one parameter family of closed orbits with stabilizers G_2, defined by an equation $g(x) = t$ where g is an invariant quadratic form. It follows that the induced action of SO_7 on the projective space $\mathbb{P}^7$ has two orbits, one open SO_7/G_2 and one closed which is a smooth projective quadric.

(2) For the ring $\mathbb{Z}$, van der Blij-Springer showed that there are only two octonions algebras and having distinct norm forms [1, §4] (see also [3]). Hence octonion algebras over $\mathbb{Z}$ are determined by their norms. For other rings of integers, it seems to be an open question.

References

[1] F. van der Blij, T. A. Springer, *The arithmetics of octaves and of the group G_2* Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math. **21** (1959), 406–418.

[2] R. Bix, *Isomorphism theorems for octonion planes over local rings*, Trans. Amer. Math. Soc. **266** (1981), 423–439.

[3] B. Conrad, B. Gross, *Non-split reductive groups over $\mathbb{Z}$*, preprint (2011).

[4] J.–L. Colliot–Thélène, J.–J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Annalen **244** (1979), 105–134.

[5] M. Demazure, P. Gabriel, *Groupes algébriques*, North-Holland (1970).

[6] P. Gille and T. Szamuely, *Central simple algebras and Galois cohomology*, Cambridge Studies in Advanced Mathematics **101** (2006), Cambridge University Press.

[7] J. Giraud, *Cohomologie non-abélienne*, Springer (1970).

[8] J.-I. Igusa, *A classification of spinors up to dimension twelve*, Amer. J. Math. **92** (1970), 997–1028.

[9] M.-A. Knus, *Quadratic and hermitian forms over rings*, Grundlehren der mathematischen Wissenschaften **294** (1991), Springer.

[10] M.-A. Knus, M. Ojanguren, R. Sridharan, *Quadratic forms and Azumaya algebras*, J. reine angew. math. **303/304** (1978), 231–248.

[11] O. Loos, H.P Petersson, M.L. Racine, *Inner derivations of alternative algebras over commutative rings*, Algebra and Number Theory **2** (2008), 927–968.

[12] M. Mimura, *Homotopy theory of Lie groups*, Handbook of algebraic topology, North-Holland, Amsterdam (1995), 951–991.

[13] H. Petersson, *Composition algebras over algebraic curves of genus zero*, Trans. Amer. Math. Soc. **337** (1993), 473–493.

[14] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).

[15] T.A. Springer, F.D. Veldkamp, *Octonions algebras, Jordan algebras and exceptional groups*, Springer Monographs in Mathematics (2000).

[16] A. Steinmetz-Zikesch, *Algèbres de Lie de dimension infinie et théorie de la descente*, à paraître aux Mémoires de la Société Mathématique de France.

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